Lecture 4 Handout

Low and High Algebra in the "u" Case

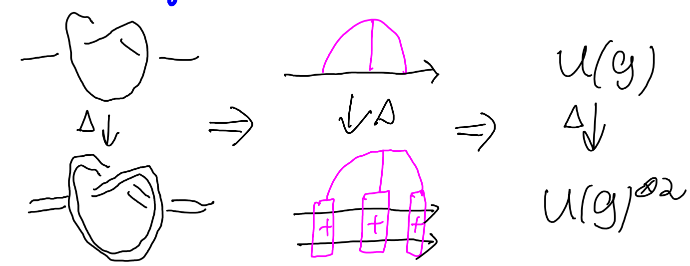
Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

The big picture, "u" Case.

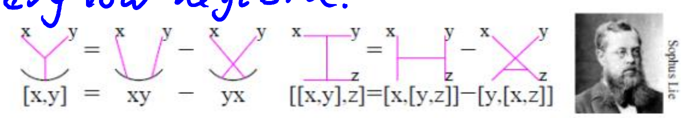
$$\underbrace{\overset{\text{Topology}}{\mathcal{K}}}_{\{\text{knots etc.}\}} \xrightarrow[\text{expansion}]{Z} \underbrace{\overset{\text{Combinatorics}}{\mathcal{A}}}_{\{\text{chord diagrams}\}/\text{IHX, STU, 4T}} \xrightarrow{\text{choose } \mathfrak{g}} \mathcal{U}(\mathfrak{g}) \qquad \mathcal{A} \xrightarrow{\text{choose } \mathfrak{g}, R} \mathbb{Q}$$

high algebra | low algebra

What's Δ?



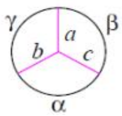
Very low algebra.



More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [a,b], c\rangle \qquad X_a v_\beta = \sum_{\cancel{\gamma}} r^\beta_{a\gamma} v_\gamma$$

and then

$$W_{\mathfrak{g},R}: \quad$$  $$\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

Exercise. Find a fast method to find $W_{\mathfrak{g},R}(D)$ when $\mathfrak{g} = \mathfrak{gl}_n$, $R = \mathbb{R}^n$? Is it related to the Conway polynomial?

Universal Representation Theory.

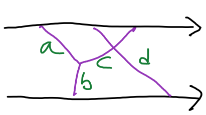
Inspired by $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$, set $\mathcal{U}(\mathfrak{g}) = \langle \text{words in } \mathfrak{g}\rangle / [x,y] = xy - yx$

* Every rep of $\mathfrak{g}$ extends to $\mathcal{U}(\mathfrak{g})$.
* $\exists\, \Delta: \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ by "word splitting", as must be for $R_1 \otimes R_2$.

Exercise. With $\mathfrak{g} = \langle x, y\rangle / [x,y] = x$, determine $\mathcal{U}(\mathfrak{g})$. Guess a generalization.

Low algebra.

$\mathcal{A}(\uparrow\uparrow) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ via



$$\longrightarrow \sum_{a\text{-}d} f_{abc} \begin{pmatrix} X_a X_d X_c \\ \otimes \\ X_b X_d \end{pmatrix}$$

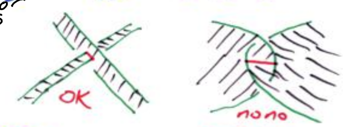
& likewise, $\mathcal{A}(\uparrow_n) \to \mathcal{U}(\mathfrak{g})^{\otimes n} \Rightarrow$ $\mathcal{A}(\uparrow_n)$ is "universal universal rep. theory"!

A "Homomorphic Expansion" $Z: \mathcal{K} \to \mathcal{A}$

is an expansion that intertwines all relevant algebraic ops. If $\mathcal{K}$ is finitely presented, finding Z is **High Algebra**.

$$\{\text{Ribbon knots}\} \underset{\text{more or less}}{=} \{u\gamma : \gamma \in \mathcal{K}(\text{O-O}),\ d\gamma = \text{OO}\}$$

where Ribbon means



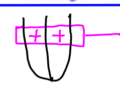
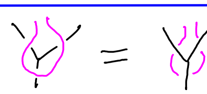
and

Algebraic knot theory:

$$\begin{array}{ccccc} & \overset{d}{\nearrow} & \mathcal{K}(\text{OO}) & \xrightarrow{Z} & \mathcal{A}(\text{OO}) \ni \text{OO} \\ \mathcal{K}(\text{O-O}) & & \xrightarrow{Z} & & \mathcal{A}(\text{O-O}) \\ & \underset{u}{\searrow} & \mathcal{K}(\text{O}) & \xrightarrow{Z} & \mathcal{A}(\text{O}) \end{array}$$

So

$$Z(\{\text{Ribbon knots}\}) \subset \{u\alpha : d\alpha = Z(\text{OO})\} \subset \mathcal{A}(\text{O-O})$$

VI  $= 0$, follows from 

An Associator:

Quantum Algebra's "root object"

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$$\begin{array}{ccc} ((AB)C)D & \longrightarrow & (AB)(CD) \\ \downarrow \Phi 1 & (\Delta 11)\Phi & \downarrow (11\Delta)\Phi \\ (A(BC))D & & A(B(CD)) \\ & \searrow (1\Delta 1)\Phi \quad \nearrow 1\Phi & \\ & A((BC)D) & \end{array}$$

$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

The hexagon? Never heard of it.

See Also. B-N & Dancso, arXiv: 1103.1896

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107/