



Local Differentials and Matrix Factorizations



Dror Bar-Natan at UIUC, March 11, 2004, <http://www.math.toronto.edu/~drorbn/Talks/UIUC-050311/>

Quantum algebra:

where

Claim. If $ba=qab$ then

$$(n)_q := 1 + q + \dots + q^{n-1},$$

$$(n)!_q := (1)_q(2)_q \cdots (n)_q,$$

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k}_q a^k b^{n-k}$$

$$\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q (n-k)!_q}.$$

Conjecture:

(I. Frenkel, though he may disown this version)

1. Every object in mathematics is the Euler characteristic of a complex.
2. Every operation in mathematics lifts to an operation between complexes.
3. Every identity in mathematics is true up to homotopy at complex-level.



I. Frenkel

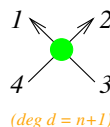
Local state spaces:

$$V = \left\langle \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right\rangle$$

$$V^{\otimes(4 \times 5)} = \left\langle \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} ; \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} ; \dots \right\rangle$$

Likewise, set $Q=d$ with:

$$\text{In}[4]= Q := \begin{pmatrix} 0 & 0 & v_1 & v_2 \\ 0 & 0 & u_2 & -u_1 \\ u_1 & v_2 & 0 & 0 \\ u_2 & -v_1 & 0 & 0 \end{pmatrix};$$



(deg d = n+1)

$$\{v_1, v_2\} = \{x_1 + x_2 - x_3 - x_4, x_1 x_2 - x_3 x_4\};$$

$\text{In}[6]= g[s_, p_] :=$

$$s^{n+1} + (n+1) \sum_{i=1}^{(n+1)/2} \frac{(-1)^i}{i} \text{Binomial}[n-i, i-1] s^{n+1-2i} p^i;$$

$g[x+y, x y] // \text{Expand}$

$$\text{Out}[6]= x^3 + y^3$$

$\text{In}[7]= \{u_1, u_2\} =$

$$\text{Cancel} \left[\left\{ \frac{g[x_1 + x_2, x_1 x_2] - g[x_3 + x_4, x_1 x_2]}{v_1}, \frac{g[x_3 + x_4, x_1 x_2] - g[x_3 + x_4, x_3 x_4]}{v_2} \right\} \right]$$

$$\text{Out}[7]= \{x_1^2 - x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2 + x_1 x_4 + x_2 x_4 + 2 x_3 x_4 + x_4^2, -3(x_3 + x_4)\}$$

$\text{In}[8]= \omega = u_1 v_1 + u_2 v_2 // \text{Expand}$

$$\text{Out}[8]= x_1^3 + x_2^3 - x_3^3 - x_4^3$$

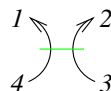
$\text{In}[9]= \text{Simplify}[Q.Q == \omega \text{IdentityMatrix}[4]]$

$$\text{Out}[9]= \text{True}$$

$\text{In}[10]=$

Example: Set $P=d$ with:

$$P = \begin{pmatrix} 0 & 0 & x_1 - x_4 & x_2 - x_3 \\ 0 & 0 & \pi_{2,3} & -\pi_{1,4} \\ \pi_{1,4} & x_2 - x_3 & 0 & 0 \\ \pi_{2,3} & x_4 - x_1 & 0 & 0 \end{pmatrix};$$



$\text{In}[11]=$

$\text{Simplify}[P.P == \omega \text{IdentityMatrix}[4]]$

$$\text{Out}[11]= \text{True}$$

Theorem: (Kh-Ro) Taking homology and then the graded Euler characteristics, we get the [MOY] relations:

$$\uparrow = \uparrow \quad \text{---} \text{---} \text{---} = [2] \quad \text{---} \text{---} \text{---} = [n-1]$$

$$\text{---} \text{---} \text{---} = \text{---} \text{---} \text{---} + [n-2]$$

$$\text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} = \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---}$$

[MOY] := Murakami, Ohtsuki, Yamada, Enseignement Math. 44 (1998)

$$[k] := \frac{q^k - q^{-k}}{q - q^{-1}}$$

Local differentials:

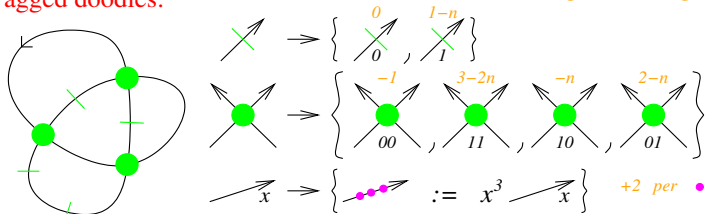
$$d \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline d & \\ \hline & \\ \hline \end{array} - \begin{array}{|c|c|} \hline & d \\ \hline & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline d & \\ \hline \end{array} - \begin{array}{|c|c|} \hline & \\ \hline & d \\ \hline \end{array}$$

where

$$d^2 \begin{array}{|c|c|} \hline \text{---} \text{---} \\ \hline \text{---} \text{---} \\ \hline \end{array} = 0 \quad \text{or} \quad d^2 \begin{array}{|c|c|} \hline \text{---} \text{---} \\ \hline \text{---} \text{---} \\ \hline \end{array} = \begin{array}{|c|c|} \hline \text{---} \text{---} \\ \hline \text{---} \text{---} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{---} \text{---} \\ \hline \text{---} \text{---} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{---} \text{---} \\ \hline \text{---} \text{---} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{---} \text{---} \\ \hline \text{---} \text{---} \\ \hline \end{array}$$

Tagged doodles:

(degrees in orange)



$$d \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} := \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} - \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = (x-y) \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \quad d \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} := \pi \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$$

$$\text{In}[1]= n = 2; \pi_{i_-, j_-} := \text{Cancel} \left[\frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j} \right]; \pi_{1,2}$$

$$\text{Out}[1]= x_1^2 + x_1 x_2 + x_2^2$$

$$\text{In}[2]= L = \begin{pmatrix} 0 & x_1 - x_2 \\ \pi_{1,2} & 0 \end{pmatrix};$$

$\text{Expand}[L.L] // \text{MatrixForm}$

(deg d = n+1)

$$\text{Out}[3]/\text{MatrixForm} = \begin{pmatrix} x_1^3 - x_2^3 & 0 \\ 0 & x_1^3 - x_2^3 \end{pmatrix}$$

Matrix factorizations:

$$\begin{array}{ccccc} M^0 & \xrightarrow{A} & M^1 & \xrightarrow{B} & M^0 \\ U^0 \downarrow V^0 & & U^1 \downarrow V^1 & & U^0 \downarrow V^0 \\ N^0 & \xrightarrow{A'} & N^1 & \xrightarrow{B'} & N^0 \end{array}$$

A category, with "complexes", morphisms, homotopies, direct sums and tensor products.

D. Eisenbud



See Khovanov and Rozansky, arXiv:math.QA/0401268