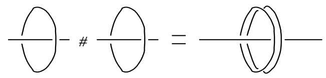
# Day 2 – u, v, w: combinatorics, low and high algebra

Dror Bar-Natan, Goettingen, April 2010
http://www.math.toronto.edu/~drorbn/Talks/Goettingen-1004/

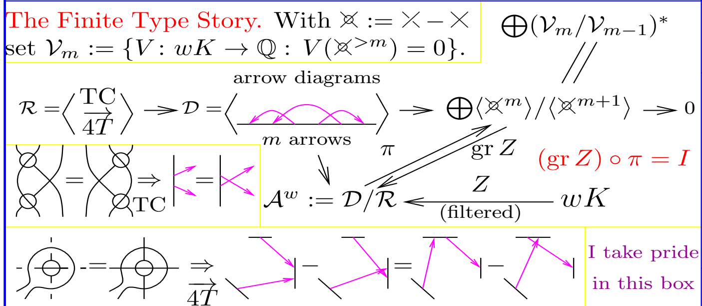
**The Scheme.** Topology → Combinatorics → Lie Theory via

$$\mathcal{K}\xrightarrow[\text{equations, unknowns}]{Z:\text{ high algebra}}\mathcal{A}=\operatorname{proj}\mathcal{K}=\bigoplus\mathcal{I}^m/\mathcal{I}^{m+1}\xrightarrow[\text{pictures}\to\text{formulas}]{\mathcal{T}_\mathfrak{g}:\text{ low algebra}}\text{“}\mathcal{U}(\mathfrak{g})\text{”}$$

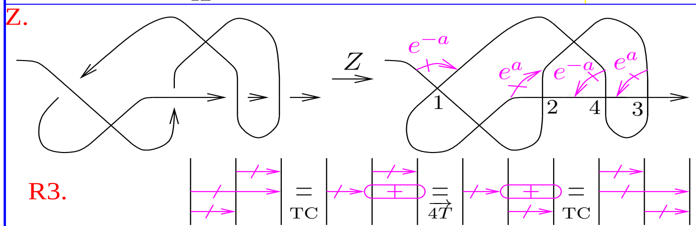
**$1+1=2$,** on an abacus, implies Duflo's $\mathcal{U}(\mathfrak{g})^\mathfrak{g}\cong\mathcal{S}(\mathfrak{g})^\mathfrak{g}$ (with T. Le and D. Thurston).



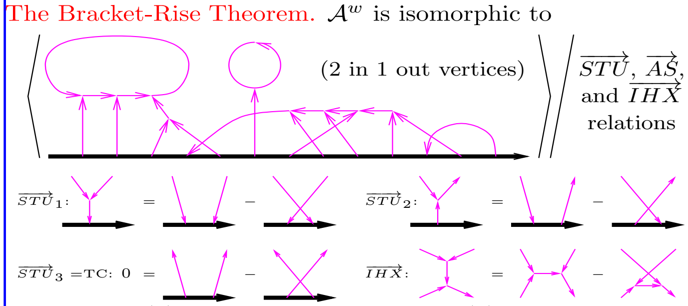
**The Finite Type Story.** With $\bowtie := \times - \times$ set $\mathcal{V}_m := \{V : wK \to \mathbb{Q} : V(\bowtie^{>m}) = 0\}$.




**Z.**

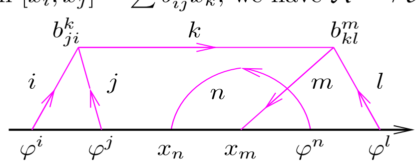



**The Bracket-Rise Theorem.** $\mathcal{A}^w$ is isomorphic to

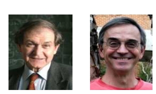



**Corollaries.** (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

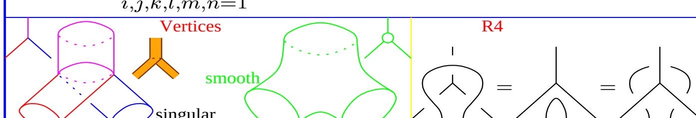
**Low Algebra.** With $(x_i)$ and $(\varphi^j)$ dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i,x_j]=\sum b_{ij}^k x_k$, we have $\mathcal{A}^w\to\mathcal{U}$ via




$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b_{ij}^k b_{kl}^m \varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^*\rtimes\mathfrak{g})$$

