THE ÅRHUS INTEGRAL OF RATIONAL HOMOLOGY 3-SPHERES III: RELATION WITH THE LE MURAKAMI OHTSUKI INVARIANT

DROR BAR-NATAN, STAVROS GAROUFALIDIS, LEV ROZANSKY, AND DYLAN P. THURSTON

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Abstract. Continuing the work started in [Å-I] and [Å-II], we prove the relationship between the Århus integral and the invariant Ω (henceforth called LMO) defined by T.Q.T. Le, J. Murakami and T. Ohtsuki in [LMO]. The basic reason for the relationship is that both constructions afford an interpretation as "integrated holonomies". In the case of the Århus integral, this interpretation was the basis for everything we did in [Å-I] and [Å-II]. The main tool we used there was "formal Gaussian integration". For the case of the LMO invariant, we develop an interpretation of a key ingredient, the map j_m, as "formal negative dimensional integration". The relation between the two constructions is then an immediate corollary of the relationship between the two integration theories.

Contents

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1. Introduction

This paper is the third in a four-part series on "the Århus integral of rational homology 3-spheres". In Part I of this series [Å-I], we gave the definition of a diagram-valued invariant $\AA$ of rational homology spheres.[1] In Part II ([Å-II]) we proved that $\AA$ is a well-defined invariant of rational homology 3-spheres and that it is universal in the class of finite type invariants of integral homology spheres. In this paper we show that $\AA$, when defined, is essentially equal to the invariant LMO defined earlier by Le, Murakami and Ohtsuki in [LMO].

Both invariants LMO and $\AA$ take values in the space $\mathcal{A}(\emptyset)$, the completed graded space of manifold diagrams modulo the AS and IHX relations, as defined in detail in [Å-I, Definition 2.3] and recalled briefly in Figure 1; the precise statement of their near-equality is as follows:

Theorem 1. (Proof in Section 5) *Let M be a rational homology sphere and let $|H^1(M)|$ denote the number of elements in its first cohomology group (over $\mathbb{Z}$). Then*

$$\AA(M) = |H^1(M)|^{-\deg}\mathrm{LMO}(M), \tag{1}$$

[1]A precise definition of $\AA$ appears in [Å-I]. It is a good idea to have [Å-I] as well as [LMO] handy while reading this paper, as many of the definitions introduced and explained in those articles will only be repeated here in a very brief manner.

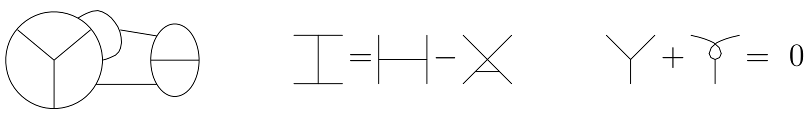


Figure 1. A connected manifold diagram of degree 6 (half the number of vertices in it) and the IHX and AS relations.

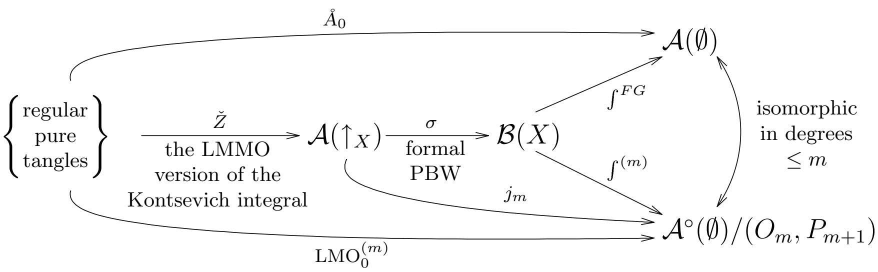


Figure 2. The main ingredients in the definitions of LMO_0 and of $\AA_0$: The top half of this diagram is the definition of $\AA_0$. The bottom half shows the two definitions of LMO_0 — the original involving j_m and our variant using $\int^{(m)}$; the two are equivalent by Lemma 1.1. Finally, the triangle on the right is nearly commutative, as detailed in Proposition 1.3.

where in the graded space $\mathcal{A}(\emptyset)$, $|H^1(M)|^{-\deg}$ denotes the operation that multiplies any degree m element by $|H^1(M)|^{-m}$.

In particular, if M is an integral homology sphere (that is, if $|H^1(M)| = 1$) then simply $\AA(M) = \mathrm{LMO}(M)$. Also note that Equation (1) implies that $\AA(M) = \hat{\Omega}(M)$, where $\hat{\Omega}$ is the invariant defined in [LMO, Section 6.2].

The definitions of LMO and $\AA$ are very similar. Let us trace this similarity to where the two definitions diverge, which is of course the key point of our paper, for it is at that point that we have to prove something non-trivial. In reading the following paragraphs on the similarity and differences between the definitions of LMO and $\AA$, the reader may find it helpful to refer to Figure 2 which summarizes the maps and the spaces involved.

The definitions of LMO and $\mathring{A}$ start with the definitions of their "pre-normalized" versions LMO_0 and $\mathring{A}_0$, which are invariants of regular links (framed links having a non-degenerate linking matrix) that are also invariant under the second Kirby move and both use the same renormalization procedure to ensure invariance also under the first Kirby move. The resulting LMO and $\mathring{A}$ are therefore invariants of rational homology spheres presented by surgery over regular links.

The definitions of LMO_0 and $\mathring{A}_0$ are also similar. For both it is beneficial to start with regular pure tangles (framed pure tangles having a non-degenerate linking matrix, see [Å-I, Definition 2.2]) rather than with regular links. By closure, every regular pure tangle defines a regular link and every regular link is obtained in this way. Both LMO_0 and $\mathring{A}_0$ descend from invariants of regular pure tangles to invariants of links; LMO_0 is defined that way in [LMO] to start with, and for $\mathring{A}_0$ it is shown in [Å-II, Section 3.1].

