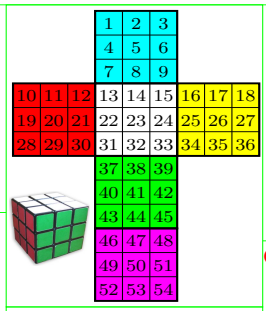
# Non-Commutative Gaussian Elimination and Rubik's Cube

**The Problem.** Let $G = \langle g_1, \dots, g_\alpha \rangle$ be a subgroup of $S_n$, with $n = O(100)$. Before you die, understand $G$:

1. Compute $|G|$.
2. Given $\sigma \in S_n$, decide if $\sigma \in G$.
3. Write a $\sigma \in G$ in terms of $g_1, \dots, g_\alpha$.
4. Produce *random* elements of $G$.




**The Commutative Analog.** Let $V = \text{span}(v_1, \dots, v_\alpha)$ be a subspace of $\mathbb{R}^n$. Before you die, understand $V$.

**Solution: Gaussian Elimination.** Prepare an empty table,

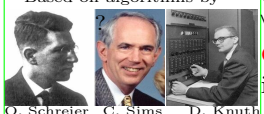
| 1 | 2 | 3 | 4 | $\cdots$ | $n-1$ | $n$ |
|---|---|---|---|---|---|---|
| | | | | | | |

Space for a vector $u_4 \in V$, of the form $u_4 = (0, 0, 0, 1, *, \dots, *)$; 1 :="the pivot".

**Feed $v_1, \dots, v_\alpha$ in order.** To feed a non-zero $v$, find its pivotal position $i$.

1. If box $i$ is empty, put $v$ there.
2. If box $i$ is occupied, find a combination $v'$ of $v$ and $u_i$ that eliminates the pivot, and feed $v'$.

Based on algorithms by

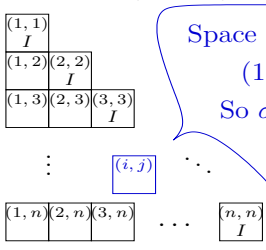


O. Schreier C. Sims D. Knuth

See also *Permutation Group Algorithms* by Á. Seress, *Efficient Representation of Perm Groups* by D. Knuth.

## Non-Commutative Gaussian Elimination

Prepare a mostly-empty table,




Space for a $\sigma_{i,j} \in S_n$ of the form $(1, 2, \dots, i-2, i-1, j, *, *, \dots, *)$
So $\sigma_{i,j}$ fixes $1, \dots, i-1$,
sends "the pivot" $i$ to $j$ and
goes wild afterwards, and
$\sigma_{i,j}^{-1}$ "does sticker $j$".

**Feed $g_1, \dots, g_\alpha$ in order.** To feed a non-identity $\sigma$, find its pivotal position $i$ and let $j := \sigma(i)$.

1. If box $(i, j)$ is empty, put $\sigma$ there.
2. If box $(i, j)$ contains $\sigma_{i,j}$, feed $\sigma' := \sigma_{i,j}^{-1}\sigma$.

**The Twist.** When done, for every occupied $(i, j)$ and $(k, l)$, feed $\sigma_{i,j}\sigma_{k,l}$. Repeat until the table stops changing.

**Claim 1.** The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table $T$.

**Claim 2.** Every $\sigma_{i,j}$ in $T$ is in $G$.

**Claim 3.** Anything fed in $T$ is now a monotone product in $T$:

$f$ was fed $\Rightarrow$ $f \in M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} : \forall i,\ j_i \geq i \ \&\ \sigma_{i,j_i} \in T\}$

## The Generators

```
n = 54;
g1 = Cycles[{{1, 18, 45, 28}, {2, 27, 44, 19}, {3, 36, 43, 10}, {46, 52, 54, 48},
    {47, 49, 53, 51}}];
g2 = Cycles[{{7, 16, 39, 30}, {8, 25, 38, 21}, {9, 34, 37, 12}, {13, 15, 33, 31},
    {14, 24, 32, 22}}];
g3 = Cycles[{{28, 31, 34, 48}, {29, 32, 35, 47}, {30, 33, 36, 46}, {37, 39, 45, 43},
    {38, 42, 44, 40}}];
g4 = Cycles[{{1, 3, 9, 7}, {2, 6, 8, 4}, {10, 54, 16, 13}, {11, 53, 17, 14},
    {12, 52, 18, 15}}];
g5 = Cycles[{{1, 13, 37, 46}, {4, 22, 40, 49}, {7, 31, 43, 52}, {10, 12, 30, 28},
    {11, 21, 29, 19}}];
g6 = Cycles[{{3, 48, 39, 15}, {6, 51, 42, 24}, {9, 54, 45, 33}, {16, 18, 36, 34},
    {17, 27, 35, 25}}];
```

Enter

**Claim 4.** If two monotone products are equal,

$$\sigma_{1,j_1}\cdots\sigma_{n,j_n} = \sigma_{1,j'_1}\cdots\sigma_{n,j'_n},$$

then all the indices that appear in them are equal, $\forall i,\ j_i = j'_i$.

