Cosmic Coincidences and Several Other Stories, 1

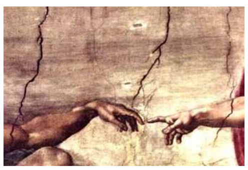
Dror Bar-Natan at the University of Tennessee
March 4, 2011, http://www.math.toronto.edu/~drorbn/Talks/Tennessee-1103/

Abstract. In the first half of my talk I will tell a cute and simple story — how given a knot in $\mathbb{R}^3$ one may count all possible "cosmic coincidences" associated with that knot, and how this count, appropriately packaged, becomes an invariant Z with values in some space $\mathcal{A}$ of linear combinations of certain trivalent graphs.

In the second half of my talk I will describe (rather sketchily, I'm afraid) a part of the story surrounding Z and $\mathcal{A}$: How the same Z also comes from quantum field theory, Feynman diagrams, and configuration space integrals. How $\mathcal{A}$ is a space of universal formulas which make sense in every metrized Lie algebra and how specific choices for that Lie algebra correspond to various famed knot invariants. How Z solves a universal topological problem, and how solving for Z is solving some universal Lie-algebraic problem. All together, this is the u-story.

In the remaining time I will mention several other Z's and $\mathcal{A}$'s and the parallel (yet sometimes interwoven) stories surrounding them — the v-story, and w-story, and perhaps also the p-story. Each of these stories is clearly still missing some chapters.

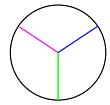
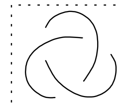
Creation of Adam

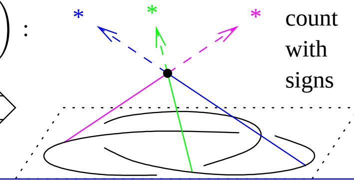


Michelangelo

Disclaimer
We'll concentrate on the beauty and ignore the cracks.

$\langle D, K\rangle_{\text{Stonehenge}} :=$ (The signed Stonehenge pairing of D and K) : count with signs

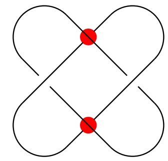
$D=$  $K=$  $\Rightarrow$



The Gaussian linking number

$lk(\text{two linked circles}) = \sum_{\text{vertical chopsticks}} (\text{signs})$

$= \langle \text{O—O}, \text{two linked circles} \rangle_{\text{Stonehenge}}$



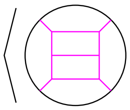
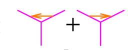
C.F. Gauss

The generating function of all cosmic coincidences:

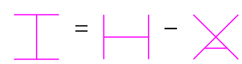
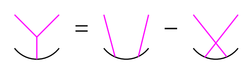
D. Thurston

$$Z(K) := \lim_{N\to\infty} \sum_{3\text{-valent } D} \frac{\langle D,K\rangle_{\text{Stonehenge}} D}{2^c c! \binom{N}{e}} \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

