The $sl_3^{/\epsilon^2}$ Example [BV3]. Here we have two formal variables T_1 and T_2, we set $T_3 := T_1T_2$, we integrate over 6 variables for each edge: p_{1i}, p_{2i}, p_{3i}, x_{1i}, x_{2i}, and x_{3i}.

```
T3 = T1 T2;  i_+ := i + 1;
$π =
  (CF@Normal[# + O[ϵ]^2] /.
      {π_{is__} ⇒ B^-1 π_{is}, x_{is__} ⇒ B^-1 x_{is}, p_{is__} ⇒ B p_{is}} /.
    ϵ B^{b_} /; b < 0 → 0 /. B → 1) &;
```

```
vs_{i_} := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
𝓕[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
ℒ[K_] := CF[ℒ /@ Features[K][[2]]];
vs[K_] :=
 Union @@ Table[{vs_i}, {i, Features[K][[1]]}]
```

The Lagrangian.

```
ℒ[X_{i_,j_}[s_]] := T3^s 𝔼[CF@Plus[
    Σ_{ν=1}^3 (x_{νi} (p_{νi+} - p_{νi}) + x_{νj} (p_{νj+} - p_{νj}) + (T_ν^s - 1) x_{νi} (p_{νi+} - p_{νj+})),
    (T1^s - 1) p_{3j} x_{1i} (T2^s x_{2i} - x_{2j}),
    ϵ s (T3^s - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T2^s - 1),
    ϵ s (1/2 + T2^s p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} -
       (T2^s - 1) p_{2j} p_{3i} x_{2i} x_{3i} + (T3^s - 1) p_{2j} p_{3j} x_{2i} x_{3i} +
       2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} -
       T2^s p_{2j} p_{3j} x_{2i} x_{3j} +
       ((T1^s - 1) p_{1j} x_{1i} (T2^{2s} p_{2j} x_{2i} - T2^s p_{2j} x_{2j} -
            (T2^s + 1) (T3^s - 1) p_{3j} x_{3i} + T2^s p_{3j} x_{3j}) +
         (T3^s - 1) p_{3j} x_{3i} (1 - T2^s p_{1i} x_{1i} + p_{2i} x_{2j} + (T2^s - 2) p_{2j} x_{2j})) /
       (T2^s - 1))]]
```

```
ℒ[C_{i_}[φ_]] := T3^φ 𝔼[Σ_{ν=1}^3 x_{νi} (p_{νi+} - p_{νi}) + ϵ φ (p_{3i} x_{3i} - 1/2)]
```

Reidemeister 3.

```
Short[
  lhs = ∫ 𝓕[i, j, k] × ℒ /@ (X_{i,j}[1] X_{i+,k}[1] X_{j+,k+}[1])
     d{vs_i, vs_j, vs_k, vs_{i+}, vs_{j+}, vs_{k+}}]
```

```
T1^3 T2^3
 𝔼[3ϵ/2 + T1^2 p_{1,2+i} π_{1,i} - (-1 + T1) T1 p_{1,2+j} π_{1,i} + <<150>>]
```

```
rhs = ∫ 𝓕[i, j, k] × ℒ /@ (X_{j,k}[1] X_{i,k+}[1] X_{i+,j+}[1])
    d{vs_i, vs_j, vs_k, vs_{i+}, vs_{j+}, vs_{k+}};
lhs == rhs
```

```
True
```

The Trefoil.

```
K = Knot[3, 1];  ∫ ℒ[K] d vs[K]
```

```
-((i T1^2 T2^2
   𝔼[-((ϵ (1 - T1 + T1^2 - T2 - T1^3 T2 + T2^2 + T1^4 T2^2 - T1 T2^3 -
          T1^4 T2^3 + T1^2 T2^4 - T1^3 T2^4 + T1^4 T2^4)) / ((1 - T1 + T1^2)
        (1 - T2 + T2^2) (1 - T1 T2 + T1^2 T2^2)))]) /
  ((1 - T1 + T1^2) (1 - T2 + T2^2) (1 - T1 T2 + T1^2 T2^2)))
```