Both LMO_0 and $\mathring{A}_0$ are defined as compositions of several maps. In both cases the first map is $\check{Z}$, the Kontsevich integral in its Le Murakami Murakami Ohtsuki [LMMO] normalization. If X is the set of components of a given regular pure tangle (or more elegantly, a set of "labels" or "colors" for these components), the first map $\check{Z}$ takes its values in $\mathcal{A}(\uparrow_X)$, the completed graded space of chord diagrams for X-labeled pure tangles (modulo the usual $4T/STU$ relations; see a precise definition in [Å-I, Definition 2.4] and a brief reminder in Figure 3). Let us recall the definition of $\check{Z}$:

> **Definition 2.6 of** [Å-I]**.** (condensed) The map $\check{Z}$ is the usual framed version of the Kontsevich integral Z, normalized in a funny way. Namely, let $\nu = Z(\bigcirc) \in \mathcal{A}$ be the Kontsevich integral of the unknot, and let $\Delta_X : \mathcal{A} \to \mathcal{A}(\uparrow_X)$ be the "X-cabling" map that replaces the single directed line in $\mathcal{A}$ by n directed lines labeled by the elements of X, and sums over all possible ways of lifting each vertex on the directed line to its n clones. Set
>
> $$\check{Z}(T) = \nu^{\otimes n} \cdot \Delta_X(\nu) \cdot Z(L)$$

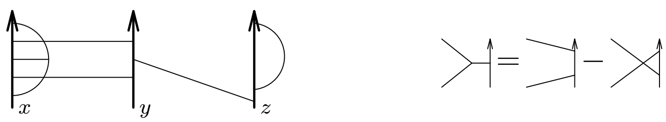


Figure 3. A diagram in $\mathcal{A}(\uparrow_{\{x,y,z\}})$ and the STU relation.

for any X-marked framed pure tangle L, using the action of $\mathcal{A}^{\otimes n}$ on $\mathcal{A}(\uparrow_X)$ defined by sticking any n diagrams in $\mathcal{A}$ on the n components of the skeleton of a diagram in $\mathcal{A}(\uparrow_X)$.

After that the definitions of LMO and $\AA$ diverge, although a part of this divergence is rather minor.

- $\AA_0$ is defined to be the composition $\AA_0 = \int^{FG} \circ\, \sigma \circ \check{Z}$. Here σ denotes the diagrammatic version of the Poincaré Birkhoff Witt theorem with values in $\mathcal{B}(X)$, the completed graded space of X-marked uni-trivalent diagrams modulo AS and IHX relations ("Chinese characters" in [B-N1, B-N2]; see a precise definition in [Å-I, Definition 2.5] and a brief reminder in Figure 4). The map σ was first defined in [B-N1, B-N2] (see also [Å-I, Definition 2.7]); it is more easily described through its inverse χ. If $C \in \mathcal{B}(X)$ is an X-marked uni-trivalent diagram with k_x legs marked x for any $x \in X$, then $\chi(C) \in \mathcal{A}(\uparrow_X)$ is the average of the $\prod_x k_x!$ ways of attaching the legs of C to n labeled vertical arrows (labeled by the elements of X), attaching legs marked by x only to the x-labeled arrow, for all $x \in X$.

  In [Å-I, Å-II] we discussed extensively how the space $\mathcal{B}(X)$ can be viewed as a space of functions, and how the partially defined map $\int^{FG}$ can be viewed as "formal Gaussian integration", (see [Å-I, Definition 2.9] and Section 4 of this article for a definition, and Appendix A.1 for an example of how to apply it). In a sense detailed in [Å-I], $\int^{FG}$ is a diagrammatic analogue of the usual notion of perturbed Gaussian integration — it is defined by breaking the "integrand" into "quadratic" and "higher order" terms, inverting the quadratic, and gluing the higher order terms to each other using the inverse quadratic as glue, in the spirit of Feynman diagrams.

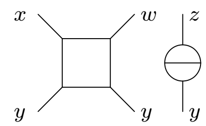


Figure 4. An $\{x, y, z, w\}$-marked uni-trivalent diagram.

- LMO_0 is defined to be an "assembly" of maps $\mathrm{LMO}_0^{(m)}$, where $m \geq 0$ is an integer. That is, the degree m piece of LMO_0 is defined to be the degree m piece of $\mathrm{LMO}_0^{(m)}$ for each m. Each map $\mathrm{LMO}_0^{(m)}$ is a composition $j_m \circ \check{Z}$ where each j_m, and hence each map $\mathrm{LMO}_0^{(m)}$, has a different target space. These target spaces are the spaces $\mathcal{A}^\circ(\emptyset)/(O_m, P_{m+1})$, where $\mathcal{A}^\circ(\emptyset)$ is the same as the space $\mathcal{A}(\emptyset)$ except that the diagrams may contain components with no vertices (closed circles). The relations O_m and P_{m+1} were introduced by Le, Murakami and Ohtsuki in [LMO]. O_m says that disjoint union with a closed circle is equivalent to multiplication by $(-2m)$ and P_{m+1} says that the sum of all ways of pairing up $2m+2$ stubs attaching to the rest of the diagram is 0 (see Figure 5). By [LMO, Lemma 3.3], when we restrict to the space $\mathcal{A}^\circ{}_{\leq m}(\emptyset)$ of diagrams of degree $\leq m$, the quotient by these relations is isomorphic to the corresponding restriction $\mathcal{A}_{\leq m}(\emptyset)$ of $\mathcal{A}(\emptyset)$. Hence the assembly LMO_0 can be regarded as taking values in $\mathcal{A}(\emptyset)$.

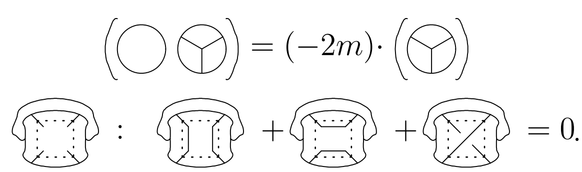


Figure 5. The relation O_m and the relation P_2. In the figure for P_2, the dashed square marks the parts of the diagrams where the relation is applied.

A part of the divergence between the two definitions can be easily remedied. In Section 2.1 we will define a map $\int^{(m)} : \mathcal{B}(X) \to \mathcal{A}^\circ(\emptyset)/(O_m, P_{m+1})$ (called "negative-dimensional formal integration" for reasons to be explained in Section 3) and prove the following lemma:

Lemma 1.1. (Proof in Section 2.1, and see also [Le, Lemma 6.3]) *The composition* $\int^{(m)} \circ \sigma$ *is equal to* $j_m/(O_m, P_{m+1})$.

Using this lemma we can redefine $\mathrm{LMO}_0^{(m)}$ to be the composition $\int^{(m)} \circ \sigma \circ \check{Z}$. Comparing with $\mathring{A}_0 = \int^{FG} \circ \sigma \circ \check{Z}$ we see that the major difference between LMO_0 and $\mathring{A}_0$ is in the use of the different "integrals" $\int^{(m)}$ and $\int^{FG}$. Thus the main technical challenge in this paper is to compare the two integration theories. This is fully achieved by Proposition 1.3 below, which says that whenever $\int^{FG}$ is defined, the two "integrals" differ only by a normalization, and hence ultimately, the same holds for $\mathring{A}$ and LMO.