N := # of stars
c := # of chopsticks
e := # of edges of D

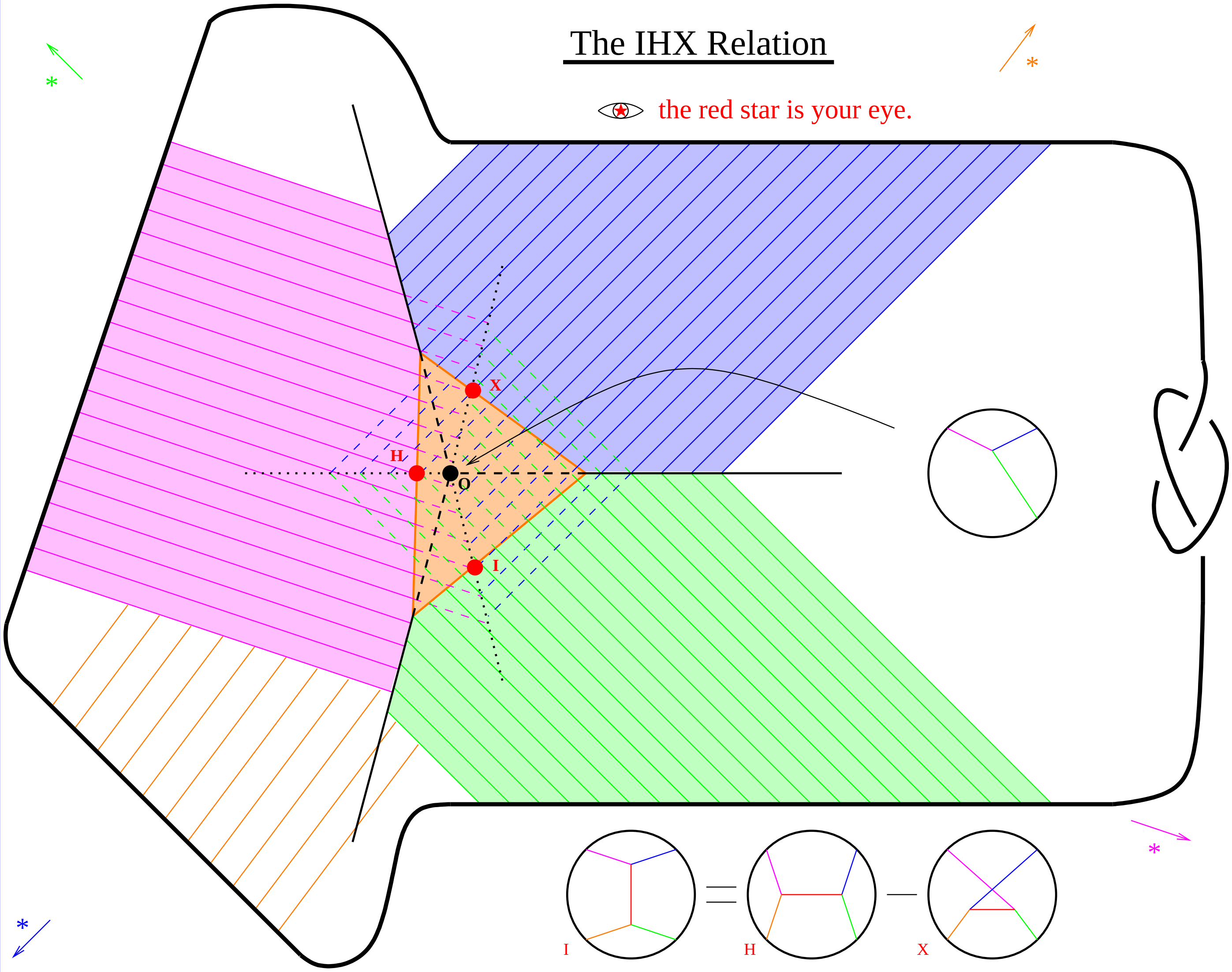
$\mathcal{A}(\circlearrowleft)$:= Span ⟨  ⟩ / (oriented vertices; AS:  =0 & more relations)

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,	An intersection line cuts through the knot – Solution: Impose STU,	The Gauss curve slides over a star – Solution: Multiply by a framing-dependent counter-term.
		
(see below)	(similar argument)	(not shown here)

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

The IHX Relation



The Cast
in rough historical order

The Neolithic People

Carl Friedrich Gauss
Edward Witten
Victor Vassiliev
Mikhail Goussarov

Maxim Kontsevich

Raoul Bott

Clifford Taubes

Thang Le

Jun Murakami

Tomotada Ohtsuki

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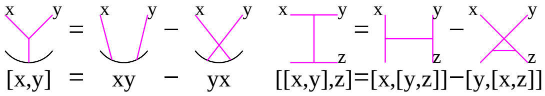
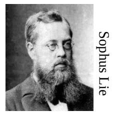
"Low Algebra" and universal formulae in Lie algebras.



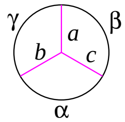


More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [X_a, X_c], X_c \rangle \qquad X_a v_\beta = \sum_\gamma r^\beta_{a\gamma} v_\gamma$$

and then

$$W_{\mathfrak{g},R} :$$



$$\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

$W_{\mathfrak{g},R} \circ Z$ is often interesting:

$\mathfrak{g} = sl(2) \longrightarrow$  The Jones polynomial

$\mathfrak{g} = sl(N) \longrightarrow$  The HOMFLYPT polynomial

Przytycki

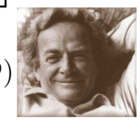
$\mathfrak{g} = so(N) \longrightarrow$  The Kauffman polynomial

Chern-Simons-Witten theory and Feynman diagrams.

