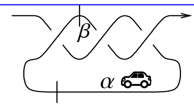
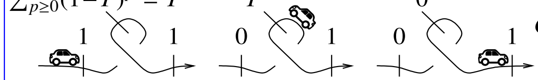
Theorem. The Green function $g_{\alpha\beta}$ is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point).



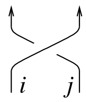
Example.

$\sum_{p\geq 0}(1-T)^p = T^{-1}$



$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof. Near a crossing c with sign s, incoming upper edge i and incoming lower edge j, both sides satisfy the *g-rules*:



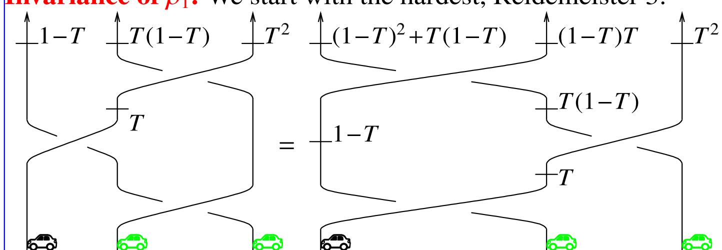
$$g_{i\beta} = \delta_{i\beta} + T^s g_{i+1,\beta} + (1-T^s)g_{j+1,\beta}, \quad g_{j\beta} = \delta_{j\beta} + g_{j+1,\beta},$$

and always, $g_{\alpha,2n+1} = 1$: use common sense and $AG = I\ (= GA)$.

Bonus. Near c, both sides satisfy the further *g-rules*:

$$g_{\alpha i} = T^{-s}(g_{\alpha,i+1} - \delta_{\alpha,i+1}), \quad g_{\alpha j} = g_{\alpha,j+1} - (1-T^s)g_{\alpha i} - \delta_{\alpha,j+1}.$$

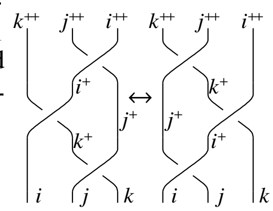
Invariance of ρ_1. We start with the hardest, Reidemeister 3:



⇒ Overall traffic patterns are unaffected by Reid3!

⇒ Green's $g_{\alpha\beta}$ is unchanged by Reid3, provided the cars injection site α and the traffic counters β are away.

⇒ Only the contribution from the R_1 terms within the Reid3 move matters, and using g-rules the relevant $g_{\alpha\beta}$'s can be pushed outside of the Reid3 area:

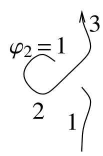



```
δ_{i_,j_} := If[i === j, 1, 0];
gRules_{s_,i_,j_} :=
  {g_{iβ_} ⧴ δ_{iβ} + T^s g_{i⁺,β} + (1 - T^s) g_{j⁺,β}, g_{jβ_} ⧴ δ_{jβ} + g_{j⁺,β},
   g_{α_,i} ⧴ T^{-s} (g_{α,i⁺} - δ_{α,i⁺}),
   g_{α_j} ⧴ g_{α,j⁺} - (1 - T^s) g_{αi} - δ_{α,j⁺}}
lhs = R₁[1, j, k] + R₁[1, i, k⁺] + R₁[1, i⁺, j⁺] //.
    gRules_{1,j,k} ⋃ gRules_{1,i,k⁺} ⋃ gRules_{1,i⁺,j⁺};
rhs = R₁[1, i, j] + R₁[1, i⁺, k] + R₁[1, j⁺, k⁺] //.
    gRules_{1,i,j} ⋃ gRules_{1,i⁺,k} ⋃ gRules_{1,j⁺,k⁺};
Simplify[lhs == rhs]
True
```

Next comes Reid1, where we use results from an earlier example:

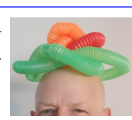
$$R_1[1, 2, 1] - 1\,(g_{22} - 1/2)\ \ /.\ \ g_{\alpha\_,\beta\_} \mapsto \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}[\![\alpha, \beta]\!]$$

$$\frac{1}{T^2} - \frac{1}{T} - \frac{-1+\frac{1}{T}}{T} \;=\; 0$$



Invariance under the other moves is proven similarly.

Wearing my Topology hat the formula for R_1, and even the idea to look for R_1, remain a complete mystery to me.



