

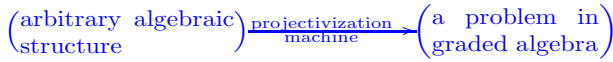
Day 1 – u, v, w: topology and philosophy

Dror Bar-Natan, Goettingen, April 2010

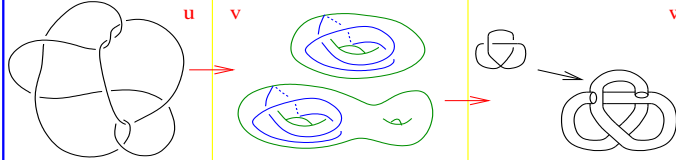
u, v, and w-Knots: Topology, Combinatorics and Low and High Algebra

<http://www.math.toronto.edu/~drorbn/Talks/Goettingen-1004/>

Plans and Dreams



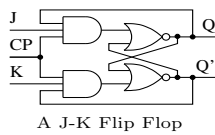
- Feed **knot-things**, get Lie algebra things.
- Feed **u-knots**, get Drinfel'd associators.
- Feed **w-knots**, get Kashiwara-Vergne-Alekseev-Torossian.
- Dream: Feed **v-knots**, get Etingof-Kazhdan.
- Dream: Knowing the question whose answer is 42, or E-K, will be useful to algebra and topology.



u-Knots (PA := Planar Algebra)

$$\{\text{knots} \ \& \ \text{links}\} = \text{PA} \left\langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \middle| \text{R123: } \begin{array}{c} \bigcirc \\ \bigcirc \end{array} = \begin{array}{c} \bigcirc \\ \bigcirc \end{array}, \begin{array}{c} \bigcirc \\ \bigcirc \end{array} = \begin{array}{c} \bigcirc \\ \bigcirc \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle_{0 \text{ legs}}$$

Circuit Algebras



v-Knots (CA := Circuit Algebra)

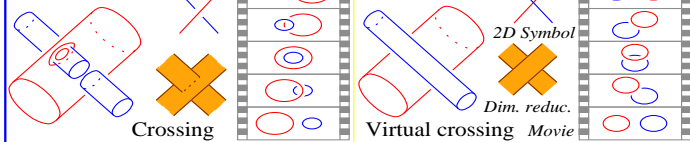
$$\{\text{v-knots} \ \& \ \text{links}\} = \text{CA} \left\langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \middle| \text{R23: } \begin{array}{c} \bigcirc \\ \bigcirc \end{array} = \begin{array}{c} \bigcirc \\ \bigcirc \end{array}, \begin{array}{c} \bigcirc \\ \bigcirc \end{array} = \begin{array}{c} \bigcirc \\ \bigcirc \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle_{0 \text{ legs}}$$

$$= \text{PA} \left\langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \middle| \text{VR123: } \begin{array}{c} \bigcirc \\ \bigcirc \end{array} = \begin{array}{c} \bigcirc \\ \bigcirc \end{array}, \begin{array}{c} \bigcirc \\ \bigcirc \end{array} = \begin{array}{c} \bigcirc \\ \bigcirc \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}; \right. \\ \left. \text{R23; D: } \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle_{0 \text{ legs}}$$

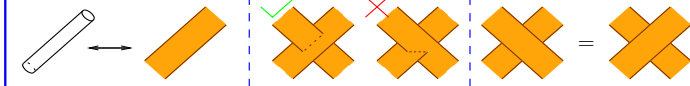
w-Tangles

$$\{\text{w-Tangles}\} = \text{v-Tangles} / \text{OC} : \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}$$

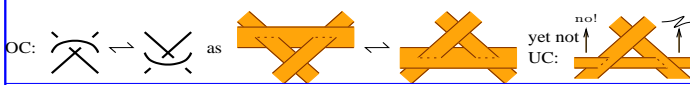
The w-generators.



A **Ribbon 2-Knot** is a surface S embedded in \mathbb{R}^4 that bounds an immersed handlebody B , with only “ribbon singularities”; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

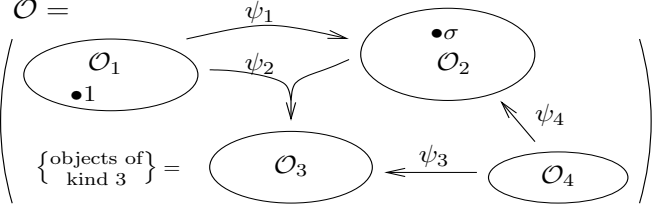


The **w-relations** include R234, VR1234, M, Overcrossings Commute (OC) but not UC:



Also see <http://www.math.toronto.edu/~drorbn/papers/WKO/>

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Homomorphic expansions for a filtered algebraic structure \mathcal{K} :

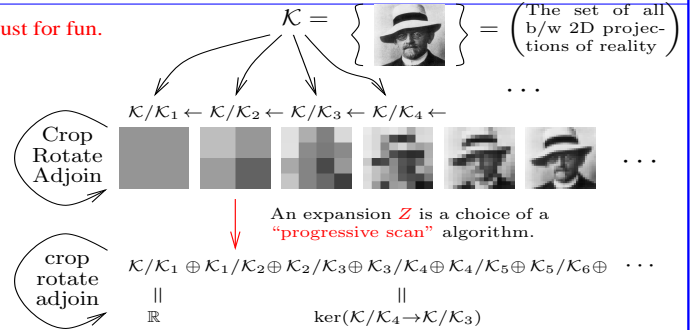
$$\text{ops} \curvearrowright \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

$$\downarrow \quad \quad \quad \downarrow z$$

$$\text{ops} \curvearrowright \text{gr } \mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An **expansion** is a filtration respecting $Z : \mathcal{K} \rightarrow \text{gr } \mathcal{K}$ that “covers” the identity on $\text{gr } \mathcal{K}$. A **homomorphic expansion** is an expansion that respects all relevant “extra” operations.

Just for fun.



Filtered algebraic structures are cheap and plenty. In any \mathcal{K} , allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the “augmentation ideal”), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available “products”).

Examples. 1. The projectivization of a group is a graded associative algebra. 2. Quandle: a set Q with an op \wedge s.t.

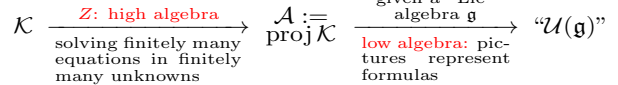
$$1 \wedge x = 1, \quad x \wedge 1 = x, \quad (\text{appetizers})$$

$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \quad (\text{main})$$

$\text{proj } Q$ is a graded Leibniz algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate \mathcal{I} !), feed $1 + \bar{x}, 1 + \bar{y}, 1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

Our case(s).



\mathcal{K} is knot theory or **topology**; $\text{proj } \mathcal{K} = \bigoplus \mathcal{I}^m / \mathcal{I}^{m+1}$ is finite **combinatorics**: bounded-complexity diagrams modulo simple relations.



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)



www.katlas.org