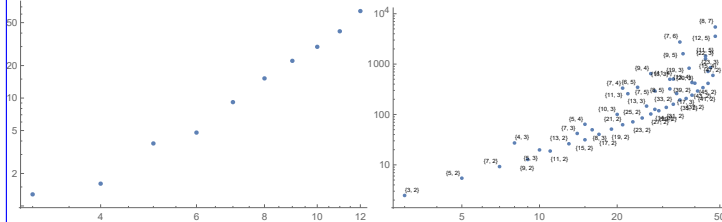


The Dogma is Wrong

Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: “quantize and use representation theory”. We present an alternative and better procedure: “centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra”. While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract ([oebf/kiw](#)). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

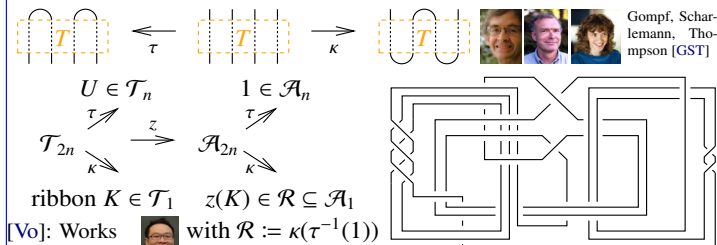
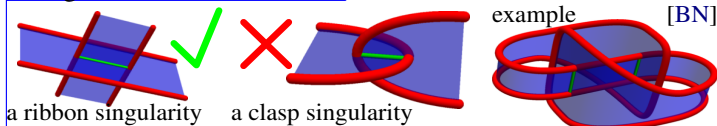
Experimental Analysis ([oebf/Exp](#)). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 crossings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1 , always $\deg \rho_1^+ \leq 2g - 1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-tring Alexander failures it does give the right answer.

Ribbon Knots.



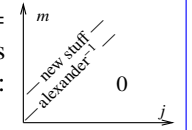
[Vo]: Works for Alexander! $A^+ = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$
 $\rho_1^+ = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 + 108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$
Faster is better, leaner is meaner!

dog·ma (dɔg'mə, dɔg' -) The Free Dictionary, [oebf/TFD](#)
n. pl. **dog·mas** or **dog·ma·ta** (-mə-tə)
1. A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.
2. A principle or statement of ideas, or a group of such principles or statements especially when considered to be authoritative or accepted uncritically: “Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry” (Bertrand Russell).

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K , in the d -dimensional representation of sl_2 . Writing

$$\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}} \Big|_{q=e^h} = \sum_{j,m \geq 0} a_{jm}(K) d^j h^m,$$

“below diagonal” coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and “on diagonal” coefficients give the inverse of the Alexander polynomial: $(\sum_{m=0}^{\infty} a_{mm}(K) h^m) \cdot \omega(K)(e^h) = 1$.



“Above diagonal” we have **Rozansky’s Theorem** [Ro3, (1.2)]:
$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1})\omega(K)(q^d)} \left(1 + \sum_{k=1}^{\infty} \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)} \right).$$

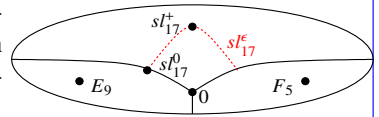
The Yang-Baxter Technique. Given an algebra A (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements $R = \sum a_i \otimes b_i \in A \otimes A$ and $C \in A$, form $Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C$.

Problem. Extract information from Z .

The Dogma. Use representation theory. In principle finite, but slow.

The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional “space of formulas”. $m_k^{ij} \{ \mathcal{F}_S \} \xrightarrow{\mathbb{E}} \{ A^{\otimes S} \} \xleftarrow{m_k^{ij}}$

The (fake) moduli of Lie algebras on V , a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^e / (e^{k+1} = 0)$.



Why are “solvable algebras” any good? Contrary to common beliefs, computations in semi-simple Lie algebras are just awful:

```
ln[1]= MatrixExp [ { a b } ] // FullSimplify // MatrixForm
```

Yet in solvable algebras, exponentiation is fine and even BCH, $z = \log(e^x e^y)$, is bearable:

```
ln[2]= MatrixExp [ { a b } ] // MatrixForm
ln[3]= MatrixExp [ { a1 b1 } ] . MatrixExp [ { a2 b2 } ] //
MatrixLog // PowerExpand // Simplify //
MatrixForm
```

Recomposing gl_n . Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(\nabla, b, \delta)$:
 $b(\nabla) = b: \nabla \otimes \nabla \rightarrow \nabla$
 $b(\Delta) \sim \delta: \nabla \rightarrow \nabla \otimes \nabla$

Now define $gl_n^e := \mathcal{D}(\nabla, b, e\delta)$. Schematically, this is $[\nabla, \nabla] = \nabla$, $[\Delta, \Delta] = e\Delta$, and $[\nabla, \Delta] = \Delta + e\nabla$. In detail, it is

$$\begin{matrix} & i & j \\ i & \begin{matrix} e_{ij} \\ h_i \end{matrix} & \\ j & \begin{matrix} f_{ji} \\ g_j \end{matrix} & \end{matrix} \quad \begin{matrix} [e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj} & [f_{ij}, f_{kl}] = e\delta_{jk} f_{il} - e\delta_{il} f_{kj} \\ [e_{ij}, f_{kl}] = \delta_{jk} (e\delta_{i<k} e_{il} + \delta_{il} (h_i + e\epsilon_i) / 2 + \delta_{i>l} f_{il}) \\ \quad - \delta_{il} (e\delta_{k<j} e_{kj} + \delta_{kj} (h_j + e\epsilon_j) / 2 + \delta_{k>j} f_{kj}) \\ [g_i, e_{jk}] = (\delta_{ij} - \delta_{ik}) e_{jk} & [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) e_{jk} \\ [g_i, f_{jk}] = (\delta_{ij} - \delta_{ik}) f_{jk} & [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) f_{jk} \end{matrix}$$