Definition 1.2. A "function" $G \in \mathcal{B}(X)$ is said to be a perturbed non-degenerate Gaussian (in the variables in X) if it is of the form

$$G = P \exp\Big(\frac{1}{2} \sum_{x,y \in X} l_{xy} \, {}^{x}\frown^{y}\Big)$$

for some invertible symmetric matrix $\Lambda = (l_{xy})$ and some $P \in \mathcal{B}^+(X)$. Here and throughout this paper, the product on diagrams is the disjoint union product, ${}^{x}\frown^{y}$ denotes a strut (a diagram in $\mathcal{B}(X)$ made of a single edge with no internal vertices) with ends labeled x and y and $\mathcal{B}^+(X) \subset \mathcal{B}(X)$ is the space of "strutless" diagrams in which each component has at least one internal vertex (cf. [Å-II, Section 2.2]).

Proposition 1.3. (Proof in Section 4) *If* $G \in \mathcal{B}(X)$ *is a perturbed non-degenerate Gaussian, then*

$$\int^{(m)} G \, dX = (-1)^{m|X|} (\det \Lambda)^m \int^{FG} G \, dX$$

in $\mathcal{A}^\circ{}_{\le m}(\emptyset)/(O_m, P_{m+1}) \simeq \mathcal{A}_{\le m}(\emptyset)$.

Note that $\int^{(m)}$ is defined in more cases than $\int^{FG}$, but when they are both defined, they are related in a simple way.

1.1. **Organization.** In Section 2, we define $\int^{(m)}$, prove Lemma 1.1, and give an alternate formulation of the P_{m+1} relation. In Section 3 we prove some properties of $\int^{(m)}$ which justify the name "negative-dimensional formal integration". These properties are useful in Section 4, where we prove the central Proposition 1.3 which is shown to imply Theorem 1 in Section 5. Section 6 contains remarks on negative-dimensional spaces, sign choices for diagrams, and the Rozansky Witten invariants. Appendix A compares the definitions of $\int^{FG}$ and $\int^{(m)}$ by working out the first two terms in a non-trivial integral.

2. A reformulation of the Le Murakami Ohtsuki invariant

In this section we give a (minor) reformulation of the Le Murakami Ohtsuki invariant $\mathrm{LMO}^{(m)}$. In Section 2.1 we present our definition of $\int^{(m)}$ and prove equivalence with the definition of j_m in [LMO]. In Section 2.2 we state and prove an alternate form C_{2m+1} of the relation P_{m+1}.

2.1. Definition and notation.

Definition 2.1. Let "negative-dimensional integration" be defined by

$$\int^{(m)} : \mathcal{B}(X) \to \mathcal{A}(\emptyset)/(O_m, P_{m+1})$$

$$\int^{(m)} G\,dX = \Big\langle \prod_{x\in X} \frac{1}{m!}\left(\frac{\partial_x\smile\partial_x}{2}\right)^m, G\Big\rangle_X \Big/ (O_m, P_{m+1}). \tag{2}$$

Here the pairing $\langle \cdot\,,\cdot\rangle_X : \mathcal{B}(\partial_X)\otimes\mathcal{B}(X)\to\mathcal{A}^\circ(\emptyset)$ is defined (as in [Å-I, Definition 2.9]) by

$$\langle D_1\,,D_2\rangle_X = \begin{pmatrix}\text{sum of all ways of gluing the } \partial_x\text{-marked legs of}\\ D_1 \text{ to the } x\text{-marked legs of } D_2, \text{ for all } x\in X\end{pmatrix},$$

where, as there, $\partial_X = \{\partial_x : x\in X\}$ denotes a set of labels "dual" to the ones in X, and the sum is declared to be 0 if the numbers of appropriately marked legs do not match. If it is clear which legs are to be attached, the subscript X may be omitted.

In other words, $\int^{(m)} G$ is the composition of:

- projection of G to the component with exactly $2m$ legs of each color in X
- sum over all $\big((2m-1)!!\big)^{|X|} = \big(\frac{(2m)!}{2^m m!}\big)^{|X|}$ ways of pairing up the legs of each color in X
- quotient by the O_m and P_{m+1} relations.

An example of how to apply this definition is given in Appendix A.2.

Our definition of $\int^{(m)}$ is slightly different in appearance than the definition of the corresponding object, j_m, in [LMO]. For example, j_m is defined on $\mathcal{A}(\uparrow_X)$ while $\int^{(m)}$ is defined on the different but isomorphic space $\mathcal{B}(X)$. We now prove Lemma 1.1, which says that this is the only difference between the two maps.

Proof of Lemma 1.1. We prove that $j_m \circ \chi = \int^{(m)}$, where $\chi : \mathcal{B}(X) \to \mathcal{A}(\uparrow_X)$ is the inverse of σ as in the previous section.

First recall from [LMO] how j_m is defined. If D is a diagram representing a class in $\mathcal{A}(\uparrow_X)$, then $j_m(D)$ is computed by removing the n arrows from D so that n groups of stubs remain, and then by gluing certain (linear combinations of) forests on these stubs, so that each tree in each forest gets glued only to the stubs within some specific group. It is not obvious that j_m is well defined; it may not respect the *STU* relation. With some effort, it is proved in [LMO] that for the specific combinations of forests used there, j_m is indeed well defined.

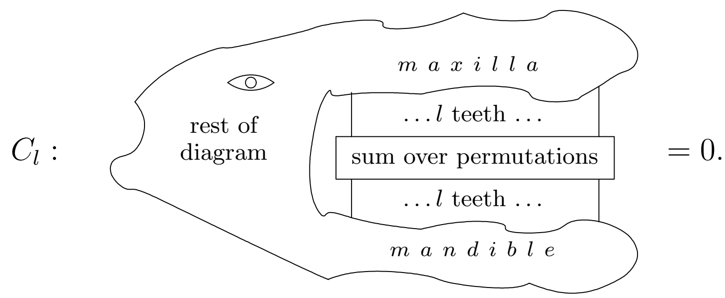
Now every tree that has internal vertices has some two leaves that connect to the same internal vertex, and hence (modulo *AS*), every such tree is anti-symmetric modulo some transposition of its leaves. Thus gluing such a tree to symmetric combinations of diagrams, such as those in the image of χ, we always get 0. Hence in the computation of $j_m \circ \chi$ it is enough to consider forests of trees that have no internal vertices; that is, forests of struts. Extracting the precise coefficients from [LMO] one easily sees that they are the same as in (2), and hence $j_m \circ \chi = \int^{(m)}$, as required. □

2.2. **The C_l relations.** It will be convenient for use in Proposition 3.1 to give another reformulation of the definitions of [LMO]. Instead of their P_{m+1} relation, we may use another relation, the C_{2m+1} relation. For motivation, see Section 3.2.

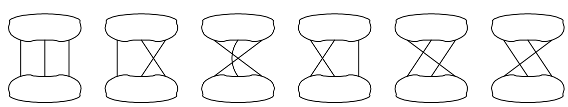
Definition 2.2. The C_l relation[2] applies when we have a diagram, with two sets of l stubs (or teeth), and says that the sum of the diagrams obtained by attaching the two sets of stubs to each other in

[2] C for Crocodile.

all $l!$ possible ways is 0, as in the following diagram.



