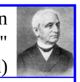
Convolutions on Lie Groups and Lie Algebras and Ribbon 2–Knots

Dror Bar-Natan, Trieste May 2009, http://www.math.toronto.edu/~drorbn/Talks/Trieste-0905

Disclaimer: Rough edges remain!

"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)



Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j:\mathfrak{g}\to\mathbb{R}$ be the Jacobian of the exponential map $\exp:\mathfrak{g}\to G$, and let $\Phi:\operatorname{Fun}(G)\to\operatorname{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x):=j^{1/2}(x)f(\exp x)$. Then if $f,g\in\operatorname{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f)\star\Phi(g)=\Phi(f\star g).$$

Group-Ring statement. There exists $\omega^2\in\operatorname{Fun}(\mathfrak{g})^G$ so that for every $\phi,\psi\in\operatorname{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2=j^{1/2}$)

$$\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)\omega^2_{x+y}e^{x+y}=\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)\omega^2_x\omega^2_y e^xe^y.$$

(shhh, this is Duflo)

Unitary statement. There exists $\omega\in\operatorname{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\operatorname{Fun}(\mathfrak{g}_x\times\mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}}=\widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

(2) $VV^*=I$ (3) $V\omega_{x+y}=\omega_x\omega_y$

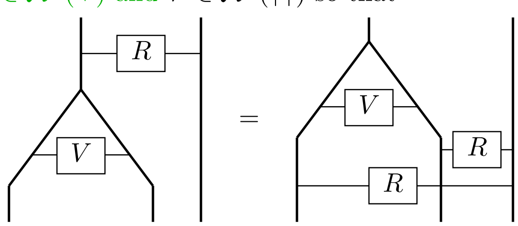
Algebraic statement. With $I\mathfrak{g}:=\mathfrak{g}^*\rtimes\mathfrak{g}$, with $c:\hat{\mathcal{U}}(I\mathfrak{g})\to\hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g})=\hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r\in\mathfrak{g}^*\otimes\mathfrak{g}$ the identity element and with $R=e^r\in\hat{\mathcal{U}}(I\mathfrak{g})\otimes\hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega\in\hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V\in\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta\otimes 1)(R)=R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}\otimes\hat{\mathcal{U}}(\mathfrak{g})$

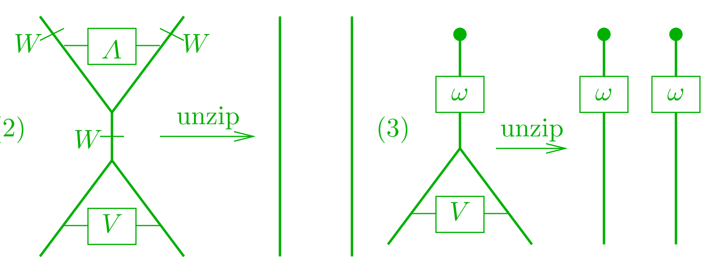
(2) $V\cdot SWV=1$ (3) $(c\otimes c)(V\Delta(\omega))=\omega\otimes\omega$

Diagrammatic statement. Let $R=\exp\,\text{⟼}\in\mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega\in\mathcal{A}^w(\uparrow)$ and $V\in\mathcal{A}^w(\uparrow\uparrow)$ so that

(1)

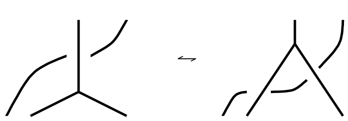


(2), (3)



Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:

(1)



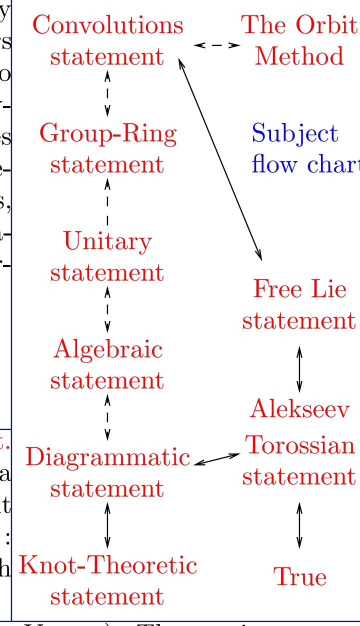
(2), (3)

The Orbit Method. By Fourier analysis, the characters of $(\operatorname{Fun}(\mathfrak{g})^G,\star)$ correspond to coadjoint orbits in $\mathfrak{g}^*$. By averaging representation matrices and using Schur's lemma to replace intertwiners by scalars, to every irreducible representation of G we can assign a character of $(\operatorname{Fun}(G)^G,\star)$.

