ALGEBRAIC TOPOLOGY, PROBLEM SET 2

1. Further applications of van Kampen theorem

Solve Exercises 2, 3, 4, 7, 8 from pages 52–53 in the Hatcher's book.

2. Properties of CW-complexes

Definition 2.1. A space X is called a *cell complex* or *CW-complex* if it is the colimit of a sequence (= functor from $\mathbb{N}$ to topological spaces) $\emptyset = X^{-1} \hookrightarrow X^0 \hookrightarrow X^1 \hookrightarrow \ldots \hookrightarrow X^n \ldots$, such that X^{n+1} is obtained from X^n by "gluing cells of dimension $n+1$", i.e. the following diagram is a pushout

$$\begin{tikzcd} \coprod_{\alpha\in A_n} S^n \arrow[r] \arrow[d, hook] & X^n \arrow[d] \\ \coprod_{\alpha\in A_n} D^{n+1} \arrow[r] & X^{n+1} \end{tikzcd}$$

A cellular complex is finite if A_n is finite for all $n \in \mathbb{N}$ and $A_n = \emptyset$ for $n \geq N \in \mathbb{N}$. The notion of a cellular subcomplex is obvious, an inclusion of a cellular subcomplex is denoted by "$\hookrightarrow$".

Remark 2.2. Compare the above definition with the definition on page 5 in the Hatcher's book and make sure that you understand that they are the same.

Exercise 2.3. Given a group $G =< g_1, \ldots, g_n, \ldots \mid r_1, \ldots, r_n, \ldots >$, construct a topological space (X, x), such that $\pi_1(X, x) \cong G$. (*Hint:* 2-dimensional CW-complex will do; take the bunch of circles corresponding to the generators of G and "glue out" the relations by 2-dimensional cells.)

Definition 2.4. *Euler characteristic* of a finite CW-complex X is the integer that equals to the alternating sum $\chi(X) = \sum_{n=0}^{N}(-1)^n|A_n|$.

Exercise 2.5. Show that the Euler characteristic of a finite CW-complex is a homotopy invariant.

The standard proof of this fact requires the knowledge of some (co)homology functor. The following suggestion for the proof generalizes the classical inductive argument which shows that for any planar connected graph $V - E + F = 2$.

<u>Outline of a proof (due to Ch. Watts).</u>

Definition 2.6. An *additive invariant* on the category of finite CW-complexes is a pair (A, a), where A is an abelian group and a is an assignment: for any finite CW-complex X, $a(X) \in A$, subject to the following axioms:

(1) $X \sim Y \Rightarrow a(X) = a(Y)$;

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(2) For any pushout of CW-complexes

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

$$a(D) = a(B) + a(C) - a(A);$$

(3) $a(\emptyset) = 0 \in A$.

The additive invariant (U, u) is called *universal* (UAI) if for any other additive invariant (A, a) there exists a *unique* homomorphism $\phi : U \to A$, s.t. for any finite CW-complex X, $a(X) = \phi(u(X))$.

First show the existence of the universal additive invariant. The proof is 'by construction': consider the free abelian group F generated by all homotopy types $[X]$ of finite CW-complexes, then define $U = F/N$, where N is the subgroup generated by all the relations of the form $[D] = [B]+[C]-[A]$ for any pushout as above; $u(X)$ is the image of $[X]$ in U. Check the universal property ! Can you recognise U as an abstract group ?

Next, calculate any additive invariant (A, a) of a finite CW-complex X in terms of $a(*)$. The calculation is by induction on the CW-structure of X. Internal induction is required to understand what is $a(S^n)$.

Conclude that $(U, u) \cong (\mathbb{Z}, \chi)$. In particular, $\chi(X)$ is a homotopy invariant, by definition of an additive invariant.