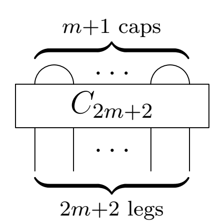
For example, an instance of the C_3 relation says that the sum of the following 6 diagrams is 0:

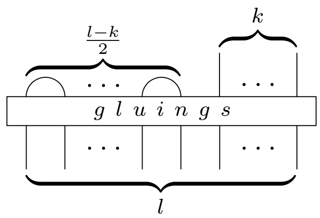


Note that both sets of relations, the P_m's and the C_l's, are decreasing in power. Namely, P_m implies P_{m+1} and C_l implies C_{l+1}, for every l and m (one easily sees that P_{m+1} is a sum of instances of P_m, and likewise for C_{l+1} and C_l). The lemma below says that up to an index-doubling, the two chains of relations are equivalent.

Lemma 2.3. *The relations C_{2m+1}, C_{2m+2}, and P_{m+1} are equivalent. (All of these relations may be applied inside any space of diagrams, regardless of the IHX, STU, or any other relations, as long as closed circles are allowed).*

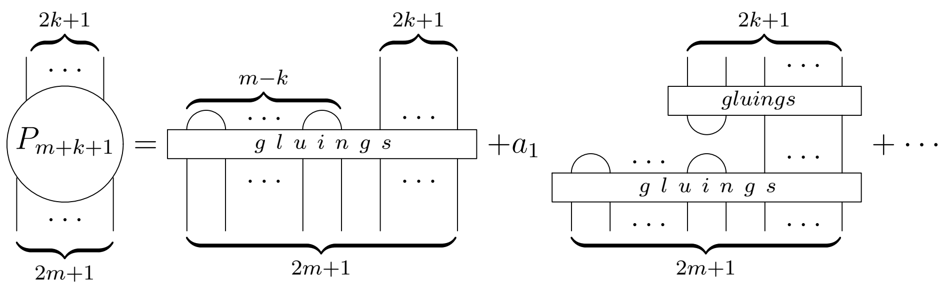
Proof. It was already noted that C_{2m+1} implies C_{2m+2}. Next, it is easy to see that C_{2m+2} implies P_{m+1}: apply the relation C_{2m+2} in the diagram shown on the right, and you get (a positive multiple of) the relation P_{m+1}.





The proof of the implication $P_{m+1} \Rightarrow C_{2m+1}$ is essentially the proof of Lemma 3.1 of [LMO], though the result is different. First, a definition: for $k \leq l$ and $l-k$ even, the diagram part C_l^k has k legs pointing up and l legs pointing down, and it is the sum of all ways of attaching all of the k legs to some of the l legs and then pairing up the remaining $(l-k)$ legs, as illustrated on the left.

We now prove by induction that for all $0 \leq k \leq m$, $C_{2m+1}^{2k+1} = 0$ modulo P_{m+1}. For $k = 0$, C_{2m+1}^1 is a version of P_{m+1}. For $k > 0$, apply P_{m+k+1} (a consequence of P_{m+1}) to a diagram with $2m+1$ legs pointing down and $2k+1$ legs pointing up, like this:



As shown in the diagram, the result splits into a sum over the number $2l$ of the upwards pointing legs that get paired with each other; for $l = 0$, we get C_{2m+1}^{2k+1}; for $l > 0$, the result can be considered to split into two diagrams, a (positive multiple of a) reversed $C_{2k+1}^{2(k-l)+1}$ on top of a $C_{2m+1}^{2(k-l)+1}$. But by the induction hypothesis this latter term $C_{2m+1}^{2(k-l)+1}$ vanishes modulo P_{m+1}, and we are left with only the first term, which is therefore also a consequence of P_{m+1}. This completes the inductive proof. To conclude the proof of Lemma 2.3, note that $C_{2m+1} = C_{2m+1}^{2m+1}$. □

3. Negative-dimensional formal integration

In this section, we give several justifications of the name "negative-dimensional formal integration" for the map $\int^{(m)}$ defined above. While doing this we prove several properties of $\int^{(m)}$ (Propositions 3.1 and 3.2) that are used in the proof of Proposition 1.3 in Section 4.

3.1. **Why integration?** First, why should $\int^{(m)}$ be called an integral? In general, an integral is (more or less) a linear map from some space of functions to the corresponding space of scalars. In our case, the appropriate space of "functions" is $\mathcal{B}(X)$ and the appropriate space of "scalars" is $\mathcal{A}(\emptyset)$.[3] The linearity of $\int^{(m)}$ is immediate.

But $\int^{(m)}$ is not just any integral; it is a Lebesgue integral. The defining property of the usual Lebesgue integral on $\mathbb{R}^n$ is translation invariance by a vector $(\bar{x}^i)$. We show that the parallel property holds for $\int^{(m)}$:

Proposition 3.1 (Translation Invariance). *For any diagram $D \in \mathcal{B}(X)$, we have*

$$\int^{(m)} D\,dX = \int^{(m)} D/(x \mapsto x + \bar{x})\,dX. \tag{3}$$

The notation $D/(x \mapsto x+\bar{x})$ means (as in [Å-II, Section 2.1]), for each leg of D colored x for $x \in X$, sum over coloring the leg by x or by $\bar{x}$. (So we end up with a sum of 2^t terms, where t is the number of X colored legs in D.) The set $\bar{X} = \{\bar{x} \mid x \in X\}$ is an independent set of variables for the formal translation.

Proof. By the relation P_{m+1} in the definition of $\int^{(m)}$ (or rather by C_{2m+1}), any diagram $D \in \mathcal{B}(X)$ with more than $2m$ legs on any component gets mapped to 0 on either side of (3), so we may assume that D has $2m$ legs or fewer of any color. But, for the right-hand side to be non-zero, $D/(x \mapsto x + \bar{x})$ must have exactly $2m$ legs colored x for each $x \in X$; this can only happen from diagrams in D with exactly $2m$ legs of each color and when none of them get converted to $\bar{x}$. But these are exactly the diagrams appearing in the integral on the left hand side. □

[3] More on the interpretation of diagrams as functions and/or scalars appears in [Å-I, Section 1.3] and [Å-II, Section 2].

3.2. **The relations O_m and C_{2m+1}.** Now that you are convinced that $\int^{(m)}$ is an integral, why do we call it a "negative-dimensional" integral? Recall that $\int^{(m)}$ is the sum over all ways of gluing in some struts, followed by the quotient by the O_m and P_{m+1} relations. This quotient is crucial; otherwise $\int^{(m)}$ is some random map without particularly nice properties. But what are these relations?

