

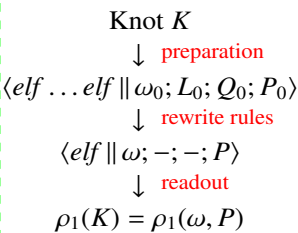


On Elves and Invariants

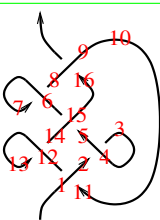
Abstract. Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

Three steps to the computation of ρ_1 :

- 1. Preparation.** Given K , results $\langle \text{long word} \parallel \text{simple formulas} \rangle$.
- 2. Rewrite rules.** Make the word simpler and the formulas more complicated, until the word “elf” is reached.
- 3. Readout.** The invariant ρ_1 is read from the last formulas.



Preparation. Draw K using a 0-framed 0-rotation planar diagram D where all crossings are pointing up. Walk along D labeling features by $1, \dots, m$ in order: over-passes, under-passes, and right-heading cups and caps (“±-cuaps”). If x is a xing, let i_x and j_x be the labels on its over/under strands, and let s_x be 0 if it right-handed and -1 otherwise. If c is a cuap, let i_c be its label and s_c be its sign. Set



$$(L; Q; P) = \sum_{x: (i,j,s)} (-)^s \left(l_j; t^s e_i f_j; (-t)^s e_i l_{(1+s)i-sj} f_j + l_i l_j + \frac{t^{2s} e_i^2 f_j^2}{4} \right) + \sum_{c: (i,s)} (0; 0; s \cdot l_i).$$

This done, output $\langle e_1 l_1 f_1 e_2 l_2 f_2 \dots e_m l_m f_m \parallel 1; L; Q; P \rangle$.

In formulas. L is always \mathbb{Z} -linear in $\{l_i\}$, Q is an R -linear combination of $\{e_i f_j\}$ where $R := \mathbb{Q}[t^{\pm 1}]$, and P is an R -linear combination of $\{1, l_i, l_i l_j, e_i f_j, e_i l_j f_k, e_i e_j f_k f_l\}$. (The key to computability!)

Rewrite Rules. Manipulate $\langle \text{word} \parallel \text{formulas} \rangle$ expressions using the rewrite rules below, until you come to the form $\langle e_1 l_1 f_1 \parallel \omega; -; -; P \rangle$. Output (ω, P) .

Rule 1, Deletions. If a letter appears in word but not in formulas, you can delete it.

Rule 2, Merges. In word, you can replace adjacent $v_i v_j$ with v_k (for $v \in \{e, l, f\}$) while making the same changes in formulas (provided k creates no naming clashes). E.g.,

$$\langle \dots e_i e_j \dots \parallel Z \rangle \rightarrow \langle \dots e_k \dots \parallel Z|_{e_i, e_j \rightarrow e_k} \rangle.$$

Rule 3, le Sorts. Provided k introduces no clashes, given $\langle \dots l_j e_i \dots \parallel \omega; L; Q; P \rangle$, decompose $L = \lambda l_j + L'$, $Q = \alpha e_i + Q'$, write $P = P(e_i, l_j)$ (with messy coefficients), set $q = e^\gamma \beta e_k + \gamma l_k$, and output

$$\langle \dots e_k l_k \dots \parallel \omega; L|_{l_j \rightarrow l_k}; t^\lambda \alpha e_k + Q'; e^{-q} P(\partial_\beta, \partial_\gamma) e^q |_{\beta \rightarrow \alpha/\omega, \gamma \rightarrow \lambda \log t} \rangle.$$

Rule 4, fl Sorts. Provided k introduces no clashes, given $\langle \dots f_i l_j \dots \parallel \omega; L; Q; P \rangle$, decompose $L = \lambda l_j + L'$, $Q = \alpha f_i + Q'$, write $P = P(f_i, l_j)$ (with messy coefficients), set $q = e^\gamma \beta f_k + \gamma l_k$, and output

$$\langle \dots l_k f_k \dots \parallel \omega; L|_{l_j \rightarrow l_k}; t^\lambda \alpha f_k + Q'; e^{-q} P(\partial_\beta, \partial_\gamma) e^q |_{\beta \rightarrow \alpha/\omega, \gamma \rightarrow \lambda \log t} \rangle.$$



Happy Birthday, Scott!



