ALGEBRAIC TOPOLOGY, PROBLEM SET 8

1. Exact sequences and diagram chasing

In this section we continue to work with the diagram

$$\begin{array}{ccccc} M & \xrightarrow{u} & N & \xrightarrow{v} & P \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ M' & \xrightarrow[u']{} & N' & \xrightarrow[v']{} & P' \end{array} \tag{1}$$

and always assume that it is commutative and its horizontal lines are exact.

Exercise 1.1. In notations of (1) prove that

1. If u', f and h are injective, then g is injective.
2. If v, f and h are surjective, then g is surjective.

Exercise 1.2. In notations of (1) prove that

1. If g is injective and if f, v are surjective, then h is injective.
2. If g is surjective and if h, u' are injective, then f is surjective.

Exercise 1.3 (Five lemma). Consider the following commutative diagram of modules

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{u_1} & M_2 & \xrightarrow{u_2} & M_3 & \xrightarrow{u_3} & M_4 & \xrightarrow{u_4} & M_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ M_1' & \xrightarrow{u_1'} & M_2' & \xrightarrow{u_2'} & M_3' & \xrightarrow{u_3'} & M_4' & \xrightarrow{u_4'} & M_5' \end{array}$$

Suppose that the horizontal lines are exact. Prove that

1. if f_2, f_4 are injective and f_1 is surjective, then f_3 is injective.
2. if f_2, f_4 are surjective and f_5 is injective, then f_3 is surjective.

In particular, if f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.

Definition 1.4. An exact sequence of the form $0 \to A_1 \to A_2 \to A_3 \to 0$ is called *short exact sequence*. The short exact sequence is *split* if $A_2 \cong A_1 \oplus A_3$.

Exercise 1.5. Prove that for a short exact sequence $0 \to A_1 \xrightarrow{p} A_2 \xrightarrow{q} A_3 \to 0$ the following conditions are equivalent

1. The sequence splits;
2. There exists $s_1 : A_3 \to A_2$ satisfying $qs_1 = id_{A_3}$.
3. There exists $s_2 : A_2 \to A_1$ satisfying $s_2 p = id_{A_1}$.

2. Singular homology

Solve exercise 15 on page 132 in the Hatcher's book.

Exercise 2.1. Calculate $H_*(S_1 \vee S_1)$.

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