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]$$

$$\longrightarrow \sum_{D:\ \text{Feynman diagram}} W_{\mathfrak{g}}(D) \not\sum \mathcal{E}(D) \longrightarrow \sum_{D:\ \text{Feynman diagram}} D \not\sum \mathcal{E}(D)$$

Witten



Feynman

Definition. *V* is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

Theorem. All knot polynomials (Conway, Jones, etc.) are of finite type.

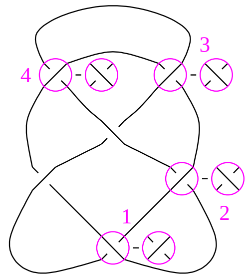
Conjecture. (Taylor's theorem) Finite type invariants separate knots.

Theorem. *Z(K)* is a universal finite type invariant!

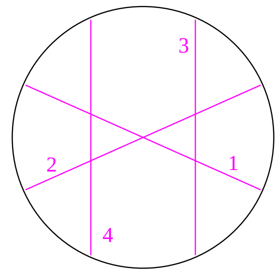
(sketch: to dance in many parties, you need many feet).

Vassiliev

Goussarov



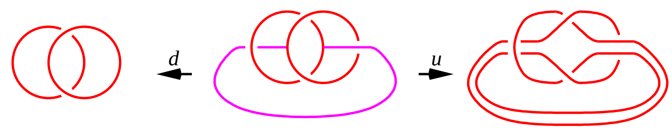
The Miller Institute knot



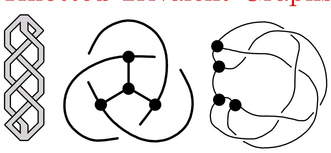
Knots are the wrong objects to study in knot theory!

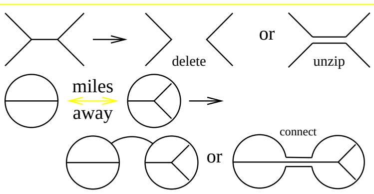
They are not finitely generated and they carry no interesting operations.

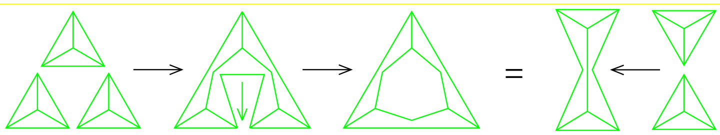
Algebraic Knot Theory



Knotted Trivalent Graphs







Theorem (∼, "High Algebra"). A homomorphic Z is the same as a "Drinfel'd Associator".

Drinfel'd

The u→v→w & p Stories

explained | sketched | could explain | could explain, gaps remain | more gaps then explains | mystery

	Topology	Combinatorics	Low Algebra	High Algebra	Counting Coincidences Conf. Space Integrals	Quantum Field Theory	Graph Homology
u-Knots	The usual Knotted Objects (KOs) in 3D — braids, knots, links, tangles, knotted graphs, etc.	Chord diagrams and Jacobi diagrams, modulo $4T$, STU, IHX, etc.	Finite dimensional metrized Lie algebras, representations, and associated spaces.	The Drinfel'd theory of associators.	Today's work. Not beautifully written, and some detour-forcing cracks remain.	Perturbative Chern-Simons-Witten theory.	The "original" graph homology.
↓ v-Knots	Virtual KOs — "algebraic", "not embedded"; KOs drawn on a surface, mod stabilization.	Arrow diagrams and v-Jacobi diagrams, modulo $6T$ and various "directed" STUs and IHXs, etc.	Finite dimensional Lie bi-algebras, representations, and associated spaces.	Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras.	No clue.	No clue.	No clue.
↓ w-Knots	Ribbon 2D KOs in 4D; "flying rings". Like v, but also with "overcrossings commute".	Like v, but also with "tails commute". Only "two in one out" internal vertices.	Finite dimensional co-commutative Lie bi-algebras ($\mathfrak{g} \ltimes \mathfrak{g}^*$), representations, and associated spaces.	The Kashiwara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups / algebras.	No clue.	Probably related to 4D BF theory.	Studied.
p-Objects	No clue.	"Acrobat towers" with 2-in many-out vertices.	Poisson structures.	Deformation quantization of poisson manifolds.	Configuration space integrals are key, but they don't reduce to counting.	 Work of Cattaneo.	Studied. Hyperbolic geometry ?