Lecture 3 Handout

The Basics of Finite-Type Invariants of Knots

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

Definition. A knot invariant is any function whose domain is {knots}. Really, we mean a computable function whose target space is understandable; e.g.

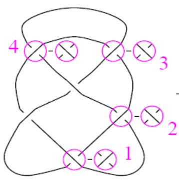
$$C : \{\text{knot diagrams}\} / \text{(Reidemeister moves)} \to \mathbb{Z}[z]$$

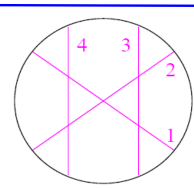
Example. The Conway polynomial is given by $C(\overcrossing) - C(\undercrossing) = zC(\text{smoothing})$ and

$$C(\underbrace{\bigcirc\bigcirc\cdots\bigcirc}_{k}) = \begin{cases} 1 & k=1 \\ 0 & k>1 \end{cases}$$

Exercise. Pick your favourite bank and compute the Conway polynomial of its logo.

Definition. Any $V : \{\text{knots}\} \to$ Abelian Group A can be extended to "knots w/ double points" using $V(\times\!\!\!\bullet) = V(\overcrossing) - V(\undercrossing)$. (Think "differentiation")





Definition. V is of type m if always

$$V(\underbrace{\times\!\!\!\bullet\ \times\!\!\!\bullet \cdots \times\!\!\!\bullet}_{m+1}) = 0 \qquad \text{(think "polynomial")}$$

Conjecture. Finite type invariants separate knots.

Theorem. If $C(K) = \sum_{m=0}^{\infty} V_m(K) z^m$ then V_m is of type m.

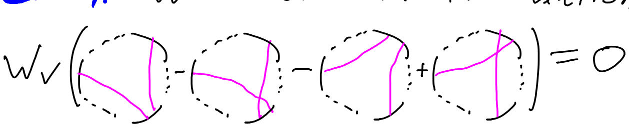
Proof. $C(\times\!\!\!\bullet) = C(\overcrossing) - C(\undercrossing) = zC(\text{smoothing})$ $\square$

Let V be of type m; then $V^{(m)}$ is constant:

$$V(\underbrace{\times\!\!\!\bullet \cdots \times\!\!\!\bullet}_{m}\ \overcrossing) = V(\underbrace{\times\!\!\!\bullet \cdots \times\!\!\!\bullet}_{m}\ \undercrossing)$$

So $W_V := V^{(m)} = V|_{m\text{-singular knots}}$ is really a function on m-chord diagrams: $W_V : \{\text{chord diagrams}\} \to A$

Claim. W_V satisfies the 4T relation:



Proof. $V(\ldots) = V(\ldots)$, $\underbrace{\times\cdot\times}_{m-2}$, $\underbrace{\times\cdots\times}_{m-2}$ $\square$

Exercise for Lecture 2. Use $\int_{\mathbb{R}} e^{-x^2/2} = \sqrt{2\pi}$, Fubini's theorem, and polar coordinates to compute $\int_{\mathbb{R}^n} e^{-\|x\|^2/2} d^n x$ in two different ways and hence to deduce the volume of S^{n-1}, the $(n-1)$-dimensional sphere.

Exercise. 1. Determine the "weight system" W_{V_m} of the m-th coefficient of the Conway polynomial and verify that is satisfies 4T.
2. Learn somewhere about the Jones polynomial, and do the same for its coefficients.

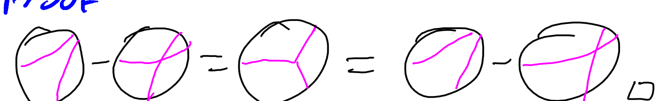
Theorem. (The Fundamental Theorem) Every "weight system", i.e. every linear functional W on $\mathcal{A} := \{\text{chord diagrams}\}/4T$ is the m'th derivative of a type m invariant: $\forall W\ \exists V$ s.t. $W = W_V$

M. Kontsevich

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

Theorem. $\mathcal{A}^{\text{today}} \cong \mathcal{A}^{\text{Monday}}$

Proof



$\square$

Proposition. The fundamental thm holds iff there exists an expansion $Z : \mathcal{K} \to \hat{\mathcal{A}}$ s.t. if K is m-singular, then $Z(K) = D_K +$ higher degrees.

Proof.

$$\begin{tikzcd} \mathcal{K} \arrow[r, "Z"] \arrow[dr, "V"'] & \mathcal{A} \arrow[d, "W"] \\ & \mathbb{Q} \end{tikzcd}$$

$\square$

Also see my old paper, "On the Vassiliev knot invariants" (google will find..)

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In[1]:= << KnotTheory`

Loading KnotTheory` version of August 22, 2010, 13:36:57.55.
Read more at http://katlas.org/wiki/KnotTheory.

In[2]:= Column[{
      Import[
       "C:\\drorbn\\AcademicPensieve\\2011-07\\RolfsenKnots\\"
        <> ToString@#[[1]] <>
        "." <> ToString@#[[2]] <> "_240.gif"],
      Conway[#][z]
     }, Center
   ] & /@ AllKnots[{0, 7}]

KnotTheory::loading : Loading precomputed data in PD4Knots`.
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J.H. Conway