ALGEBRAIC TOPOLOGY, PROBLEM SET 7

1. Exact sequences and diagram chasing

Throughout this section $A, B, C, \ldots$ are (left) modules over a ring R, and $f, g, \ldots$ are homomorphisms of modules (if you do not like modules, then think of abelian groups and homomorphisms).

Definition 1.1. Let $A \xrightarrow{f} B$ be a morphism of modules, then

$$\begin{aligned} \mathrm{Ker}(f) &= \{a \in A \mid f(a) = 0\}, \\ \mathrm{Im}(f) &= \{b \in B \mid \exists a \in A:\ f(a) = b\}, \\ \mathrm{Coker}(f) &= B/\mathrm{Im}(f). \end{aligned}$$

A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called exact in B if $\mathrm{Im}(f) = \mathrm{Ker}(g)$. A sequence is exact if it is exact in each (intermediate) node. A diagram is commutative if all paths with beginning and end in the same nodes represent equal maps.

Exercise 1.2. In the following commutative diagram of modules,

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "f"'] & N \arrow[r, "v"] \arrow[d, "g"'] & P \arrow[d, "h"'] \\ M' \arrow[r, "u'"'] & N' \arrow[r, "v'"'] & P' \end{tikzcd}$$

horizontal lines are exact. Show that

1. If h is injective ($\mathrm{Ker}(h) = 0$), then $\mathrm{Im}(g) \cap \mathrm{Im}(u') = \mathrm{Im}(u' \circ f) = \mathrm{Im}(g \circ u)$.
2. If g is surjective ($\mathrm{Coker}(g) = 0$), then $\mathrm{Ker}(g) + \mathrm{Im}(u) = \mathrm{Ker}(v' \circ g) = \mathrm{Ker}(h \circ v)$.

Exercise 1.3. Consider the following commutative diagram of modules,

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "f"'] & N \arrow[d, "g"'] \\ M' \arrow[r, "u'"'] & N'. \end{tikzcd}$$

Show that there exist and unique homomorphisms $u_1 : \mathrm{Ker}(f) \to \mathrm{Ker}(g)$ and $u_2 \colon \mathrm{Coker}(f) \to \mathrm{Coker}(g)$, such that the diagrams

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "u_1"] \arrow[d, "i"'] & \mathrm{Ker}(g) \arrow[d, "j"'] \\ M \arrow[r, "u"'] & N \end{tikzcd} \qquad \begin{tikzcd} M' \arrow[r, "u'"] \arrow[d, "p"'] & N' \arrow[d, "q"'] \\ \mathrm{Coker}(f) \arrow[r, "u_2"'] & \mathrm{Coker}(g) \end{tikzcd}$$

are commutative (i, j are canonical inclusion; p, q are canonical surjections).

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Exercise 1.4 (Snake lemma)**.** Apply Exercise 1.3 on the diagram of Exercise 1.2 to obtain the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ker}(f) & \xrightarrow{u_1} & \mathrm{Ker}(g) & \xrightarrow{v_1} & \mathrm{Ker}(h) \\
{\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle k}\downarrow \\
M & \xrightarrow{u} & N & \xrightarrow{v} & P \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\
M' & \xrightarrow[u']{} & N' & \xrightarrow[v']{} & P' \\
{\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow \\
\mathrm{Coker}(f) & \xrightarrow[u_2]{} & \mathrm{Coker}(g) & \xrightarrow[v_2]{} & \mathrm{Coker}(h).
\end{array}$$

Suppose that two middle horizontal lines are exact. Prove that

1. $v_1 \circ u_1 = 0$; if u' is injective, then the upper line is exact.
2. $v_2 \circ u_2 = 0$; if v is surjective, then the lower line is exact.
3. Suppose that u' is injective and v is surjective, then there exists and unique homomorphism $d\colon \mathrm{Ker}(h) \to \mathrm{Coker}(f)$, such that for any $z \in \mathrm{Ker}(h)$, $y \in N$, $x' \in M'$ which satisfy $v(y) = k(z)$ and $u'(x') = g(y)$, then $d(z) = p(x')$. Besides, the sequence

   $$\mathrm{Ker}(f) \xrightarrow{u_1} \mathrm{Ker}(g) \xrightarrow{v_1} \mathrm{Ker}(h) \xrightarrow{d} \mathrm{Coker}(f) \xrightarrow{u_2} \mathrm{Coker}(g) \xrightarrow{v_2} \mathrm{Coker}(h)$$

   is exact.

2. Singular chains and chain homotopy

Solve exercises 11, 12 on page 132 in the Hatcher's book.