Dror Bar-Natan: Classes: 2001-02: Fundamental Concepts in Algebraic Topology:

Last Class on Covering Spaces, April 25, 2002

Throughout this page X will be a connected, locally connected and semi-locally simply connected topological space, and all covering spaces will be connected. Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be such a covering map with and let H be $p_\star \pi_1(\tilde{X}, \tilde{x}_0) < \pi_1(X, x_o)$. As we have seen in class, $\tilde{X}$ is homeomorphic to the space

$$\tilde{M}_H := \{[\gamma : I \to X] : \ \gamma(0) = x_0\} / [\gamma] \sim [\eta\gamma] \text{ for } [\eta] \in H$$

of head-and-tail-preserving homotopy classes of snakes with tails at x_0, modulo "reattaching the tails by elements of H", with the topology induced by "moving the head". It is nice to interpret some of the statements below from the perspective of $\tilde{M}_H$.

Furthermore, we found that the correspondence $H \leftrightarrow \tilde{X} = \tilde{X}_H$ is a bijective correspondence between subgroups of $\pi_1(X, x_0)$ and coverings of (X, x_0). Let us learn some more about this correspondence:

Caravaggio's Medusa, circa 1598. According to legend, the shock from seeing Medusa the Gorgon (or her sisters Sthenno and Euryale) would turn anyone to stone.

Claim 1. *If $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering with group $H < \pi_1(X, x_0)$ then the decks $p^{-1}(U)$ of $\tilde{X}$ over a small enough $U \subset X$ are enumerated (not canonically) by the left cosets $H \backslash \pi_1(X)$ of H in $\pi_1(X)$.*

Claim 2. *If $H_1 < H_2 < \pi_1(X, x_0)$ then $\tilde{X}_{H_1}$ is a covering of $\tilde{X}_{H_2}$.*

Definition 3. An automorphism (aka "deck transformation") of a covering $p : \tilde{X} \to X$ is a (not necessarily basepoint-preserving) homeomorphism $\alpha : \tilde{X} \to \tilde{X}$ which covers p: $p \circ \alpha = \alpha$. Let $\mathrm{Aut}(\tilde{X})$ be the group (!) of all automorphisms of $\tilde{X}$.

Claim 4. *If $\tilde{X}_U$ denotes the universal cover of X, then there is a natural identification of $\pi_1(X, x_0)$, of $\mathrm{Aut}(\tilde{X}_U)$ and of $p^{-1}(x_0)$.*

Claim 5. *For all $H < \pi_1(X) = \mathrm{Aut}(\tilde{X}_U)$, the covering $\tilde{X}_H$ is homeomorhic to the quotient space $\tilde{X}_U / H$.*

Definition 6. A covering $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is called *normal* if $\mathrm{Aut}(\tilde{X})$ acts transitively on $p^{-1}(x_0)$.

Claim 7. *For $H < \pi_1(X)$, the covering $\tilde{X}_H$ is normal iff the subgroup H is normal in $\pi_1(X)$.*

Claim 8. *For any $H < \pi_1(X)$ we have $\mathrm{Aut}\, \tilde{X}_H = N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X)$, the group of all $\gamma \in \pi_1(X)$ for which $\gamma^{-1} H \gamma = H$. In particular, if H (equivalently $\tilde{X}_H$) is normal, then $\mathrm{Aut}(\tilde{X}_H) = \pi_1(X)/H$.*

Definition 9. We say that a group action of a group G on a space X is a *covering action* if X can be covered with open sets $U \subset X$ for which for any given U the collection $\{g(U) : g \in G\}$ is a collection of disjoint sets.

Claim 10. *If a group G acts via a covering action on a space X then the quotient map $p : X \to X/G$ is a covering map and G is the group of its automorphisms.*