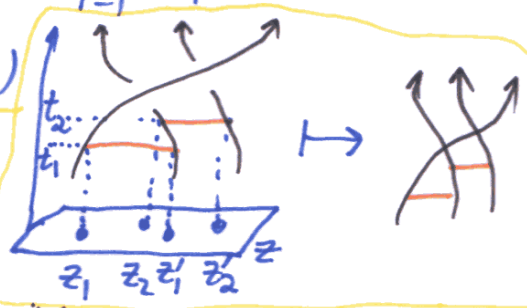
Math 1352 Algebraic Knot Theory – The Knizhnik-Zamolodchikov Connection

Theorem 1. The following is an invariant of braids in $\mathbb{R}_t \times \mathbb{C}_z$ (Fixed endpoints)

$$Z(B) = \sum\!\!\!\!\!\!\int_{\substack{t_1 \le \dots \le t_m \\ p = (\{z_i, z_i'\})}} \frac{D_p}{(2\pi i)^m} \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \quad \text{in} \quad \mathcal{A}(\uparrow_n) := \langle t^{ij} : 1 \le i \ne j \le n \rangle \Big/ \begin{matrix} t^{ij} = t^{ji}; \\ [t^{ij}, t^{kl}] = 0 \\ [t^{ij}, t^{ik} + t^{jk}] = 0 \end{matrix}$$



$= \langle$  $\rangle / 4T$

horizontal chords.

= "Chord diagrams for braids".

Formal Connections & Curvature.

Let $\Omega \in \Omega^1(M, \overset{\text{graded unital.}}{A})$ with $\deg \Omega = 1$.

$\gamma : [0,1] = I \to M$ induces

$\phi : \Delta^m = \{0 \le t_1 \le \dots \le t_m \le 1\} \to M^m$.

Set $\mathrm{hol}_\gamma(\Omega) = P\exp \int_\gamma \Omega = \sum\!\!\!\!\!\!\int_{\Delta^m} \phi^* \Omega^m$

where $\Omega^m := \pi_1^* \Omega \wedge \dots \wedge \pi_m^* \Omega$

Theorem 2. If $F_\Omega := d\Omega + \Omega \wedge \Omega = 0$, then $\mathrm{hol}_\gamma(\Omega)$ is invariant under end-point preserving homotopies of γ.

Proof 2. Let $\Gamma : I_s \times I_t \to M$, $\Phi : I_s \times \Delta^m \to M^m$;

By Stokes',

$$\int_{\Delta^m} \Phi_1^* \Omega^m - \int_{\Delta^m} \Phi_0^* \Omega^m = \int_{I \times \Delta^m} d\Phi^* \Omega^m - \int_{I \times \partial\Delta^m} \Phi^* \Omega^m =: A_m - B_m$$

Now

$$A_m = \sum_{k=1}^{m} (-1)^{k+1} \int_{I \times \Delta^m} \pi_1^* \Omega \wedge \dots \wedge \pi_k^* d\Omega \wedge \dots \wedge \pi_m^* \Omega$$

and

$$B_m = \underbrace{\int_{I \times [t_1 = 0]} \Phi^* \Omega^m}_{=0} \pm \underbrace{\int_{I \times [t_m = 1]} \Phi^* \Omega^m}_{=0} + \sum_{k=1}^{m-1} (-1)^k \int_{I \times [t_k = t_{k+1}]} \Phi^* \Omega^m$$

(fixed endpoints)

$$= \sum_{k=1}^{m-1} (-1)^k \int_{I \times \Delta^{m-1}} \pi_1^* \Omega \wedge \dots \wedge \pi_k^* (\Omega \wedge \Omega) \wedge \dots \wedge \pi_{m-1}^* \Omega$$

and now $\sum A_m = \sum B_m$ by telescopic summation & $F_\Omega = 0$.

The KZ connection.

$M = \mathbb{C}^n \setminus \{\text{diagonals}\}$, $A = \mathcal{A}(\uparrow_n)$,

and $\Omega = \sum_{i<j} t^{ij} w_{ij}$ where $w_{ij} := \frac{dz_i - dz_j}{z_i - z_j} \underset{\text{locally}}{=} d\log(z_i - z_j)$

Compute $F_\Omega = d\Omega + \Omega \wedge \Omega$: $dw_{ij} = 0$ so $d\Omega = 0$.

$$\Omega \wedge \Omega = \sum_{\substack{i<j \\ k<l}} t^{ij} t^{kl} w_{ij} \wedge w_{kl} = \underbrace{A}_{|\{ijkl\}| = 2} + \underbrace{B}_{=3} + \underbrace{C}_{=4} \quad \text{where}$$

$A = C = 0$ as $[t^{ij}, t^{kl}] = 0$ if $|\{ijkl\}| = 2$ or 4 and

$$B = \sum_{\alpha < \beta < \gamma} [t^{\alpha\beta}, t^{\beta\gamma}] w_{\alpha\beta} \wedge w_{\beta\gamma} + \text{cyclic perms}$$

$$= \sum_{\alpha < \beta < \gamma} Y^{\alpha\beta\gamma} (w_{\alpha\beta} \wedge w_{\beta\gamma} + \text{cyc perms}) = 0$$

$= 0$ by "Arnold's identity"

Note: by 4T, $[t^{\alpha\beta}, t^{\beta\gamma}] = Y^{\alpha\beta\gamma} =$ 

Proof of 1

Simply take in theorem 2,

γ = the braid

and

Ω = the KZ connection.

Dror Bar-Natan, Feb 13, 2007

More at http://drorbn.net/index.php?title=07-1352/Class_Notes_for_February_13 and at http://www.math.toronto.edu/~drorbn/Talks/Aarhus-1305/ (day 2)