Wearing my Quantum Algebra hat, I spy a Heisenberg algebra $\mathbb{H} = A\langle p, x\rangle/([p,x] = 1)$:

cars $\leftrightarrow p$ traffic counters $\leftrightarrow x$

Where did it come from? Consider $\mathfrak{g}_\epsilon := sl^\epsilon_{2+} := L\langle y, b, a, x\rangle$ with relations

$$[b,x] = \epsilon x, \quad [b,y] = -\epsilon y, \quad [b,a] = 0,$$
$$[a,x] = x, \quad [a,y] = -y, \quad [x,y] = b + \epsilon a.$$

At invertible ϵ, it is isomorphic to sl_2 plus a central factor, and it can be quantized à la Drinfel'd [Dr] much like sl_2 to get an algebra $QU = A\langle y, b, a, x\rangle$ subject to (with $q = e^{\hbar\epsilon}$):

$$[b,a] = 0, \quad [b,x] = \epsilon x, \quad [b,y] = -\epsilon y,$$
$$[a,x] = x, \quad [a,y] = -y, \quad xy - qyx = \frac{1 - e^{-\hbar(b+\epsilon a)}}{\hbar}.$$

Now QU has an R-matrix solving Yang-Baxter (meaning Reid3),

$$R = \sum_{m,n\geq 0} \frac{y^n b^m \otimes (\hbar a)^m (\hbar x)^n}{m![n]_q!}, \qquad ([n]_q! \text{ is a "quantum factorial"})$$

and so it has an associated "universal quantum invariant" à la Lawrence and Ohtsuki [La, Oh1], $Z_\epsilon(K) \in QU$.

Now $QU \cong \mathcal{U}(\mathfrak{g}_\epsilon)$ (only as algebras!) and $\mathcal{U}(\mathfrak{g}_\epsilon)$ represents into $\mathbb{H}$ via

$$y \to -tp - \epsilon\cdot xp^2, \quad b \to t + \epsilon\cdot xp, \quad a \to xp, \quad x \to x,$$

(abstractly, $\mathfrak{g}_\epsilon$ acts on its Verma module

$$\mathcal{U}(\mathfrak{g}_\epsilon)/(\mathcal{U}(\mathfrak{g}_\epsilon)\langle y, a, b - \epsilon a - t\rangle) \cong \mathbb{Q}[x]$$

by differential operators, namely via $\mathbb{H}$), so R can be pushed to $\mathcal{R} \in \mathbb{H}\otimes\mathbb{H}$.

Everything still makes sense at $\epsilon = 0$ and can be expanded near $\epsilon = 0$ resulting with $\mathcal{R} = \mathcal{R}_0(1 + \epsilon\mathcal{R}_1 + \cdots)$, with $\mathcal{R}_0 = e^{t(xp\otimes 1 - x\otimes p)}$ and $\mathcal{R}_1$ a quartic polynomial in p and x. So p's and x's get created along K and need to be pushed around to a standard location ("normal ordering"). This is done using

$$(p\otimes 1)\mathcal{R}_0 = \mathcal{R}_0(T(p\otimes 1) + (1-T)(1\otimes p)),$$
$$(1\otimes p)\mathcal{R}_0 = \mathcal{R}_0(1\otimes p),$$

and when the dust settles, we get our formulas for ρ_1. But QU is a quasi-triangular Hopf algebra, and hence **ρ_1 is homomorphic**. Read more at [BV1, BV2] and hear more at ωεβ/SolvApp, ωεβ/Dogma, ωεβ/DoPeGDO, ωεβ/FDA, ωεβ/AQDW.

Also, we can (and know how to) look at higher powers of ϵ and we can (and more or less know how to) replace sl_2 by arbitrary semi-simple Lie algebra (e.g., [Sch]). So **ρ_1 is not alone!**


Schaveling

These constructions are very similar to Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] and hence to the "loop expansion" of the Kontsevich integral and the coloured Jones polynomial [Oh2].

If this all reads like **insanity** to you, it should (and you haven't seen half of it). Simple things should have simple explanations. Hence, **Homework.** Explain ρ_1 with no reference to quantum voodoo and find it a topology home (large enough to house generalizations!). Make explicit the homomorphic properties of ρ_1. Use them to do topology!

P.S. As a friend of Δ, ρ_1 gives a genus bound, sometimes better than Δ's. How much further does this friendship extend?

Video: http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210. Handout: http://www.math.toronto.edu/~drorbn/Talks/Nara-2308.