The relation O_m is simple. Recall [Å-II, Section 2] that we think of diagrams as representing tensors and/or functions in/on some vector space V. Since a strut corresponds to the identity tensor in $V^\star \otimes V$ (cf. [Å-II, Figure 2]), its closure, a circle, should correspond to the trace of the identity or the dimension of V. Hence O_m, which says that a circle is equivalent to the constant $(-2m)$, is the parallel of "$\dim V = -2m$".

The relation P_{m+1} is more subtle. It is easier to look at the equivalent relation C_{2m+1}, which implies the relation C_l for every $l > 2m$. If a single vertex corresponds to some space V, then a collection of l vertices corresponds to $V^{\otimes l}$; and, when we sum over all permutations without signs, we get (a multiple of) the projection onto the symmetric subspace, $S^l(V)$. The relation C_l says that this projection (and hence the target, $S^l(V)$) is 0. Compare this with the following statement about $\mathbb{R}^k$ for $k \geq 0$:

$$\dim S^l(\mathbb{R}^k) = \begin{pmatrix} l+k-1 \\ l \end{pmatrix} = \frac{(k+l-1)(k+l-2)\cdots(k+1)k}{l(l-1)\cdots 2\cdot 1}.$$

We can see from this formula that if a space V formally has a dimension $k = -2m$, then $\dim S^l(V)$ vanishes precisely when $l > 2m$. This is in complete agreement with what we just found about P_{m+1}.

3.3. **An example: Gaussian integration.** Let us compute! Consider the well-known Gaussian integral over $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} e^{q(x,x)/2}\, d^n x = \frac{(2\pi)^{\frac{n}{2}}}{(\det\, -q)^{\frac{1}{2}}}$$

where q is an arbitrary negative-definite quadratic form q. The factor $(2\pi)^{n/2}$ is just a normalization factor that could be absorbed into the measure $d^n x$. (Recall that we identified $\int^{(m)}$ as Lebesgue integration by translation invariance, which only determines the measure up to an overall scale factor.) The remaining factor, $(\det\, -q)^{-1/2}$, is more fundamental. Does a similar result hold for $\int^{(m)}$? To answer this question, one would first have to know what a "determinant" of a quadratic form on a negative-dimensional space is. While there is a good answer to this question (called the "superdeterminant" or the "Berezinian"; see, e.g., [Be, page 82]), it would take us too far afield to discuss it in full. Instead, let us take a slightly different tack. Fix a negative-definite quadratic form Λ on $\mathbb{R}^n$, and consider the quadratic form $q = \Lambda \otimes (\delta_{ij})$ on $\mathbb{R}^n \otimes \mathbb{R}^k \cong \mathbb{R}^{nk}$. We have $\det q = (\det \Lambda)^k$ and so we find that

$$\int_{\mathbb{R}^{nk}} e^{q(x,x)/2}\, d^{nk}x = C(\det -\Lambda)^{-k/2} \tag{4}$$

for some constant C.

Consider now the sum $\sum_{x,y\in X} l_{xy}\, {}^{x}\frown^{y}$ in $\mathcal{B}(X)$. According to the voodoo of diagrammatic calculus [Å-II, Section 2], it is in analogy with a quadratic form on $V^{\otimes n}$, where $n = |X|$ and V is some vector space that plays a role similar to $\mathbb{R}^k$ in the above discussion. The proposition below is then the diagrammatic analog of (4), taking $k = \dim V = -2m$.

Proposition 3.2. *For any set X with $|X| = n$ and $\Lambda = (l_{xy})$ a symmetric matrix on $\mathbb{R}^X$,*

$$\int^{(m)} \exp\left(\frac{1}{2}\sum_{x,y\in X} l_{xy}\, {}^{x}\frown^{y}\right) = (\det -\Lambda)^m = (-1)^{nm}(\det \Lambda)^m.$$

Note that other than symmetry there is no restriction on the matrix Λ. This proposition is a consequence of Lemma 4.2 of [LMO] and a computation for $m = 1$ (given in [LMMO]), but we give our own proof for completeness, and also to provide a more direct link to typical determinant calculations.

Proof. We are to calculate the reduction modulo O_m and P_{m+1} of

$$D_1 := \left\langle \prod_{x\in X} \frac{1}{m!}\left(\frac{\partial_x\smile\partial_x}{2}\right)^m, \exp\left(\frac{1}{2}\sum_{x,y\in X} l_{xy}\, {}^{x}\frown^{y}\right)\right\rangle_X.$$

The only terms that can appear in D_1 are closed loops. The relation O_m replaces each of these by a number, reducing the result to $\mathbb{Q}$. The relation P_{m+1} is irrelevant and will be ignored in the remainder of the proof.

Introduce a new set of variables A (and dual variables ∂_A) with $|A| = m$, and consider

$$D_2 := \left\langle \prod_{x\in X} \frac{1}{m!}\left(\sum_{a\in A} \overset{(x,a)}{\underset{(\partial_x,\partial_a)}{\downarrow}}\right)^m, \exp\left(\sum_{x,y\in X}\sum_{a\in A} l_{xy} \overset{(\partial_x,\partial_a)}{\underset{(y,a)}{\uparrow}}\right)\right\rangle_{XA}. \tag{5}$$

Let us compare D_1 and D_2. After all relevant gluings, D_1 becomes a sum (with coefficients) of disjoint unions of unoriented loops, each of which is a polygon of struts whose vertices are colored by elements of X. Similarly, D_2 is also a sum (with coefficients) of a disjoint union of loops, only that now the loops are oriented and the struts they are made of are colored by the elements of XA, keeping the A part of the coloring constant along each loop. In both cases the coefficients come from the same simple rule, which involves only the X part of the coloring. We see that each term in D_1 with c circles corresponds to $(2m)^c$ terms of D_2: for each loop in a given term of D_1, choose a color $a \in A$ and an orientation, and you get a term in D_2. So we find that

$$D_2/(\bigcirc = -1) \;=\; D_1/(\bigcirc = -2m).$$

Recall that $\frac{1}{4!}(a+b+c+d)^4 \;=\; abcd +$ (non-multilinear terms). Similarly,

$$\prod_{x\in X} \frac{1}{m!}\left(\sum_{a\in A} \overset{(x,a)}{\underset{(\partial_x,\partial_a)}{\downarrow}}\right)^m = \prod_{(x,a)\in XA} \overset{(x,a)}{\underset{(\partial_x,\partial_a)}{\downarrow}} + (\text{terms with strut repetitions}).$$

We assert that terms with strut repetitions can be ignored in the computation of $D_2/(\bigcirc = -1)$. Indeed, for some fixed $x_0 \in X$ and $a_0 \in A$ set $\alpha = (x_0, a_0)$ and $\partial_\alpha = (\partial_{x_0}, \partial_{a_0})$, and suppose a strut repetition like $\downarrow^{\alpha}_{\partial_\alpha}\downarrow^{\alpha}_{\partial_\alpha}$ occurs within the left operand of a pairing as in (5). Then, as illustrated in Figure 6, the gluings in the evaluation of the pairings come in pairs. One easily sees that the number of cycles differs by 1 for the gluings within each pair, and hence modulo $(\bigcirc = -1)$ the whole sum of gluings vanishes.

