

M. Khovanov

Local Differentials and Matrix Factorizations

L. Rozansky



Dror Bar–Natan at UIUC, March 11, 2004, http://www.math.toronto.edu/~drorbn/Talks/UIUC–050311/

Quantum algebra:

Claim. If $ba=qab$ then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}$$

where

$$(n)_q := 1 + q + \ldots + q^{n-1},$$

$$(n)!_q := (1)_q (2)_q \cdots (n)_q,$$

$$\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q (n-k)!_q}.$$

Conjecture:

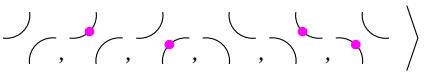
(I. Frenkel, though he may disown this version)

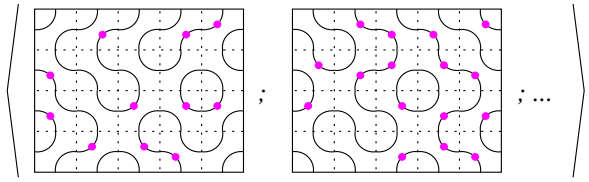
1. Every object in mathematics is the Euler characteristic of a complex.
2. Every operation in mathematics lifts to an operation between complexes.
3. Every identity in mathematics is true up to homotopy at complex–level.



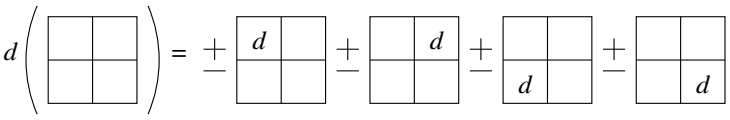
I. Frenkel

Local state spaces:

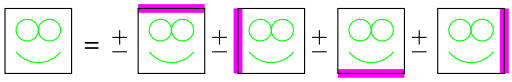
$V=$ 

$V^{\otimes(4\times5)} =$ 

Local differentials:

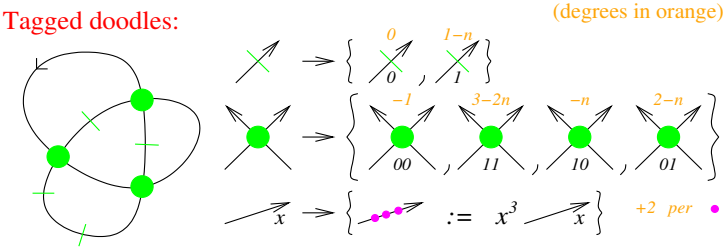


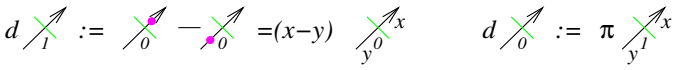
where

$d^2 =0$ **or** $d^2 =$ 

Tagged doodles:

(degrees in orange)

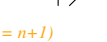




$$\ln[1]:= n = 2;\ \pi_{i_,j_} := \text{Cancel}\left[\frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j}\right];\ \pi_{1,2}$$

$$\text{Out}[1]= x_1^2 + x_1 x_2 + x_2^2$$

$$\ln[2]:= L = \begin{pmatrix} 0 & x_1 - x_2 \\ \pi_{1,2} & 0 \end{pmatrix};$$

Set $L=d$ 

$$\text{Expand}[L.L] \text{ // MatrixForm}$$

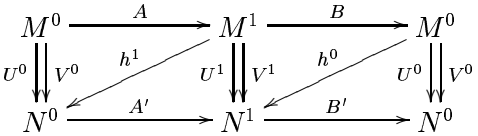
(deg d = $n+1$)

$$\text{Out}[3]//\text{MatrixForm}= \begin{pmatrix} x_1^3 - x_2^3 & 0 \\ 0 & x_1^3 - x_2^3 \end{pmatrix}$$

Matrix factorizations:

$$D = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

$$AB = BA = \omega I$$



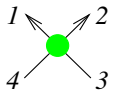
A category, with "complexes", morphisms, homotopies, direct sums and tensor products.



D. Eisenbud

See Khovanov and Rozansky, arXiv:math.QA/0401268

Likewise, set $Q=d$ with:



(deg d = $n+1$)

$$\ln[4]:= Q := \begin{pmatrix} 0 & 0 & v_1 & v_2 \\ 0 & 0 & u_2 & -u_1 \\ u_1 & v_2 & 0 & 0 \\ u_2 & -v_1 & 0 & 0 \end{pmatrix};$$

$$\{v_1, v_2\} = \{x_1 + x_2 - x_3 - x_4,\ x_1 x_2 - x_3 x_4\};$$

$$\ln[6]:= g[s_, p_] := s^{n+1} + (n+1) \sum_{i=1}^{(n+1)/2} \frac{(-1)^i}{i} \text{Binomial}[n-i, i-1]\, s^{n+1-2i}\, p^i;$$

$$g[x+y, x\,y] \text{ // Expand}$$

$$\text{Out}[6]= x^3 + y^3$$

$$\ln[7]:= \{u_1, u_2\} = \text{Cancel}\left[\left\{\frac{g[x_1+x_2, x_1 x_2] - g[x_3+x_4, x_1 x_2]}{v_1}, \frac{g[x_3+x_4, x_1 x_2] - g[x_3+x_4, x_3 x_4]}{v_2}\right\}\right]$$

$$\text{Out}[7]= \{x_1^2 - x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2 + x_1 x_4 + x_2 x_4 + 2 x_3 x_4 + x_4^2,\ -3(x_3 + x_4)\}$$

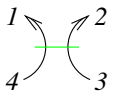
$$\ln[8]:= \omega = u_1 v_1 + u_2 v_2 \text{ // Expand}$$

$$\text{Out}[8]= x_1^3 + x_2^3 - x_3^3 - x_4^3$$

$$\ln[9]:= \text{Simplify}[Q.Q == \omega\, \text{IdentityMatrix}[4]]$$

$$\text{Out}[9]= \text{True}$$

Example: Set $P=d$

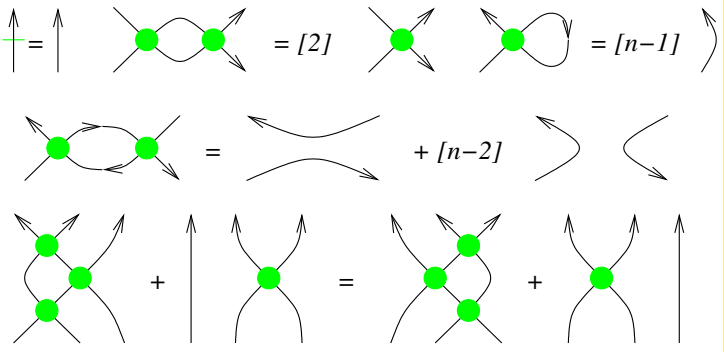


$$\ln[10]:= P = \begin{pmatrix} 0 & 0 & x_1 - x_4 & x_2 - x_3 \\ 0 & 0 & \pi_{2,3} & -\pi_{1,4} \\ \pi_{1,4} & x_2 - x_3 & 0 & 0 \\ \pi_{2,3} & x_4 - x_1 & 0 & 0 \end{pmatrix};$$

$$\ln[11]:= \text{Simplify}[P.P == \omega\, \text{IdentityMatrix}[4]]$$

$$\text{Out}[11]= \text{True}$$

Theorem: (Kh–Ro) Taking homology and then the graded Euler characteristics, we get the [MOY] relations:



[MOY] := Murakami, Ohtsuki, Yamada, Enseignement Math. 44 (1998)

$$[k] := \frac{q^k - q^{-k}}{q - q^{-1}}$$

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