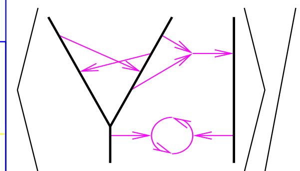


Penrose Cvitanovic

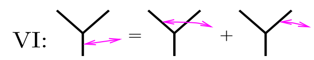



Kashiwara Vergne Alekseev Torossian

**w-Jacobi diagrams and $\mathcal{A}$.** $\mathcal{A}^w(Y\uparrow)\cong\mathcal{A}^w(\uparrow\uparrow\uparrow)$ is

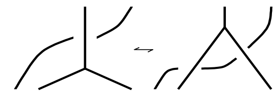


same relations, plus

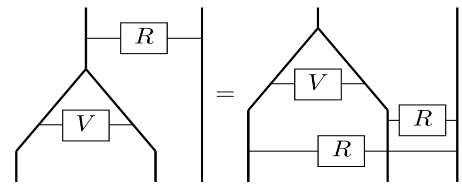



$\deg=\frac{1}{2}\#\{\text{vertices}\}=6$

**Knot-Theoretic statement (simplified).** There exists a homomorphic expansion $Z$ for trivalent w-tangles. In particular, $Z$ should respect $R4$.



**Diagrammatic statement (simplified).** Let $R = \exp \text{↦} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $V\in\mathcal{A}^w(\uparrow\uparrow)$ so that




**Algebraic statement (simplified).** With $r\in\mathfrak{g}^*\otimes\mathfrak{g}$ the identity element and with $R=e^r\in\hat{\mathcal{U}}(I\mathfrak{g})\otimes\hat{\mathcal{U}}(\mathfrak{g})$ there exist $V\in\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that $V(\Delta\otimes 1)(R)=R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}\otimes\hat{\mathcal{U}}(\mathfrak{g})$

**Unitary statement (simplified).** There exists a unitary tangential differential operator $V$ defined on $\operatorname{Fun}(\mathfrak{g}_x\times\mathfrak{g}_y)$ so that $V\widehat{e^{x+y}}=\widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

**Unitary $\Longleftrightarrow$ Algebraic.** Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\operatorname{Fun}(\mathfrak{g})$: $\varphi\in\mathfrak{g}^*$ becomes a multiplication operator, and $x\in\mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\operatorname{ad}x$: $(x\varphi)(y):=\varphi([x,y])$.

**Group-Algebra statement (simplified).** For every $\phi,\psi\in\operatorname{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$:

$$\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)e^{x+y}=\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)e^xe^y.$$

(shhh, this is Duflo)

**Unitary $\Longrightarrow$ Group-Algebra.** $\iint e^{x+y}\phi(x)\psi(y) = \langle 1, e^{x+y}\phi(x)\psi(y)\rangle = \langle V1, Ve^{x+y}\phi(x)\psi(y)\rangle = \langle 1, e^xe^yV\phi(x)\psi(y)\rangle = \langle 1, e^xe^y\phi(x)\psi(y)\rangle = \iint e^xe^y\phi(x)\psi(y)$.

**Convolutions statement (Kashiwara-Vergne, simplified).** Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let $G$ be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, and let $\Phi:\operatorname{Fun}(G)\to\operatorname{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x):=f(\exp x)$. Then if $f,g\in\operatorname{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f)\star\Phi(g)=\Phi(f\star g)$.

**Convolutions and Group Algebras** (ignoring all Jacobians). If $G$ is finite, $A$ is an algebra, $\tau:G\to A$ is multiplicative then $(\operatorname{Fun}(G),\star)\to(A,\cdot)$ via $L:f\mapsto\sum f(a)\tau(a)$. For Lie $(G,\mathfrak{g})$,

$$\begin{array}{ccc}(\mathfrak{g},+)\ni x & \xrightarrow{\tau_0=\exp_\mathcal{S}} & e^x\in\hat{\mathcal{S}}(\mathfrak{g})\\ \downarrow\exp_G & \searrow^{\exp_\mathcal{U}} & \downarrow\chi\\ (G,\cdot)\ni e^x & \xrightarrow{\tau_1} & e^x\in\hat{\mathcal{U}}(\mathfrak{g})\end{array}\qquad\text{so}\qquad\begin{array}{ccc}\operatorname{Fun}(\mathfrak{g}) & \xrightarrow{L_0} & \hat{\mathcal{S}}(\mathfrak{g})\\ \downarrow\Phi^{-1} & & \downarrow\chi\\ \operatorname{Fun}(G) & \xrightarrow{L_1} & \hat{\mathcal{U}}(\mathfrak{g})\end{array}$$

with $L_0\psi=\int\psi(x)e^xdx\in\hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi=\int\psi(x)e^x\in\hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i\in\operatorname{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1)\star\Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1\star\psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

$\star$ in $G$: $\iint\psi_1(x)\psi_2(y)e^xe^y$ $\star$ in $\mathfrak{g}$: $\iint\psi_1(x)\psi_2(y)e^{x+y}$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Goettingen-1004/