Now compare D_2 to

$$D_3 := \left\langle \prod_{(x,a)\in XA} \overset{(x,a)}{\underset{(\partial_x,\partial_a)}{\downarrow}}, \exp\left(\sum_{x,y\in X}\sum_{a\in A} l_{xy} \overset{(\partial_x,\partial_a)}{\underset{(y,a)}{\uparrow}}\right)\right\rangle_{XA}.$$

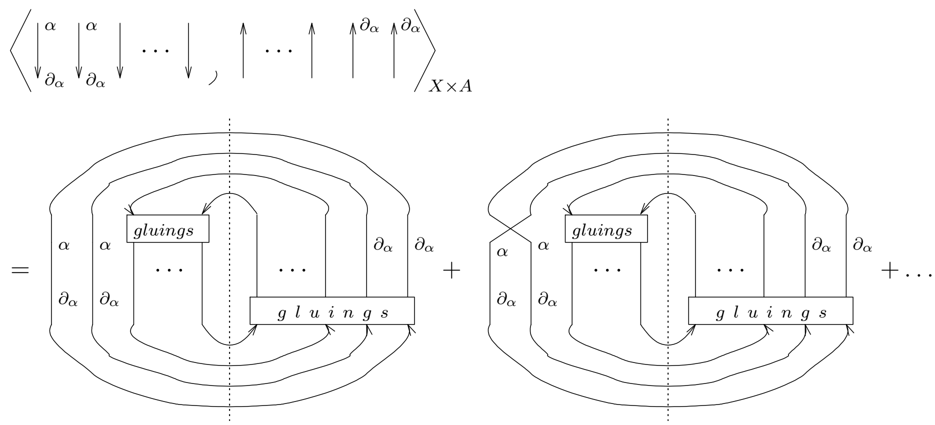


Figure 6. Two ways of gluing a repeating strut.

By our assertion, $D_2 = D_3$ modulo $(\bigcirc = -1)$. But D_3 looks very much like the usual formula for the determinant, reducing to

$$D_3/(\bigcirc = -1) = \sum_{\pi \in S(XA)} \prod_{(x,a)\in XA} l_{xa,\,\pi(xa)}(-1)^{\mathrm{cycles}(\pi)}$$

where $(l_{xa,yb}) = \Lambda \otimes (\delta_{ab})$. Using the relationship between the number of cycles of a permutation $\pi \in S(XA)$ and its signature, $(-1)^{\mathrm{cycles}(\pi)} = (-1)^{nm}\,\mathrm{sgn}(\pi)$, we find that

$$\int^{(m)} G = (-1)^{nm} \det(l_{xa,yb}) = (-1)^{nm}(\det \Lambda)^m$$

as required. $\square$

4. Relating the two integration theories

The classical computation of perturbed Gaussian integration uses only translation invariance, and a single non-perturbed computation to determine the normalization coefficient. For negative dimensional integration translation invariance was proved in Proposition 3.1, and the non-perturbed computation is in Section 3.3. So the proof of Proposition 1.3 proceeds just as in the classical computation of perturbed Gaussian integration:

Proof of Proposition 1.3. Recall that we are comparing $\int^{(m)}$ to $\int^{FG}$ on a perturbed Gaussian with variables in X and a non-degenerate quadratic part $\Lambda = (l_{xy})$:

$$G = P \exp\left(\frac{1}{2}\, l_{xy}\; {}^{x}\!\frown^{y}\right).$$

(Here and throughout this proof, repeated variables should be summed over X.) From [Å-I, Definition 2.9] we have

$$\int^{FG} G\, dX = \left\langle \exp\left(-\frac{1}{2}\, l^{xy}\; {}_{\partial_x}\!\smile_{\partial_y}\right), P \right\rangle, \tag{6}$$

with, as usual, $(l^{xy}) = \Lambda^{-1}$. Indeed, Equation (6) is precisely the definition of formal Gaussian integration, which in itself imitates the standard recipe for evaluating perturbed Gaussian integrals, where one sums over all pairings of multiple copies of the inverse quadratic form with the perturbation term P.

We need to evaluate $\int^{(m)} G\, dX$. First separate the strutless part, P, using a standard trick: (We note that in the first line below we slightly extend the definition of $\langle\cdot,\cdot\rangle_{\bar X}$, allowing it to have values in $\mathcal{A}(X)$ rather than just $\mathcal{A}(\emptyset)$ as in the original Definition 2.1)

$$\begin{aligned}\int^{(m)} G\, dX &= \int^{(m)} \left\langle P/(x \mapsto \partial_{\bar x})\,,\, \exp\left(\frac{1}{2}\, l_{xy}\; {}^{x}\!\frown^{y} + \overset{x}{\underset{\bar x}{|}}\right)\right\rangle_{\bar X} dX \\ &= \left\langle P/(x \mapsto \partial_{\bar x})\,,\, \int^{(m)} \exp\left(\frac{1}{2} l_{xy}\; {}^{x}\!\frown^{y} + \overset{x}{\underset{\bar x}{|}}\right) dX \right\rangle_{\bar X}.\end{aligned}$$

Now we complete the square in the integral:

$$\begin{aligned}&\int^{(m)} \exp\left(\frac{1}{2}\, l_{xy}\; {}^{x}\!\frown^{y} + \overset{x}{\underset{\bar x}{|}}\right) dX \\ &\quad = \exp\left(-\frac{1}{2}\, l^{xy}\; {}_{\bar x}\!\smile_{\bar y}\right) \int^{(m)} \exp\left(\frac{1}{2}\, l_{xy}\; {}^{x}\!\frown^{y} + \overset{x}{\underset{\bar x}{|}} + \frac{1}{2}\, l^{xy}\; {}_{\bar x}\!\smile_{\bar y}\right) dX.\end{aligned}$$

A short computation (left to the reader) shows that we have completed the square:

$$\exp\left(\frac{1}{2}\, l_{xy}\; {}^{x}\!\frown^{y} + \overset{x}{\underset{\bar x}{|}} + \frac{1}{2}\, l^{xy}\; {}_{\bar x}\!\smile_{\bar y}\right) = \exp\left(\frac{1}{2}\, l_{xy}\; {}^{x}\!\frown^{y}\right) \Big/ (x \mapsto x + l^{xy}\bar y).$$

