Solution of Term Exam 4

Problem 1. Agents of CSIS have secretly developed a function $E(x)$ that has the following properties:

- $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$.
- $E(0) = 1$
- E is differentiable at 0 and $E'(0) = 1$.

Prove the following:

1. E is everywhere differentiable and $E' = E$.
2. $E(x) = e^x$ for all $x \in \mathbb{R}$. The only lemma you may assume is that if a function f satisfies $f'(x) = 0$ for all x then f is a constant function.

Solution. (Graded by Brian Pigott)

1. The fact that $1 = E'(0)$ means that $1 = \lim_{h\to 0} \frac{E(h)-E(0)}{h} = \lim_{h\to 0}\frac{E(h)-1}{h}$. Hence, using $E(x+h) = E(x)E(h)$ we get
$$\lim_{h\to 0}\frac{E(x+h)-E(x)}{h} = \lim_{h\to 0}\frac{E(x)E(h)-E(x)}{h} = E(x)\lim_{h\to 0}\frac{E(h)-1}{h} = E(x).$$
This proves both that E is differentiable at x and that $E'(x) = E(x)$.
2. Consider $q(x) = E(x)e^{-x}$. Differentiating we get
$$q'(x) = E'(x)e^{-x} + E(x)(e^{-x})' = E(x)e^{-x} - E(x)e^{-x} = 0.$$
Hence $q(x)$ is a constant function. But $q(0) = E(0)e^0 = 1 \cdot 1 = 1$, hence this constant must be 1. So $E(x)e^{-x} = 1$ and thus $E(x) = e^x$.

Problem 2. Compute the following integrals: (a few lines of justification are expected in each case, not just the end result.)

1. $\displaystyle\int \frac{x^2+1}{x+2}dx.$
2. $\displaystyle\int e^{ax}\sin bx\,dx$ (assume that $a, b \in \mathbb{R}$ and that $a \neq 0$ and $b \neq 0$).
3. $\displaystyle\int x\log\sqrt{1+x^2}\,dx.$
4. $\displaystyle\int_0^\infty e^{-x}\,dx$. (This, of course, is $\displaystyle\lim_{T\to\infty}\int_0^T e^{-x}\,dx$).

Solution. (Graded by Shay Fuchs)

1. $\displaystyle\int \frac{x^2+1}{x+2}dx = \int \left(x - 2 + \frac{5}{x+2}\right) dx = \frac{x^2}{2} - 2x + 5\log|x+2| + C.$

2. Let F denote the anti-derivative we are interested in; i,e,, $F = \displaystyle\int e^{ax}\sin bx\,dx$. Integrating by parts twice we get

$$\begin{aligned} F &= \int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx \\ &= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx \\ &= \frac{e^{ax}}{a^2}(a\sin bx - b\cos bx) - \frac{b^2}{a^2}F, \end{aligned}$$

so

$$\left(1 + \frac{b^2}{a^2}\right) F = \frac{e^{ax}}{a^2}(a\sin bx - b\cos bx),$$

or

$$F = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx).$$

(with the $+C$ neglected).

3. Taking $u = 1 + x^2$ hence $du = 2xdx$ and using a formula from class, $\int \log u\,du = u\log u - u + C$, we get

$$\int x\log\sqrt{1+x^2}\,dx = \frac{1}{4}\int 2x\log(1+x^2)\,dx = \frac{1}{4}\int \log u\,du = \frac{1}{4}(u\log u - u) + C$$

$$= \frac{1}{4}((1+x^2)\log(1+x^2) - 1 - x^2) + C.$$

4. $\displaystyle\int_0^\infty e^{-x}\,dx = \lim_{T\to\infty}\int_0^T e^{-x}\,dx = \lim_{T\to\infty} -e^{-x}\Big|_0^T = \lim_{T\to\infty} e^{-0} - e^{-T} = 1.$

Problem 3.

1. State (without proof) the formula for the surface area of an object defined by spinning the graph of a function $y = f(x)$ (for $a \le x \le b$) around the x axis.

2. Compute the surface area of a sphere of radius 1.

Solution. (Graded by Brian Pigott)

1. That surface area, excluding the "caps", is $2\pi\displaystyle\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$. Including the caps it is the same plus the area of the caps, $\pi f(a)^2 + \pi f(b)^2$. For the purpose of this exam, both solutions are acceptable.

2. Here f is the function whose graph is a semi-circle, so $f(x) = \sqrt{1-x^2}$, and $a = -1$ and $b = 1$. By direct computation, $f(x)\sqrt{1+(f'(x))^2} = 1$ so the surface area of a sphere of radius 1 is $2\pi \int_{-1}^{1} 1dx = 4\pi$.

Problem 4.

1. State and prove the remainder formula for Taylor polynomials (it is sufficient to discuss just one form for the remainder, no need to mention all the available forms).

2. It is well known (and need not be reproven here) that the nth Taylor polynomial $P_n = P_{n,0,e^x}$ of e^x around 0 is given by $P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$. It is also well known (and need not be reproven here) that factorials grow faster than exponentials, so for any fixed c we have $\lim_{n\to\infty} c^n/n! = 0$. Show that for large enough n,

$$\left|e^{157} - P_n(157)\right| < \frac{1}{157}.$$

Solution. (Graded by Derek Krepski)

1. Statement: If f is differentiable $n+1$ times on the interval between a and x and if $P_n(x)$ denotes the nth Taylor polynomial of f at a, then there is some t between a and x for which

$$f(x) - P_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}.$$

Proof: See your class notes from March 10, 2005.

2. By the remainder formula with $f(x) = e^x$ and with $a = 0$ we have that for any n, there is a $t \in (0, 157)$ for which

$$e^{157} - P_n(157) = \frac{e^t}{(n+1)!}157^{n+1}.$$

But $t < 157$ implies $0 < e^t < e^{157}$ and so

$$\left|e^{157} - P_n(157)\right| < \frac{e^{157}}{(n+1)!}157^{n+1}.$$

Now take n big enough so that $157^{n+1}/(n+1)! < 1/157e^{157}$ (this is possible because $157^{n+1}/(n+1)! \to 0$) and get

$$\left|e^{157} - P_n(157)\right| < \frac{1}{157},$$

as required.

The results. 66 students took the exam; the average grade was 58.2 and the standard deviation was about 24.