Kashiwara

Vergne

Measure theoretic statement. Ignoring all ω's, there exists a measure preserving and orbit preserving transformation $T:\mathfrak{g}_x\times\mathfrak{g}_y\to\mathfrak{g}_x\times\mathfrak{g}_y$ for which $e^{x+y}\circ T=e^xe^y$.



Free Lie statement (Kashiwara-Vergne). There exist convergent Lie series F and G so that

$$x+y-\log e^ye^x=(1-e^{-\operatorname{ad}x})F+(e^{\operatorname{ad}y}-1)G$$

$$\operatorname{tr}(\operatorname{ad}x)\partial_xF+\operatorname{tr}(\operatorname{ad}y)\partial_yG=\frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1\right)$$

Alekseev

Alekseev-Torossian statement. There is an element $F\in\operatorname{TAut}_2$ with

$$F(x+y)=\log e^xe^y$$

and $j(F)\in\operatorname{im}\tilde{\delta}\subset\operatorname{tr}_2$, where for $a\in\operatorname{tr}_1$, $\tilde{\delta}(a):=a(x)+a(y)-a(\log e^xe^y)$.

Torossian

Convolutions and Group Rings (ignoring all Jacobians). If G is finite, $(\operatorname{Fun}(G),\star)\cong(\mathbb{R}G,\cdot)$ via $T:f\mapsto\sum f(a)\tau(a)$. For Lie $\mathfrak{g}$ and G,

$$\begin{array}{ccc}(\mathfrak{g},+)\ni x & \xrightarrow{\tau} & e^x\in\hat{\mathcal{S}}(\mathfrak{g})\\ \downarrow\exp & & \downarrow\chi\\ (G,\cdot)\ni e^x & \xrightarrow{\tau} & e^x\in\hat{\mathcal{U}}(\mathfrak{g})\end{array}\quad\text{so}\quad\begin{array}{ccc}\psi\in\operatorname{Fun}(\mathfrak{g}) & \xrightarrow{T} & \hat{\mathcal{S}}(\mathfrak{g})\\ \downarrow\Phi^{-1} & & \downarrow\chi\\ \operatorname{Fun}(G) & \xrightarrow{T} & \hat{\mathcal{U}}(\mathfrak{g})\end{array}$$

with $T\psi=\int\psi(x)e^xdx\in\hat{\mathcal{S}}(\mathfrak{g})$ and $T\Phi^{-1}\psi=\int\psi(x)e^x\in\hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i\in\operatorname{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1)\star\Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1\star\psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, T is a "Fourier transform")

$$\star\text{ in }G:\iint\psi_1(x)\psi_2(y)e^xe^y\qquad\star\text{ in }\mathfrak{g}:\iint\psi_1(x)\psi_2(y)e^{x+y}$$

Unitary $\Longrightarrow$ Group-Ring. $\iint\omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

$$=\langle\omega_{x+y},\omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle=\langle V\omega_{x+y},Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$$
$$=\langle\omega_x\omega_y,e^xe^yV\phi(x)\psi(y)\omega_{x+y}\rangle=\langle\omega_x\omega_y,e^xe^y\phi(x)\psi(y)\omega_x\omega_y\rangle$$
$$=\iint\omega^2_x\omega^2_ye^xe^y\phi(x)\psi(y).$$

Unitary $\Longleftrightarrow$ Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\operatorname{Fun}(\mathfrak{g})$:

- $\varphi\in\mathfrak{g}^*$ becomes a multiplication operator.
- $x\in\mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\operatorname{ad}x$: $(x\varphi)(y):=\varphi([x,y])$.
- c is now "the constant term".