But now, by Propositions 3.1 and 3.2,

$$\int^{(m)} \exp\left(\frac{1}{2}\, l_{xy}\, {}^{x}\frown^{y} + \underset{\bar{x}}{\overset{x}{|}} + \frac{1}{2}\, l^{xy}\, {}_{\bar{x}}\smile_{\bar{y}}\right) dX$$
$$= \int^{(m)} \exp\left(\frac{1}{2}\, l_{xy}\, {}^{x}\frown^{y}\right) dX = (-1)^{nm}(\det\Lambda)^m,$$

and so

$$\int^{(m)} G\, dX = \left\langle P\big/(x \to \partial_{\bar{x}})\, , \exp\left(-\frac{1}{2}\, l^{xy}\, {}_{\bar{x}}\smile_{\bar{y}}\right) \cdot (-1)^{nm}(\det\Lambda)^m \right\rangle_{\bar{X}}$$
$$= (-1)^{nm}(\det\Lambda)^m \int^{FG} G\, dX \bigg/ (O_m, C_{2m+1}). \qquad \square$$

Remark 4.1. As in the classical case (see, e.g., the Appendix of [Å-I]), this proof can be recast in the language of Laplace (or Fourier) transforms.

5. Proposition 1.3 implies Theorem 1

Let L be an n-component regular link with linking matrix Λ having σ_+ positive eigenvalues and σ_- negative eigenvalues. Lemma 1.1 and Proposition 1.3 imply that in the quotient $\mathcal{A}^{\circ}{}_{\leq m}(\emptyset)/(O_m, P_{m+1}) \simeq \mathcal{A}_{\leq m}(\emptyset)$ we have

$$\mathring{A}_0(L) = (-1)^{nm}(\det\Lambda)^{-m}\mathrm{LMO}_0^{(m)}(L). \tag{7}$$

Applying this equality to $U_x^{\pm}$, the x-labeled unknot with ±1 framing, we find that

$$\mathring{A}_0(U_x^+) = (-1)^m\mathrm{LMO}_0(U_x^+) \qquad \text{and} \qquad \mathring{A}_0(U_x^-) = \mathrm{LMO}_0(U_x^-).$$

These are the renormalization factors used in the definition of $\mathring{A}$ and $\mathrm{LMO}^{(m)}$, respectively. Using them and Equation (7) once again, we can compare $\mathring{A}(M)$ and $\mathrm{LMO}^{(m)}(M)$ as follows:

$$\begin{aligned}
\mathring{A}(M) &= \mathring{A}_0(U_x^+)^{-\sigma_+}\mathring{A}_0(U_x^-)^{-\sigma_-}\mathring{A}_0(L)\\
&= (-1)^{(n-\sigma_+)m}(\det\Lambda)^{-m}\mathrm{LMO}_0(U_x^+)^{-\sigma_+}\mathrm{LMO}_0(U_x^-)^{-\sigma_-}\mathrm{LMO}_0(L)\\
&= |\det\Lambda|^{-m}\,\mathrm{LMO}^{(m)}(M) = |H^1(M)|^{-m}\,\mathrm{LMO}^{(m)}(M).
\end{aligned}$$

Finally, Equation (1) of Theorem 1 now follows from the fact that $\mathrm{LMO}(M)$ takes its degree m part from $\mathrm{LMO}^{(m)}(M)$.

6. Some philosophy

The impatient mathematical reader may skip this section; there is nothing with rigorous mathematical content here. For the moment, the material in this section is purely philosophy, and not very well-developed philosophy at that.

This interpretation of $\int^{(m)}$ as negative-dimensional formal integration probably seems somewhat strange. After all Chern Simons theory, the basis for the theory of trivalent graphs (the spaces $\mathcal{A}$ and $\mathcal{B}$), and much of the theory of Vassiliev invariants, takes place very definitely in positive dimensions: the vector space associated to a vertex is some Lie algebra $\mathfrak{g}$. To integrate over these positive dimensional spaces, a different theory is necessary, as developed in Part II of this series [Å-II].

One potential answer to this problem is to forget about the Lie algebra for the moment and just look at the structure of diagrams. There are at least three different reasonably natural sign conventions for the diagrams under consideration [Ko1, Th]. The standard choice is to give an orientation (ordering up to even permutations) of the edges around each vertex. But another natural (though usually less convenient) choice is to leave the edges around a vertex unordered and, instead, give a direction on each edge and a sign ordering of the *set* of all vertices.[4] But now look what happens to the space $\mathcal{B}$: because of the ordering on the vertices, the diagrams are no longer completely symmetric under the action of permuting the legs; they are now completely *anti-symmetric*. This anti-symmetry of legs is exactly what we would expect for functions of fermionic variables or functions on a negative-dimensional space. Furthermore, the integration map $\int^{(m)}$ is quite suggestive from this point of view: it looks like evaluation against a top exterior power of a symplectic form on a vector space, which is a correct analogue of integration.

Alternatively, we could try to keep the connection with physics and find a physical theory that exhibits this negative-dimensional behavior. Fortunately, such a theory has been found: it is the Rozansky

[4]In this discussion, we assume that all vertices of the graphs have odd valency (as holds for all diagrams considered in this paper). See [Ko1, Th] for details on dealing with diagrams with vertices of even valency.

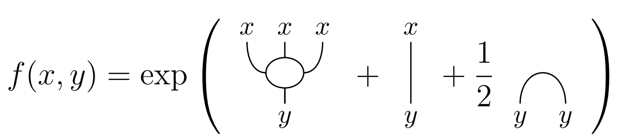
Witten theory [RW, Ko2, Ka]. In this theory weight systems are constructed from a hyper-Kähler manifold Y of dimension $4m$. The really interesting thing for present purposes is that the factors assigned to the vertices are holomorphic one-forms on Y, which anti-commute (using the wedge product on forms). (In keeping with the above remarks about signs, each edges is assigned a symplectic form on Y, which is anti-symmetric.) So in this case, there *is* a kind of $(-2m)$-dimensional space associated to vertices. (But note that this space is "spread out" over Y: it is the parity-reversed holomorphic tangent bundle.)

Finally, it is interesting to note that the definition of $\int^{(m)}$ is more general than that of $\int^{FG}$, and the proofs are equally simple. On the other hand, $\int^{FG}$ has some advantages. It is easier to compute, as you can see in Appendix A. Its philosophical meaning is much clearer and it takes values in $\mathcal{A}(\emptyset)$ directly, rather than in some quotient. Also, it makes Part IV of this series possible — its relationship with Lie algebras is clearer.

Appendix A. A computational example

In order to make more concrete the definitions of the two types of integrals we consider in this appendix we will integrate a formal power series with $\int^{(m)}$ and with $\int^{FG}$ and check that the answer obeys Proposition 1.3. We will also see how the combinatorial factors work in practice.

