



# From Stonehenge to Witten Skipping all the Details

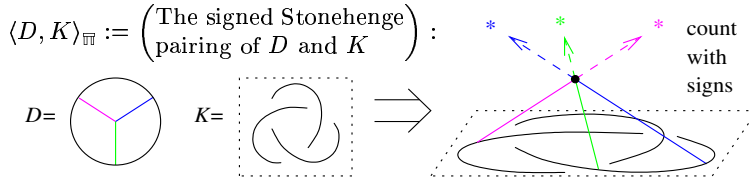
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It is well known that when the Sun rises on midsummer's morning over the "Heel Stone" at Stonehenge, its first rays shine right through the open arms of the horseshoe arrangement. Thus astrological lineups, one of the pillars of modern thought, are much older than the famed Gaussian linking number of two knots.

Recall that the latter is itself an astrological construct: one of the standard ways to compute the Gaussian linking number is to place the two knots in space and then count (with signs) the number of shade points cast on one of the knots by the other knot, with the only lighting coming from some fixed distant star.

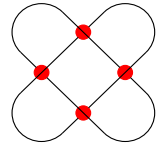


The Gaussian linking number

$$lk(\bigcirc) = \frac{1}{2} \sum_{\text{vertical chopsticks}} (\text{signs})$$



Carl Friedrich Gauss



$lk=2$

Dylan Thurston

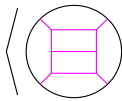


Thus we consider the generating function of all stellar coincidences:

$$Z(K) := \lim_{N \rightarrow \infty} \sum_{\substack{D \\ \text{3-valent}}} \frac{1}{2^e c! \binom{N}{e}} \langle D, K \rangle_{\overline{\mathbb{R}^3}} D \cdot \left( \begin{array}{c} \text{framing-} \\ \text{dependent} \\ \text{counter-term} \end{array} \right) \in \mathcal{A}(\odot)$$

$N$  := # of stars  
 $c$  := # of chopsticks  
 $e$  := # of edges of  $D$

$\mathcal{A}(\odot)$



oriented vertices  
 AS:  $\text{Y} + \text{Y} = 0$   
 & more relations

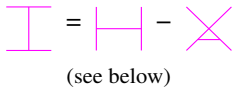
**Theorem.** Modulo Relations,  $Z(K)$  is a knot invariant!

When deforming, catastrophes occur when:

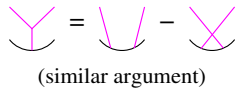
A plane moves over an intersection point –  
 Solution: Impose IHX,

An intersection line cuts through the knot –  
 Solution: Impose STU,

The Gauss curve slides over a star –  
 Solution: Multiply by a framing-dependent counter-term.



(see below)

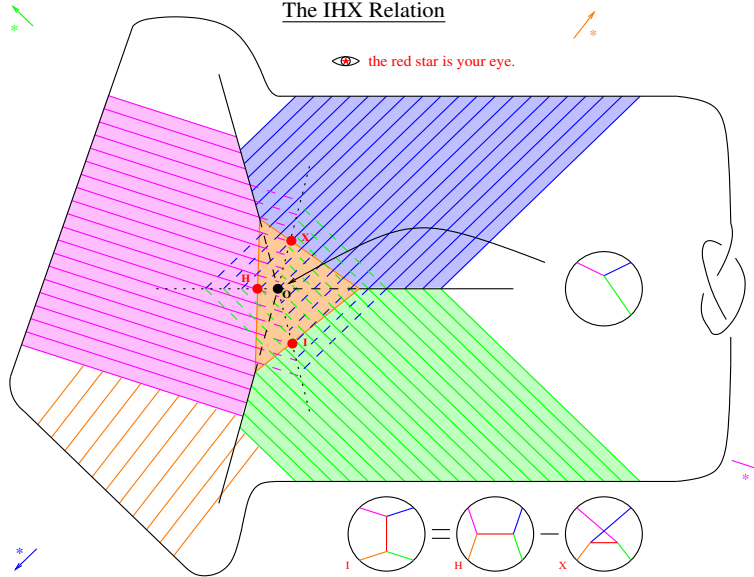


(similar argument)

(not shown here)

The IHX Relation

the red star is your eye.



$V$ : vector space  
 $dV$ : Lebesgue's measure on  $V$ .  
 $Q$ : A quadratic form on  $V$ ;  
 $Q(V) = \langle L^2 V, V \rangle$  where  
 $L: V \rightarrow V^*$  is linear  
**Compute**  $I = \int_V dV e^{\pm Q + P}$   
 $= \int_V \frac{1}{m!} dV P^m e^{Q/2}$   
 $\sim \sum_{m=0}^{\infty} \frac{1}{m!} P^m (2V) e^{-\frac{1}{2} Q^*(V)}$   
 $= \sum_{m=0}^{\infty} \frac{\epsilon^{11} P^m (2V) (Q^*)^m}{2^m m!} \Big|_{V=0}$

In our case,  
 $\star Q$  is  $d$ , so  $Q^{-1}$  is an integral operator.  
 $\star P$  is  $\frac{2}{3} A^3 A^3 A^3$   
 $\star H$  is the homonomy, itself a sum of integrals along the knot  $K$ ,  
 & when the dust settles, we get  $Z(K)$ !

The Fourier Transform:  
 $(F: V \rightarrow C) \Rightarrow (F: V^* \rightarrow C)$   
 via  $F(V) = \int_V F(V) e^{-i \langle V, V \rangle} dV$ .  
 Simple Facts:  
 1.  $F(0) = \int_V F(V) dV$ .  
 2.  $\frac{\partial}{\partial V_i} F \sim \sqrt{i} F$ .  
 3.  $(e^{Q/2}) \sim e^{-Q^*/2}$   
 where  $Q^*(V) = \langle V, L^{-1} V \rangle$   
 (That's the heart of the Fourier Inversion Formula).

So  $\int_V H(V) e^{\pm Q + P} dV$   
 $\sim H(2V) e^{Q/2} e^{-Q^*(V)/2} \Big|_{V=0}$   
 is  $\sum \text{products of } Q^{-1}\text{'s, } P\text{'s and one } H$   
 $= \sum_{\text{Diagrams}} c(D) \left( \text{products of } Q^{-1}\text{'s, } P\text{'s and one } H \right)$

Differentiation and Pairings:  
 $\frac{\partial^3}{\partial x^2 \partial y} x^3 y^2 = 3! 2! j$  indeed,  
 $\left\{ \frac{\partial x}{x} \frac{\partial x}{x} \frac{\partial x}{x} \frac{\partial y}{y} \frac{\partial y}{y} \right\}$   
 $(\lambda_{ijk} \partial_i \partial_j \partial_k)^2 (\lambda^{lmn} \psi_l \psi_m \psi_n)^3$  is  
 $\left\{ \frac{\lambda_{ijk}}{x^i} \frac{\lambda_{lmn}}{x^l} \frac{\lambda_{pqr}}{x^p} \right\}$  (2 possible)

"God created the knots, all else in topology is the work of man."



Leopold Kronecker (modified)

It all is perturbative Chern-Simons-Witten theory:

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A \text{hol}_K(A) \exp \left[ \frac{ik}{4\pi} \int_{\mathbb{R}^3} \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right]$$

$$\rightarrow \sum_{D: \text{Feynman diagram}} W_{\mathfrak{g}}(D) \int \mathcal{E}(D) \rightarrow \sum_{D: \text{Feynman diagram}} D \int \mathcal{E}(D)$$



Shiing-shen Chern



James H Simons

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