Dror Bar-Natan: Classes: 2004-05: Math 157 - Analysis I:

Visualization

web version:
http://www.math.toronto.edu/~drorbn/classes/0405/157AnalysisI/Visualization/Visualization.html

Our task for today (and probably tomorrow as well) is to master the axiomatically meaningless task of visualization of numbers and functions. We will learn how to interpret graphically all of the following:

1. A number a, the order relation $a < b$ and the absolute value of a difference $|a - b|$.
2. Intervals such as $(a, b) := \{x : a < x < b\}$, $[a, b) := \{x : a \leq x < b\}$, $[a, b] := \{x : a \leq x \leq b\}$, $(a, \infty) := \{x : x > a\}$ and $(-\infty, a] := \{x : x \leq a\}$.
3. A point (a, b) in the plane. (Notice the sad clash of notation).
4. The graphs of the functions $f_1(x) = c$, $f_2(x) = cx$ and $f_3(x) = cx + d$.
5. The parabola $y = x^2$ and the graphs of $f(x) = x^n$ for several n's.
6. The graphs of $f_1(x) = \frac{1}{x}$, $f_2(x) = \frac{1}{x^2}$, $f_3(x) = \frac{1}{1+x^2}$ and $f_4(x) = \frac{x}{1+x^2}$.
7. The graphs of $f_1(x) = \sin x$, $f_2(x) = \sin\frac{1}{x}$, $f_3(x) = x\sin\frac{1}{x}$ and $f_4(x) = x^2\sin\frac{1}{x}$.
8. The graphs of $f_1(x) = \begin{cases} x^2 & x < 1 \\ 2 & x \geq 1 \end{cases}$, $f_2(x) = \begin{cases} x^2 & x \leq 1 \\ 2 & x > 1 \end{cases}$ and $f_3(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$.
9. The Euclidean distance function $d((a, b), (c, d)) := \sqrt{(a - c)^2 + (b - d)^2}$ (that's a "metric"!).
10. The circle $(x-a)^2+(y-b)^2 = r^2$, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Just for fun. For $x \in [0, 1]$, the number $f(x)$ is defined to be the result of the following process: Write x in binary, replace every 1 in the resulting expansion by a 2, and interpret the result as a number written in base 3. For example, $x = \frac{1}{3} = 0.01010101_2\ldots \to 0.02020202_3\ldots = \frac{1}{4} = f(x)$.

- Draw the graph of f.
- Draw the range of f as a subset of $[0, 1]$. (The answer, called "the Cantor set" plays a major role in much of analysis and in particular in the theory of fractals. In some sense its dimension is the irrational number $\frac{\log 2}{\log 3}$.)

Deep thought questions. Why, *really* why, is $3 \cdot 5 = 5 \cdot 3$? Why, *really* why, is $(-3) \cdot (-5) = 3 \cdot 5$?