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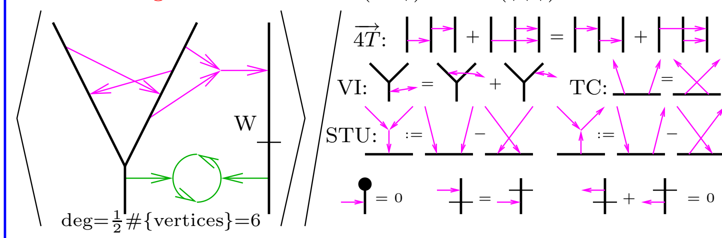
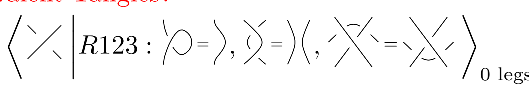
w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow)\cong\mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



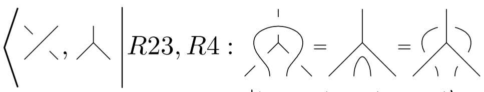
Δ acts by double and sum, S by reverse and negate.

What are w-Trivalent Tangles?

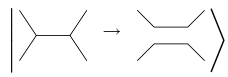
$\left\{\begin{matrix}\text{knots}\\ \text{\&links}\end{matrix}\right\} = PA\langle$ (crossing) $\mid R123:$ … $\rangle$ 0 legs



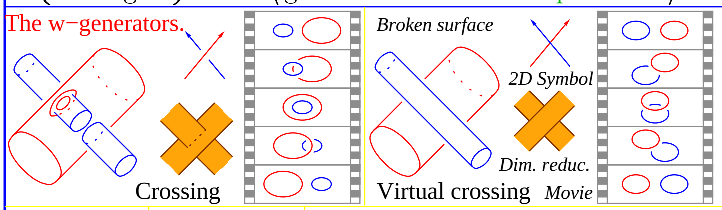
$\left\{\begin{matrix}\text{trivalent}\\ \text{tangles}\end{matrix}\right\} = PA\langle$ (crossing), (vertex) $\mid R23, R4:$ … $\rangle$

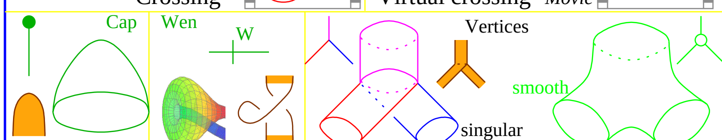


wTT=

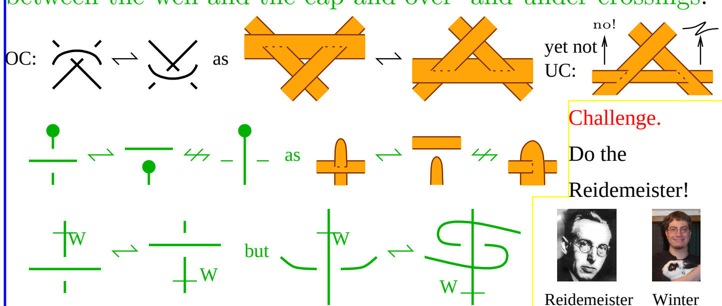


$\left\{\begin{matrix}\text{trivalent}\\ \text{w-tangles}\end{matrix}\right\} = PA\langle$ w-generators $\mid$ w-relations $\mid$ unary w-operations $\rangle$

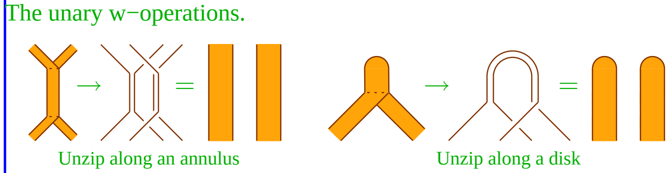




The w-realations include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2=1$, and funny interactions between the wen and the cap and over- and under-crossings:

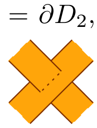


The unary w−operations.

