The sl_2 Example. Let $\mathfrak{g}^\epsilon = \langle h,e,l,f\rangle/([h,\cdot]=0,\ [e,l]=-e,\ [f,l]=f,\ [e,f]=h-2\epsilon l)$ and let $\mathfrak{g}_k=\mathfrak{g}^\epsilon/(\epsilon^{k+1}=0)$.

The Main $\mathfrak{g}_k$ Theorem. The $\mathfrak{g}_k$-invariant of any S-component tangle T can be written in the form

$$Z(T)=\mathbb{O}\left(\omega\mathbb{e}^{L+Q+P}\colon \bigotimes_{i\in S} e_il_if_i\right),$$

where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i:=\mathbb{e}^{h_i}$), where $L=\sum a_{ij}h_il_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients, where $Q=\sum b_{ij}e_if_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}, and where P is a polynomial in $\{\epsilon, e_i, l_i, f_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\{e_i,\sqrt{l_i},f_i\}$. Furthermore, after setting $h_i=h$ and $t_i=t$ for all i, the invariant $Z(T)$ is poly-time computable.

The Main $\mathfrak{g}_k$ Lemma. The following "re-ordering relations" hold:

$$\mathbb{O}\left(\mathbb{e}^{\gamma l+\beta e}\colon le\right)=\mathbb{O}\left(\mathbb{e}^{\gamma l+\mathbb{e}^\gamma\beta e}\colon el\right)\quad\text{(and similarly for } fl\to lf),$$

$$\mathbb{O}\left(\mathbb{e}^{\beta e+\alpha f+\delta ef}\colon fe\right)=\mathbb{O}\left(\nu\mathbb{e}^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta ef)+\lambda_k(\epsilon,e,l,f,\alpha,\beta,\delta)}\colon elf\right),$$

with $\nu=(1+h\delta)^{-1}$ and where $\lambda_k(\epsilon,e,l,f,\alpha,\beta,\delta)$ is some fixed polynomial of degree at most $2k+2$ in $\epsilon,e,\sqrt{l},f,\alpha,\beta,\delta$, with scalar coefficients.

Demo Programs. ωεβ/Demo

Formatting

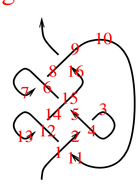
```
CF[ℰ_] := Module[{vars = Union@Cases[ℰ, e_ | l_ | f_, ∞]},
   If[vars === {}, Factor[ℰ],
    Total[CoefficientRules[ℰ, vars] /.
      (p_ → c_) ⧴ Factor[c] Times @@ (vars^p)] ]];
CF[ℰ_𝔼] := CF /@ ℰ;
```

Preparation

```
𝔼[i_, j_, s_] := 𝔼[1, (-1)^s l_j, (-t)^s e_i f_j,
   t^s e_i l_(1+s) i-s j f_j + (-1)^s l_i l_j + (-t^2)^s e_i^2 f_j^2/4];
𝔼[i_, s_] := 𝔼[1, 0, 0, s l_i];
𝔼 /: 𝔼[1, L1_, Q1_, P1_] 𝔼[1, L2_, Q2_, P2_] :=
  𝔼[1, L1 + L2, Q1 + Q2, P1 + P2];
```

Preparing the Trefoil

```
z1 = (𝔼[1, 11, 0] 𝔼[4, 2, -1] 𝔼[15, 5, 0] ×
   𝔼[6, 8, -1] 𝔼[9, 16, 0] 𝔼[12, 14, -1] ×
   𝔼[3, -1] 𝔼[7, +1] 𝔼[10, -1] 𝔼[13, +1])
```



$$\mathbb{E}\Big[1,\ -l_2+l_5-l_8+l_{11}-l_{14}+l_{16},\ -\frac{e_4f_2}{t}+e_{15}f_5-\frac{e_6f_8}{t}+e_1f_{11}-\frac{e_{12}f_{14}}{t}+e_9f_{16},\ -\frac{e_4^2f_2^2}{4t^2}+\frac14e_{15}^2f_5^2-\frac{e_6^2f_8^2}{4t^2}+\frac14e_1^2f_{11}^2-\frac{e_{12}^2f_{14}^2}{4t^2}+\frac14e_9^2f_{16}^2+e_1f_{11}l_1+\frac{e_4f_2l_2}{t}-l_3-l_2l_4+l_7+\frac{e_6f_8l_8}{t}-l_6l_8+e_9f_{16}l_9-l_{10}+l_1l_{11}+l_{13}+\frac{e_{12}f_{14}l_{14}}{t}-l_{12}l_{14}+e_{15}f_5l_{15}+l_5l_{15}+l_9l_{16}\Big]$$