We will integrate the following "function" in $\mathcal{B}(x, y)$:

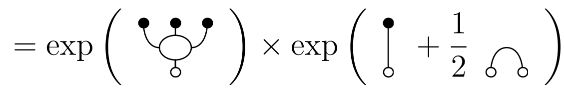


Since our drawings will get crowded, we will replace the labels x and $\partial/\partial x$ by a solid circle $\bullet$, and the labels y and $\partial/\partial y$ by a open circle $\circ$. We will write I^{FG} for the formal Gaussian integral of f, and $I^{(m)}$ for the negative-dimensional integral of f.

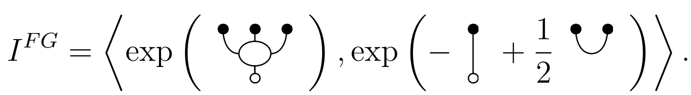
A.1. **Computing the formal Gaussian integral.**

To compute the first few terms of $I^{FG} = \int^{FG} f(x, y)\, dx\, dy$, first note that f takes the form of a perturbed Gaussian

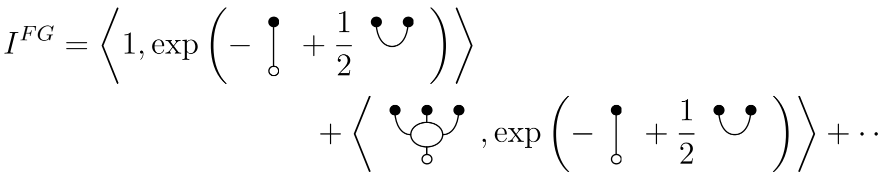
$$f = P \times \exp\left(l_{ij}\, {}^{i}\frown{}^{j}\right)$$



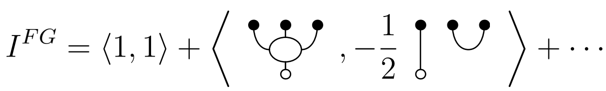
with quadratic part $\Lambda = (l_{ij}) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Then the prescription for the formal Gaussian integral of f involves pairing the perturbation with a quadratic part given by $-\Lambda^{-1} = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 0 \end{smallmatrix}\right)$:



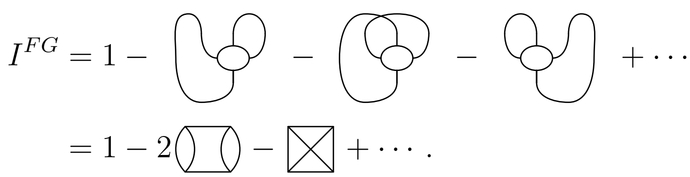
Expanding the exponential on the left of the pairing, we get



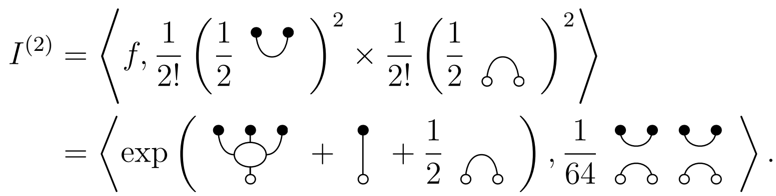
Each left side has a definite number of vertices of each color and has a non-zero pairing with only one term in the exponential on the right, so we can simplify this to



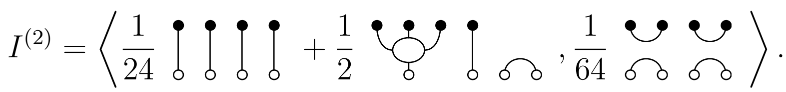
Now we evaluate the pairing. Note how the combinatorics of the pairing with exponentials of struts works out: we end up summing over all ways of pairing the end points of P, with coefficients given by products of appropriate entries of $-\Lambda^{-1}$. All other combinatorial factors cancel.



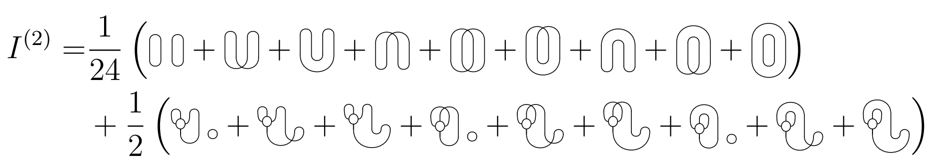
A.2. **Computing the negative-dimensional integral $\int^{(m)}$.** We now turn to the negative-dimensional integral of f. To be able to compare the results, we must have m at least 2; let us therefore set $m = 2$.



The right side has four x-vertices and four y-vertices. There are only two terms in the exponential on the left which have non-zero pairing with the right side:

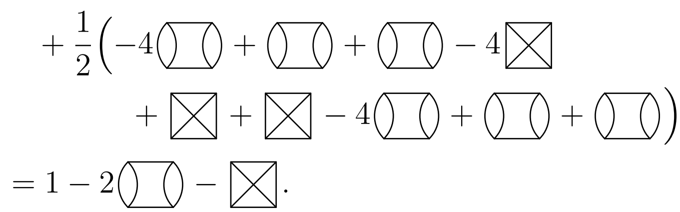


To evaluate the pairing, notice that the combinatorial factors on the right side cancel, and we end up summing over all ways of pairing the vertices of the same color on the left, each appearing with coefficient 1.



Now apply the O_m relation with $m = 2$, which replaces each circle by -4.

$$I^{(2)} = \frac{1}{24}\Big(16 - 4 - 4 - 4 + 16 - 4 - 4 - 4 + 16\Big)$$



Note that this matches the result we found from the formal Gaussian integration, which is expected since $\det \Lambda = -1$ and the degree of the non-trivial elements is even.

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Institute of Mathematics, The Hebrew University, Giv'at-Ram, Jerusalem 91904, Israel

Current address: Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 3G3

E-mail address: `drorbn@math.toronto.edu`

URL: `http://www.math.toronto.edu/~drorbn`

Department of Mathematics, Harvard University, Cambridge MA 02138, USA

Current address: School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA

E-mail address: `stavros@math.gatech.edu`

URL: `http://www.math.gatech.edu/~stavros`

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago IL 60607-7045, USA

Current address: Department of Mathematics, UNC-CH, CB 3250 Phillips Hall, Chapel Hill, NC 27599-3250, USA

E-mail address: `rozansky@email.unc.edu`

Department of Mathematics, University of California at Berkeley, Berkeley CA 94720-3840, USA

Current address: Department of Mathematics, Harvard University, Cambridge MA 02138, USA

E-mail address: `dpt@math.harvard.edu`