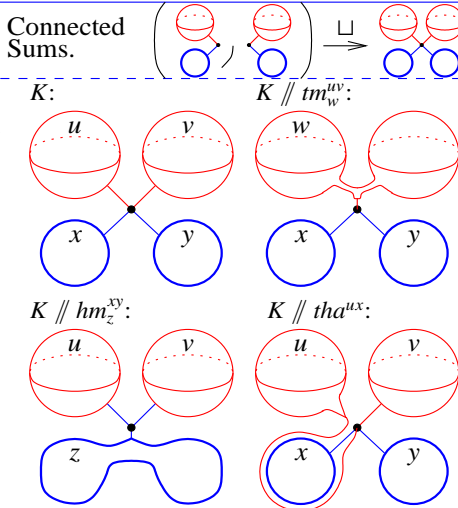
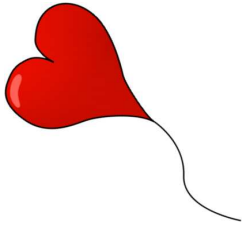


Operations

Punctures & Cuts

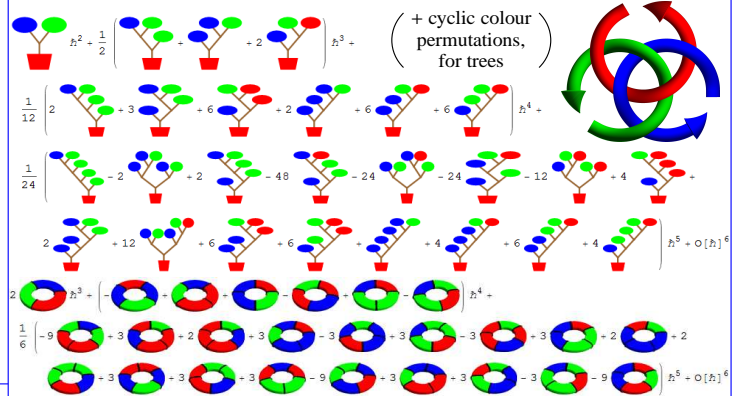
If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2 .

Proposition. The generators generate.



Theorem 3 [BND, BN]. $\exists!$ a homomorphic expansion, aka a homomorphic universal finite type invariant Z of w -knotted balloons and hoops. $\zeta := \log Z$ takes values in $FL(T)^H \times CW(T)$.

ζ is computable! ζ of the Borromean tangle, to degree 5:



Definition. l_{xu} is the linking number of hoop x with balloon u . For $x \in H$, $\sigma_x := \prod_{u \in T} T_u^{l_{xu}} \in R = R_T = \mathbb{Z}((T_a)_{a \in T})$, the ring of rational functions in T variables.

Theorem 2 [BNS]. $\exists!$ an invariant $\beta: w\mathcal{K}^{bh}(H; T) \rightarrow R \times M_{T \times H}(R)$, intertwining

$$1. \left(\begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & A_1 \end{array}, \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & A_2 \end{array} \right) \xrightarrow{\sqcup} \begin{array}{c|cc} \omega_1 \omega_2 & H_1 & H_2 \\ \hline T_1 & A_1 & 0 \\ T_2 & 0 & A_2 \end{array}$$

$$2. \begin{array}{c|c} \omega & H \\ \hline u & \alpha \\ v & \beta \\ T & \Xi \end{array} \xrightarrow{tm_w^{uv}} \begin{array}{c|c} \omega & H \\ \hline w & \alpha + \beta \\ T & \Xi \end{array}_{T_u, T_v \rightarrow T_w}$$

$$3. \begin{array}{c|cc|c} \omega & x & y & H \\ \hline T & \alpha & \beta & \Xi \end{array} \xrightarrow{hm_z^{xy}} \begin{array}{c|c|c} \omega & z & H \\ \hline T & \alpha + \sigma_x \beta & \Xi \end{array}$$

$$4. \begin{array}{c|c|c} \omega & x & H \\ \hline u & \alpha & \theta \\ T & \phi & \Xi \end{array} \xrightarrow[\nu:=1+\alpha]{tha^{ux}} \begin{array}{c|c|c} \nu\omega & x & H \\ \hline u & \sigma_x \alpha / \nu & \sigma_x \theta / \nu \\ T & \phi / \nu & \Xi - \phi \theta / \nu \end{array}$$

and satisfying $(\epsilon_x; \epsilon_u; \rho_{ux}^\pm) \xrightarrow{\beta} \left(\begin{array}{c|c} 1 & x \\ \hline u & \end{array}; \begin{array}{c|c} 1 & \\ \hline u & T_u^{\pm 1} - 1 \end{array} \right)$.

Proposition. If T is a u -tangle and $\beta(\delta T) = (\omega, A)$, then $\gamma(T) = (\omega, \sigma - A)$, where $\sigma = \text{diag}(\sigma_a)_{a \in S}$. Under this, $m_c^{ab} \leftrightarrow tha^{ab} // tm_c^{ab} // hm_c^{ab}$.

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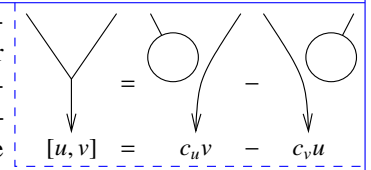
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Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, ζ reduces to β and the KBH operations on ζ reduce to the formulas in Theorem 2.



A Big Question. Does it all extend to arbitrary 2-knots (not necessarily “simple”)? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A, B) := \int_M \langle B, F_A \rangle.$$

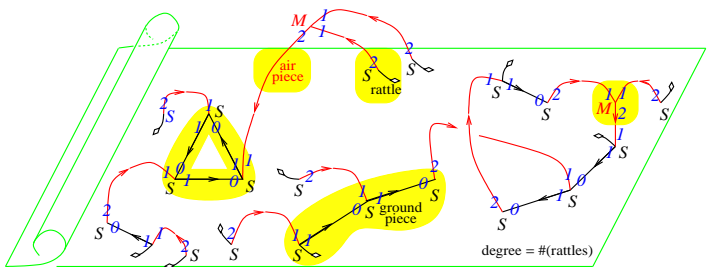
With $\kappa: (S = \mathbb{R}^2) \rightarrow M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

$$O(A, B, \kappa) := \int \mathcal{D}\beta \mathcal{D}\alpha \exp\left(\frac{i}{\hbar} \int_S \langle \beta, d_{\kappa^*} \alpha + \kappa^* B \rangle\right).$$

The BF Feynman Rules. For an edge e , let Φ_e be its direction, in S^3 or S^1 . Let ω_3 and ω_1 be volume forms on S^3 and S^1 . Then

$$Z_{BF} = \sum_{\text{diagrams } D} \frac{|D|}{|\text{Aut}(D)|} \int_{\mathbb{R}^2} \dots \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \dots \int_{\mathbb{R}^4} \prod_{e \in D} \Phi_e^* \omega_3 \prod_{e \in D} \Phi_e^* \omega_1$$

(modulo some STU - and IHX -like relations).



Issues. • Signs don't quite work out, and BF seems to reproduce only “half” of the wheels invariant.

• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.

• I don't know how to define “finite type” for arbitrary 2-knots.



“God created the knots, all else in topology is the work of mortals.”

Leopold Kronecker (modified)

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