Differential Polynomials (Implementing $P(\partial_\alpha,\partial_\beta)(f)$)

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] :=
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

le and *fl* Sorts

```
S_{l_j_ (x:e|f)_i_ → k_}[𝔼[ω_, L_, Q_, P_]] :=
 With[{λ = ∂_{l_j} L, α = ∂_{x_i} Q, q = e^γ β x_k + γ l_k}, CF[
   𝔼[ω, L /. l_j → l_k, t^λ α x_k + (Q /. x_i → 0),
    e^-q DP_{l_j→D_γ, x_i→D_β}[P][e^q] /. {β → α/ω, γ → λ Log[t]}] ]];
```

The Λόγος

```
Λ[k_] := ((t - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 e_k l_k f_k δ^2 μ^2 -
    δ (1 + μ) (f_k^2 α^2 + e_k^2 β^2) - e_k^2 f_k^2 δ^3 (1 + 3 μ) -
    2 (α β + 2 δ μ + e_k f_k δ^2 (1 + 2 μ) + 2 l_k δ μ^2) (f_k α + e_k β) -
    4 (l_k μ^2 + e_k f_k δ (1 + μ)) (α β + δ μ)) (1 + t) / 4;
```

fe Sorts

```
S_{f_i_ e_j_ → k_}[𝔼[ω_, L_, Q_, P_]] :=
 With[{q = ((1 - t) α β + β e_k + α f_k + δ e_k f_k) / μ}, CF[
   𝔼[μ ω, L, μ ω q + μ (Q /. f_i | e_j → 0),
     μ^4 e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + ω^4 Λ[k]] /. μ → 1 + (t - 1) δ /.
    {α → ω^-1 (∂_{f_i} Q /. e_j → 0), β → ω^-1 (∂_{e_j} Q /. f_i → 0),
     δ → ω^-1 ∂_{f_i,e_j} Q}]];
```

Elf Merges

```
m_{i_,j_→k_}[Z_𝔼] := Module[{x, z},
  CF[(Z // S_{f_i e_j→x} // S_{l_i e_x→x} // S_{f_x l_j→x}) /. z_{i|j|x} → z_k]]
```

Rewriting the Trefoil (by merging 16 elves)

```
(Do[z1 = z1 // m_{1,k→1}, {k, 2, 16}]; z1)
```

$$\mathbb{E}\left[\frac{1-t+t^2}{t},\ 0,\ 0,\ \frac{(-1+t)\,(1-t+t^2)^2\,(1-t+2t^2)}{t^3}-\frac{2\,(1+t)\,(1-t+t^2)^3\,e_1f_1}{t^4}-\frac{2\,(-1+t)\,(1+t)\,(1-t+t^2)^3\,l_1}{t^4}\right]$$

Readout

```
ρ1[𝔼[ω_, _, _, P_]] := CF[ t ((P /. e_ | l_ | f_ → 0) - t ω^3 (∂_t ω)) / ((t - 1)^2 ω^2) ]
```

$\rho_1(3_1)$

```
ρ1[z1] // Expand
```

$\frac{1}{t}+t$

References.

[BN] D. Bar-Natan, *Polynomial Time Knot Polynomial,* research proposal for the 2017 Killam Fellowship, ωεβ/K17.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

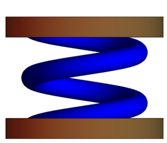
[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Vo] H. Vo, University of Toronto Ph.D. thesis, in preparation.



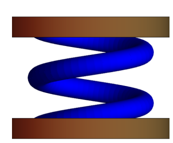
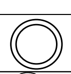
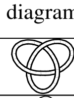
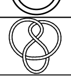
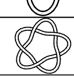


diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon; unknotting number / amphicheiral	diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon; unknotting number / amphicheiral
	0_1^a 0	1	0 / ✔ 0 / ✔		3_1^a t	$t-1$	1 / ✘ 1 / ✘
	4_1^a 0	$3-t$	1 / ✘ 1 / ✔		5_1^a $2t^3+3t$	t^2-t+1	2 / ✘ 2 / ✘

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Toulouse-1705/`