M.Sc. Math Workshop — Assignment #3

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(15) Prove: If $\lambda > 0$ is irrational and $\epsilon > 0$ then there exists 5 continuous functions $\phi_i : [0,1] \to [0,1]$ ($1 \leq i \leq 5$) so that for every continuous function $f : [0,1] \times [0,1] \to \mathbf{R}$ there exists a continuous function $g : [0, 1+\lambda] \to \mathbf{R}$ so that

$$|f(x,y) - \sum_{i=1}^{5} g(\phi_i(x) + \lambda\phi_i(y))| < \left(\frac{2}{3} + \epsilon\right) |f(x,y)|$$

for every $x, y \in [0,1]$.

(16) A *quantum probability space* is a pair $(\mathcal{H}, v)$ where $\mathcal{H}$ is a Hilbert space and $v \in \mathcal{H}$ is a unit vector. A *random variable* on $\mathcal{H}$ is a self-adjoint operator $\mathcal{H} \to \mathcal{H}$. We say that $\langle v, A^n v\rangle$ is the expectation value of the nth power of the random variable A (if this quantity exists, namely if v is in the domain of definition of A^n). In particular, we set $E(A) = \langle v, Av\rangle$ to be the *expectation* of A, and $V(A) = \langle v, A^2 v\rangle - \langle v, Av\rangle^2$ to be the *variance* of A. Prove that if P and Q are random variables on some quantum probability space $(\mathcal{H}, v)$, and P and Q satisfy $[P,Q] = PQ - QP = iI$, then $V(P)V(Q) \geq \frac{1}{4}$. It is a good idea to start with the simplifying assumption $E(P) = E(Q) = 0$.

(17) Prove that a finite group of affine transformations always has a fixed point.

(18) Prove that the area of any planar section of a perfect tetrahedron is at most the area of a face of that tetrahedron.

(19) Prove that any knot in $\mathbf{R}^3$ is the boundary of some double-sided (non-Möbius) surface embeded in $\mathbf{R}^3$.

(20) A rectangle R is tiled (presented as a disjoint union, not minding about 1-dimensional boundaries) with (possibly different) semi-integral rectangles — rectangles at least one of whose sides is of integral length. Prove that R itself is semi-integral.