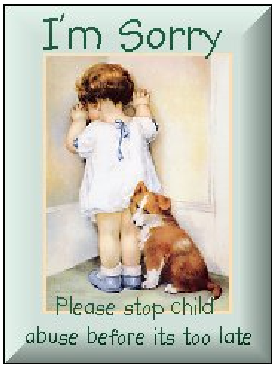
Covering Spaces Done Right

April 30, 2002



http://members.tripod.com/ ibkimmi1/page36.html

Let B be a topological space and let $\mathcal{C}(B)$ be the category of covering spaces of B: The category whose objects are coverings $X \to B$ and whose morphisms are maps between such coverings that commute with the covering projections — a morphism between $p_X : X \to B$ and $p_Y : Y \to B$ is a map $\alpha : X \to Y$ so that the diagram

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & B & \end{tikzcd}$$

is commutative.

Every topologists' wettest dream is to find that her/his favorite category of topological objects is equivalent to some category of easily understood algebraic objects. The following theorem realizes this dream in full in the case of the category $\mathcal{C}(B)$ of covering spaces of any reasonable base space B:

Theorem 1 (Classification of covering spaces)**.**

- *If B has base point b and fundamental group $G = \pi_1(B, b)$, then the map which assigns to every covering $p : X \to B$ its fiber $p^{-1}(b)$ over the basepoint b induces a functor $\mathcal{F}$ from the category $\mathcal{C}(B)$ of coverings of B to the category $\mathcal{S}(G)$ of G-sets — sets with a right G-action and set maps that respect the G action.*
- *If in addition B is connected, locally connected and semi-locally simply connected then the functor $\mathcal{F}$ is an equivalence of categories. (In fact, this is iff).*

If indeed the categories $\mathcal{C}(B)$ and $\mathcal{S}(G)$ are equivalent, one should be able to extract everything topological about a covering $p : X \to B$ from its associated G-set $\mathcal{F}(X) = p^{-1}(b)$. The following theorem shows this to be right in at least two cases:

Theorem 2. *For B connected, locally connected and semi-locally simply connected and X a covering of B:*

- *The set of connected components of X is in a bijective correspondence with the set of orbits of G in $\mathcal{F}(X)$.*
- *Let $x \in \mathcal{F}(X) = p^{-1}(b)$ be a basepoint for X that covers the basepoint b of B. Then the fundamental group $\pi_1(X, x)$ is isomorphic via the projection $p_\star$ into $G = \pi_1(B, b)$ to the the stabilizer group $\{h \in G : xh = x\}$ of x in $\mathcal{F}(X)$.*

(Both assertions of this theorem can be sharpened to deal with morphisms as well, but we will not bother to do so).

Ok. Every math technician can spend some time and effort and understand the statements and (only then) the proofs of these two theorems. Your true challenge is to digest the following statement:

Pretty much all there is to know about covering spaces is in these two theorems

In particular, most facts we learned about covering spaces are simple algebraic corollaries of these theorems:

Corollary 3. *If X is connected then its covering number (= "number of decks") is equal to the index of $H = p_\star\pi_1(X)$ in $G = \pi_1(B)$, and the decks of X are in a non-canonical correspondence with the left cosets $H\backslash G$ of H in G.*

Corollary 4. *If B is semi-locally simply connected, there exists a unique (up to base-point-preserving isomorphism) "universal covering space U of B" (a connected and simply connected covering U).*

Corollary 5. *The group of automorphisms of the universal covering U is equal to $G = \pi_1(B)$.*

Corollary 6. *If B is semi-locally simply connected, then for every $H < G = \pi_1(B)$ there is a unique (up to base-point-preserving isomorphism) connected covering space X with $p_\star\pi_1(X) = H$.*

Corollary 7. *If X_i for $i = 1, 2$ are connected coverings of B with groups $H_i = p_{i\star}\pi_1(X_i)$ and if $H_1 < H_2$ then X_1 is a covering of X_2 of covering number $(H_2 : H_1)$.*

Corollary 8. *If B is semi-locally simply connected there is a bijection between conjugacy classes of subgroups of $G = \pi_1(B)$ and unbased connected coverings of B.*

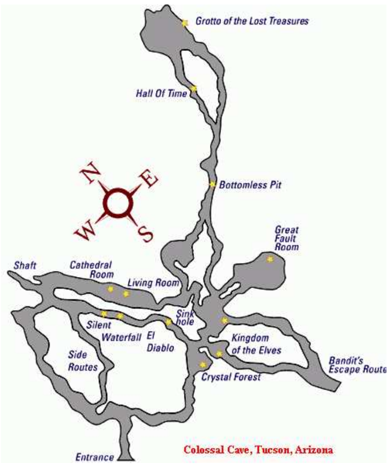
Corollary 9. *A connected covering X is normal iff its group $p_\star\pi_1(X)$ is normal in $G = \pi_1(B)$.*

Corollary 10. *If X is a connected covering of B and $H = p_\star\pi_1(X)$, then $\mathrm{Aut}(X) = N_G(H)/H$ where $N_G(H)$ is the normalizer of H in G.*

Corollary 11. *If I forgot anything, it follows too.*

Steps in the proofs of Theorem 1 and 2.

(1) Use path lifting to construct a right action of G on $p^{-1}(b)$.
(2) Show that this is indeed a group action and that morphisms of coverings induce morphisms of right G-sets.
(3) Start the construction of an "inverse" functor $\mathcal{G}$ of $\mathcal{F}$: Use spelunking (cave exploration) to construct a universal covering U of B, if B is semi-locally simply connected.
(4) Show that $\mathcal{F}(U) = G$.
(5) Use the construction of U or the general lifting property for covering spaces to show that there is a left action of G on U.
(6) For a general right G-set S set $\mathcal{G}(S) = S \times_G U = \{(s, u) \in S \times U\}/(sg, u) \sim (g, su)$ and show that $\mathcal{G}(S)$ is a covering of B and $\mathcal{F}(\mathcal{G}(S)) = S$.
(7) Show that $\mathcal{G}$ is compatible with maps between right G-sets.
(8) Understand the relationship between connected components and orbits.
(9) Prove Theorem 2.
(10) Use the existence and uniqueness of lifts to show that $\mathcal{G} \circ \mathcal{F}$ is equivalent to the identity functor (working connected component by connected component).



Thanks, Ronen Katz, for relevant comments.