

Cars, Interchanges, Traffic Counters, and some Pretty Darned Good Knot Invariants

More at ωεβ/APAI

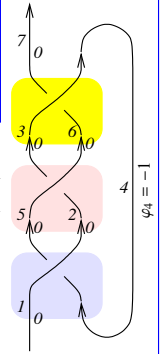


Abstract. Reporting on joint work with Roland van der Veen, I'll tell you some stories about ρ_1 , an easy to define, strong, fast to compute, homomorphic, and well-connected knot invariant. ρ_1 was first studied by Rozansky and Overbay [Ro1, Ro2, Ro3, Ov] and Ohtsuki [Oh2], it has far-reaching generalizations, it is elementary and dominated by the coloured Jones polynomial, and I wish I understood it.



Jones:

Formulas stay; interpretations change with time.

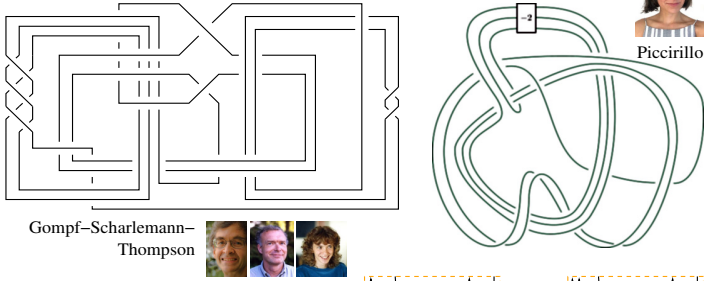


Common misconception. Dominated, elementary \Rightarrow lesser.

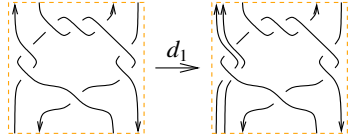
We seek strong, fast, homomorphic knot and tangle invariants.

Strong. Having a small "kernel".

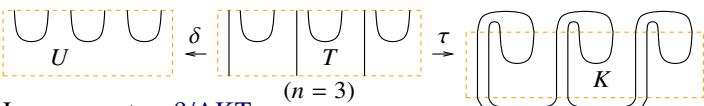
Fast. Computable even for large knots (best: poly time).



Homomorphic. Extends to tangles and behaves under tangle operations; especially gluings and doublings:



Why care for "Homomorphic"? **Theorem.** A knot K is ribbon iff there exists a $2n$ -component tangle T with skeleton as below such that $\tau(T) = K$ and where $\delta(T) = U$ is the *untangle*:



Hear more at ωεβ/AKT.

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[BV2] D. Bar-Natan and R. van der Veen, *Perturbed Gaussian Generating Functions for Universal Knot Invariants*, arXiv:2109.02057.

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[Jo] V. F. R. Jones, *Hecke Algebra Representations of Braid Groups and Link Polynomials*, Annals Math., **126** (1987) 335–388.

[La] R. J. Lawrence, *Universal Link Invariants using Quantum Groups*, Proc. XVII Int. Conf. on Diff. Geom. Methods in Theor. Phys., Chester, England, August 1988. World Scientific (1989) 55–63.

[LTW] X-S. Lin, F. Tian, and Z. Wang, *Burau Representation and Random Walk on String Links*, Pac. J. Math., **182-2** (1998) 289–302, arXiv:q-alg/9605023.

[Oh1] T. Ohtsuki, *Quantum Invariants*, Series on Knots and Everything **29**, World Scientific 2002.

[Oh2] T. Ohtsuki, *On the 2-loop Polynomial of Knots*, Geom. Top. **11** (2007) 1357–1475.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial*, Ph.D. thesis, University of North Carolina, August 2013, ωεβ/Ov.

[Ro1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I*, Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial*, Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture*, arXiv:math/0201139.

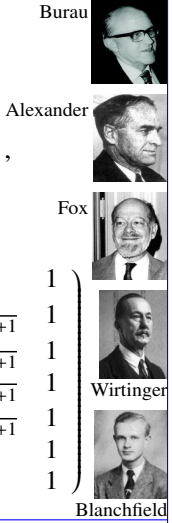
[Sch] S. Schaveling, *Expansions of Quantum Group Invariants*, Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Formulas. Draw an n -crossing knot K as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \dots, 2n + 1\}$ and with rotation numbers φ_k . Let A be the $(2n + 1) \times (2n + 1)$ matrix constructed by starting with the identity matrix I , and adding a 2×2 block for each crossing:

$$c : \begin{matrix} s = +1 \\ j+1 \uparrow & i+1 \uparrow \\ & j \downarrow & i \downarrow \end{matrix} \quad \begin{matrix} s = -1 \\ i+1 \uparrow & j+1 \uparrow \\ & j \downarrow & i \downarrow \end{matrix} \quad \longrightarrow \quad \begin{matrix} A & \text{col } i+1 & \text{col } j+1 \\ \text{row } i & -T^s & T^s - 1 \\ \text{row } j & 0 & -1 \end{matrix}$$

Let $G = (g_{\alpha\beta}) = A^{-1}$. For the trefoil example, it is:

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{1-T} & \frac{1}{1-T} & \frac{1}{1-T} & \frac{1}{1-T} & 1 \\ 0 & 0 & \frac{T^2-T+1}{1-T} & \frac{T^2-T+1}{1-T} & \frac{T^2-T+1}{1-T} & \frac{T^2-T+1}{1-T} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

"The Green Function"

Note. The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2} \det(A), \quad \text{with } \varphi = \sum_k \varphi_k, \quad w = \sum_c s.$$

Classical Topologists: This is boring. Yawn.

Formulas, continued. Finally, set

$$R_1(c) := s(g_{ji}(g_{j+1,j} + g_{j,j+1} - g_{ij}) - g_{ii}(g_{j,j+1} - 1) - 1/2)$$

$$\rho_1 := \Delta^2 \left(\sum_c R_1(c) - \sum_k \varphi_k (g_{kk} - 1/2) \right).$$

In our example $\rho_1 = -T^2 + 2T - 2 + 2T^{-1} - T^{-2}$.

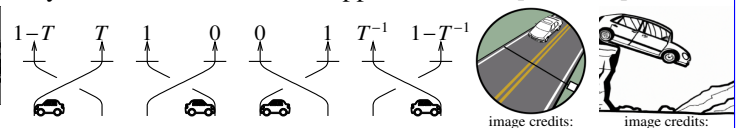
Theorem. ρ_1 is a knot invariant.

Proof: later.

Classical Topologists: Whiskey Tango Foxtrot?

Cars, Interchanges, and Traffic Counters.

Cars always drive forward. When a car crosses over a bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0^*$. At the very end, cars fall off and disappear. See also [Jo, LTW].



* In algebra $x \sim 0$ if for every y in the ideal generated by x , $1 - y$ is invertible.