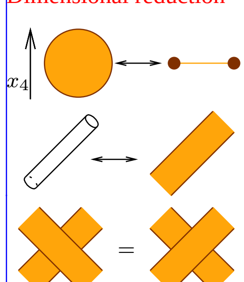


A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2\cap\partial B=\partial D_2$, modulo isotopies of S alone.

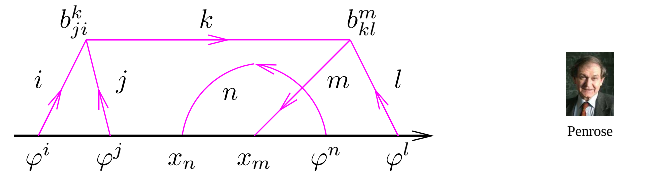
Example.



Dimensional reduction

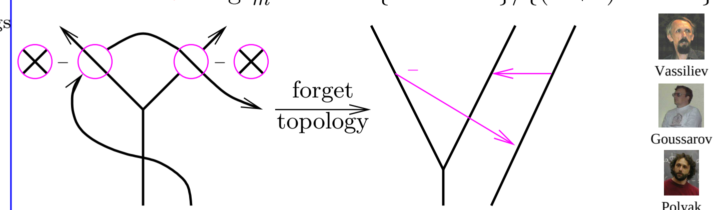


Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i,x_j]=\sum b_{ij}^k x_k$, we have $\mathcal{A}^w\to\mathcal{U}$ via



$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij}b^m_{kl}\varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g})$$

From wTT to $\mathcal{A}^w$. $\operatorname{gr}_m \text{wTT} := \{m\text{−cubes}\}/\{(m+1)\text{−cubes}\}$:



Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\text{ops} \circlearrowleft \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

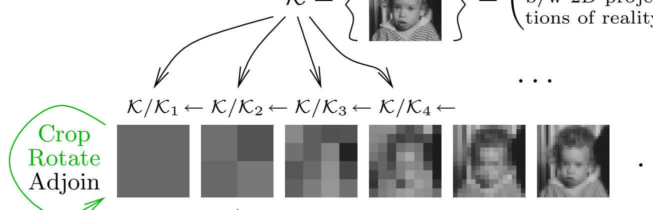
$$\Downarrow \qquad \downarrow Z$$

$$\text{ops} \circlearrowleft \operatorname{gr}\mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An expansion is a filtration respecting $Z:\mathcal{K}\to\operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

A concrete example.

$\mathcal{K} = \{$  $\} = \left(\text{The set of all b/w 2D projections of reality}\right)$



An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin

$$\mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \quad \cdots$$

$\mathcal{K}/\mathcal{K}_1 = \mathbb{R}$, $\quad \mathcal{K}_2/\mathcal{K}_3 = \ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)$

Our case(s).

$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z:\text{ high algebra}} \mathcal{A} := \operatorname{gr}\mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a "Lie" algebra } \mathfrak{g}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$

$\mathcal{K}$ is knot theory or topology; $\operatorname{gr}\mathcal{K}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.

But we have (at least) three knot theories, $u\to v\to w$, and thus their "high algebras" are related!

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle(\mathcal{K}_1)^m\rangle$ (using all available "products").

We skipped...
- The Alexander polynomial and Milnor numbers.
- u-Knots and Drinfel'd associators.
- v-Knots, quantum groups and Etingof-Kazhdan.
- BF theory and the successful religion of path integrals.

Unitary $\Longrightarrow$ Group-Ring. $\iint \omega_{x+y}^2 e^{x+y}\phi(x)\psi(y)$

$$=\langle\omega_{x+y},\omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle=\langle V\omega_{x+y},Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$$
$$=\langle\omega_x\omega_y,e^xe^yV\phi(x)\psi(y)\omega_{x+y}\rangle=\langle\omega_x\omega_y,e^xe^y\phi(x)\psi(y)\omega_x\omega_y\rangle$$
$$=\iint\omega_x^2\omega_y^2e^xe^y\phi(x)\psi(y).$$