UNIVERSITY OF TORONTO
Faculty of Arts and Science

APRIL/MAY EXAMINATIONS 2002

MAT 157Y1Y
ANALYSIS I

Duration: 3 hours

No Aids Allowed (calculators, books, etc)

There are five problems, each worth 20 points.

Problem 1

Evaluate the following integrals:

(a) $\displaystyle\int \arctan(\sqrt{x})\,dx$

(b) $\displaystyle\int \frac{\cos x}{\sin^3 x}\,dx$

(c) $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$

(d) $\displaystyle\int \frac{x+2}{x^2+x}\,dx$

(e) $\displaystyle\int \frac{1}{x^2}\,\sin\frac{1}{x}\,dx$

Problem 2

For each of the following series, determine whether it converges or diverges. If it converges, determine whether the convergence is absolute or conditional. If the expression contains an x, determine the largest interval of x (if any) within which the series converges; you do not have to consider convergence at the ends of the interval.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n-2)/2}}{2^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n^3+1}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{(n+2)!}$

(d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n\,\frac{n^2}{1+n^2}$

(e) $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

Problem 3

Evaluate the following limits, if they exist, and briefly justify your answer:

(a) $\displaystyle\lim_{n\to\infty} \frac{a^{n+1}+b^{n+1}}{a^n+b^n} \quad (0<a<b)$

(b) $\displaystyle\lim_{x\to 0} \frac{1-\sqrt{1-4x^2}}{x^2}$

(c) $\displaystyle\lim_{t\to 0} \frac{f(a+2t)-f(a+t)}{2t} \quad (f \text{ differentiable at } a)$

(d) $\displaystyle\lim_{x\to 0} \frac{\sin 5x - \sin 3x}{x}$

(e) $\displaystyle\lim_{x\to 0} \frac{\sqrt{x^2}}{x}$

Problem 4

(a) Prove that if f is a continuous function on $[0,1]$, then

$$\int_0^\pi x\, f(\sin x)\, dx \;=\; \frac{\pi}{2}\int_0^\pi f(\sin x)\, dx.$$

(b) Use this result to prove that

$$\int_0^\pi \frac{x\, \sin x\, dx}{1+\cos^2 x} \;=\; \frac{\pi}{2}\int_0^\pi \frac{\sin x\, dx}{1+\cos^2 x}$$

and evaluate the integral.

Problem 5

Define the function $f(x)$ for $x \geq 0$ by

$$f(x) \;=\; \int_0^x \frac{dt}{\sqrt{1+t^3}}.$$

(Don't try to evaluate this integral.)

(a) Prove that

(i) $f(0) = 0$,

(ii) f is continuous and strictly increasing for $0 \leq x < \infty$, and

(iii) $\lim_{x\to\infty} f(x) = M$ where M is a finite number.

(b) Show that

$$\frac{f''(x)}{f'(x)^3\, x^2}$$

is a constant.

(c) By (a), we know that $f(x)$ has an inverse $g(y)$ defined on $0 \leq y < M$. Show that $g'' = c\, g^2$ where c is a constant. What is the value of c?