“God created the knots, all else in topology is the work of mortals.”

Leopold Kronecker (modified)



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Rule 5, fe Sorts. Provided k introduces no clashes, given $\langle \dots f_i e_j \dots \parallel \omega; L; Q; P \rangle$, decompose $Q = Q_{fe} f_i e_j + Q_f f_i + Q_e e_j + Q'$ write $P = P(f_i, e_j)$ (with messy coefficients), set $\mu = 1 + (t-1)\delta$ and $q = ((1-t)\alpha\beta + \beta e_k + \alpha f_k + \delta e_k f_k)/\mu$, and output

$$\left\langle \dots e_k f_k \dots \parallel \begin{matrix} \mu\omega; L; \mu\omega q + \mu Q'; \\ \omega^4 \Lambda_k + e^{-q} P(\partial_\alpha, \partial_\beta) (e^q) \end{matrix} \right\rangle \Big|_{\substack{\alpha \rightarrow Q_f / \omega, \beta \rightarrow Q_e / \omega, \\ \delta \rightarrow Q_{fe} / \omega}}$$

where Λ_k is the Λόγος, “a principle of order and knowledge”:

$$\Lambda_k = \frac{t+1}{4} \left(-\delta(\mu+1)(\beta^2 e_k^2 + \alpha^2 f_k^2) - \delta^3(3\mu+1)e_k^2 f_k^2 - 2(\beta e_k + \alpha f_k)(\alpha\beta + 2\delta\mu + \delta^2(2\mu+1)e_k f_k + 2\delta\mu^2 l_k) - 4(\alpha\beta + \delta\mu)(\delta(\mu+1)e_k f_k + \mu^2 l_k) - 4\delta^2 \mu^2 e_k f_k l_k + (t-1)(2(\alpha\beta + \delta\mu)^2 - \alpha^2 \beta^2) \right).$$

elf merges, m_k^{ij} , are defined as compositions

$$e_i l_i \overline{f_i e_j} l_j f_j \xrightarrow{S_x^{f_i e_j}} e_i \overline{l_i e_x} \overline{f_x l_j} f_j \xrightarrow{S_x^{l_i e_x} // S_x^{f_x l_j}} \overline{e_i e_x} \overline{l_x l_x} \overline{f_x f_j} \xrightarrow{i, j, x \rightarrow k} e_k l_k f_k$$

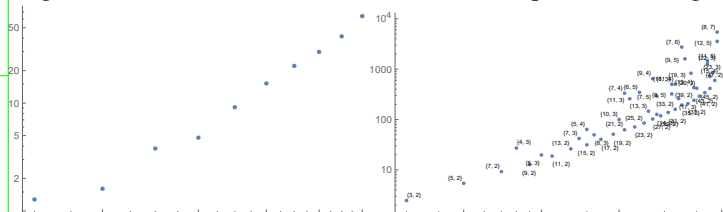
Readout. Given $\langle \text{elf} \parallel \omega; -; -; P \rangle$, output

$$\rho_1(K) := \frac{t(P|_{e, l, f \rightarrow 0} - t\omega^3)}{(t-1)^2 \omega^2}.$$

(ω is the Alexander polynomial, L and Q are not interesting).



Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1 , always $\deg \rho_1^+ \leq 2g - 1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Why Works? The Lie algebra \mathfrak{g}_1 (below) is a “solvable approximation of \mathfrak{sl}_2 ”.

Theorem. The map (as defined below)

$\langle w \parallel \omega; L; Q; P \rangle \mapsto \mathbb{O} \left(\omega^{-1} e^{L \log t + \omega^{-1} Q} (1 + \epsilon \omega^{-4} P) : w \right) \in \hat{\mathcal{U}}(\mathfrak{g}_1)$ is well defined modulo the sorting rules. It maps the initial preparation to a product of “ R -matrices” and “cuap values” satisfying the usual moves for Morse knots (R3, etc.). (And hence the result is a “quantum invariant”, except computed very differently; no representation theory!).