A faster program, in which the Feynman diagrams are "pre-computed" (see theta.nb at ωεβ/ap):

```
F1[{s_, i_, j_}] := CF[
  s (1/2 - g_{3ii} + T2^s g_{1ii} g_{2ji} - g_{1ii} g_{2jj} - (T2^s - 1) g_{2ji} g_{3ii} +
     2 g_{2jj} g_{3ii} - (1 - T3^s) g_{2ji} g_{3ji} - g_{2ii} g_{3jj} - T2^s g_{2ji} g_{3jj} +
     g_{1ii} g_{3jj} +
     ((T1^s - 1) g_{1ji} (T2^{2s} g_{2ji} - T2^s g_{2jj} + T2^s g_{3jj}) +
        (T3^s - 1) g_{3ji} (1 - T2^s g_{1ii} - (T1^s - 1) (T2^s + 1) g_{1ji} +
           (T2^s - 2) g_{2jj} + g_{2ij})) / (T2^s - 1))]
```

```
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
 CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g_{1,j1,i0} g_{3,j0,i1}
   ((T2^s0 g_{2,i1,i0} - g_{2,i1,j0}) - (T2^s0 g_{2,j1,i0} - g_{2,j1,j0}))]
```

```
F3[φ_, k_] = φ g_{3kk} - φ/2;
```

We call the invariant computed θ:

```
Θ[K_] := Θ[K] = Module[{X, φ, n, A, Δ, G, ev, θ},
   {X, φ} = Rot[K]; n = Length[X];
   A = IdentityMatrix[2 n + 1];
   Cases[X, {s_, i_, j_} ⇒
     (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                        0     -1   ))];
   Δ = T^((-Total[φ] - Total[X[[All,1]]])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] := Factor[ℰ /. g_{ν_,α_,β_} ⇒ (G[[α, β]] /. T → T_ν)];
   θ = ev[Σ_{k=1}^n F1[X[[k]]]];
   θ += ev[Σ_{k1=1}^n Σ_{k2=1}^n F2[X[[k1]], X[[k2]]]];
   θ += ev[Σ_{k=1}^{2n} F3[φ[[k]], k]];
   Factor@{Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}
  ];
```

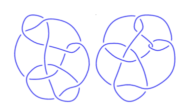
Some Knots.

```
Expand[Θ[Knot[3, 1]]]
```

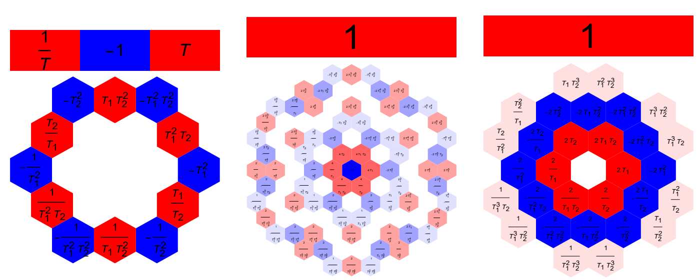
```
{-1 + 1/T + T, -1/T1^2 - T1^2 - 1/T2^2 - 1/(T1^2 T2^2) + 1/(T1 T2^2) +
  1/(T1^2 T2) + T1/T2 + T2/T1 + T1^2 T2 - T2^2 + T1 T2^2 - T1^2 T2^2}
```

```
(* PolyPlot suppressed *)
```

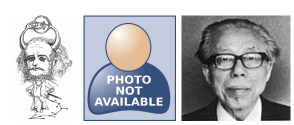
```
GraphicsRow[PolyPlot[Θ[Knot[#]],
    Labeled → True] &
  /@ {"3_1", "K11n34", "K11n42"}]
```



K11n34 K11n42



So θ detects knot mutation and separates the Conway knot K11n34 from the Kinoshita-Terasaka knot K11n42!



Conway Kinoshita Terasaka

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505.