Dror Bar-Natan: Talks: MSRI-0808:

Projectivization, w-Knots, Kashiwara-Vergne and Alekseev-Torossian

"An Algebraic Structure"

The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

The Projectivization Tentative Speculative Paradigm.

Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

Graded Equations Examples

- $e(x+y)=e(x)e(y)$ in $\mathbb{Q}[[x,y]]$.
- The pentagon and hexagons in $\mathcal{A}(\uparrow_{3,4})$.
- The equations defining a QUEA, the work of Etingof and Kazhdan.

- The Alekseev-Torossian equations in $\mathcal{U}(\mathrm{sder}_n)$ and $\mathcal{U}(\mathrm{tder}_n)$.

sder $\leftrightarrow$ tree-level $\mathcal{A}$
tder $\leftrightarrow$ more

$$F\in\mathcal{U}(\mathrm{tder}_2);\quad F^{-1}e(x+y)F=e(x)e(y)\iff F\in\mathrm{Sol}_0$$

$$\Phi=\Phi_F:=(F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23}\in\mathcal{U}(\mathrm{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4}=\Phi^{12,3,4}\Phi^{1,2,34}\qquad\text{"the pentagon"}$$

$t=\frac{1}{2}(y,x)\in\mathrm{sder}_2$ satisfies $4T$ and $r=(y,0)\in\mathrm{tder}_2$ satisfies $6T$

$R:=e(r)$ satisfies Yang-Baxter: $R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$

also $R^{12,3}=R^{13}R^{23}$ and $F^{23}R^{1,23}(F^{23})^{-1}=R^{12}R^{13}$

$\tau(F):=RF^{21}e(-t)$ is an involution, $\Phi_{\tau(F)}=(\Phi_F^{321})^{-1}$

$\mathrm{Sol}_0^\tau:=\{F:\tau(F)=F\}$ is non-empty; for $F\in\mathrm{Sol}_0^\tau$,

$$e(t^{13}+t^{23})=\Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

and $$e(t^{12}+t^{13})=(\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$


Alekseev

This is just a part of the Alekseev-Torossian work!

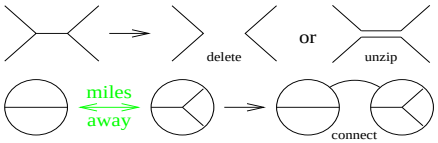

Torossian

So What?

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

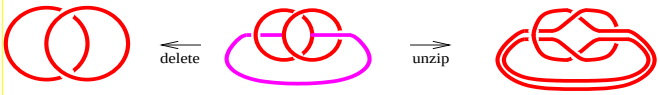
Knotted Trivalent Graphs

$\mathcal{O}(\text{⧋})=\{$  $,\dots\}$



Theorem. KTG is generated by the unknotted ⧋ and the Möbius band, with identifiable relations between them.

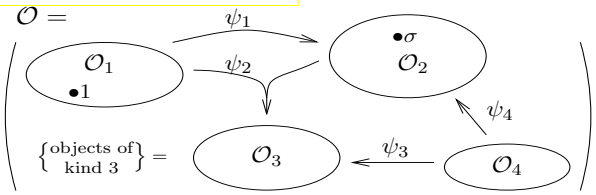
Theorem. $Z(\text{⧋})$ is equivalent to an associator Φ.



Algebraic Knot Theory

Theorem. {ribbon knots} $\sim\{u\gamma:\gamma\in\mathcal{O}(\circ\!\!-\!\!\circ),\ d\gamma=\bigcirc\!\!\bigcirc\}$.

Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Defining proj $\mathcal{O}$.

The augmentation "ideal":

$$I=I_{\mathcal{O}}:=\left\{\begin{array}{l}\text{formal differences of ob-}\\ \text{jects "of the same kind"}\end{array}\right\}.$$

Then $I^n:=\left\{\begin{array}{l}\text{all outputs of algebraic}\\ \text{expressions at least } n \text{ of}\\ \text{whose inputs are in } I\end{array}\right\}$, and

$$\mathrm{proj}\,\mathcal{O}:=\bigoplus_{n\ge 0}I^n/I^{n+1}\qquad\left(\begin{array}{l}\text{has same kinds and opera-}\\ \text{tions, but different objects}\\ \text{and axioms}\end{array}\right).$$

Knot Theory Anchors.


Vassiliev


Goussarov

- $(\mathcal{O}/I^{n+1})^\star$ is "type n invariants".
- $(I^n/I^{n+1})^\star$ is "weight systems".
- proj $\mathcal{O}$ is $\mathcal{A}$, "chord diagrams".

Warmup Examples.

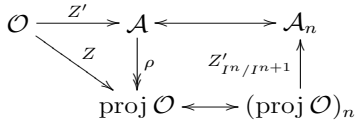
- The projectivization of a group is a graded associative algebra.
- A quandle: a set Q with a binary op $\wedge$ s.t.

$$1\wedge x=1,\quad x\wedge 1=x\wedge x=x,\qquad\text{(appetizers)}$$
$$(x\wedge y)\wedge z=(x\wedge z)\wedge(y\wedge z).\qquad\text{(main)}$$

proj Q is a graded Lie algebra: set $\bar v:=(v-1)$ (these generate I!), feed $1+\bar x$, $1+\bar y$, $1+\bar z$ in (main), collect the surviving terms of lowest degree:

$$(\bar x\wedge\bar y)\wedge\bar z=(\bar x\wedge\bar z)\wedge\bar y+\bar x\wedge(\bar y\wedge\bar z).$$

An Expansion is $Z:\mathcal{O}\to\mathrm{proj}\,\mathcal{O}$ s.t. $Z(I^n)\subset(\mathrm{proj}\,\mathcal{O})_{\ge n}$ and $Z_{I^n/I^{n+1}}=Id_{I^n/I^{n+1}}$ (A "universal finite type invariant"). In practice, it is hard to determine proj $\mathcal{O}$, but easy to guess a surjection $\rho:\mathcal{A}\to\mathrm{proj}\,\mathcal{O}$. So find $Z':\mathcal{O}\to\mathcal{A}$ with $Z'(I^n)\subset\mathcal{A}_{\ge n}$ and $Z'_{I^n/I^{n+1}}\circ\rho_n=Id_{\mathcal{A}_n}$:



Can you make this diagram less confusing?


X-S. Lin

Homomorphic Expansions are expansions that intertwine the algebraic structure on $\mathcal{O}$ and proj $\mathcal{O}$. They provide finite / combinatorial handles on global problems.

The Key Point. If $\mathcal{O}$ is finitely presented, finding a homomorphic expansion is solving finitely many equations with finitely many unknowns, in some graded spaces.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MSRI-0808/