**Claim 5.** Let $M_k$ denote the set of monotone products in $T$ starting in column $k$:

$$M_k := \{\sigma_{k,j_k}\cdots\sigma_{n,j_n} : \forall i \geq k, j_i \geq i \text{ and } \sigma_{i,j_i} \in T\}.$$

then for every $k$, $M_k M_k \subset M_k$ (and so each $M_k$ is a subgroup of $G$).

**Proof.** By backwards induction. Clearly $M_n M_n \subset M_n$. Now assume that $M_5 M_5 \subset M_5$ and show that $M_4 M_4 \subset M_4$. Start with $\sigma_{8,j} M_4 \subset M_4$:

$$\sigma_{8,j}(\sigma_{4,j_4} M_5) \overset{1}{=} (\sigma_{8,j}\sigma_{4,j_4}) M_5 \overset{2}{\subset} M_4 M_5$$

$$\overset{3}{=} \cup_j \sigma_{4,j}(M_5 M_5) \overset{4}{\subset} \cup_j \sigma_{4,j} M_5 \subset M_4$$

(1: associativity, 2: thank the twist, 3: associativity and tracing $i_4$, 4: induction). Now the general case

$$(\sigma_{4,j'_4}\sigma_{5,j'_5}\cdots)(\sigma_{4,j_4}\sigma_{5,j_5}\cdots)$$

falls like a chain of dominos.

**Theorem.** $G = M_1$ and we have achieved our goals.

## A Demo Program

```
σ_ ∘ τ_ := PermutationProduct[τ, σ];
Feed[Cycles[{}]] := Null;
Feed[τ_] := Module[{i, j, k, l},
  i = Min[PermutationSupport[τ]];
  j = PermutationReplace[i, τ];
  If[Head[σ_{i,j}] === Cycles,
   Feed[InversePermutation[σ_{i,j}] ∘ τ],
   (*Else*) σ_{i,j} = τ;
   For[k = 1, k < n, ++k,
    For[l = k + 1, l ≤ n, ++l,
     If[Head[σ_{k,l}] === Cycles,
      Feed[σ_{i,j} ∘ σ_{k,l}]; Feed[σ_{k,l} ∘ σ_{i,j}]]
    ]
   ]];
$RecursionLimit = ∞;
```

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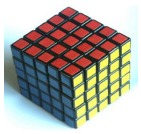

www.geocities.com/jaapsch/puzzles/

www.powerstrike.net/puzzles/

http://lodelnic.free.fr/

**Homework Problem 1.**
Can you do cosets?

www.powerstrike.net/puzzles/

## The Results

```
Table[Feed[g_α]; ∏_{i=1}^{n} (1 + Count[Range[n], j_ /; Head[σ_{i,j}] == Cycles]), {α, 6}]
```

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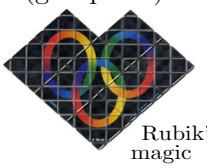
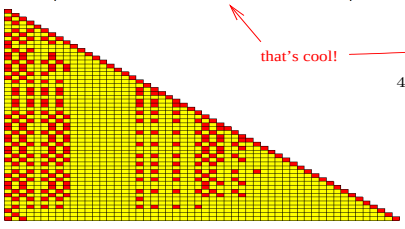
```
{4, 16, 159 993 501 696 000, 21 119 142 223 872 000, 43 252 003 274 489 856 000, 43 252 003 274 489 856 000}
```

**Homework Problem 2.**
Can you do categories (groupoids)?

| 7 | 9 | 2 | 5 |
|---|---|---|---|
| 1 | 4 | 8 | 3 |
| 6 | 10 | 11 | 12 |
| 13 | 14 | 15 | |



Rubik's magic




43, 252, 003, 274, 489, 856, 000

$$= \frac{8!\,3^8\,12!\,2^{12}}{12}$$

In Inuit: