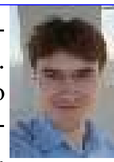
Dror Bar-Natan — Handout Portfolio

As of June 7, 2025 — see also http://drorbn.net/hp — paper copies are available from the author, at a muffin plus cappuccino each or the monetary equivalent (to offset printing costs). This document has 38 pages.

Warning. This is a reduced quality version with some content removed to control file size. My full handout portfolio, about 6 times longer, is at the URL above.

The Strongest Genuinely Computable Knot Invariant in 2024

Abstract. "Genuinely computable" means we have computed it for random knots with over 300 crossings. "Strongest" means it separates prime knots with up to 15 crossings better than the less-computable HOMFLY-PT and Khovanov homology taken together. And hey, it's also meaningful and fun.


van der Veen

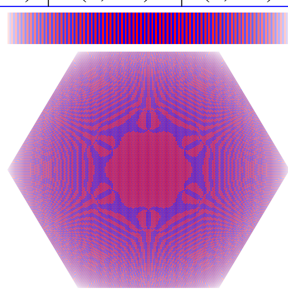
Continues Rozansky, Garoufalidis, Kricker, and Ohtsuki, joint with van der Veen.

Acknowledgement. This work was supported by NSERC grants RGPIN-2018-04350 and RGPIN-2025-06718 and by the Chu Family Foundation (NYC).

Strongest. Testing $\Theta = (\Delta, \theta)$ on prime knots up to mirrors and reversals, counting the number of distinct values (with deficits in parenthesis): (ρ_1: [Ro1, Ro2, Ro3, Ov, BV1])

	knots	(H, Kh)	(Δ, ρ_1)	$\Theta = (\Delta, \theta)$	together
reign		2005-22	2022-24	2024-	
xing ≤ 10	249	248 (1)	249 (0)	249 (0)	249 (0)
xing ≤ 11	801	771 (30)	787 (14)	798 (3)	798 (3)
xing ≤ 12	2,977	(214)	(95)	(19)	(18)
xing ≤ 13	12,965	(1,771)	(959)	(194)	(185)
xing ≤ 14	59,937	(10,788)	(6,253)	(1,118)	(1,062)
xing ≤ 15	313,230	(70,245)	(42,914)	(6,758)	(6,555)

Genuinely Computable. Here's Θ on a random 300 crossing knot (from [DHOEBL]). For almost every other invariant, that's science fiction.

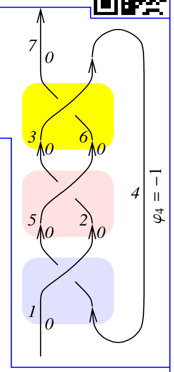


Fun. There's so much more to see in 2D pictures than in 1D ones! Yet almost nothing of the patterns you see we know how to prove. We'll have fun with that over the next few years. Would you join?

Meaningful. θ gives a genus bound (unproven yet with confidence). We hope (with reason) it says something about ribbon knots.

Conventions. T, T_1, and T_2 are indeterminates and $T_3 := T_1T_2$.

Preparation. Draw an n-crossing knot K as a diagram D as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \ldots, 2n+1\}$ and with rotation numbers φ_k.



Model T Traffic Rules. Cars always drive forward. When a car crosses over a sign-s bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0$. At the very end, cars fall off and disappear. On various edges *traffic counters* are placed. See also [Jo, LTW].

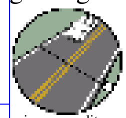
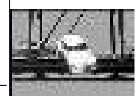


image credits: diamondtraffic.com


$p = 1 - T^s$

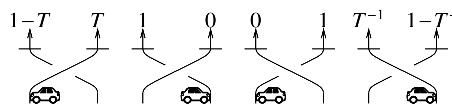
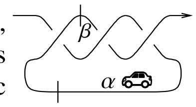


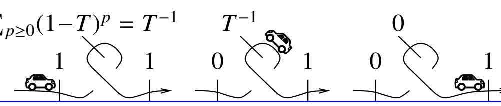

image credits: Dall-E

Definition. The *traffic function* $G = (g_{\alpha\beta})$ (also, the *Green function* or the *two-point function*) is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point). There are also model-T_ν traffic functions $G_\nu = (g_{\nu\alpha\beta})$ for $\nu = 1, 2, 3$.



Example.

$\sum_{p\geq 0}(1-T)^p = T^{-1}$



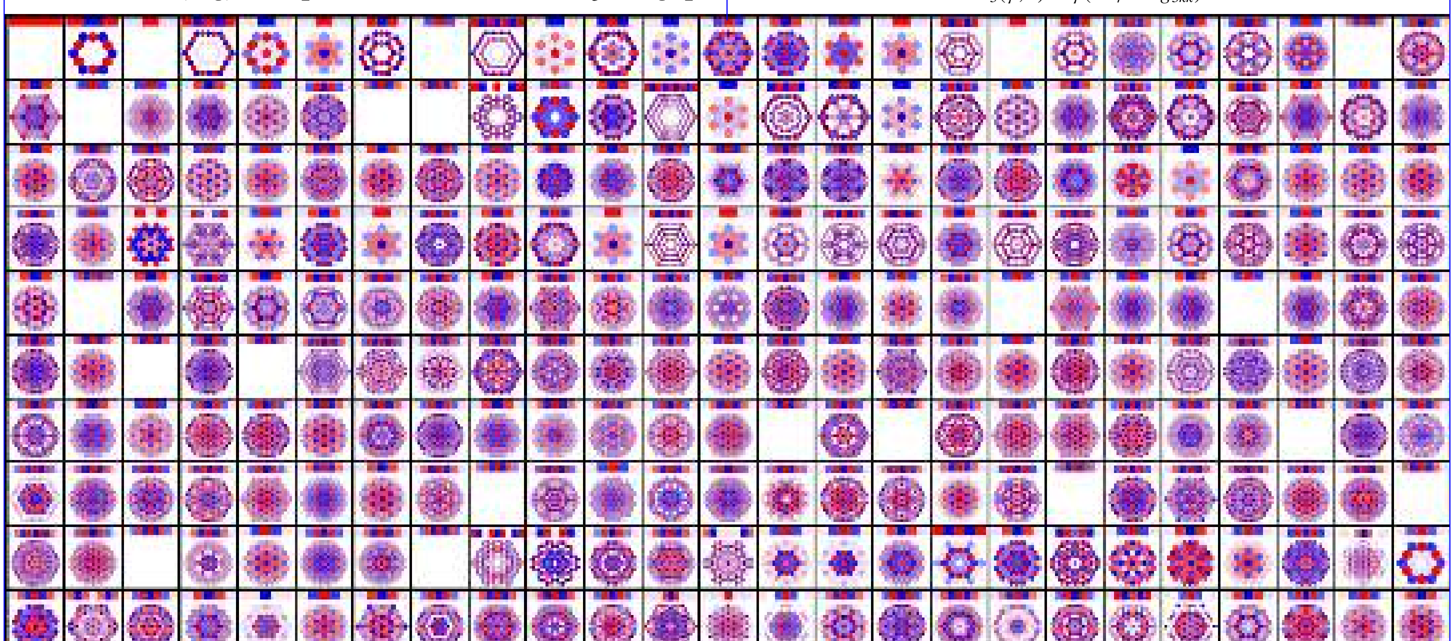
$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Don't Look.

$$\begin{aligned} F_1(c) = s\Big[&1/2 - g_{3ii} + T_2^s g_{1ii} g_{2ji} - T_2^s g_{3jj} g_{2ji} - (T_2^s - 1) g_{3ii} g_{2ji} \\ &+ (T_3^s - 1) g_{2ji} g_{3ji} - g_{1ii} g_{2jj} + 2 g_{3ii} g_{2jj} + g_{1ii} g_{3jj} - g_{2ii} g_{3jj}\Big] \\ &+ \frac{s}{T_2^s - 1}\Big[(T_1^s - 1) T_2^s \left(g_{3jj} g_{1ji} - g_{2jj} g_{1ji} + T_2^s g_{1ji} g_{2ji}\right) \\ &+ (T_3^s - 1)\left(g_{3ji} - T_2^s g_{1ii} g_{3ji} + g_{2ij} g_{3ji} + (T_2^s - 2) g_{2jj} g_{3ji}\right) \\ &- (T_1^s - 1)(T_2^s + 1)(T_3^s - 1) g_{1ji} g_{3ji}\Big] \end{aligned}$$

$$F_2(c_0, c_1) = \frac{s_1 (T_1^{s_0} - 1)(T_3^{s_1} - 1) g_{1 j_1 i_0} g_{3 j_0 i_1}}{T_2^{s_1} - 1}\left(T_2^{s_0} g_{2 i_1 i_0} + g_{2 j_1 j_0} - T_2^{s_0} g_{2 j_1 i_0} - g_{2 i_1 j_0}\right)$$

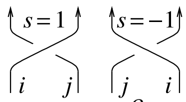
$$F_3(\varphi, k) = \varphi(-1/2 + g_{3kk})$$



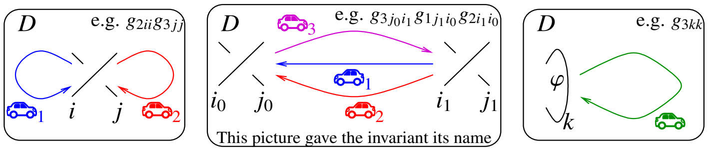
2025/06/04@23:23

Video and more at http://www.math.toronto.edu/~drorbn/Talks/PhuQuoc-2506.

Theorem [BV3]. With $c = (s,i,j)$, $c_0 = (s_0,i_0,j_0)$, and $c_1 = (s_1,i_1,j_1)$ denoting crossings, there is a quadratic $F_1(c) \in \mathbb{Q}(T_\nu)[g_{\nu\alpha\beta} : \alpha,\beta \in \{i,j\}]$, a cubic $F_2(c_0,c_1) \in \mathbb{Q}(T_\nu)[g_{\nu\alpha\beta} : \alpha,\beta \in \{i_0,j_0,i_1,j_1\}]$, and a linear $F_3(\varphi,k)$ such that θ is a knot invariant:

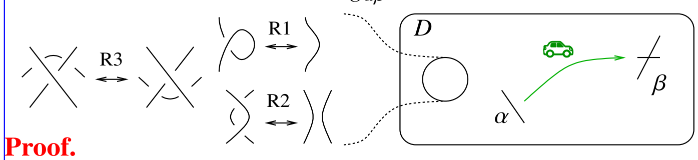


$$\theta(D) := \underbrace{\Delta_1\Delta_2\Delta_3}_{\substack{\text{normalization,}\\ \text{see later}}}\left(\sum_c F_1(c) + \sum_{c_0,c_1} F_2(c_0,c_1) + \sum_k F_3(\varphi_k,k)\right),$$

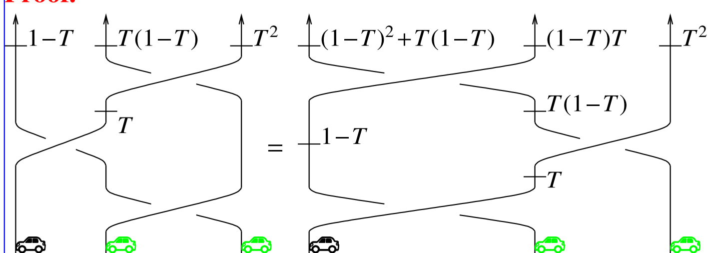


If these pictures remind you of Feynman diagrams, it's because they are Feynman diagrams [BN2].

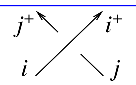
Lemma 1. The traffic function $g_{\alpha\beta}$ is a "relative invariant":



Proof.



Lemma 2. With $k^+ := k+1$, the "g-rules" hold near a crossing $c = (s,i,j)$:



$$g_{j\beta} = g_{j^+\beta} + \delta_{j\beta} \qquad g_{i\beta} = T^s g_{i^+\beta} + (1-T^s)g_{j^+\beta} + \delta_{i\beta} \qquad g_{2n^+,\beta} = \delta_{2n^+,\beta}$$
$$g_{\alpha i^+} = T^s g_{\alpha i} + \delta_{\alpha i^+} \qquad g_{\alpha j^+} = g_{\alpha j} + (1-T^s)g_{\alpha i} + \delta_{\alpha j^+} \qquad g_{\alpha,1} = \delta_{\alpha,1}$$

Corollary 1. G is easily computable, for $AG = I$ $(= GA)$, with A the $(2n+1)\times(2n+1)$ identity matrix with additional contributions:

$$c = (s,i,j) \mapsto \begin{array}{c|cc} A & \text{col } i^+ & \text{col } j^+ \\ \hline \text{row } i & -T^s & T^s-1 \\ \text{row } j & 0 & -1 \end{array}$$

For the trefoil example, we have:

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note. The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2}\det(A), \qquad \text{with } \varphi = \textstyle\sum_k \varphi_k,\ w = \sum_c s.$$

We also set $\Delta_\nu := \Delta(T_\nu)$ for $\nu = 1,2,3$.

Questions, Conjectures, Expectations, Dreams.

Question 1. What's the relationship between Θ and the Garoufalidis-Kashaev invariants [GK, GL]?

Conjecture 2. On classical (non-virtual) knots, θ always has hexagonal (D_6) symmetry.

Conjecture 3. θ is the ϵ^1 contribution to the "solvable approximation" of the sl_3 universal invariant, obtained by running the quantization machinery on the double $\mathcal{D}(\mathfrak{b}, b, \epsilon\delta)$, where $\mathfrak{b}$ is the Borel subalgebra of sl_3, b is the bracket of $\mathfrak{b}$, and δ the cobracket. See [BV2, BN1, Sch]

Conjecture 4. θ is equal to the "two-loop contribution to the Kontsevich Integral", as studied by Garoufalidis, Rozansky, Kricker, and in great detail by Ohtsuki [GR, Ro1, Ro2, Ro3, Kr, Oh].

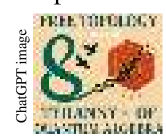
Fact 5. θ has a perturbed Gaussian integral formula, with integration carried out over over a space $6E$, consisting of 6 copies of the space of edges of a knot diagram D. See [BN2].

Conjecture 6. For any knot K, its genus $g(K)$ is bounded by the T_1-degree of θ: $2g(K) \geq \deg_{T_1}\theta(K)$.

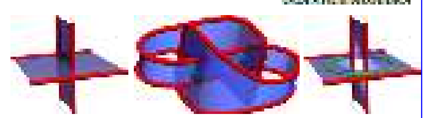
Conjecture 7. $\theta(K)$ has another perturbed Gaussian integral formula, with integration carried out over over the space $6H_1$, consisting of 6 copies of $H_1(\Sigma)$, where Σ is a Seifert surface for K.

Expectation 8. There are many further invariants like θ, given by Green function formulas and/or Gaussian integration formulas. One or two of them may be stronger than θ and as computable.

Dream 9. These invariants can be explained by something less foreign than semisimple Lie algebras.



Dream 10. With Conjecture 7 in mind, θ will have something to say about ribbon knots.



References.

[BN1] D. Bar-Natan, *Everything around sl_{2+}^ϵ is DoPeGDO. So what?,* talk in Da Nang, May 2019. Handout and video at ωεβ/DPG.

[BN2] —, *Knot Invariants from Finite Dimensional Integration,* talks in Beijing (July 2024, ωεβ/icbs24) and in Geneva (August 2024, ωεβ/ge24).

[BV1] —, R. van der Veen, *A Perturbed-Alexander Invariant,* Quantum Topology **15** (2024) 449–472, ωεβ/APAI.

[BV2] —, —, *Perturbed Gaussian Generating Functions for Universal Knot Invariants,* arXiv:2109.02057.

[BV3] —, —, *A Very Fast, Very Strong, Topologically Meaningful and Fun Knot Invariant,* in preparation, draft at ωεβ/Theta.

[DHOEBL] N. Dunfield, A. Hirani, M. Obeidin, A. Ehrenberg, S. Bhattacharyya, D. Lei, and others, *Random Knots: A Preliminary Report,* lecture notes at ωεβ/DHOEBL. Also a data file at ωεβ/DD.

[GK] S. Garoufalidis, R. Kashaev, *Multivariable Knot Polynomials from Braided Hopf Algebras with Automorphisms,* arXiv:2311.11528.

[GL] —, S. Y. Li, *Patterns of the V_2-polynomial of knots,* arXiv:2409.03557.

[GR] —, L. Rozansky, *The Loop Expansion of the Kontsevich Integral, the Null-Move, and S-Equivalence,* arXiv:math.GT/0003187.

[Jo] V. F. R. Jones, *Hecke Algebra Representations of Braid Groups and Link Polynomials,* Annals Math., **126** (1987) 335-388.

[Kr] A. Kricker, *The Lines of the Kontsevich Integral and Rozansky's Rationality Conjecture,* arXiv:math/0005284.

[LTW] X-S. Lin, F. Tian, Z. Wang, *Burau Representation and Random Walk on String Links,* Pac. J. Math., **182-2** (1998) 289–302, arXiv:q-alg/9605023.

[Oh] T. Ohtsuki, *On the 2–loop Polynomial of Knots,* Geom. Top. **11** (2007) 1357–1475.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, Aug. 2013, ωεβ/Ov.

[Ro1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

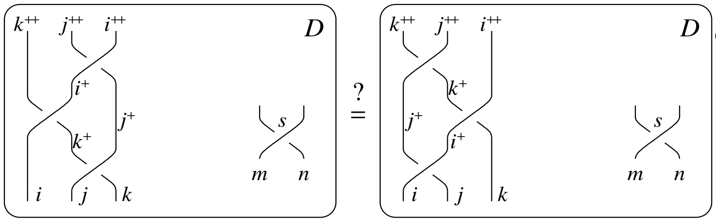
[Ro2] —, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] —, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Sch] S. Schaveling, *Expansions of Quantum Group Invariants,* Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/PhuQuoc-2506`.

Corollary 2. Proving invariance is easy:



Invariance under R3

This is Theta.nb of http://drorbn.net/v25/ap.

```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

```
T3 = T1 T2;
```

```
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```

```
F1[{s_, i_, j_}] =
  CF[
   s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj -
       (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
       g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
       ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
         (T3^s - 1) g3ji
          (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji +
            (T2^s - 2) g2jj + g2ij)) / (T2^s - 1))];
```

```
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
  CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1
    ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
```

```
F3[φ_, k_] = -φ/2 + φ g3kk;
```

```
δi_,j_ := If[i === j, 1, 0];
gRs_,i_,j_ := {
   gν_jβ_ :> gνj⁺β + δjβ,
   gν_iβ_ :> Tν^s gνi⁺β + (1 - Tν^s) gνj⁺β + δiβ,
   gν_α_i⁺ :> Tν^s gναi + δαi⁺,
   gν_α_j⁺ :> gναj + (1 - Tν^s) gναi + δαj⁺
  }
```

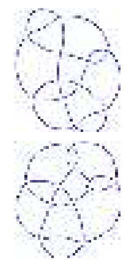
```
DSum[Cs___] := Sum[F1[c], {c, {Cs}}] +
  Sum[F2[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
lhs = DSum[{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺},
    {s, m, n}] //. gR1,j,k ⋃ gR1,i,k⁺ ⋃ gR1,i⁺,j⁺;
rhs = DSum[{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺},
    {s, m, n}] //. gR1,i,j ⋃ gR1,i⁺,k ⋃ gR1,j⁺,k⁺;
Simplify[lhs == rhs]
```

```
True
```

The Main Program

```
Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, θ},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :>
     (A[[{i, j}, {i + 1, j + 1}]] += (-T^s  T^s - 1
                                        0     -1  ))];
   Δ = T^((-Total[φ] - Total[Cs[[All,1]]])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] :=
    Factor[ℰ /. gν_,α_,β_ :> (G[[α, β]] /. T → Tν)];
   θ = ev[Σ_{k=1}^n F1[Cs[[k]]]];
   θ += ev[Σ_{k1=1}^n Σ_{k2=1}^n F2[Cs[[k1]], Cs[[k2]]]];
   θ += ev[Σ_{k=1}^{2n} F3[φ[[k]], k]];
   Factor@
    {Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}];
```

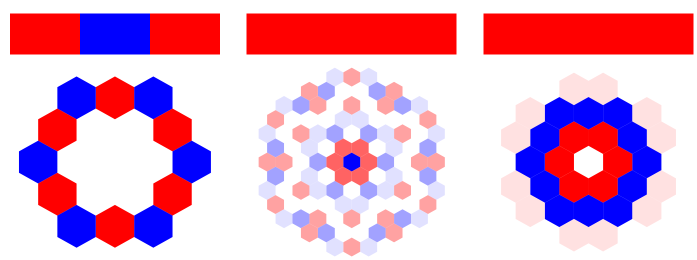
The Trefoil, Conway, and Kinoshita-Terasaka



```
Θ[Knot[3, 1]] // Expand
```

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2T_2^2}+\frac{1}{T_1T_2^2}+\frac{1}{T_1^2T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2T_2-T_2^2+T_1T_2^2-T_1^2T_2^2\right\}$$

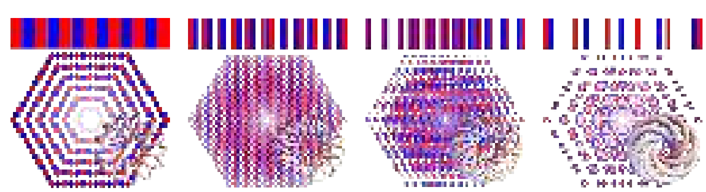
```
GraphicsRow[PolyPlot[Θ[Knot[#]]] & /@
  {"3_1", "K11n34", "K11n42"}]
```



(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

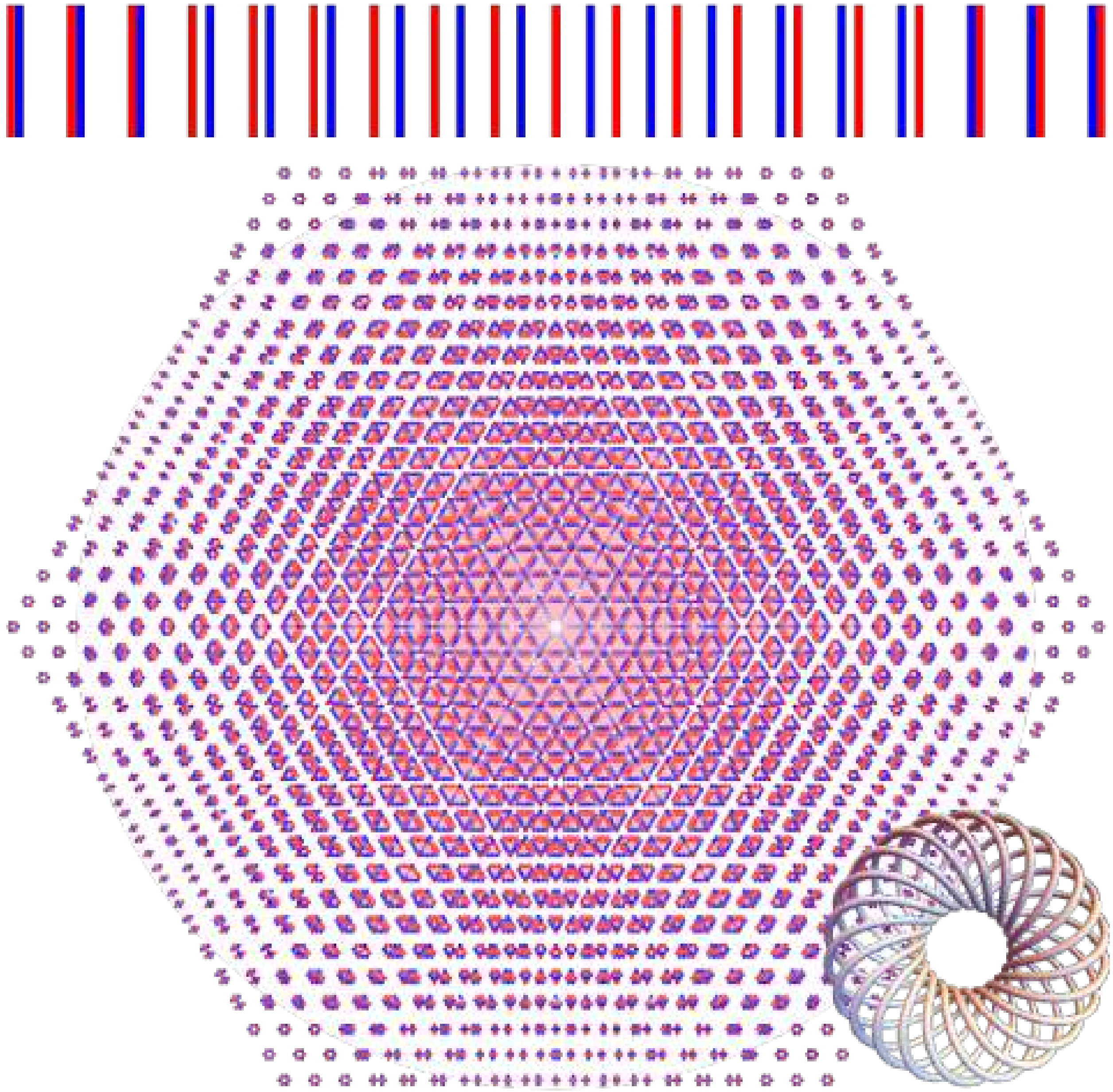
Some Torus Knots

```
GraphicsRow[ImageCompose[
    PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
    TubePlot[TorusKnot @@ #, ImageSize → 240],
    {Right, Bottom}, {Right, Bottom}
   ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```

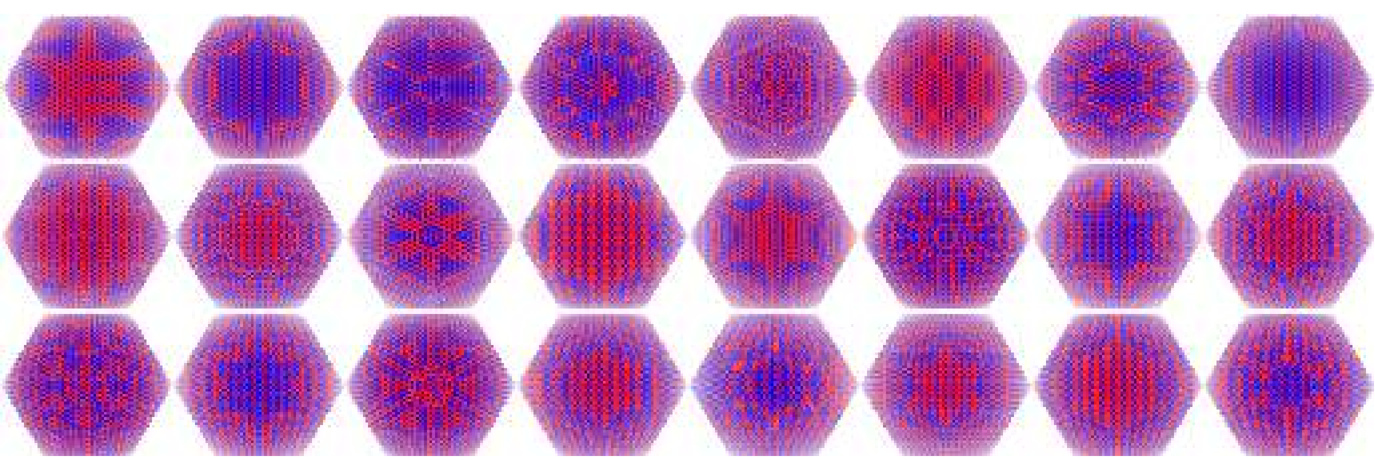


Video and more at http://www.math.toronto.edu/~drorbn/Talks/PhuQuoc-2506.

The 132-crossing torus knot $T_{22/7}$: (many more at ωεβ/TK)

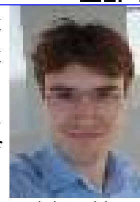


Random knots from [DHOEBL], with 50-73 crossings: (many more at ωεβ/DK)



Video and more at `http://www.math.toronto.edu/~drorbn/Talks/PhuQuoc-2506`.

Knot Invariants from Zero-Dimensional QFT



joint with R. van der Veen

Abstract. For the purpose of today, an "I-Type Knot Invariant" is a knot invariant computed from a knot diagram by integrating the exponential of a *perturbed Gaussian* Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.

Q. Are there any such things? **A.** Yes.

Q. Are they any good? **A.** They are the strongest we know per CPU cycle, and are excellent in other ways too.

Q. Didn't Witten do that back in 1988 with path integrals?

A. No. His constructions are infinite dimensional and far from rigorous.

Q. But integrals belong in analysis!

A. Ours only use squeaky-clean algebra.

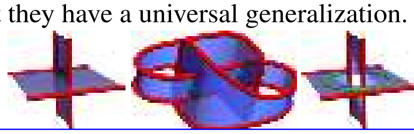
ChatGPT image

Dreams. Given a knot K with a Seifert surface Σ, we dream that there is a 0D Lagrangian $L_\Sigma\colon 6H_1(\Sigma;\mathbb{R})\to\mathbb{R}$ whose coefficients are (low degree) finite type invariants of graphs representing multiple homology classes, such that

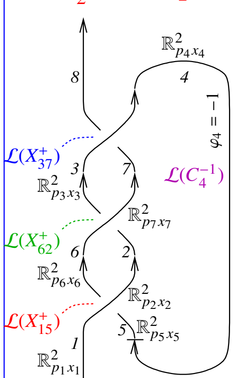
$$Z=\oint_{6H_1(\Sigma;\mathbb{R})}\exp\left(L_\Sigma\right)$$

is the invariant θ presented below. We dream the formulas are simple and natural, and that they have a universal generalization.

Sweet. Even sweeter: May say something about ribbon knots!



The $sl_2^{/\epsilon^2}$ Example. With T an indeterminate and with $\epsilon^2=0$:



$$\longrightarrow Z=\oint_{\mathbb{R}^{14}_{p_ix_i}}\mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C_4^{-1})$$

measure on $\mathbb{R}$ is $(2\pi)^{-1/2}\cdot$*standard*

where $\mathcal{L}(X^s_{ij})=T^{s/2}\mathbb{e}^{L(X^s_{ij})}$ and $\mathcal{L}(C^\varphi_i)=T^{\varphi/2}\mathbb{e}^{L(C^\varphi_i)}$, and

$$\begin{aligned}L(X^s_{ij})&=x_i(p_{i+1}-p_i)+x_j(p_{j+1}-p_j)\\&\qquad+(T^s-1)x_i(p_{i+1}-p_{j+1})\\&\qquad+\frac{\epsilon s}{2}\left(x_i(p_i-p_j)\begin{pmatrix}(T^s-1)x_ip_j\\+2(1-x_jp_j)\end{pmatrix}-1\right)\end{aligned}$$

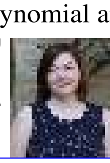
$$L(C^\varphi_i)=x_i(p_{i+1}-p_i)+\epsilon\varphi(1/2-x_ip_i)$$

So $Z=T\oint\mathbb{e}^{L(\text{knot})}dp_1\dots dp_7dx_1\dots dx_7$, where $L(\text{knot})=$

$$\sum_{i=1}^7x_i(p_{i+1}-p_i)+(T-1)(x_1(p_2-p_6)+x_6(p_7-p_3)+x_3(p_4-p_8))+\frac{\epsilon}{2}\begin{pmatrix}x_1(p_1-p_5)((T-1)x_1p_5+2(1-x_5p_5))-1\\+x_6(p_6-p_2)((T-1)x_6p_2+2(1-x_2p_2))-1\\+x_3(p_3-p_7)((T-1)x_3p_7+2(1-x_7p_7))-1\\+2x_4p_4-1\end{pmatrix},$$

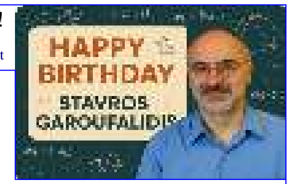
and so $Z=(T-1+T^{-1})^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right)=\Delta^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})\rho_1}{\Delta^2}\right)$. Here Δ is the Alexander polynomial and ρ_1 is the Rozansky-Overbay polynomial [R1, R2, R3, Ov, BV1, BV2].

Rozansky



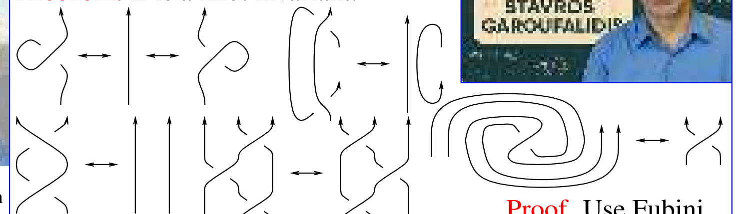
Overbay

Happy birthday, dear Stavros!



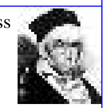
ChatGPT image, ωεβ/chat

Theorem. Z is a knot invariant.



Proof. Use Fubini.

(Alternative) Gaussian Integration.



Gauss

Goal. Compute (if convergent) $\displaystyle\int_{\mathbb{R}^n}dx\exp\left(-\frac12a^{ij}x_ix_j+V(x)\right)$.

Solution. Set $\displaystyle\mathcal{Z}_\lambda(x):=\lambda^{-n/2}\int_{\mathbb{R}^n}dy\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$.

Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x)=(\det A)^{-1/2}\exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y=0$,

$$\begin{aligned}\frac12g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x)&=\frac{\lambda^{-n/2}}{2}\int_{\mathbb{R}^n}dy\,g_{ij}(\partial_{x_i}-\partial_{y_i})(\partial_{x_j}-\partial_{y_j})\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)\\&=\frac{\lambda^{-n/2}}{2\lambda^2}\int_{\mathbb{R}^n}dy\left(g_{ij}a^{ii'}a^{jj'}y_{i'}y_{j'}-\lambda g_{ij}a^{ji}\right)\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)\\&=\frac{\lambda^{-n/2}}{2\lambda^2}\int_{\mathbb{R}^n}dy\left(a^{ij}y_iy_j-\lambda n\right)\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)\\&=\partial_\lambda\mathcal{Z}_\lambda(x).\end{aligned}$$

Hence $(*)\quad\partial_\lambda\mathcal{Z}_\lambda(x)=\frac12g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x)$,

and therefore $\mathcal{Z}_\lambda(x)=(\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x)$.

We've just witnessed the birth of "Feynman Diagrams".

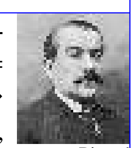
Feynman

Even better. With $Z_\lambda:=\log(\sqrt{\det A}\mathcal{Z}_\lambda)$, by a simple substitution into $(*)$, we get the "Synthesis Equation":

$$Z_0=V,\quad\partial_\lambda Z_\lambda=\frac12\sum_{i,j=1}^ng_{ij}\left(\partial_{x_i,x_j}Z_\lambda+(\partial_{x_i}Z_\lambda)(\partial_{x_j}Z_\lambda)\right)=:F(Z_\lambda),$$

an ODE (in λ) whose solution is pure algebra.

Picard Iteration (used to prove the existence and uniqueness of solutions of ODEs). To solve $\partial_\lambda f_\lambda=F(f_\lambda)$ with a given f_0, start with f_0, iterate $f\mapsto f_0+\int_0^\lambda F(f_\lambda)d\lambda$, and seek a fixed point. In our cases, it is always reached after finitely many iterations!



Picard

Definition. $\oint$: The result of this process, ignoring the convergence of the actual integral.

Strong. The pair (Δ,ρ_1) attains 270,316 distinct values on the 313,230 prime knots with up to 15 crossings (a deficit of 42,914), whereas the pair (H = HOMFLYPT polynomial, Kh = Khovanov Homology) attains only 242,985 distinct values on the same knots (a deficit of 70,245). The pair (Δ,θ), discussed later, has a deficit of only 6,758, and the triple (Δ,θ,ρ_2), of only 6,341.

Yet better than (H,Kh) and other Reshetikhin-Turaev-Witten invariants and knot homologies, Δ, ρ_1, ρ_2 and θ can be computed in **polynomial time** (and hence, even for very large knots).

So ugly as the formulas may be (and θ's formulas are uglier), these invariants are possibly **the best we have!**

Acknowledgement. This work was supported by NSERC grants RGPIN-2018-04350 and RGPIN-2025-06718 and by the Chu Family Foundation (NYC).

2025/06/03@00:35

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505.

Implementation (see IType.nb of ωεβ/ap).

```
Once[<< KnotTheory`; << Rot.m];
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301. Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505 to compute rotation numbers.

```
CF[ω_. ℰ_𝔼] := CF[ω] × CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs, ps, c},
   vs = Cases[ℰ, (x | p | ξ | π | g)__, ∞] ⋃ {ϵ};
   Total[CoefficientRules[Expand[ℰ], vs] /.
     (ps_ → c_) :> Factor[c] (Times @@ vs^ps)] ];
```

Integration using Picard iteration. The core is in yellow and hacks are in pink.

```
𝔼 /: 𝔼[A_] × 𝔼[B_] := 𝔼[A + B];
$π = Identity; (* The Wisdom Projection *)
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) :=
  Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, DDZ,
    FZ, a, b},
   n = Length@vs; L0 = L /. ϵ → 0;
   Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /.
      (p | x)__ → 0, {a, n}, {b, n}];
   If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
   Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
   FixedPoint[(DZ = Table[∂_v Z, {v, vs}];
       DDZ = Table[∂_u DZ, {u, vs}];
       FZ = Sum[G⟦a, b⟧ (DDZ⟦a, b⟧ + DZ⟦a⟧ × DZ⟦b⟧),
          {a, n}, {b, n}] / 2;
       Z = CF[Z0 + ∫_0^λ $π[FZ] ⅆλ]) &, Z];
   PowerExpand@Factor[ω Δ^(-1/2)] ×
    𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

$$\int \mathbb{E}\left[-\mu x^2/2 + i\,\xi\, x\right] d\{x\}$$

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

Joseph Fourier

$$\text{FofG} = \int \mathbb{E}\left[-\mu\,(x-a)^2/2 + i\,\xi\, x\right] d\{x\}$$

$$\frac{\mathbb{E}\left[\frac{i\,(2 a \mu + i\,\xi)\,\xi}{2\mu}\right]}{\sqrt{\mu}}$$

$$\int \text{FofG}\;\mathbb{E}[-i\,\xi\, x]\, d\{\xi\}$$

$$\mathbb{E}\left[-\frac{1}{2}\,(a-x)^2\,\mu\right]$$

So we've tested and nearly proven the Fourier inversion formula!

$$L = -\frac{1}{2}\{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c \end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\};$$

$$\text{Z12} = \int \mathbb{E}[L]\, d\{x_1, x_2\}$$

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\,(-b^2+a\,c)} + \frac{b\,\xi_1\,\xi_2}{b^2-a\,c} + \frac{a\,\xi_2^2}{2\,(-b^2+a\,c)}\right]}{\sqrt{-b^2+a\,c}}$$

Guido Fubini

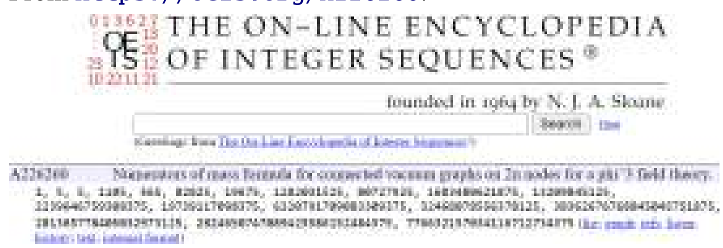
$$\left\{\text{Z1} = \int \mathbb{E}[L]\, d\{x_1\},\ \text{Z12} == \int \text{Z1}\, d\{x_2\}\right\}$$

$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2+a\,c)\,x_2^2}{2\,a} - \frac{b\,x_2\,\xi_1}{a} + \frac{\xi_1^2}{2\,a} + x_2\,\xi_2\right]}{\sqrt{a}},\ \text{True}\right\}$$

$$\$\pi = \text{Normal}\left[\# + O[\epsilon]^{13}\right]\ \&;\ \int \mathbb{E}\left[-\phi^2/2 + \epsilon\,\phi^3/6\right] d\{\phi\}$$

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24} + \frac{5\,\epsilon^4}{16} + \frac{1105\,\epsilon^6}{1152} + \frac{565\,\epsilon^8}{128} + \frac{82\,825\,\epsilon^{10}}{3072} + \frac{19\,675\,\epsilon^{12}}{96}\right]$$

From https://oeis.org/A226260:



The Right-Handed Trefoil.

```
K = Mirror@Knot[3, 1]; Features[K]
```

$$\text{Features}[7,\ C_4[-1]\ X_{1,5}[1]\ X_{3,7}[1]\ X_{6,2}[1]]$$

```
ℒ[X_{i_,j_}[s_]] := T^(s/2) 𝔼[
    x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) +
     (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
     (ϵ s / 2) ×
      (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
ℒ[C_i_[φ_]] := T^(φ/2) 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧]
vs[K_] :=
  Join @@ Table[{p_i, x_i}, {i, Features[K]⟦1⟧}]
{vs[K], ℒ[K]}
```

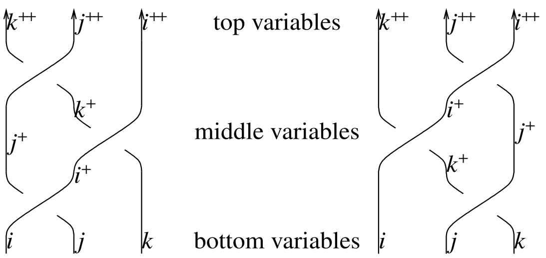
$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[-2\epsilon - p_1 x_1 + \epsilon\, p_1 x_1 + T\, p_2 x_1 - \epsilon\, p_5 x_1 + (1-T)\, p_6 x_1 + \frac{1}{2}(-1+T)\,\epsilon\, p_1 p_5 x_1^2 + \frac{1}{2}(1-T)\,\epsilon\, p_5^2 x_1^2 - p_2 x_2 + p_3 x_2 - p_3 x_3 + \epsilon\, p_3 x_3 + T\, p_4 x_3 - \epsilon\, p_7 x_3 + (1-T)\, p_8 x_3 + \frac{1}{2}(-1+T)\,\epsilon\, p_3 p_7 x_3^2 + \frac{1}{2}(1-T)\,\epsilon\, p_7^2 x_3^2 - p_4 x_4 + \epsilon\, p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - \epsilon\, p_1 p_5 x_1 x_5 + \epsilon\, p_5^2 x_1 x_5 - \epsilon\, p_2 x_6 + (1-T)\, p_3 x_6 - p_6 x_6 + \epsilon\, p_6 x_6 + T\, p_7 x_6 + \epsilon\, p_2^2 x_2 x_6 - \epsilon\, p_2 p_6 x_2 x_6 + \frac{1}{2}(1-T)\,\epsilon\, p_2^2 x_6^2 + \frac{1}{2}(-1+T)\,\epsilon\, p_2 p_6 x_6^2 - p_7 x_7 + p_8 x_7 - \epsilon\, p_3 p_7 x_3 x_7 + \epsilon\, p_7^2 x_3 x_7\Big]\Big\}$$

$$\$\pi = \text{Normal}\left[\# + O[\epsilon]^2\right]\ \&;\ \int \mathcal{L}[K]\, \mathrm{d}\, vs[K]$$

$$-\frac{\mathbb{i}\, T\, \mathbb{E}\left[-\frac{(-1+T)^2 (1+T^2)\, \epsilon}{(1-T+T^2)^2}\right]}{1-T+T^2}$$

A faster program to compute ρ_1, and more stories about it, are at [BV2].

Invariance Under Reidemeister 3.



Reidemeister

$$\text{lhs} = \int (\mathcal{L} \,/@\, (X_{i,j}[1]\, X_{i+1,k}[1]\, X_{j+1,k+1}[1]))\ \mathrm{d}\{p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$$
$$\text{rhs} = \int (\mathcal{L} \,/@\, (X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1]))\ \mathrm{d}\{x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$$
$$\text{lhs} === \text{rhs}$$

False

Invariance Under Reidemeister 3, Take 2.

$$\text{lhs} = \int (\mathcal{L} \,/@\, (X_{i,j}[1]\, X_{i+1,k}[1]\, X_{j+1,k+1}[1]))\ \mathrm{d}\{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$$
$$\text{rhs} = \int (\mathcal{L} \,/@\, (X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1]))\ \mathrm{d}\{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$$
$$\text{lhs} === \text{rhs}$$

True

$$\text{lhs}$$

Degenerate Q!

Invariance Under Reidemeister 3, Take 3.

$$\text{lhs} = \int (\mathbb{E}[\mathbb{i}\,\pi_i\, p_i + \mathbb{i}\,\pi_j\, p_j + \mathbb{i}\,\pi_k\, p_k] \times \mathcal{L} \,/@\, (X_{i,j}[1]\, X_{i+1,k}[1]\, X_{j+1,k+1}[1]))\ \mathrm{d}\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$$
$$\text{rhs} = \int (\mathbb{E}[\mathbb{i}\,\pi_i\, p_i + \mathbb{i}\,\pi_j\, p_j + \mathbb{i}\,\pi_k\, p_k] \times \mathcal{L} \,/@\, (X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1]))\ \mathrm{d}\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$$
$$\text{lhs} == \text{rhs}$$

True

$$\text{lhs}$$

$$\begin{aligned}T^{3/2}\, \mathbb{E}\Big[& -\frac{3\epsilon}{2} + \mathbb{i}\, T^2 p_{2+i}\pi_i - \mathbb{i}\,(-1+T)\, T\, p_{2+j}\pi_i + \mathbb{i}\, T^2 \epsilon\, p_{2+j}\pi_i - \mathbb{i}\,(-1+T)\, p_{2+k}\pi_i + \\ & \mathbb{i}\, T \epsilon\, p_{2+k}\pi_i - \frac{1}{2}(-1+T)\, T^3 \epsilon\, p_{2+i}\, p_{2+j}\, \pi_i^2 + \frac{1}{2}(-1+T)\, T^3 \epsilon\, p_{2+j}^2 \pi_i^2 - \\ & \frac{1}{2}(-1+T)\, T^2 \epsilon\, p_{2+i}\, p_{2+k}\, \pi_i^2 + \frac{1}{2}(-1+T)^2\, T \epsilon\, p_{2+j}\, p_{2+k}\, \pi_i^2 + \\ & \frac{1}{2}(-1+T)\, T \epsilon\, p_{2+k}^2 \pi_i^2 + \mathbb{i}\, T\, p_{2+j}\pi_j - \mathbb{i}\, T \epsilon\, p_{2+j}\pi_j - \mathbb{i}\,(-1+T)\, p_{2+k}\pi_j + \\ & \mathbb{i}\,(-1+2T)\, \epsilon\, p_{2+k}\pi_j + T^3 \epsilon\, p_{2+i}\, p_{2+j}\, \pi_i \pi_j - T^3 \epsilon\, p_{2+j}^2 \pi_i \pi_j - \\ & (-1+T)\, T^2 \epsilon\, p_{2+i}\, p_{2+k}\, \pi_i \pi_j + (-1+T)^2\, T \epsilon\, p_{2+j}\, p_{2+k}\, \pi_i \pi_j + \\ & (-1+T)\, T \epsilon\, p_{2+k}^2 \pi_i \pi_j - \frac{1}{2}(-1+T)\, T \epsilon\, p_{2+j}\, p_{2+k}\, \pi_j^2 + \frac{1}{2}(-1+T)\, T \epsilon\, p_{2+k}^2 \pi_j^2 + \\ & \mathbb{i}\, p_{2+k}\pi_k - 2\,\mathbb{i}\,\epsilon\, p_{2+k}\pi_k + T^2 \epsilon\, p_{2+i}\, p_{2+k}\, \pi_i \pi_k - (-1+T)\, T \epsilon\, p_{2+j}\, p_{2+k}\, \pi_i \pi_k - \\ & T \epsilon\, p_{2+k}^2 \pi_i \pi_k + T \epsilon\, p_{2+j}\, p_{2+k}\, \pi_j \pi_k - T \epsilon\, p_{2+k}^2 \pi_j \pi_k \Big]\end{aligned}$$

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at ωεβ/ap.

There's more! To get sl_2 invariants mod ϵ^3, add the following to $L(X_{ij}^+)$, $L(X_{ij}^-)$, and $L(C_i^\varphi)$, respectively (and see More.nb at ωεβ/ap for the verifications):

$$\epsilon^2\, r_2[1, i, j]$$

$$\begin{aligned}\frac{1}{12}\epsilon^2 \Big(& -6 p_i x_i + 6 p_j x_i - 3(-1+3T)\, p_i p_j x_i^2 + \\ & 3(-1+3T)\, p_j^2 x_i^2 + 4(-1+T)\, p_i^2 p_j x_i^3 - 2(-1+T)(5+T)\, p_i p_j^2 x_i^3 + \\ & 2(-1+T)(3+T)\, p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j - 6 p_i^2 p_j x_i^2 x_j + \\ & 6(2+T)\, p_i p_j^2 x_i^2 x_j - 6(1+T)\, p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2\Big)\end{aligned}$$

$$\epsilon^2\, r_2[-1, i, j]$$

$$\begin{aligned}\frac{1}{12 T^2}\epsilon^2 \Big(& -6 T^2 p_i x_i + 6 T^2 p_j x_i + \\ & 3(-3+T)\, T\, p_i p_j x_i^2 - 3(-3+T)\, T\, p_j^2 x_i^2 - 4(-1+T)\, T\, p_i^2 p_j x_i^3 + \\ & 2(-1+T)(1+5T)\, p_i p_j^2 x_i^3 - 2(-1+T)(1+3T)\, p_j^3 x_i^3 + \\ & 18 T^2 p_i p_j x_i x_j - 18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T(1+2T)\, p_i p_j^2 x_i^2 x_j - \\ & 6 T(1+T)\, p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2\Big)\end{aligned}$$

$$\epsilon^2\, \gamma_2[\varphi, i]$$

$$-\frac{1}{2}\epsilon^2 \varphi^2 p_i x_i$$

Even more! • The sl_2 formulas mod ϵ^4 are in the last page of the handout of [BN3].

- Using [GPV] we can show that every finite type invariant is I-Type.
- Probably, ⟨Reshetikhin-Turaev⟩ ⊂ ⟨I-Type⟩ efficiently.
- Possibly, ⟨Rozansky Polynomials⟩ ⊂ ⟨I-Type⟩ efficiently.
- Knot signatures are I-Type, at least mod 8.
- We already have some work on sl_3, and it leads to the strongest genuinely-computable knot invariant presently known.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505.

The $sl_3^{/\epsilon^2}$ Example [BV3]. Here we have two formal variables T_1 and T_2, we set $T_3 := T_1T_2$, we integrate over 6 variables for each edge: p_{1i}, p_{2i}, p_{3i}, x_{1i}, x_{2i}, and x_{3i}.

```
☺ T3 = T1 T2;  i_^+ := i + 1;
$π =
  (CF@Normal[# + O[ϵ]^2] /.
        {π_is__ :→ B^-1 π_is, x_is__ :→ B^-1 x_is, p_is__ :→ B p_is} /.
      ϵ B^b_ /; b < 0 → 0 /. B → 1) &;
```

```
☺ vs_i_ := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
𝓕[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧];
vs[K_] :=
 Union @@ Table[{vs_i}, {i, Features[K]⟦1⟧}]
```

The Lagrangian.

```
☺ ℒ[X_{i_,j_}[s_]] := T3^s 𝔼[CF@Plus[
    Sum_{ν=1}^3 (x_{νi} (p_{νi+} - p_{νi}) + x_{νj} (p_{νj+} - p_{νj}) + (T_ν^s - 1) x_{νi} (p_{νi+} - p_{νj+})),
    (T1^s - 1) p_{3j} x_{1i} (T2^s x_{2i} - x_{2j}),
    ϵ s (T3^s - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T2^s - 1),
    ϵ s (1/2 + T2^s p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} -
       (T2^s - 1) p_{2j} p_{3i} x_{2i} x_{3i} + (T3^s - 1) p_{2j} p_{3j} x_{2i} x_{3i} +
       2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} -
       T2^s p_{2j} p_{3j} x_{2i} x_{3j} +
       ((T1^s - 1) p_{1j} x_{1i} (T2^{2s} p_{2j} x_{2i} - T2^s p_{2j} x_{2j} -
            (T2^s + 1) (T3^s - 1) p_{3j} x_{3i} + T2^s p_{3j} x_{3j}) +
         (T3^s - 1) p_{3j} x_{3i} (1 - T2^s p_{1i} x_{1i} + p_{2i} x_{2j} + (T2^s - 2) p_{2j} x_{2j})) /
      (T2^s - 1))]]
```

```
☺ ℒ[C_i_[φ_]] := T3^φ 𝔼[Sum_{ν=1}^3 x_{νi} (p_{νi+} - p_{νi}) + ϵ φ (p_{3i} x_{3i} - 1/2)]
```

Reidemeister 3.

```
☺ Short[
   lhs = ∫ 𝓕[i, j, k] × ℒ /@ (X_{i,j}[1] X_{i+,k}[1] X_{j+,k+}[1])
       d{vs_i, vs_j, vs_k, vs_{i+}, vs_{j+}, vs_{k+}}]
```

```
□ T1^3 T2^3
   𝔼[3ϵ/2 + T1^2 p_{1,2+i} π_{1,i} - (-1 + T1) T1 p_{1,2+j} π_{1,i} + <<150>>]
```

```
☺ rhs = ∫ 𝓕[i, j, k] × ℒ /@ (X_{j,k}[1] X_{i,k+}[1] X_{i+,j+}[1])
      d{vs_i, vs_j, vs_k, vs_{i+}, vs_{j+}, vs_{k+}};
lhs == rhs
```

```
□ True
```

The Trefoil.

```
☺ K = Knot[3, 1];  ∫ ℒ[K] d vs[K]
```

```
□ -((i T1^2 T2^2
      𝔼[-((ϵ (1 - T1 + T1^2 - T2 - T1^3 T2 + T2^2 + T1^4 T2^2 - T1 T2^3 -
              T1^4 T2^3 + T1^2 T2^4 - T1^3 T2^4 + T1^4 T2^4)) / ((1 - T1 + T1^2)
            (1 - T2 + T2^2) (1 - T1 T2 + T1^2 T2^2)))]) /
    ((1 - T1 + T1^2) (1 - T2 + T2^2) (1 - T1 T2 + T1^2 T2^2)))
```

A faster program, in which the Feynman diagrams are "pre-computed" (see theta.nb at ωεβ/ap):

```
☺ F1[{s_, i_, j_}] := CF[
   s (1/2 - g_{3ii} + T2^s g_{1ii} g_{2ji} - g_{1ii} g_{2jj} - (T2^s - 1) g_{2ji} g_{3ii} +
       2 g_{2jj} g_{3ii} - (1 - T3^s) g_{2ji} g_{3ji} - g_{2ii} g_{3jj} - T2^s g_{2ji} g_{3jj} +
       g_{1ii} g_{3jj} +
       ((T1^s - 1) g_{1ji} (T2^{2s} g_{2ji} - T2^s g_{2jj} + T2^s g_{3jj}) +
          (T3^s - 1) g_{3ji} (1 - T2^s g_{1ii} - (T1^s - 1) (T2^s + 1) g_{1ji} +
             (T2^s - 2) g_{2jj} + g_{2ij})) / (T2^s - 1))]
```

```
☺ F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
  CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g_{1,j1,i0} g_{3,j0,i1}
    ((T2^s0 g_{2,i1,i0} - g_{2,i1,j0}) - (T2^s0 g_{2,j1,i0} - g_{2,j1,j0}))]
```

```
☺ F3[φ_, k_] = φ g_{3kk} - φ/2;
```

We call the invariant computed θ:

```
☺ Θ[K_] := Θ[K] = Module[{X, φ, n, A, Δ, G, ev, θ},
     {X, φ} = Rot[K]; n = Length[X];
     A = IdentityMatrix[2 n + 1];
     Cases[X, {s_, i_, j_} :→
        (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                          0     -1    ))];
     Δ = T^((-Total[φ] - Total[X⟦All,1⟧])/2) Det[A];
     G = Inverse[A];
     ev[ℰ_] := Factor[ℰ /. g_{ν_,α_,β_} :→ (G⟦α, β⟧ /. T → T_ν)];
     θ = ev[Sum_{k=1}^n F1[X⟦k⟧]];
     θ += ev[Sum_{k1=1}^n Sum_{k2=1}^n F2[X⟦k1⟧, X⟦k2⟧]];
     θ += ev[Sum_{k=1}^{2n} F3[φ⟦k⟧, k]];
     Factor@{Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}
   ];
```

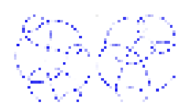
Some Knots.

```
☺ Expand[Θ[Knot[3, 1]]]
```

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2T_2^2}+\frac{1}{T_1T_2^2}+\frac{1}{T_1^2T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2T_2-T_2^2+T_1T_2^2-T_1^2T_2^2\right\}$$

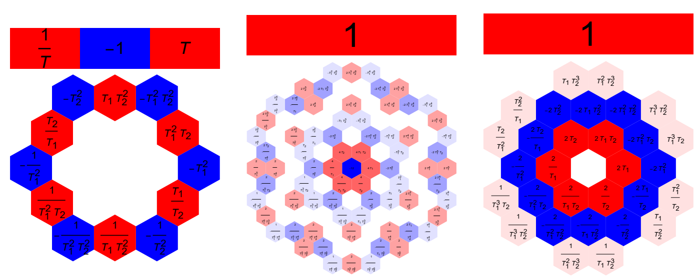
```
☺ (* PolyPlot suppressed *)
```

```
☺ GraphicsRow[PolyPlot[Θ[Knot[#]]],
     Labeled → True] &
   /@ {"3_1", "K11n34", "K11n42"}]
```



K11n34 K11n42

```
□
```



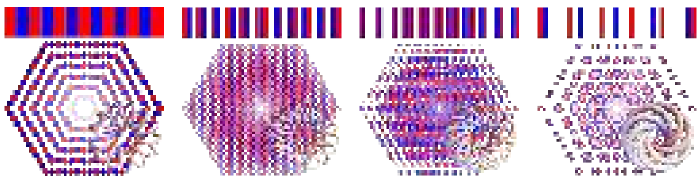
So θ detects knot mutation and separates the Conway knot K11n34 from the Kinoshita-Terasaka knot K11n42!



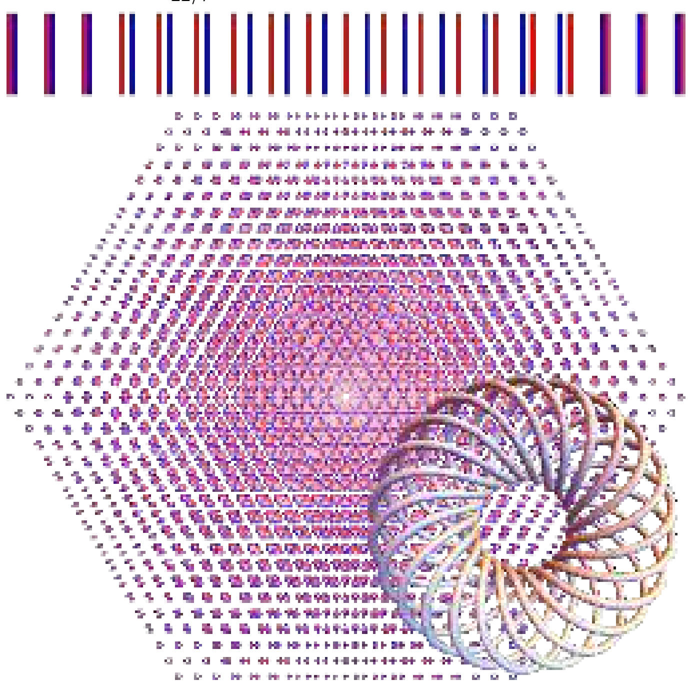
Conway Kinoshita Terasaka

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505`.

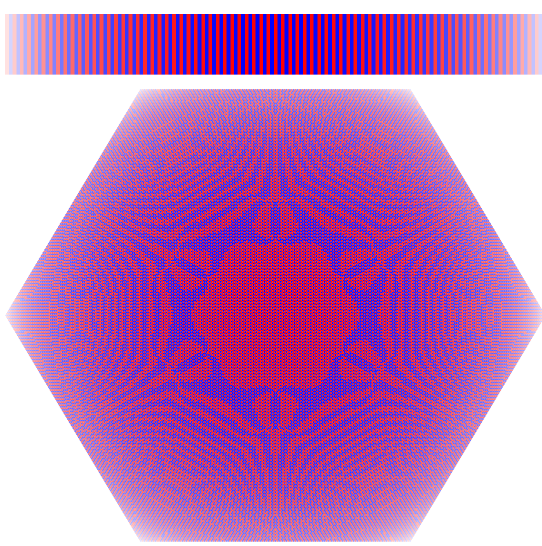
```
GraphicsRow[ImageCompose[
    PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
    TubePlot[TorusKnot @@ #, ImageSize → 240],
    {Right, Bottom}, {Right, Bottom}
  ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```



The torus knot $T_{22/7}$:

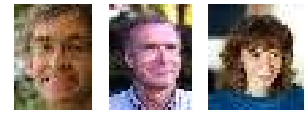


Next, a random 300 crossing knot from [DHOEBL] (more at ωεβ/DK):

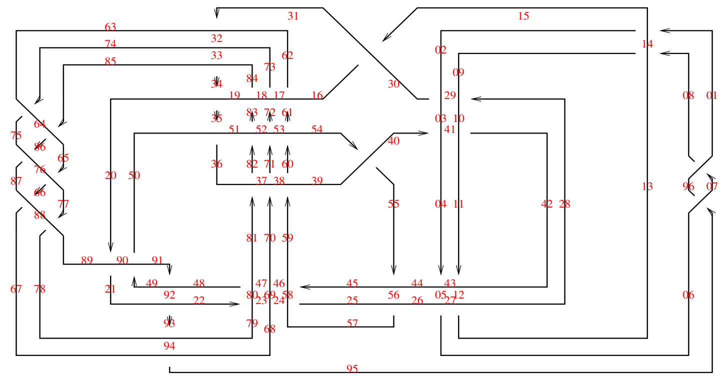


Unproven Fact. For any knot K, twice its genus $g(K)$ bounds the T_1 degree of θ: $\deg_{T_1}\theta(K) \le 2g(K)$.

The 48-crossing Gompf-Scharlemann-Thompson GST_{48} knot [GST] is significant because it may be a counterexample to the slice-ribbon conjecture:

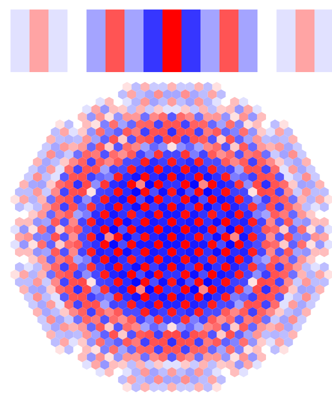


Gompf Scharlemann Thompson



$$GST_{48} = \mathrm{EPD}[X_{14,1}, \overline{X}_{2,29}, X_{3,40}, X_{43,4}, \overline{X}_{26,5}, X_{6,95}, X_{96,7}, X_{13,8}, \overline{X}_{9,28}, X_{10,41}, X_{42,11}, \overline{X}_{27,12}, X_{30,15}, \overline{X}_{16,61}, \overline{X}_{17,72}, \overline{X}_{18,83}, X_{19,34}, \overline{X}_{89,20}, \overline{X}_{21,92}, \overline{X}_{79,22}, \overline{X}_{68,23}, \overline{X}_{57,24}, \overline{X}_{25,56}, X_{62,31}, X_{73,32}, X_{84,33}, \overline{X}_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, \overline{X}_{39,54}, X_{44,55}, X_{58,45}, X_{69,46}, X_{80,47}, X_{48,91}, X_{90,49}, X_{51,82}, X_{52,71}, X_{53,60}, \overline{X}_{63,74}, \overline{X}_{64,85}, \overline{X}_{76,65}, \overline{X}_{87,66}, \overline{X}_{67,94}, \overline{X}_{75,86}, \overline{X}_{88,77}, \overline{X}_{78,93}];$$

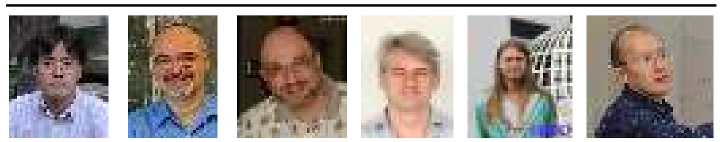
```
AbsoluteTiming[PolyPlot[Θ48 = Θ@GST48, ImageSize → Small]]
```

{17.1103,  }

```
{Exponent[Θ48[[1]], T], Floor[Exponent[Θ48[[2]], T2] / 2]}
```

{8, 10}

So Θ knows things about GST_{48} that Δ doesn't!

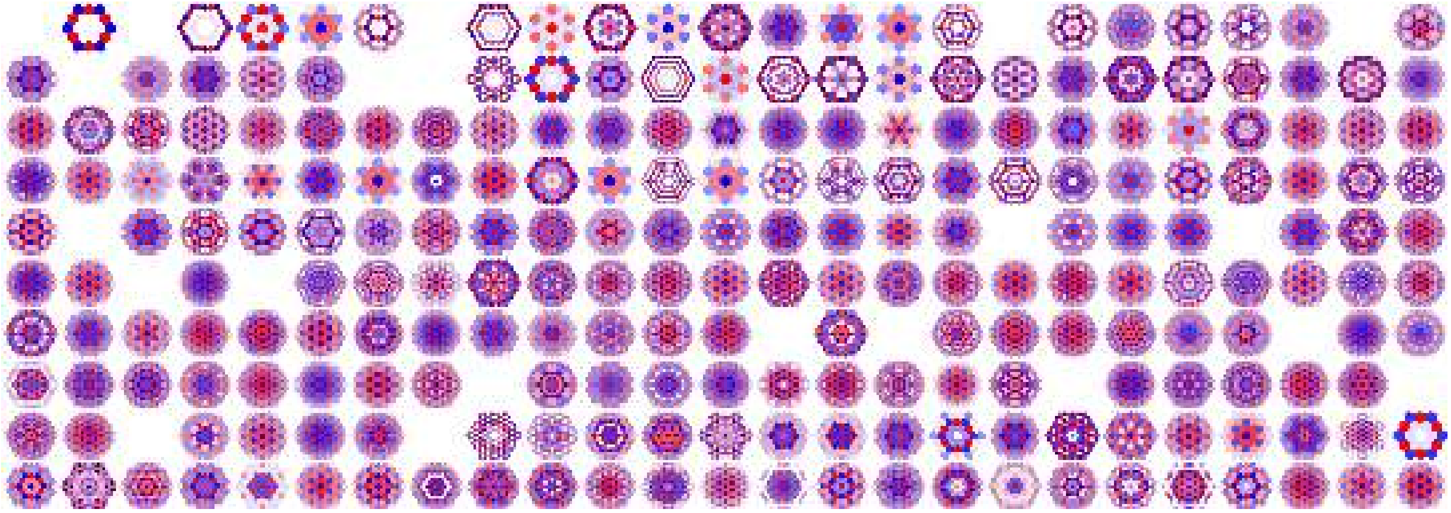


Ohtsuki Garoufalidis Rozansky Kricker Schaveling Kashaev

Prior Art. θ is probably equal to the "2-loop polynomial" studied by Ohtsuki [Oh2] (at greater difficulty, with harder computations), continuing B-N, Garoufalidis, Rozansky, Kricker, and Schaveling [BNG, GR, R1, R2, R3, Kr, Sch]. θ is related, but probably not equivalent, to the invariant studied by Garoufalidis–Kashaev [GK].

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505.

The Rolfsen Table of Knots.



Where is it coming from? The most honest answer is "we don't know" (and that's good!). The second most, "undetermined coefficients for an ansatz that made sense". The ansatz comes from the following principles / earlier work:

Morphisms have generating functions. Indeed, there is an isomorphism

$$\mathcal{G}\colon \operatorname{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j]) \to \mathbb{Q}[y_j][\![\xi_i]\!],$$

and by PBW, many relevant spaces are polynomial rings, though only as vector spaces.

Composition is integration. Indeed, if $f \in \operatorname{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j])$ and $g \in \operatorname{Hom}(\mathbb{Q}[y_j], \mathbb{Q}[z_k])$, then

$$\mathcal{G}(g \circ f) = \int \mathbb{e}^{-y \cdot \eta} f g \, dy \, d\eta$$

Use universal invariants. These take values in a universal enveloping algebra (perhaps quantized), and thus they are expressible as long compositions of generating functions. See [La, Oh1].

"Solvable approximation" $\rightsquigarrow$ perturbed Gaussians. Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{h}$ be its Cartan subalgebra, and let $\mathfrak{b}^u$ and $\mathfrak{b}^l$ be its upper and lower Borel subalgebras. Then $\mathfrak{b}^u$ has a bracket β, and as the dual of $\mathfrak{b}^l$ it also has a cobracket δ, and in fact, $\mathfrak{g} \oplus \mathfrak{h} \equiv \operatorname{Double}(\mathfrak{b}^u, \beta, \delta)$. Let $\mathfrak{g}^+_\epsilon := \operatorname{Double}(\mathfrak{b}^u, \beta, \epsilon\delta)$ (mod ϵ^{d+1} it is solvable for any d). Then by [BV3, BN1] (in the case of $\mathfrak{g} = sl_2$) all the interesting tensors of $\mathcal{U}(\mathfrak{g}^+_\epsilon)$ (quantized or not) are perturbed Gaussian with perturbation parameter ϵ with with understood bounds on the degrees of the perturbations.

References.

[BN1] D. Bar-Natan, *Everything around sl^ϵ_{2+} is DoPeGDO. So what?,* talk given in "Quantum Topology and Hyperbolic Geometry Conference", Da Nang, Vietnam, May 2019. Handout and video at ωεβ/DPG.

[BN2] D. Bar-Natan, *Algebraic Knot Theory,* talk given in Sydney, September 2019. Handout and video at ωεβ/AKT.

[BN3] D. Bar-Natan, *Cars, Interchanges, Traffic Counters, and some Pretty Darned Good Knot Invariants,* talk given in "Using Quantum Invariants to do Interesting Topology", Oaxaca, Mexico, October 2022. Handout and video at ωεβ/Cars.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BV1] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* Proc. Amer. Math. Soc. **147** (2019) 377–397, arXiv:1708.04853.

[BV2] D. Bar-Natan and R. van der Veen, *A Perturbed-Alexander Invariant,* Quantum Topology **15** (2024) 449–472, ωεβ/APAI.

[BV3] D. Bar-Natan and R. van der Veen, *A Very Fast, Very Strong, Topologically Meaningful and Fun Knot Invariant,* in preparation, draft at ωεβ/Theta.

[BV3] D. Bar-Natan and R. van der Veen, *Perturbed Gaussian Generating Functions for Universal Knot Invariants,* arXiv:2109.02057.

[BG] J. Becerra Garrido, *Universal Quantum Knot Invariants,* Ph.D. thesis, University of Groningen, ωεβ/BG.

[DHOEBL] N. Dunfield, A. Hirani, M. Obeidin, A. Ehrenberg, S. Bhattacharyya, D. Lei, and others, *Random Knots: A Preliminary Report,* lecture notes at ωεβ/DHOEBL. Also a data file at ωεβ/DD.

[GK] S. Garoufalidis and R. Kashaev, *Multivariable Knot Polynomials from Braided Hopf Algebras with Automorphisms,* arXiv:2311.11528.

[GR] S. Garoufalidis and L. Rozansky, *The Loop Expansion of the Kontsevich Integral, the Null-Move, and S-Equivalence,* arXiv:math.GT/0003187.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[Kr] A. Kricker, *The Lines of the Kontsevich Integral and Rozansky's Rationality Conjecture,* arXiv:math/0005284.

[La] R. J. Lawrence, *Universal Link Invariants using Quantum Groups,* Proc. XVII Int. Conf. on Diff. Geom. Methods in Theor. Phys., Chester, England, August 1988. World Scientific (1989) 55–63.

[LV] D. López Neumann and R. van der Veen, *Genus Bounds from Unrolled Quantum Groups at Roots of Unity,* arXiv:2312.02070.

[Oh1] T. Ohtsuki, *Quantum Invariants,* Series on Knots and Everything **29**, World Scientific 2002.

[Oh2] T. Ohtsuki, *On the 2–Loop Polynomial of Knots,* Geom. Topol. **11-3** (2007) 1357–1475.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, August 2013, ωεβ/Ov.

[R1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[R2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[R3] L. Rozansky, *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Sch] S. Schaveling, *Expansions of Quantum Group Invariants,* Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Thanks for bearing with me!

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505`.

A Seifert Dream

Thanks for inviting me to Pitzer College!

Abstract. Given a knot K with a Seifert surface Σ, I dream that the well-known Seifert linking form Q, a quadratic form on $H_1(\Sigma)$, has plenty docile local perturbations P_ϵ such that the formal Gaussian integrals of $\exp(Q+P_\epsilon)$ are invariants of K.
In my talk I will explain what the above means, why this dream is oh so sweet, and why it is in fact closer to a plan than to a delusion. Joint with Roland van der Veen.

The Seifert-Alexander Formula. With $P, Q \in H_1(\Sigma)$,

$$Q(P,G) = T^{1/2} lk(P^+, G) - T^{-1/2} lk(P, G^+)$$

$$\Delta(K) = \det(Q)$$

$$\int_{2H_1(\Sigma)} dp\, dx \, \exp Q(p,x) \doteq \det(Q)^{-1}$$

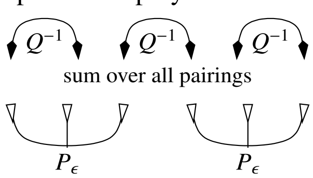
(where $\doteq$ means "ignoring silly factors").

From Mexico City, tariffs exempt

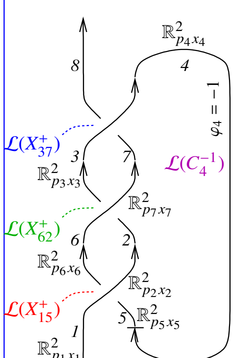
Perturbed Gaussian Integration. We say that $P_\epsilon \in \epsilon\mathbb{Q}[x_1, \dots x_n][\![\epsilon]\!]$ is M-docile (for some $M\colon \mathbb{N} \to \mathbb{N}$) if for every monomial m in P_ϵ we have $\deg_{x_1,\dots,x_n}(m) \le M(\deg_\epsilon(m))$.

Theorem (Feynman). If Q is a quadratic in $x_1, \dots, x_n$ and P_ϵ is docile, set $Z_\epsilon = \int_{\mathbb{R}^n} dx_1 \cdots x_n \exp(Q+P_\epsilon)$. Then every coefficient in the ϵ-expansion of Z_ϵ is computable in polynomial time in n. in fact,

$$\Delta^{1/2} Z_\epsilon \doteq \left\langle \exp Q^{-1}(\partial_{x_i}), \exp P_\epsilon \right\rangle =$$



$\theta(T,1)$ is like that! With $\epsilon^2 = 0$,



$$\longrightarrow Z \doteq \oint_{2E=\mathbb{R}^{14}_{p_i x_i}} \mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C_4^{-1})$$

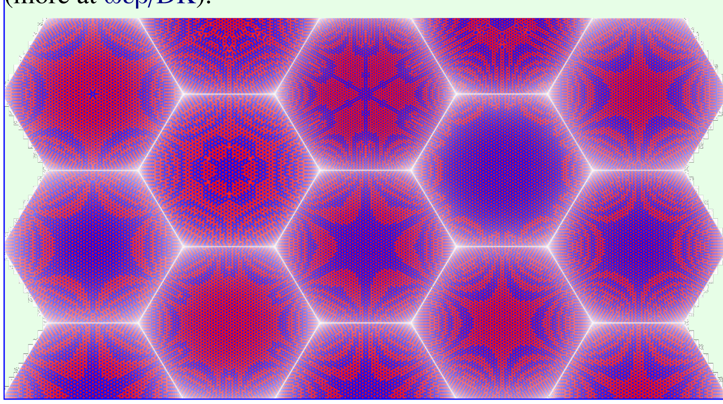
where $\mathcal{L}(X^s_{ij}) \doteq \mathbb{e}^{L(X^s_{ij})}$, $\mathcal{L}(C^\varphi_i) \doteq \mathbb{e}^{L(C^\varphi_i)}$,

$$L(X^s_{ij}) = x_i(p_{i+1}-p_i) + x_j(p_{j+1}-p_j) + (T^s-1)x_i(p_{i+1}-p_{j+1}) + \frac{\epsilon s}{2}\left(x_i(p_i-p_j)\begin{pmatrix} (T^s-1)x_i p_j \\ +2(1-x_j p_j)\end{pmatrix} - 1\right)$$

$$L(C^\varphi_i) = x_i(p_{i+1}-p_i) + \epsilon\varphi(1/2 - x_i p_i)$$

$\theta(T_1,T_2)$ is likewise, with harder formulas and integration over $6E$.

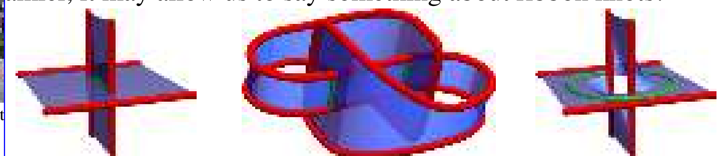
Right. The 132-crossing torus knot $T_{22/7}$ (more at ωεβ/TK).
Below. Random knots from [DHOEBL], with 101-115 crossings (more at ωεβ/DK).



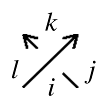
Dream. There is a similar perturbed Gaussian integral formula for θ, but with integration over $6H_1(\Sigma)$. The quadratic Q will be the same as in the Seifert-Alexander formula (but repeated 3 times, for each T_ν). The perturbation P_ϵ will be given by low-degree finite type invariants of curves on Σ (possibly also dependent on the intersection points of such curves, or on other information coming from Σ).

Evidence. Experimentally (yet undeniably), $\deg\theta$ is bounded by the genus of Σ. How else could such a genus bound arise? Further very strong evidence comes from the conjectural (yet undeniable) understanding of θ as the two-loop contribution to the Kontsevich integral [Oh] and/or as the "solvable approximation" of the universal sl_3 invariant [BN1, BV2].

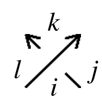
Why so sweet? It will allow us to prove the aforementioned genus bound and likely, the hexagonal symmetry. Sweeter and dreamier, it may allow us to say something about ribbon knots!



What's "local"? How will we compute? The Będlewo Alexander formula: Let F be the faces of a knot diagram. Make an $F\times F$ matrix A by adding for each crossing contributions

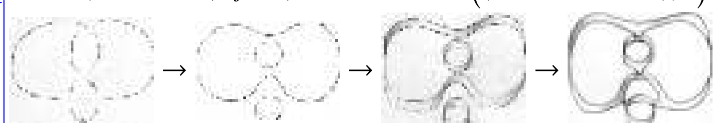


$$\to \begin{pmatrix} -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$



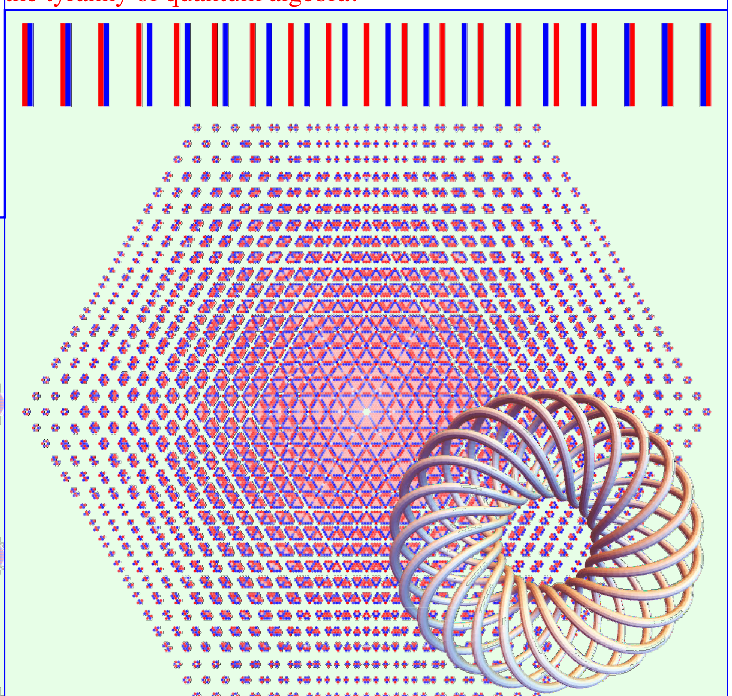
$$\to \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

at rows / columns (i,j,k,l). Then $\Delta = \det'\left((T^{1/2}A - T^{-1/2}A)/2\right)$.



(the Seifert algorithm by Emily Redelmeier)

Expect the like for θ! Expect more like θ! Topology first! Resist the tyranny of quantum algebra!



2025/03/11@09:36

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Pitzer-250308.

The Strongest Genuinely Computable Knot Invariant ~~Since~~ In 2024

The First International On-line Knot Theory Congress
February 1-5, 2025

Dror Bar-Natan

Abstract. "Genuinely computable" means we have computed it for random knots with over 300 crossings. "Strongest" means it separates prime knots with up to 15 crossings better than the less-computable HOMFLY-PT and Khovanov homology taken together. And hey, it's also meaningful and fun.
Continues Rozansky, Garoufalidis, Kricker, and Ohtsuki, joint with van der Veen.

ωεβ:=http://drorbn.net/ktc25

These slides and the code within are online at ωεβ:=http://drorbn.net/ktc25

(I wish all speakers were making their slides available **before** / **for** their talks).

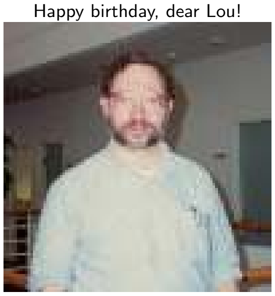
(I'll post the video there too)

A paper-in-progress is at ωεβ/Theta.

If you can, please turn your video on!

ωεβ:=http://drorbn.net/ktc25

Happy birthday, dear Lou!



Lou Kauffman at MSRI, March 1991

ωεβ:=http://drorbn.net/ktc25

Acknowledgement.

This work was supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

ωεβ:=http://drorbn.net/ktc25

The Strongest Genuinely Computable Knot Invariant ~~Since~~ In 2024

Strongest? Genuinely Computable?

ωεβ:=http://drorbn.net/ktc25

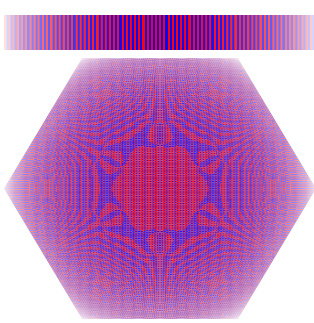
Strongest.

Testing $\Theta = (\Delta, \theta)$ on prime knots up to mirrors and reversals, counting the number of distinct values (with deficits in parenthesis): (ρ_1: [Ro1, Ro2, Ro3, Ov, BV1])

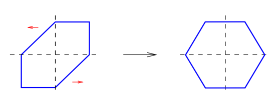
	knots	(H, Kh)	(Δ, ρ_1)	$\Theta = (\Delta, \theta)$	(Δ, θ, ρ_2)	all together
reign		2005-22	2022-24	2024	2025-	
xing ≤ 10	249	248 (1)	249 (0)	249 (0)	249(0)	249 (0)
xing ≤ 11	801	771 (30)	787 (14)	798 (3)	798 (3)	798 (3)
xing ≤ 12	2,977	(214)	(95)	(19)	(10)	(10)
xing ≤ 13	12,965	(1,771)	(959)	(194)	(169)	(169)
xing ≤ 14	59,937	(10,788)	(6,253)	(1,118)	(982)	(981)
xing ≤ 15	313,230	(70,245)	(42,914)	(6,758)	(6,341)	(6,337)

ωεβ:=http://drorbn.net/ktc25

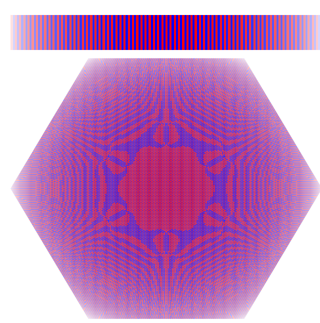
Genuinely Computable. Here's Θ on a random 300 crossing knot (from [DHOEBL]). For almost every other knot invariant, that's science fiction.
Gukov: Should take 300 years if Moore's law persists.
Us: A few hours on a laptop, 0 GPUs.



ωεβ:=http://drorbn.net/ktc25

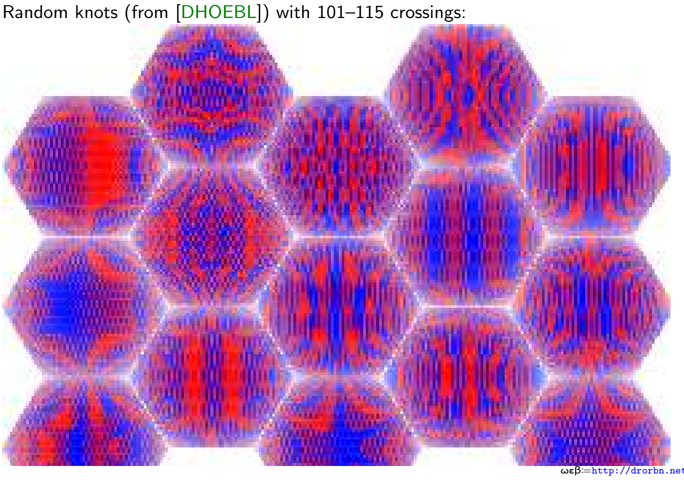


Fun. There's so much more to see in 2D pictures than in 1D ones! Yet almost nothing of the patterns you see we know how to prove. We'll have fun with that over the next few years. Would you join?

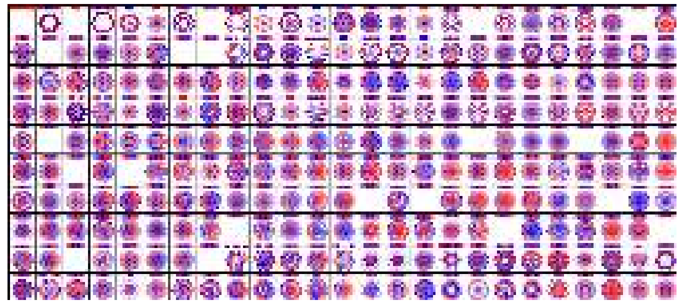


Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.

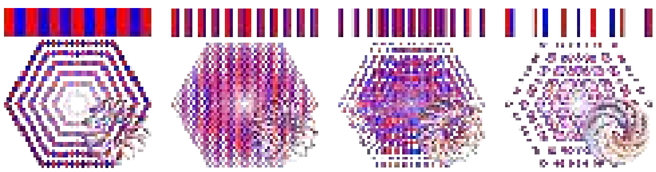
Random knots (from [DHOEBL]) with 101–115 crossings:

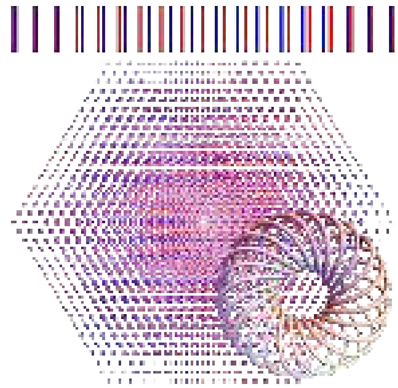


The Rolfsen Table:



The torus knots $TK_{13/2}$, $TK_{17/3}$, $TK_{13/5}$, and $TK_{7/6}$:





The torus knot $TK_{22/7}$:

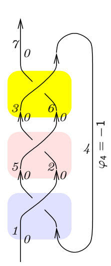
Meaningful.

θ gives a genus bound (unproven yet with confidence). We hope (with reason) it says something about ribbon knots.

Convention.

T, T_1, and T_2 are indeterminates and $T_3 := T_1T_2$.

Preparation. Draw an n-crossing knot K as a diagram D as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \dots, 2n+1\}$ and with rotation numbers φ_k.



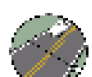
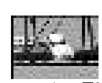


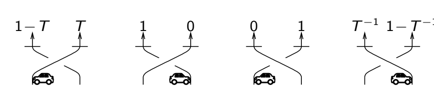
image credits: diamondtraffic.com

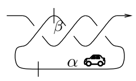


$p = 1 - T^s$

image credits: Dall-E

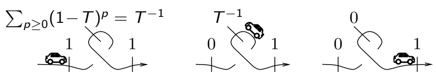
Model T Traffic Rules. Cars always drive forward. When a car crosses over a sign-s bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0$. At the very end, cars fall off and disappear. On various edges traffic counters are placed. See also [Jo, LTW].





Definition. The traffic function $G = (g_{\alpha\beta})$ (also, the Green function or the two-point function) is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is after the injection point). There are also model-T_ν traffic functions $G_\nu = (g_{\nu\alpha\beta})$ for $\nu = 1, 2, 3$.

Example.



$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Given crossings $c = (s, i, j)$, $c_0 = (s_0, i_0, j_0)$, and $c_1 = (s_1, i_1, j_1)$, let

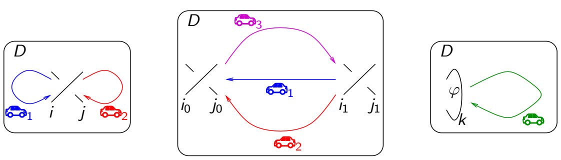
$$\begin{aligned} F_1(c) = s\,[1/2 &- g_{3ii} + T_2^s g_{1ii} g_{2ji} - T_2^s g_{3jj} g_{2ji} - (T_2^s - 1) g_{3ii} g_{2ji} \\ &+ (T_3^s - 1) g_{2ji} g_{3ji} - g_{1ii} g_{2jj} + 2 g_{3ii} g_{2jj} + g_{1ii} g_{3jj} - g_{2ii} g_{3jj}] \\ &+ \frac{s}{T_2^s - 1} [(T_1^s - 1) T_2^s (g_{3jj} g_{1ji} - g_{2jj} g_{1ji} + T_2^s g_{1ji} g_{2ji}) \\ &+ (T_3^s - 1)(g_{3ji} - T_2^s g_{1ii} g_{3ji} + g_{2ij} g_{3ji} + (T_2^s - 2) g_{2jj} g_{3ji}) \\ &- (T_1^s - 1)(T_2^s + 1)(T_3^s - 1) g_{1ji} g_{3ji}] \\ F_2(c_0, c_1) &= \frac{s_1 (T_1^{s_0} - 1)(T_3^{s_1} - 1) g_{1 j_1 i_0} g_{3 j_0 i_1}}{T_2^{s_1} - 1} (T_2^{s_0} g_{2 i_1 i_0} + g_{2 j_1 j_0} - T_2^{s_0} g_{2 j_1 i_0} - g_{2 i_1 j_0}) \\ F_3(\varphi_k, k) &= \varphi_k (g_{3kk} - 1/2) \end{aligned}$$

(Computers don't care!)

Main Theorem.

The following is a knot invariant: (the Δ_ν are normalizations discussed later)

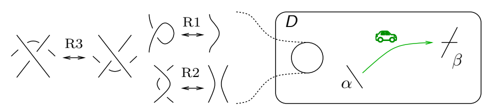
$$\theta(D) := \Delta_1 \Delta_2 \Delta_3 \left(\sum_c F_1(c) + \sum_{c_0, c_1} F_2(c_0, c_1) + \sum_k F_3(\varphi_k, k) \right).$$



If these pictures remind you of Feynman diagrams, it's because they are Feynman diagrams [BN2].

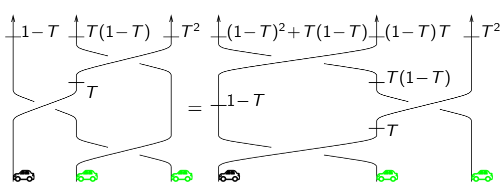
Lemma 1.

The traffic function $g_{\alpha\beta}$ is a "relative invariant":

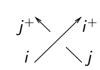


(There is some small print for R1 and R2 which change the numbering of the edges and sometimes collapse a pair of edges into one)

Proof.



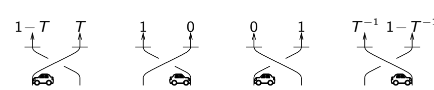
Lemma 2.



With $k^+ := k + 1$, the "g-rules" hold near a crossing $c = (s, i, j)$:

$$g_{j\beta} = g_{j^+\beta} + \delta_{j\beta} \quad g_{i\beta} = T^s g_{i^+\beta} + (1 - T^s) g_{j^+\beta} + \delta_{i\beta} \quad g_{2n^+,\beta} = \delta_{2n^+,\beta}$$

$$g_{\alpha i^+} = T^s g_{\alpha i} + \delta_{\alpha i^+} \quad g_{\alpha j^+} = g_{\alpha j} + (1 - T^s) g_{\alpha i} + \delta_{\alpha j^+} \quad g_{\alpha,1} = \delta_{\alpha,1}$$



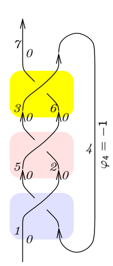
Corollary 1.

G is easily computable, for $AG = I$ $(= GA)$, with A the $(2n+1) \times (2n+1)$ identity matrix with additional contributions:

$$c = (s, i, j) \mapsto \begin{array}{c|cc} A & \text{col } i^+ & \text{col } j^+ \\ \hline \text{row } i & -T^s & T^s - 1 \\ \text{row } j & 0 & -1 \end{array}$$

For the trefoil example, we have:

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



And so,

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.

Note.

The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2}\det(A),$$

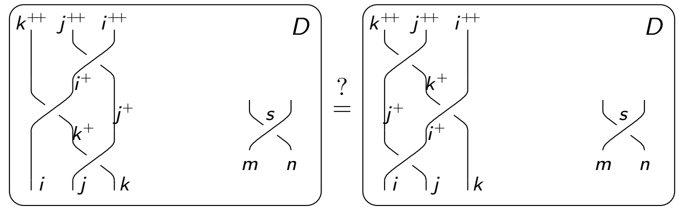
with

$$\varphi = \sum_k \varphi_k, \qquad w = \sum_c s.$$

We also set $\Delta_\nu := \Delta(T_\nu)$ for $\nu = 1, 2, 3$. This defines and explains the normalization factors in the Main Theorem.

Corollary 2.

Proving invariance is easy:



Invariance under R3

This is Theta.nb of http://drorbn.net/ktc25/ap.

```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/ktc25/ap to compute rotation numbers.
Loading PolyPlot.m from
  http://drorbn.net/ktc25/ap to plot 2-variable polynomials.
```

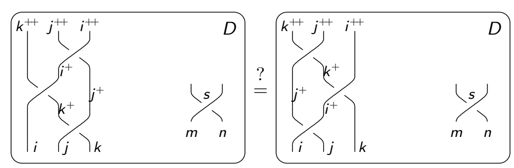
```
T3 = T1 T2;
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```



```
F1[{s_, i_, j_}] =
  CF[
   s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj - (T2^s - 1) g2ji g3ii + 2 g2jj g3ii -
      (1 - T3^s) g2ji g3ji - g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
      ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
        (T3^s - 1) g3ji (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji + (T2^s - 2) g2jj + g2ij)) /
     (T2^s - 1))];
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
 CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1
   ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
F3[φ_, k_] = -φ/2 + φ g3kk;
```



```
δi_,j_ := If[i === j, 1, 0];
gRs_,i_,j_ := {
   gν_jβ_ :→ gνj⁺β + δjβ, gν_iβ_ :→ Tν^s gνi⁺β + (1 - Tν^s) gνj⁺β + δiβ,
   gν_α_i⁺ :→ Tν^s gναi + δαi⁺, gν_α_j⁺ :→ gναj + (1 - Tν^s) gναi + δαj⁺
  }
```


```
DSum[Cs___] := Sum[F1[c], {c, {Cs}}] +
  Sum[F2[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
lhs = DSum[{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}, {s, m, n}] //.
   gR1,j,k ⋃ gR1,i,k⁺ ⋃ gR1,i⁺,j⁺;
rhs = DSum[{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}, {s, m, n}] //.
   gR1,i,j ⋃ gR1,i⁺,k ⋃ gR1,j⁺,k⁺;
Simplify[lhs == rhs]
```

```
True
```


The Main Program

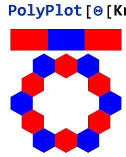
```
Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, θ},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :→ (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                                                0     -1    ))];
   Δ = T^((-Total[φ] - Total[Cs[[All,1]]])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] := Factor[ℰ /. gν_,α_,β_ :→ (G[[α, β]] /. T → Tν)];
   θ = ev[Sum_{k=1}^{n} F1[Cs[[k]]]];
   θ += ev[Sum_{k1=1}^{n} Sum_{k2=1}^{n} F2[Cs[[k1]], Cs[[k2]]]];

   θ += ev[Sum_{k=1}^{2 n} F3[φ[[k]], k]];
   Factor@{Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}];
```


The Trefoil Knot

```
Θ[Knot[3, 1]] // Expand
```

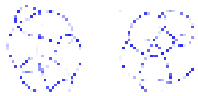
$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2T_2^2}+\frac{1}{T_1T_2^2}+\frac{1}{T_1^2T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2T_2-T_2^2+T_1T_2^2-T_1^2T_2^2\right\}$$

```
PolyPlot[Θ[Knot[3, 1]], ImageSize → Tiny]
```

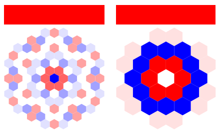


Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.

The Conway and Kinoshita-Terasaka Knots



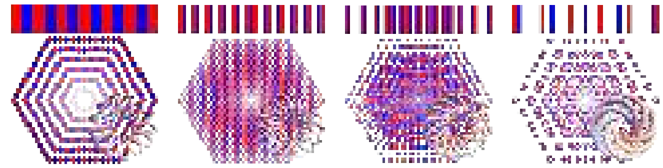
```
GraphicsRow[PolyPlot[Θ[Knot[#]], ImageSize → Tiny] & /@
  {"K11n34", "K11n42"}]
```



(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

The Torus Knots $TK_{13/2}$, $TK_{17/3}$, $TK_{13,5}$, and $TK_{7,6}$

```
GraphicsRow[ImageCompose[
    PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
    TubePlot[TorusKnot @@ #, ImageSize → 240],
    {Right, Bottom}, {Right, Bottom}
  ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```



Questions, Conjectures, Expectations, Dreams.

Question 1.

What's the relationship between Θ and the Garoufalidis-Kashaev invariants [GK, GL]?

Conjecture 2.

On classical (non-virtual) knots, θ always has hexagonal (D_6) symmetry.

Conjecture 3.

θ is the ϵ^1 contribution to the "solvable approximation" of the sl_3 universal invariant, obtained by running the quantization machinery on the double $\mathcal{D}(\mathfrak{b}, b, \epsilon\delta)$, where $\mathfrak{b}$ is the Borel subalgebra of sl_3, b is the bracket of $\mathfrak{b}$, and δ the cobracket. See [BV2, BN1, Sch]

Conjecture 4.

θ is equal to the "two-loop contribution to the Kontsevich Integral", as studied by Garoufalidis, Rozansky, Kricker, and in great detail by Ohtsuki [GR, Ro1, Ro2, Ro3, Kr, Oh].

Fact 5. θ has a perturbed Gaussian integral formula, with integration carried out over a space $6E$, consisting of 6 copies of the space of edges of a knot diagram D. See [BN2].

Conjecture 6. For any knot K, its genus $g(K)$ is bounded by the T_1-degree of θ: $2g(K) \geq \deg_{T_1} \theta(K)$.

Conjecture 7. $\theta(K)$ has another perturbed Gaussian integral formula, with integration carried out over over the space $6H_1$, consisting of 6 copies of $H_1(\Sigma)$, where Σ is a Seifert surface for K.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.

Question 8.

Is there a direct quantum field theory derivation of θ? Perhaps using the ϵ-expansion (at constant $k!$) of Chern-Simons-Witten theory with gauge group $\mathfrak{g}_+^\epsilon := \mathcal{D}(\mathfrak{b}, b, \epsilon\delta)$ with some Seifert-surface-dependent gauge fixing?

Expectation 9.

There are many further invariants like θ, given by Green function formulas and/or Gaussian integration formulas. One or two of them may be stronger than θ and as computable.

Dream 10.

These invariants can be explained by something less foreign than semisimple Lie algebras.

Dream 11.

θ will have something to say about ribbon knots.

Thank You!

References.

[BN1] D. Bar-Natan, Everything around sl_{2+}^ϵ is DoPeGDO. So what?, talk in Da Nang, May 2019. Handout and video at ωεβ/DPG.

[BN2] —, Knot Invariants from Finite Dimensional Integration, talks in Beijing (July 2024, ωεβ/icbs24) and in Geneva (August 2024, ωεβ/ge24).

[BV1] —, R. van der Veen, A Perturbed-Alexander Invariant, Quantum Topology **15** (2024) 449–472, ωεβ/APAI.

[BV2] —, —, Perturbed Gaussian Generating Functions for Universal Knot Invariants, arXiv: 2109.02057.

[DHOEBL] N. Dunfield, A. Hirani, M. Obeidin, A. Ehrenberg, S. Bhattacharyya, D. Lei, and others, Random Knots: A Preliminary Report, lecture notes at ωεβ/DHOEBL. Also a data file at ωεβ/DD.

[GK] S. Garoufalidis, R. Kashaev, Multivariable Knot Polynomials from Braided Hopf Algebras with Automorphisms, arXiv:2311.11528.

[GL] —, S. Y. Li, Patterns of the V_2-polynomial of knots, arXiv:2409.03557.

[GR] —, L. Rozansky, The Loop Expansion of the Kontsevich Integral, the Null-Move, and S-Equivalence, arXiv:math.GT/0003187.

[Jo] V. F. R. Jones, Hecke Algebra Representations of Braid Groups and Link Polynomials, Annals Math., **126** (1987) 335-388.

[Kr] A. Kricker, The Lines of the Kontsevich Integral and Rozansky's Rationality Conjecture, arXiv:math/0005284.

[LTW] X-S. Lin, F. Tian, Z. Wang, Burau Representation and Random Walk on String Links, Pac. J. Math., **182-2** (1998) 289–302, arXiv:q-alg/9605023.

[Oh] T. Ohtsuki, On the 2–loop Polynomial of Knots, Geom. Top. **11** (2007) 1357–1475.

[Ov] A. Overbay, Perturbative Expansion of the Colored Jones Polynomial, Ph.D. thesis, University of North Carolina, Aug. 2013, ωεβ/Ov.

[Ro1] L. Rozansky, A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I, Comm. Math. Phys. **175-2** (1996) 275–296, arXiv: hep-th/9401061.

[Ro2] —, The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial, Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] —, A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture, arXiv: math/0201139.

[Sch] S. Schaveling, Expansions of Quantum Group Invariants, Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.

Shifted Partial Quadratics, their Pushforwards, and Signature Invariants for Tangles

Abstract. Following a general discussion of the computation of zombians of unfinished columbaria (with examples), I will tell you about my recent joint work w/ Jessica Liu on what we feel is the "textbook" extension of knot signatures to tangles, which for unknown reasons, is not in any of the textbooks that we know.

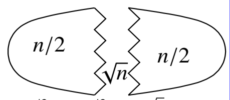

Jessica Liu


Columbaria in an East Sydney Cemetery


Image: Freepik.com
Jacobian, Hamiltonian, Zombian

Prior Art on signatures for tangles / braids. Gambaudo and Ghys [GG], Cimasoni and Conway [CC], Conway [Co], Merz [Me]. All define signatures of tangles / braids by first closing them to links and then work hard to derive composition properties.

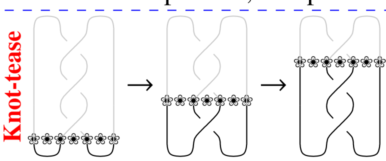
Why Tangles? • Faster!



$2^{n/2}+2^{n/2}+2^{\sqrt{n}}\ll 2^n$

- Conceptually clearer proofs of invariance (and of skein relations).
- Often fun and consequential:
  - ∘ The Jones Polynomial ⇝ The Temperley-Lieb Algebra.
  - ∘ Khovanov Homology ⇝ "Unfinished complexes", complexes in a category.
  - ∘ The Kontsevich Integral ⇝ Associators.
  - ∘ HFK ⇝ OMG, type D, type A, $\mathcal{A}_\infty$, …

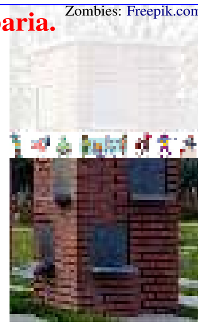
Knot-tease



Zombies: Freepik.com

Computing Zombians of Unfinished Columbaria.

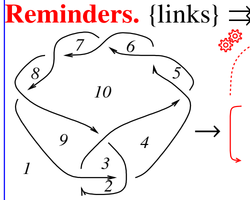
- Must be no slower than for finished ones.
- Future zombies must be able to complete the computation.
- Future zombies must not even know the size of the task that today's zombies were facing.
- We must be able to extend to ZPUCs, Zombie Processed Unfinished Columbaria!


Columbarium near Assen

Example / Exercise. Compute the determinant of a $1{,}000\times 1{,}000$ matrix in which 50 entries are not yet given.

Homework / Research Projects. • What with ZPUCs? • Use this to get an Alexander tangle invariant.

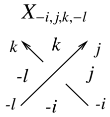
Reminders. {links} ⇉ {matrices / quadratic forms} $\xrightarrow[\sigma]{\text{signature}}\mathbb{Z}$:



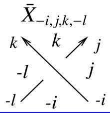
$\to$ $A=\bar{A}^T$ $\to$ $\begin{pmatrix}1 & & & & & \\ & \ddots & \sigma_+ & & & \\ & & -1 & & & \\ & & & \ddots & \sigma_- & \\ & & & & 0 & \\ & & & & & \ddots\end{pmatrix}$ (columns $1\ \dots\ 10$) $\to\sigma=\sigma_+-\sigma_-+s$

With $|\omega|=1$, $t=1-\omega$, $r=t+\bar{t}$, $v=\mathrm{Re}(\omega)$, and $u=\mathrm{Re}(\omega^{1/2})$:

Tristram-Levine (TL) | Kashaev (Kas)



$$A \mathrel{+}= \begin{pmatrix}-r & -t & 2t & \bar{t}\\ -\bar{t} & 0 & \bar{t} & 0\\ 2\bar{t} & t & -r & -\bar{t}\\ t & 0 & -t & 0\end{pmatrix}\begin{matrix}i\\j\\k\\l\end{matrix},\quad s=0 \qquad A \mathrel{+}= \begin{pmatrix}v & u & 1 & u\\ u & 1 & u & 1\\ 1 & u & v & u\\ u & 1 & u & 1\end{pmatrix}\begin{matrix}i\\j\\k\\l\end{matrix},\quad s \mathrel{-}= 1$$



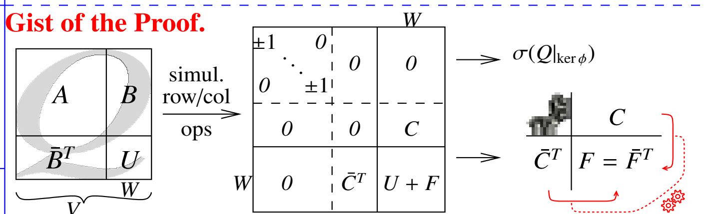
$$A \mathrel{+}= \begin{pmatrix}r & -t & -2\bar{t} & \bar{t}\\ -\bar{t} & 0 & \bar{t} & 0\\ -2t & t & r & -\bar{t}\\ t & 0 & -t & 0\end{pmatrix}\begin{matrix}i\\j\\k\\l\end{matrix},\quad s=0 \qquad A \mathrel{-}= \begin{pmatrix}v & u & 1 & u\\ u & 1 & u & 1\\ 1 & u & v & u\\ u & 1 & u & 1\end{pmatrix}\begin{matrix}i\\j\\k\\l\end{matrix},\quad s \mathrel{+}= 1$$

~~Kashaev's Conjecture~~ [Ka] **Liu's Theorem** [Li]. For links, $\sigma_{Kas}=2\sigma_{TL}$.

A *Partial Quadratic (PQ)* on V is a quadratic Q defined only on a subspace $\mathcal{D}_Q\subset V$. We add PQs with $\mathcal{D}_{Q_1+Q_2}:=\mathcal{D}_{Q_1}\cap\mathcal{D}_{Q_2}$. Given a linear $\psi\colon V\to W$ and a PQ Q on W, there is an obvious *pullback* ψ^*Q, a PQ on V.

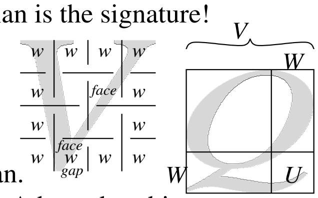
Theorem 1. Given a linear $\phi\colon V\to W$ and a PQ Q on V, there is a unique *pushforward* PQ ϕ_*Q on W such that for every *PQ* U on W, $\sigma_V(Q+\phi^*U)=\sigma_{\ker\phi}(Q|_{\ker\phi})+\sigma_W(U+\phi_*Q)$.
(If you must, $\mathcal{D}(\phi_*Q)=\phi(\mathrm{ann}_Q(\mathcal{D}(Q)\cap\ker\phi))$ and $(\phi_*Q)(w)=Q(v)$, where v is s.t. $\phi(v)=w$ and $Q(v,\mathrm{rad}\,Q|_{\ker\phi})=0$).

Gist of the Proof.



… and the quadratic $F=:\phi_*Q$ is well-defined only on $D:=\ker C$.

Exactly what we want, if the Zombian is the signature!
V: The full space of *faces*.
W: The boundary, made of *gaps*.
Q: The known parts.
U: The part yet unknown.
$\sigma_V(Q+\phi^*(U))$: The overall Zombian.
$\sigma(Q|_{\ker\phi})$: An internal bit. $U+\phi_*Q$: A boundary bit.
And so our ZPUC is the pair $S=(\sigma(Q|_{\ker\phi}),\phi_*Q)$.



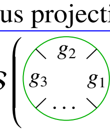
A *Shifted Partial Quadratic (SPQ)* on V is a pair $S=(s\in\mathbb{Z},Q$ a PQ on $V)$. addition also adds the shifts, pullbacks keep the shifts, yet $\phi_*S:=(s+\sigma_{\ker\phi}(Q|_{\ker\phi}),\phi_*Q)$ and $\sigma(S):=s+\sigma(Q)$.

Theorem 1' (*Reciprocity*). Given $\phi\colon V\to W$, for SPQs S on V and U on W we have $\sigma_V(S+\phi^*U)=\sigma_W(U+\phi_*S)$ (and this characterizes ϕ_*S). Note. ψ^* is additive but ϕ_* is not.

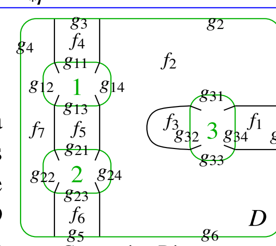
Theorem 2. ψ^* and ϕ_* are functorial.

Theorem 3. "The pullback of a pushforward scene is a pushforward scene": If, on the right, β and δ are arbitrary, $Y=\mathrm{EQ}(\beta,\gamma)=V\oplus_Z W=\{(v,w)\colon\beta v=\gamma w\}$ and μ and ν are the obvious projections, then $\gamma^*\beta_*=\nu_*\mu^*$.

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma"] \\ V \arrow[r, "\beta"'] \arrow[ur, dotted] & Z \end{tikzcd}$$

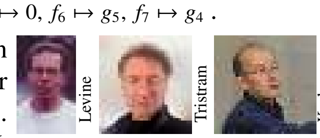
Definition. $\mathcal{S}$  $:=\left\{\begin{matrix}\text{SPQ } S\\ \text{on } \langle g_i\rangle\end{matrix}\right\}$.

Theorem 4. $\{\mathcal{S}(\text{cyclic sets})\}$ is a planar algebra, with compositions $\mathcal{S}(D)((S_i)):=\phi^D_*(\psi^*_D(\bigoplus_i S_i))$, where $\psi_D\colon\langle f_i\rangle\to\langle g_{\alpha i}\rangle$ maps every face of D to the sum of the input gaps adjacent to it and $\phi^D\colon\langle f_i\rangle\to\langle g_i\rangle$ maps every face to the sum of the output gaps adjacent to it. So for our D, $\psi_D\colon$ $f_1\mapsto g_{34}$, $f_2\mapsto g_{31}+g_{14}+g_{24}+g_{33}$, $f_3\mapsto g_{32}$, $f_4\mapsto g_{11}$, $f_5\mapsto g_{13}+g_{21}$, $f_6\mapsto g_{23}$, $f_7\mapsto g_{12}+g_{22}$ and ϕ^D: $f_1\mapsto g_1$, $f_2\mapsto g_2+g_6$, $f_3\mapsto 0$, $f_4\mapsto g_3$, $f_5\mapsto 0$, $f_6\mapsto g_5$, $f_7\mapsto g_4$.



Connection Diagram

Theorem 5. TL and Kas, defined on X and $\bar{X}$ as before, extend to planar algebra morphisms {tangles} → $\{\mathcal{S}\}$. Restricted to links, $TL=\sigma_{TL}$ and $Kas=\sigma_{Kas}$.



Video: http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201. Handout: http://www.math.toronto.edu/~drorbn/Talks/USC-240205.

Rooting the BKT for FTI

ωεβ:=http://drorbn.net/tok2309

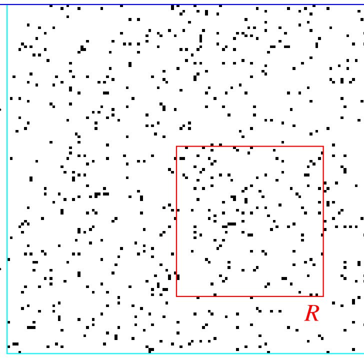
Acknowledgement. This work was partially supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

Abstract. Following joint work with Itai Bar-Natan, Iva Halacheva, and Nancy Scherich, I will show that the Best Known Time (BKT) to compute a typical Finite Type Invariant (FTI) of type d on a typical knot with n crossings is roughly equal to $n^{d/2}$, which is roughly the square root of what I believe was the standard belief before, namely about n^d.

Conventions. • $\underline{n} := \{1, 2, \ldots, n\}$. • For complexity estimates we ignore constant and logarithmic terms: $n^3 \sim 2023d!(\log n)^d n^3$.

A Key Preliminary. Let $Q \subset \underline{n}^l$ be an enumerated subset, with $1 \ll q = |Q| \ll n^l$. In time $\sim q$ we can set up a lookup table of size $\sim q$ so that we will be able to compute $|Q \cap R|$ in time ~ 1, for any rectangle $R \subset \underline{n}^l$.

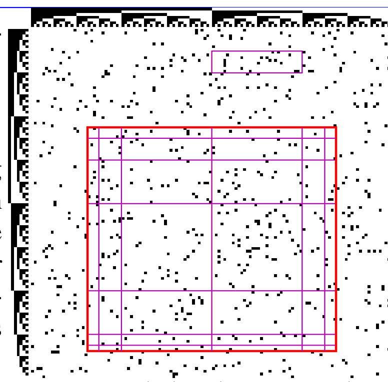
Fails. • Count after R is presented. • Make a lookup table of $|Q \cap R|$ counts for all R's.



Unfail. Make a restricted lookup table of the form

$$\left\{ \underset{\text{dyadic}}{R} \to \underset{>0}{|Q \cap R|} \right\}.$$

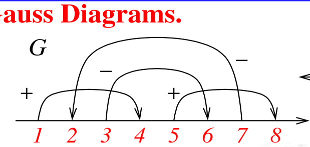
• Make the table by running through $x \in Q$, and for each one increment by 1 only the entries for dyadic $R \ni x$ (or create such an entry, if it didn't exist already). This takes $q \cdot (\log_2 n)^l \sim q$ ops.



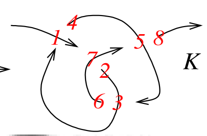
• Entries for empty dyadic R's are not needed and not created.
• Using standard sorting techniques, access takes $\log_2 q \sim 1$ ops.
• A general R is a union of at most $(2\log_2 n)^l \sim 1$ dyadic ones, so counting $|Q \cap R|$ takes ~ 1 ops.

Generalization. Without changing the conclusion, replace counts $|Q \cap R|$ with summations $\sum_R \theta$, where $\theta\colon \underline{n}^l \to V$ is supported on a sparse Q, takes values in a vector space V with $\dim V \sim 1$, and in some basis, all of its coefficients are "easy".

Gauss Diagrams.



$\longleftrightarrow$

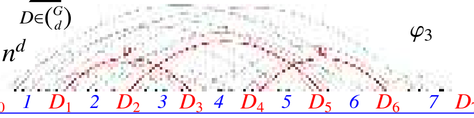


Gauss

Here's $|G| = n = 100$ (signs suppressed):

Definitions. Let $\mathcal{G} := \mathbb{Q}\langle\text{Gauss Diagrams}\rangle$, with $\mathcal{G}_d$ / $\mathcal{G}_{\leq d}$ the diagrams with exactly / at most d arrows. Let $\varphi_d\colon \mathcal{G} \to \mathcal{G}_d$ be $\varphi_d\colon G \mapsto \sum_{D\subset G,\ |D|=d} D = \sum_{D\in\binom{G}{d}} D$, and let $\varphi_{\leq d} = \sum_{e\leq d}\varphi_e$.

Naively, it takes $\binom{n}{d} \sim n^d$ ops to compute φ_d.



2023/09/11@06:00

My Primary Interest. Strong, fast, homomorphic knot and tangle invariants. ωεβ/Nara, ωεβ/Kyoto, ωεβ/Tokyo

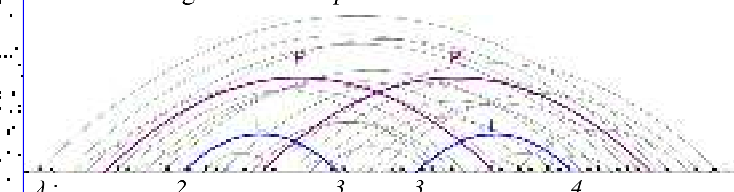
The [GPV] Theorem. A knot invariant is finite type of type d iff it is of the form $\omega \circ \varphi_{\leq d}$ for some $\omega \in \mathcal{G}^*_{\leq d}$.

Goussarov-Polyak-Viro

- $\Leftarrow$ is easy; $\Rightarrow$ is hard and IMHO not well understood.
- $\varphi_{\leq d}$ is not an invariants and not every ω gives an invariant!
- The theory of finite type invariants is very rich. Many knot invariants factor through finite type invariants, and it is possible that they separate knots.
- We need a fast algorithm to compute $\varphi_{\leq d}$!

Our Main Theorem. On an n-arrow Gauss diagram, φ_d can be computed in time $\sim n^{\lceil d/2\rceil}$.

Proof. With $d = p + l$ (p for "put", l for "lookup"), pick p arrows and look up in how many ways the remaining l can be placed in between the legs of the first p:



To reconstruct $D = P\#_\lambda L$ from P and L we need a non-decreasing "placement function" $\lambda\colon \underline{2l} \to \underline{2p+1}$.

$$\varphi_d(G) = \sum_{D\in\binom{G}{d}} D = \binom{d}{p}^{-1} \sum_{P\in\binom{G}{p}} \sum_{\substack{\text{non-decreasing} \\ \lambda\colon \underline{2l}\to\underline{2p+1}}} \sum_{\substack{L\in\binom{G}{l} \\ L_i\in(P_{\lambda(i)-1}, P_{\lambda(i)})}} P\#_\lambda L$$

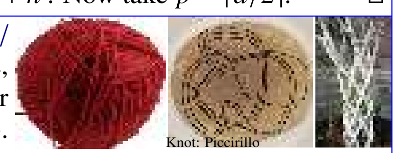
Define $\theta_G\colon \underline{2n}^{2l} \to \mathcal{G}_l$ by

$$(L_1, \ldots, L_{2l}) \mapsto \begin{cases} L & \text{if } (L_1, \ldots, L_{2l}) \text{ are the ends of some } L \subset G \\ 0 & \text{otherwise} \end{cases}$$

and now $\varphi_d(G) = \binom{d}{p}^{-1} \sum_{P\in\binom{G}{p}} \sum_{\substack{\text{non-decreasing} \\ \lambda\colon \underline{2l}\to\underline{2p+1}}} P\#_\lambda \left(\sum_{\prod_i (P_{\lambda(i)-1}, P_{\lambda(i)})} \theta_G \right)$

can be computed in time $\sim n^p + n^l$. Now take $p = \lceil d/2 \rceil$. □

Question ([BBHS], ωεβ/Fields). For computations, planar projections are better than braids (as likely $l \sim n^{3/2}$). But are yarn balls better than planar projections (here likely $n \sim L^{4/3}$)?



Length L n crossings   length l

References.

[BBHS] D. Bar-Natan, I. Bar-Natan, I. Halacheva, and N. Scherich, *Yarn Ball Knots and Faster Computations,* J. of Appl. and Comp. Topology (to appear), arXiv:2108.10923.

[GPV] M. Goussarov, M. Polyak, O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Tokyo-230911/`

Cars, Interchanges, Traffic Counters, and some Pretty Darned Good Knot Invariants

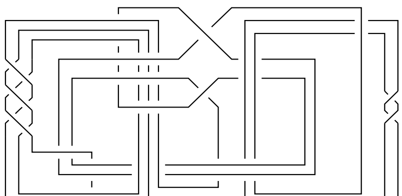
More at ωεβ/APAI

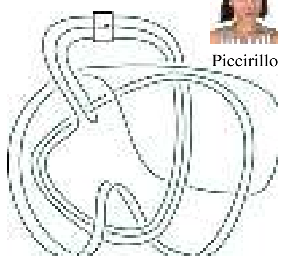
Abstract. Reporting on joint work with Roland van der Veen, I'll tell you some stories about ρ_1, an easy to define, strong, fast to compute, homomorphic, and well-connected knot invariant. ρ_1 was first studied by Rozansky and Overbay [Ro1, Ro2, Ro3, Ov] and Ohtsuki [Oh2], it has far-reaching generalizations, it is elementary and dominated by the coloured Jones polynomial, and I wish I understood it.


Rozansky Overbay Ohtsuki van der Veen

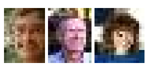
Common misconception. Dominated, elementary ⇒ lesser.

We seek strong, fast, homomorphic knot and tangle invariants.
Strong. Having a small "kernel".
Fast. Computable even for large knots (best: poly time).

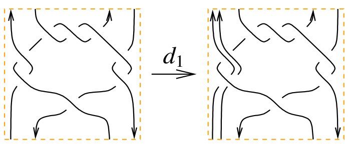




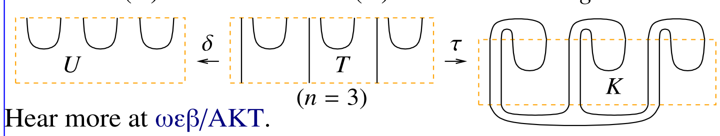

Picciriillo


Gompf–Scharlemann–Thompson

Homomorphic. Extends to tangles and behaves under tangle operations; especially gluings and doublings:

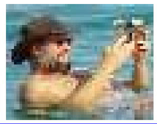


Why care for "Homomorphic"? Theorem. A knot K is *ribbon* iff there exists a $2n$-component tangle T with skeleton as below such that $\tau(T) = K$ and where $\delta(T) = U$ is the *untangle*:



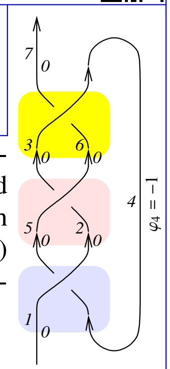
Hear more at ωεβ/AKT.

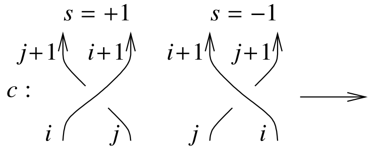
Acknowledgement. This work was supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).



Jones: Formulas stay; interpretations change with time.

Formulas. Draw an n-crossing knot K as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \dots, 2n+1\}$ and with rotation numbers φ_k. Let A be the $(2n+1)\times(2n+1)$ matrix constructed by starting with the identity matrix I, and adding a 2×2 block for each crossing:



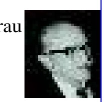


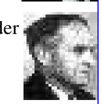
A	col $i+1$	col $j+1$
row i	$-T^s$	T^s-1
row j	0	-1

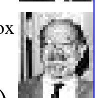
Let $G = (g_{\alpha\beta}) = A^{-1}$. For the trefoil example, it is:

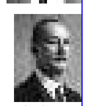
$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

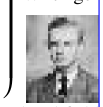
$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$


Burau


Alexander


Fox


Wirtinger


Blanchfield


"The Green Function"

Note. The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2}\det(A), \qquad \text{with } \varphi = \sum_k \varphi_k,\ w = \sum_c s.$$

Classical Topologists: This is boring. Yawn.

Formulas, continued. Finally, set

$$R_1(c) := s\left(g_{ji}\left(g_{j+1,j} + g_{j,j+1} - g_{ij}\right) - g_{ii}\left(g_{j,j+1} - 1\right) - 1/2\right)$$

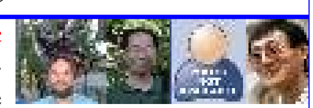
$$\rho_1 := \Delta^2\left(\sum_c R_1(c) - \sum_k \varphi_k\left(g_{kk} - 1/2\right)\right).$$

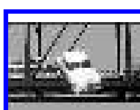
In our example $\rho_1 = -T^2 + 2T - 2 + 2T^{-1} - T^{-2}$.

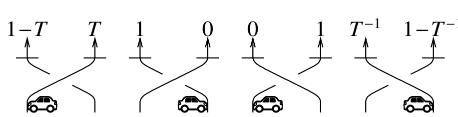
Theorem. ρ_1 is a knot invariant. Proof: later.

Classical Topologists: Whiskey Tango Foxtrot?

Cars, Interchanges, and Traffic Counters. Cars always drive forward. When a car crosses over a bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0^*$. At the very end, cars fall off and disappear. See also [Jo, LTW].


Jones Lin Tian Wang


$p = 1 - T^s$



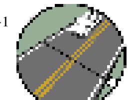


image credits: diamondtraffic.com

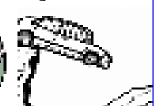


image credits: Dall-E

* In algebra $x \sim 0$ if for every y in the ideal generated by x, $1 - y$ is invertible.

References.

[BV1] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* Proc. Amer. Math. Soc. **147** (2019) 377–397, arXiv:1708.04853.
[BV2] D. Bar-Natan and R. van der Veen, *Perturbed Gaussian Generating Functions for Universal Knot Invariants,* arXiv:2109.02057.
[Dr] V. G. Drinfel'd, *Quantum Groups,* Proc. Int. Cong. Math., 798–820, Berkeley, 1986.
[Jo] V. F. R. Jones, *Hecke Algebra Representations of Braid Groups and Link Polynomials,* Annals Math., **126** (1987) 335-388.
[La] R. J. Lawrence, *Universal Link Invariants using Quantum Groups,* Proc. XVII Int. Conf. on Diff. Geom. Methods in Theor. Phys., Chester, England, August 1988. World Scientific (1989) 55–63.
[LTW] X-S. Lin, F. Tian, and Z. Wang, *Burau Representation and Random Walk on String Links,* Pac. J. Math., **182-2** (1998) 289–302, arXiv:q-alg/9605023.
[Oh1] T. Ohtsuki, *Quantum Invariants,* Series on Knots and Everything **29**, World Scientific 2002.
[Oh2] T. Ohtsuki, *On the 2–loop Polynomial of Knots,* Geom. Top. **11** (2007) 1357–1475.
[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, August 2013, ωεβ/Ov.
[Ro1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.
[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.
[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.
[Sch] S. Schaveling, *Expansions of Quantum Group Invariants,* Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Video: http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210. Handout: http://www.math.toronto.edu/~drorbn/Talks/Nara-2308.

Computing the Zombian of an Unfinished Columbarium

Confession. It's about 50% of what I do.

Apology. It's a 20 minutes talk. Necessarily, it will be superficial.

Abstract. The zombies need to compute a quantity, the zombian, that pertains to some structure — say, a columbarium. But unfortunately (for them), a part of that structure will only be known in the future. What can they compute today with the parts they already have to hasten tomorrow's computation?
That's a common quest, and I will illustrate it with a few examples from knot theory and with two examples about matrices — determinants and signatures. I will also mention two of my dreams (perhaps delusions): that one day I will be able to reproduce, and extend, the Rolfsen table of knots using code of the highest level of beauty.

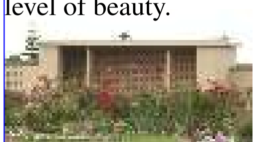
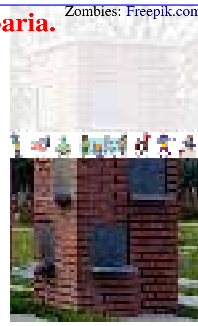


Image: Freepik.com

Columbaria in an East Sydney Cemetery — Jacobian, Hamiltonian, Zombian

Zombies: Freepik.com

Computing Zombians of Unfinished Columbaria.



Columbarium near Assen

- Future zombies must be able to complete the computation.
- Must be no slower than for finished ones.
- Future zombies must not even know the size of the task that today's zombies were facing.
- We must be able to extend to ZPUCs, Zombie Processed Unfinished Columbaria!

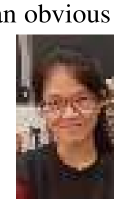
Exercise 1. Compute the sum of 1,000 numbers, the last 50 of which are still unknown.

Exercise 2. Compute the determinant of a $1,000 \times 1,000$ matrix in which 50 entries are not yet given.

Example 3. Same, for signatures of matrices / quadratic forms.

A quadratic form on a v.s. V over $\mathbb{C}$ is a quadratic $Q\colon \overline{V} \to \mathbb{C}$, or a sesquilinear Hermitian $\langle\cdot,\cdot\rangle$ on $V\times V$ (so $\langle x,y\rangle = \overline{\langle y,x\rangle}$ and $Q(y) = \langle y,y\rangle$), or given a basis η_i of V^*, a matrix $A=(a_{ij})$ with $A = \bar{A}^T$ and $Q = \sum a_{ij}\bar\eta_i\eta_j$. The signature σ of Q is $\sigma_+ - \sigma_-$, where for some P, $\bar{P}^TAP = \operatorname{diag}(1,\overset{\sigma_+}{\cdots},1,-1,\overset{\sigma_-}{\cdots},-1,0,\ldots)$.

A *Partial Quadratic (PQ)* on V is a quadratic Q defined only on a subspace $\mathcal{D}_Q \subset V$. We add PQs with $\mathcal{D}_{Q_1+Q_2} := \mathcal{D}_{Q_1}\cap\mathcal{D}_{Q_2}$. Given a linear $\psi\colon V\to W$ and a PQ Q on W, there is an obvious *pullback* ψ^*Q, a PQ on V.

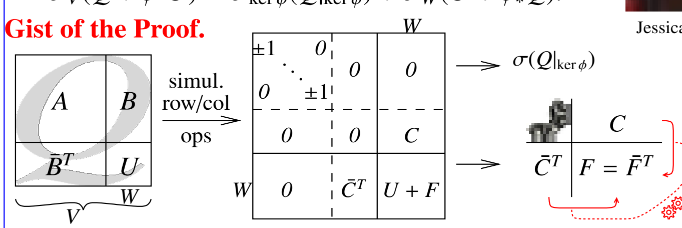


Jessica Liu

Theorem 1 (with Jessica Liu). Given a linear $\phi\colon V\to W$ and a PQ Q on V, there is a unique *pushforward* PQ ϕ_*Q on W such that for every *PQ* U on W,

$$\sigma_V(Q+\phi^*U) = \sigma_{\ker\phi}(Q|_{\ker\phi}) + \sigma_W(U+\phi_*Q).$$

Gist of the Proof.

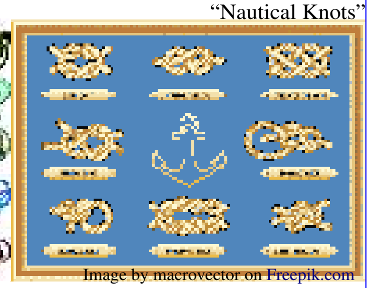


...and the quadratic $F =: \phi_*Q$ is well-defined only on $D := \ker C$.

(more at ωεβ/icerm.)

Acknowledgement. This work was partially supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

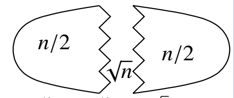
Knots and Tangles.

"Nautical Knots"

Image by macrovector on Freepik.com

Why Tangles? • As common as knots!
- Faster computations!
- Conceptually clearer proofs of invariance (and of skein relations).
- Often fun and consequential:

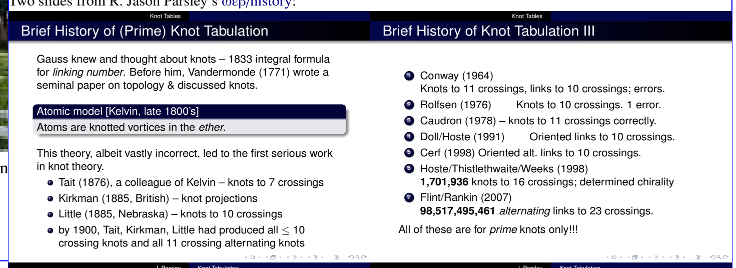


$2^{n/2} + 2^{n/2} + 2^{\sqrt{n}} \ll 2^n$

- ∘ The Alexander polynomial ⇝ Zombian = det.
- ∘ Knot signatures ⇝ Pushforwards of quadratic forms.
- ∘ The Jones Polynomial ⇝ The Temperley-Lieb Algebra.
- ∘ Khovanov Homology ⇝ "Unfinished complexes", complexes in a category.
- ∘ The Kontsevich Integral ⇝ Drinfel'd Associators. ...

One more story is left to tell, of knot tabulation.

Two slides from R. Jason Parsley's ωεβ/history:



Knot Tables

Brief History of (Prime) Knot Tabulation

Gauss knew and thought about knots – 1833 integral formula for *linking number*. Before him, Vandermonde (1771) wrote a seminal paper on topology & discussed knots.

Atomic model [Kelvin, late 1800's]
Atoms are knotted vortices in the *ether*.

This theory, albeit vastly incorrect, led to the first serious work in knot theory.

- Tait (1876), a colleague of Kelvin – knots to 7 crossings
- Kirkman (1885, British) – knot projections
- Little (1885, Nebraska) – knots to 10 crossings
- by 1900, Tait, Kirkman, Little had produced all ≤ 10 crossing knots and all 11 crossing alternating knots

J. Parsley — Knot Tabulation

Knot Tables

Brief History of Knot Tabulation III

1. Conway (1964) Knots to 11 crossings, links to 10 crossings; errors.
2. Rolfsen (1976) Knots to 10 crossings. 1 error.
3. Caudron (1978) – knots to 11 crossings correctly.
4. Doll/Hoste (1991) Oriented links to 10 crossings.
5. Cerf (1998) Oriented alt. links to 10 crossings.
6. Hoste/Thistlethwaite/Weeks (1998) **1,701,936** knots to 16 crossings; determined chirality
7. Flint/Rankin (2007) **98,517,495,461** *alternating* links to 23 crossings.

All of these are for *prime* knots only!!!

J. Parsley — Knot Tabulation

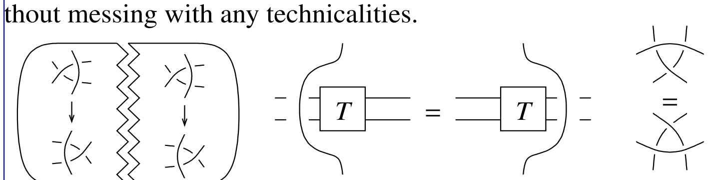
There's also Burton's tabulation to 19 crossings ωεβ/Burton, and Khesin's K250, arXiv:1705.10319.

Embarrassment 1 (personal). I don't know how to reproduce the Rolfsen table of knots! Many others can, yet I still take it on faith, contradicting one of the tenets of our practice, "thou shalt not use what thou canst not prove".

It's harder than it seems! Producing all knot diagrams is a mess, identifying all available Reidemeister moves is a mess, and you sometimes have to go up in crossing number before you can go down again.

Embarrassment 2 (communal). There isn't anywhere a tabulation of tangles! When you want to test your new discoveries, where do you go?

Dream. Conquer both embarrassments at once. Reproduce the Rolfsen table, and extend it to tangles, using code of the highest level of beauty. The algorithm should be so clear and simple that anyone should be able to easily implement it in an afternoon without messing with any technicalities.



We don't even need to look at all knot diagrams!

The dreaded slide moves, which go up in crossing number, are parametrized by tangles!

R-moves are tangle equalities!

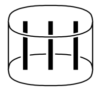
$\begin{pmatrix}A&B\\C&U\end{pmatrix} \xrightarrow{\det(A)} \begin{pmatrix}I&A^{-1}B\\C&U\end{pmatrix} \xrightarrow{1} \begin{pmatrix}I&A^{-1}B\\0&U-CA^{-1}B\end{pmatrix}$, so $\det\begin{pmatrix}A&B\\C&U\end{pmatrix} = \det(A)\det(U-CA^{-1}B)$.

Roughly, $\det(A)$ is "det on ker", $-CA^{-1}B$ is "a pushforward of $\begin{pmatrix}A&B\\C&U\end{pmatrix}$". (what if $\nexists A^{-1}$?)

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ottawa-2306/

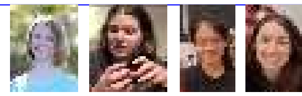
Tangles in a Pole Dance Studio: A Reading of Massuyeau, Alekseev, and Naef

Preliminary Definitions. Fix $p \in \mathbb{N}$ and $\mathbb{F} = \mathbb{Q}/\mathbb{C}$. Let $D_p := D^2\backslash(p \text{ pts})$, and let the Pole Dance Studio be $PDS_p := D_p \times I$.



PDS_3

Abstract. I will report on joint work with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich. Little of what we do is original, and much of it is simply a reading of Massuyeau [Ma] and Alekseev and Naef [AN1].



Dancso Hogan Liu Scherich

We study the pole-strand and strand-strand double filtration on the space of tangles in a pole dance studio (a punctured disk cross an interval), the corresponding homomorphic expansions, and a strand-only HOMFLY-PT relation. When the strands are transparent or nearly transparent to each other we recover and perhaps simplify substantial parts of the work of the aforementioned authors on expansions for the Goldman-Turaev Lie bi-algebra.

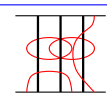
Jessica, Nancy, Tamara, Zsuzsi, & Dror in PDS_4

$\Longrightarrow$ Expansions $W\colon FG\langle X_i\rangle \to FA\langle x_i\rangle$:
Magnus: $X_i \mapsto 1 + x_i$, $X_i^{-1} \mapsto 1 - x_i + x_i^2 - \dots$
Exponential: $X_i^{\pm 1} \mapsto e^{\pm x_i}$

Definitions. Let $\pi := FG\langle X_1, \dots, X_p\rangle$ be the free group (of deformation classes of based curves in D_p), $\bar\pi$ be the framed free group (deformation classes of based immersed curves), $|\pi|$ and $|\bar\pi|$ denote $\mathbb{F}$-linear combinations of cyclic words ($|x_iw| = |wx_i|$, unbased curves), $A := FA\langle x_1, \dots, x_p\rangle$ be the free associative algebra, and let $|A| := A/(x_iw = wx_i)$ denote cyclic algebra words.

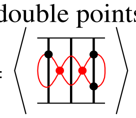
Theorem 1 (Goldman, Turaev, Massuyeau, Alekseev, Kawazumi, Kuno, Naef). $|\bar\pi|$ and $|A|$ are Lie bialgebras, and there is a "homomorphic expansion" $W\colon |\bar\pi| \to |A|$: a morphism of Lie bialgebras with $W(|X_i|) = 1 + |x_i| + \dots$.

Further Definitions. • $\mathcal{K} = \mathcal{K}_0 = \mathcal{K}_0^0 = \mathcal{K}(S) := \mathbb{F}\langle\text{framed tangles in } PDS_p\rangle$.



• $\mathcal{K}_t^s :=$(the image via $\times \to \nearrow\!\!\!\!\nwarrow - \nwarrow\!\!\!\!\nearrow$ of tangles in PDS_p that have t double points, of which s are strand-strand).

E.g.,



$$\mathcal{K}_5^2(\bigcirc) = \left\langle \quad \right\rangle / . \quad \times \to \nearrow\!\!\!\!\nwarrow - \nwarrow\!\!\!\!\nearrow$$

• $\mathcal{K}^{/s} := \mathcal{K}/\mathcal{K}^s$. Most important, $\mathcal{K}^{/1}(\bigcirc) = |\bar\pi|$, and there is $P\colon \mathcal{K}(\bigcirc) \to |\bar\pi|$.

• $\mathcal{A} := \prod \mathcal{K}_t/\mathcal{K}_{t+1}$, $\quad \mathcal{A}^s := \prod \mathcal{K}_t^s/\mathcal{K}_{t+1}^s \subset \mathcal{A}$, $\quad \mathcal{A}^{/s} := \mathcal{A}/\mathcal{A}^s$.

Fact 1. The Kontsevich Integral is an "expansion" $Z\colon \mathcal{K} \to \mathcal{A}$, compatible with several noteworthy structures.

Fact 2 (Le-Murakami, [LM1]). Z satisfies the strand-strand HOMFLY-PT relations: It descends to $Z_H\colon \mathcal{K}_H \to \mathcal{A}_H$, where

$$\mathcal{K}_H := \mathcal{K} \Big/\!\!\Big/ \left(\nearrow\!\!\!\!\nwarrow - \nwarrow\!\!\!\!\nearrow = (e^{\hbar/2} - e^{-\hbar/2}) \cdot \uparrow\uparrow \right)$$

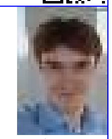
$$\mathcal{A}_H := \mathcal{A}/(\downarrow\!\!-\!\!\uparrow = \hbar \asymp \text{ or } \uparrow\!\!-\!\!\uparrow = \hbar \times)$$

and $\deg \hbar = (1, 1)$.

Proof of Fact 2. $Z(\nearrow\!\!\!\!\nwarrow) - Z(\nwarrow\!\!\!\!\nearrow) = \times \cdot \left(e^{\uparrow\!\!-\!\!\uparrow/2} - e^{-\uparrow\!\!-\!\!\uparrow/2}\right)$
$= \times \cdot \left(e^{\hbar\times/2} - e^{-\hbar\times/2}\right) = \left(e^{\hbar/2} - e^{-\hbar/2}\right) \uparrow\uparrow$. $\square$

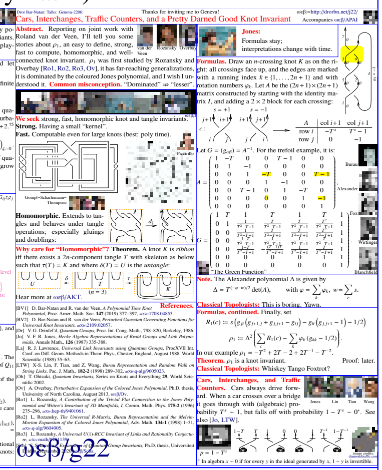
Le, Murakami

Other Passions. With Roland van der Veen, I use "solvable approximation" and "Perturbed Gaussian Differential Operators" to unveil simple, strong, fast to compute, and topologically meaningful knot invariants near the Alexander polynomial. ($\subset$ polymath!)



van der Veen

ωεβ/v19



ωεβ/g22

Key 1. $W\colon |\bar\pi| \to |A|$ is $Z_H^{/1}\colon \mathcal{K}_H^{/1}(\bigcirc) \to \mathcal{A}_H^{/1}(\bigcirc)$.

Key 2 (Schematic). Suppose $\lambda_0, \lambda_1\colon |\bar\pi| \to \mathcal{K}(\bigcirc)$ are two ways of lifting plane curves into knots in PDS_p (namely, $P \circ \lambda_i = I$). Then for $\gamma \in |\bar\pi|$,

Lemma 1. "Division by $\hbar$" is well-defined.

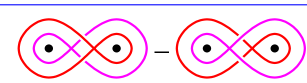
$$\eta(\gamma) := (\lambda_0(\gamma) - \lambda_1(\gamma))/\hbar \in \mathcal{K}_H^{/1}(\bigcirc\bigcirc) = |\bar\pi| \otimes |\bar\pi|$$

and we get an operation η on plane curves. If Kontsevich likes λ_0 and λ_1 (namely if there are λ_i^a with $Z^{/2}(\lambda_i(\gamma)) = \lambda_i^a(W(\gamma))$), then η will have a compatible algebraic companion η^a:

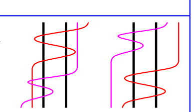
$$\eta^a(\alpha) := (\lambda_0^a(\alpha) - \lambda_1^a(\alpha))/\hbar \in \mathcal{A}_H^{/1}(\bigcirc\bigcirc) = |A| \otimes |A|.$$

For indeed, in $\mathcal{A}_H^{/2}$ we have $\hbar W(\eta(\gamma)) = \hbar Z(\eta(\gamma)) = Z(\lambda_0(\gamma)) - Z(\lambda_1(\gamma)) = \lambda_0^a(W(\gamma)) - \lambda_1^a(W(\gamma)) = \hbar\eta^a(W(\gamma))$.

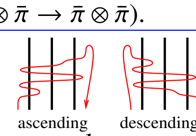
Example 1. With $\gamma_1, \gamma_2 \in |\pi|$ (or $|\bar\pi|$) set $\lambda_0(\gamma_1, \gamma_2) = \tilde\gamma_1 \cdot \tilde\gamma_2$ and $\lambda_1(\gamma_1, \gamma_2) = \tilde\gamma_2 \cdot \tilde\gamma_1$ where $\tilde\gamma_i$ are arbitrary lifts of γ_i. Then η_1 is the Goldman bracket! Note that here λ_0 and λ_1 are not well-defined, yet η_1 is.

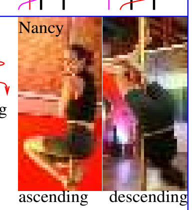


Example 2. With $\gamma_1, \gamma_2 \in \pi$ (or $\bar\pi$) and with λ_0, λ_1 as on the right, we get the "double bracket" $\eta_2\colon \pi \otimes \pi \to \pi \otimes \pi$ (or $\bar\pi \otimes \bar\pi \to \bar\pi \otimes \bar\pi$).

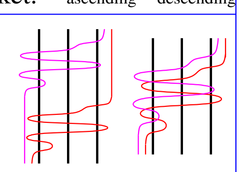


Example 3. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending realization as a bottom tangle and $\lambda_1(\gamma)$ its descending realization as a bottom tangle, we get $\eta_3\colon \bar\pi \to \bar\pi \otimes |\bar\pi|$. Closing the first component and anti-symmetrizing, this is the Turaev cobracket.





Example 4 [Ma]. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending outer double and $\lambda_1(\gamma)$ its ascending inner double we get $\eta_4\colon \bar\pi \to \bar\pi \otimes \bar\pi$. After some massaging, it too becomes the Turaev cobracket.



The rest is essentially **Exercises:** **1.** Lemma 1? **2.** $\mathcal{A}$? **3.** Fact 2? **4.** $\mathcal{A}^{/1}$? Especially, $\mathcal{A}^{/1}(\bigcirc) \cong |A|$! **5.** Explain why Kontsevich likes our λ's. **6.** Figure out η_i^a, $i = 1, \dots, 4$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-2208/

Kashaev's Signature Conjecture

CMS Winter 2021 Meeting, December 4, 2021

Dror Bar-Natan with Sina Abbasi

Agenda. Show and tell with signatures.

Abstract. I will display side by side two nearly identical computer programs whose inputs are knots and whose outputs seem to always be the same. I'll then admit, very reluctantly, that I don't know how to prove that these outputs are always the same. One program I wrote mostly in Bedlewo, Poland, in the summer of 2003 and as of recently I understand why it computes the Levine-Tristram signature of a knot. The other is based on the 2018 preprint *On Symmetric Matrices Associated with Oriented Link Diagrams* by Rinat Kashaev (arXiv:1801.04632), where he conjectures that a certain simple algorithm also computes that same signature.

If you can, please turn your video on! (And mic, whenever needed).

These slides and all the code within are available at http://drorbn.net/cms21.

(I'll post the video there too)


```
Bed[K_, ω_] :=
 Module[{t, r, XingsByArmpits, bends, faces, p, A, is},
  t = 1 - ω; r = t + t*;
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p[a,-d] p[b,-a] p[c,-b] p[d,-c];
  faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z];
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X₊,
     ( -r   -t   2t   t*  )   ( r    -t   -2t*  t*  )
     ( -t*  0    t*   0   ) , ( -t*  0    t*    0   ) ],
     ( 2t*  t    -r   -t* )   ( -2t  t    r     -t* )
     ( t    0    -t   0   )   ( t    0    -t    0   )
   {x, XingsByArmpits}];
  MatrixSignature[A]];
```

```
Kas[K_, ω_] :=
 Module[{u, v, XingsByArmpits, bends, faces, p, A, is},
  u = Re[ω^(1/2)]; v = Re[ω];
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p[a,-d] p[b,-a] p[c,-b] p[d,-c];
  faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z];
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X₊,
     ( v  u  1  u )     ( v  u  1  u )
     ( u  1  u  1 ) , - ( u  1  u  1 ) ],
     ( 1  u  v  u )     ( 1  u  v  u )
     ( u  1  u  1 )     ( u  1  u  1 )
   {x, XingsByArmpits}];
  (MatrixSignature[A] - Writhe[K]) / 2];
```


Why am I showing you 💀code💀?

- I love code — it's fun!
- Believe it or not, it is more expressive than math-talk (though I'll do the math-talk as well, to confirm with prevailing norms).
- It is directly verifiable. Once it is up and running, you'll never ask yourself "did he misplace a sign somewhere"?

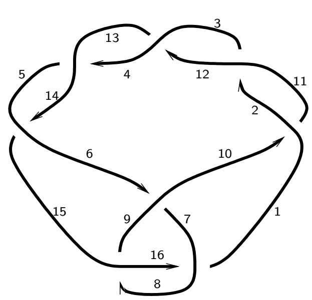

```
Bed[K_, ω_] :=
 Module[{t, r, XingsByArmpits, bends, faces, p, A, is},
  t = 1 - ω; r = t + t*;
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p[a,-d] p[b,-a] p[c,-b] p[d,-c];
  faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z];
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X₊,
     ( -r   -t   2t   t*  )   ( r    -t   -2t*  t*  )
     ( -t*  0    t*   0   ) , ( -t*  0    t*    0   ) ],
     ( 2t*  t    -r   -t* )   ( -2t  t    r     -t* )
     ( t    0    -t   0   )   ( t    0    -t    0   )
   {x, XingsByArmpits}];
  MatrixSignature[A]];
```

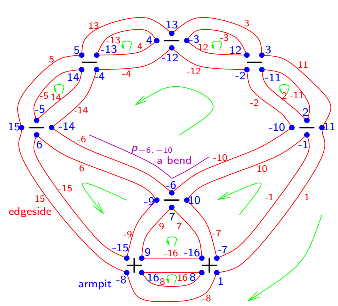
```
Kas[K_, ω_] :=
 Module[{u, v, XingsByArmpits, bends, faces, p, A, is},
  u = Re[ω^(1/2)]; v = Re[ω];
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p[a,-d] p[b,-a] p[c,-b] p[d,-c];
  faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z];
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X₊,
     ( v  u  1  u )     ( v  u  1  u )
     ( u  1  u  1 ) , - ( u  1  u  1 ) ],
     ( 1  u  v  u )     ( 1  u  v  u )
     ( u  1  u  1 )     ( u  1  u  1 )
   {x, XingsByArmpits}];
  (MatrixSignature[A] - Writhe[K]) / 2];
```


Verification.

```
Once[<< KnotTheory`]
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
MatrixSignature[A_] :=
  Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-12 &]]];
Writhe[K_] := Sum[If[PositiveQ[x], 1, -1], {x, List @@ PD@K}];
Sum[ω = e^(i RandomReal[{0,2π}]); Bed[K, ω] == Kas[K, ω], {10},
 {K, AllKnots[{3, 10}]}]
KnotTheory: Loading precomputed data in PD4Knots`.
2490 True
```


Label everything!

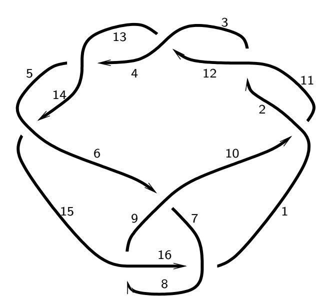




$PD[X[10, 1, 11, 2], X[2, 11, 3, 12], \ldots]$ $\{X_-[-1, 11, 2, -10], X_-[-11, 3, 12, -2], \ldots\}$

Lets run our code line by line. . .

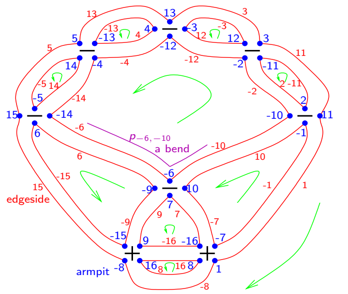
```
PD[8₂] = PD[X[10, 1, 11, 2],
   X[2, 11, 3, 12], X[12, 3, 13, 4],
   X[4, 13, 5, 14], X[14, 5, 15, 6],
   X[8, 16, 9, 15], X[16, 8, 1, 7],
   X[6, 9, 7, 10]];
K = 8₂;
```



Video and more at http://www.math.toronto.edu/~drorbn/Talks/CMS-2112/

```
XingsByArmpits =
 List @@ PD[K] /.
  x : X[i_, j_, k_, l_] :→
   If[PositiveQ[x], X+[-i, j, k, -l],
    X-[-j, k, l, -i]]
```

$\{X_-[-1, 11, 2, -10], X_-[-11, 3, 12, -2], X_-[-3, 13, 4, -12], X_-[-13, 5, 14, -4], X_-[-5, 15, 6, -14], X_+[-8, 16, 9, -15], X_+[-16, 8, 1, -7], X_-[-9, 7, 10, -6]\}$

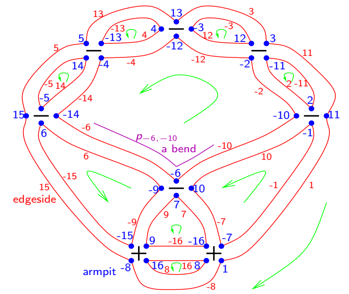



```
bends = Times @@ XingsByArmpits /.
  _[X][a_, b_, c_, d_] :→
   p[a,-d] p[b,-a] p[c,-b] p[d,-c]
```

$p_{-16,7}\, p_{-15,-9}\, p_{-14,-6}\, p_{-13,4}\, p_{-12,-4}\, p_{-11,2}\, p_{-10,-2}\, p_{-9,6}\, p_{-8,15}\, p_{-7,-1}\, p_{-6,-10}\, p_{-5,14}\, p_{-4,-14}\, p_{-3,12}\, p_{-2,-12}\, p_{-1,10}\, p_{1,-8}\, p_{2,-11}\, p_{3,11}\, p_{4,-13}\, p_{5,13}\, p_{6,-15}\, p_{7,9}\, p_{8,16}\, p_{9,-16}\, p_{10,-7}\, p_{11,1}\, p_{12,-3}\, p_{13,3}\, p_{14,-5}\, p_{15,5}\, p_{16,8}$

```
faces = bends //. p[x__,y_] p[y_,z__] :→ p[x,y,z]
```

$p_{-13,4,-13}\, p_{-11,2,-11}\, p_{-5,14,-5}\, p_{-3,12,-3}\, p_{8,16,8}\, p_{6,-15,-9,6}\, p_{9,-16,7,9}\, p_{10,-7,-1,10}\, p_{-10,-2,-12,-4,-14,-6,-10}\, p_{1,-8,15,5,13,3,11,1}$




```
A = Table[0, Length@faces, Length@faces];
A // MatrixForm
```

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

Do[is = Position[faces, #]⟦1, 1⟧ & /@ List @@ x;
 A⟦is, is⟧ += If[Head[x] === X+,

$$\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}, -\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}\Bigg],$$

 {x, XingsByArmpits}];


```
x = XingsByArmpits⟦1⟧
```

$X_-[-1, 11, 2, -10]$

```
faces
```

$p_{-13,4,-13}\, p_{-11,2,-11}\, p_{-5,14,-5}\, p_{-3,12,-3}\, p_{8,16,8}\, p_{6,-15,-9,6}\, p_{9,-16,7,9}\, p_{10,-7,-1,10}\, p_{-10,-2,-12,-4,-14,-6,-10}\, p_{1,-8,15,5,13,3,11,1}$

```
is = Position[faces, #]⟦1, 1⟧ & /@ List @@ x
```

{8, 10, 2, 9}

A⟦is, is⟧ += If[Head[x] === X+,

$$\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}, -\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}\Bigg];$$

```
A // MatrixForm
```

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&-v&0&0&0&0&0&-1&-u&-u\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&-v&-u&-u\\ 0&-u&0&0&0&0&0&-u&-1&-1\\ 0&-u&0&0&0&0&0&-u&-1&-1 \end{pmatrix}$$

Recall, $is = \{8, 10, 2, 9\}$

Do[is = Position[faces, #]⟦1, 1⟧ & /@ List @@ x;
 A⟦is, is⟧ += If[Head[x] === X+,

$$\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}, -\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}\Bigg],$$

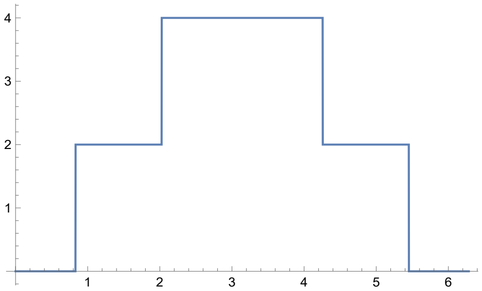
 {x, Rest@XingsByArmpits}]


```
A // MatrixForm
```

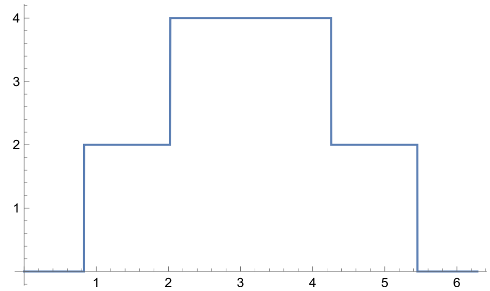
$$\begin{pmatrix} -2v&0&-1&-1&0&0&0&0&-2u&-2u\\ 0&-2v&0&-1&0&0&0&-1&-2u&-2u\\ -1&0&-2v&0&0&-1&0&0&-2u&-2u\\ -1&-1&0&-2v&0&0&0&0&-2u&-2u\\ 0&0&0&0&2&1&2u&1&0&2u\\ 0&0&-1&0&1&1-2v&0&-1&-2u&0\\ 0&0&0&0&2u&0&-1+2v&0&-1&2\\ 0&-1&0&0&1&-1&0&1-2v&-2u&0\\ -2u&-2u&-2u&-2u&0&-2u&-1&-2u&-6&-5\\ -2u&-2u&-2u&-2u&2u&0&2&0&-5&-5+2v \end{pmatrix}$$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/CMS-2112/

```
Plot[ω = ⅇ^(ⅈ t); u = Re[ω^(1/2)]; v = Re[ω];
 (MatrixSignature[A] - Writhe[K]) / 2,
 {t, 0, 2 π}]
```

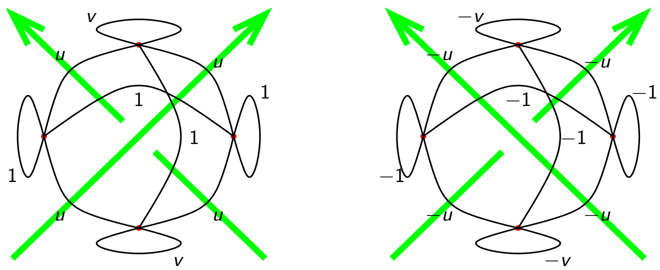



```
Plot[Bed[Knot[8, 2], ⅇ^(ⅈ t)], {t, 0, 2 π}]
```



Kashaev for Mathematicians.

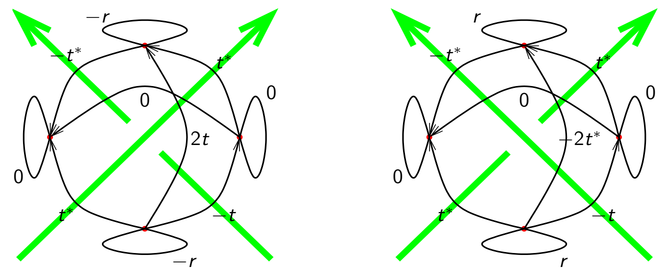
For a knot K and a complex unit ω set $u = \Re(\omega^{1/2})$, $v = \Re(\omega)$, make an $F \times F$ matrix A with contributions



and output $\frac{1}{2}(\sigma(A) - w(K))$.

Bedlewo for Mathematicians.

For a knot K and a complex unit ω set $t = 1 - \omega$, $r = 2\Re(t)$, make an $F \times F$ matrix A with contributions



(conjugate if going against the flow) and output $\sigma(A)$.

Why are they equal?

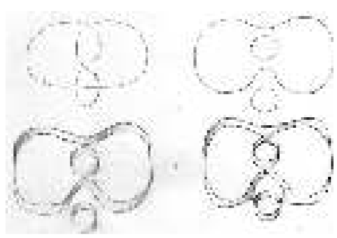
I dunno, yet note that

- Kashaev is over the $\mathbb{R}$eals, Bedlewo is over the $\mathbb{C}$omplex numbers.
- There's a factor of 2 between them, and a shift.

. . . so it's not merely a matrix manipulation.

Theorem. The Bedlewo program computes the Levine-Tristram signature of K at ω.

(Easy) **Proof.** Levine and Tristram tell us to look at $\sigma((1-\omega)L + (1-\omega^*)L^T)$, where L is the linking matrix for a Seifert surface S for K: $L_{ij} = \mathrm{lk}(\gamma_i, \gamma_j^+)$ where γ_i run over a basis of $H_1(S)$ and γ_i^+ is the pushout of γ_i. But signatures don't change if you run over and over-determined basis, and the faces make such and over-determined basis whose linking numbers are controlled by the crossings. The rest is details.



Art by Emily Redelmeier

Thank You!

The Alexander Polynomial is a Quantum Invariant in a Different Way

▶ On a chat window here I saw a comment "Alexander is the quantum $gl(1|1)$ invariant". I have an opinion about this, and I'd like to share it. First, some stories.

I left the wonderful subject of Categorification nearly 15 years ago. It got crowded, lots of very smart people had things to say, and I feared I will have nothing to add. Also, clearly the next step was to categorify all other "quantum invariants". Except it was not clear what "categorify" means. Worse, I felt that I (perhaps "we all") didn't understand "quantum invariants" well enough to try to categorify them, whatever that might mean.

I still feel that way! I learned a lot since 2006, yet I'm still not comfortable with quantum algebra, quantum groups, and quantum invariants. I still don't feel that I know what God had in mind when She created this topic.

Yet I'm not here to rant about my philosophical quandaries, but only about things that I learned about the Alexander polynomial after 2006.

Yes, the Alexander polynomial fits within the Dogma, "one invariant for every Lie algebra and representation" (it's $gl(1|1)$, I hear). But it's better to think of it as a quantum invariant arising by other means, outside the Dogma.

Alexander comes from (or in) practically any non-Abelian Lie algebra. Foremost from the not-even-semi-simple 2D "$ax+b$" algebra. You get a polynomially-sized extension to tangles using some lovely formulas (can you categorify them?). It generalizes to higher dimensions and it has an organized family of siblings. (There are some questions too, beyond categorification).

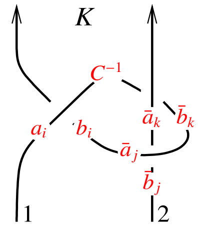
I note the spectacular existing categorification of Alexander by Ozsváth and Szabó. The theorems are proven and a lot they say, the programs run and fast they run. Yet if that's where the story ends, She has abandoned us. Or at least abandoned me: a simpleton will never be able to catch up.

If you care only about categorification, the take-home from my talk will be a challenge: Categorify what I believe is the best Alexander invariant for tangles.

The Yang-Baxter Technique. Given an algebra U (typically some $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and suitable elements R, C,

$$R=\sum a_i\otimes b_i\in U\otimes U \quad\text{with}\quad R^{-1}=\sum \bar{a}_i\otimes\bar{b}_i \quad\text{and}\quad C, C^{-1}\in U,$$

form
$$Z(K)=\sum_{i,j,k} a_iC^{-1}\bar{b}_k\bar{a}_jb_i\otimes b_j\bar{a}_k.$$



Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

Example 1. Let $\mathfrak{a} := L\langle a,x\rangle/([a,x]=x)$, $\mathfrak{b} := \mathfrak{a}^\star = \langle b,y\rangle$, and $\mathfrak{g} := \mathfrak{b}\rtimes\mathfrak{a} = \mathfrak{b}\oplus\mathfrak{a}$ with $[a,x]=x$, $[a,y]=-y$, $[b,\cdot]=0$, and $[x,y]=b$ and with $\deg(y,b,a,x)=(1,1,0,0)$. Let $U=\hat{\mathcal{U}}(\mathfrak{g})$ and

$$R := \mathbb{e}^{b\otimes a+y\otimes x}\in U\otimes U \quad\text{or better}\quad R_{ij} := \mathbb{e}^{b_ia_j+y_ix_j}\in U_i\otimes U_j, \quad\text{and}\quad C_i=\mathbb{e}^{-b_i/2}.$$

Gentle's Agreement. Everything converges!

Theorem 1. With "scalars":=power series in $\{b_i\}$ which are rational functions in $\{b_i\}$ and $\{B_i := \mathbb{e}^{b_i}\}$,

$$Z(K)=\mathbb{O}_{ybax}\left(\omega^{-1}\mathbb{e}^{\underbrace{l^{ij}b_ia_j}_{L}+\underbrace{q^{ij}y_ix_j}_{Q}}(1+\epsilon P_1+\epsilon^2P_2+\dots)\right)$$

a tangle w/o closed components

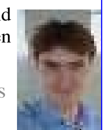
the "i over j" linking numbers (integers)

categorify us! scalars

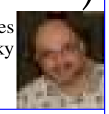
a docile perturbation for other Lie algebras; semisimple algebras have a hidden parameter ϵ!

"normal ordering" at $ybax$ order

a scalar; if K is a long knot, the Alexander poly $\Delta(T)$ categorify me!



With Roland van der Veen



Continues Lev Rozansky

Example 2. Let $\mathfrak{h} := A\langle p,x\rangle/([p,x]=1)$ be the Heisenberg algebra, with $C_i=\mathbb{e}^{t/2}$ and $R_{ij}=\mathbb{e}^{t/2}\mathbb{e}^{t(p_i-p_j)x_j}$.

I just told you the whole Alexander story! Everything else is details.

Claim. $R_{ij}=\mathbb{O}_{px}\left(\mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j}\right)$.

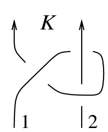
Theorem 2. $Z(K)=\mathbb{O}_{px}\left(\omega^{-1}\mathbb{e}^{q^{ij}p_ix_j}\right)$ where ω and the q^{ij} are rational functions in $T=\mathbb{e}^t$. In fact ω and ωq^{ij} are Laurent polynomials (categorify us!). When K is a long knot, ω is the Alexander polynomial.

Packaging. Write $\mathbb{O}_{px}\left(\omega^{-1}\mathbb{e}^{q^{ij}p_ix_j}\right)$ as

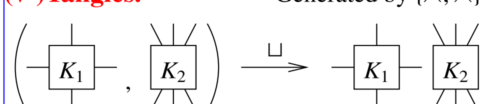
$$\mathbb{E}_{p_1,\dots,x_1,\dots}[\omega,Q]\leftrightarrow \begin{array}{c|ccc} \omega & x_1 & x_2 & \cdots \\ \hline p_1 & q^{11} & q^{12} & \cdots \\ p_2 & q^{21} & q^{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

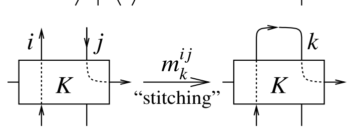
The "First Tangle". $Z(K)=$

$$\mathbb{E}_{12}\left[\frac{2T-1}{T},\frac{(T-1)(p_1-p_2)(Tx_1-x_2)}{2T-1}\right] = \begin{array}{c|cc} 2-T^{-1} & x_1 & x_2 \\ \hline p_1 & \frac{T(T-1)}{2T-1} & \frac{1-T}{2T-1} \\ p_2 & \frac{T(1-T)}{2T-1} & \frac{T-1}{2T-1} \end{array}$$



(v-)Tangles. Generated by $\{\nwarrow\!\!\nearrow, \nearrow\!\!\nwarrow\}$!





There's also strand doubling and reversal...

Theorem 3. Full evaluation via

$$\left({}_i\nearrow_j,\ {}_j\nwarrow_i\right)\to\begin{array}{c|cc} 1 & x_i & x_j \\ \hline p_i & 0 & T^{\pm1}-1 \\ p_j & 0 & 1-T^{\pm1} \end{array} \qquad (1)\square$$

$$K_1\sqcup K_2\to\begin{array}{c|cc} \omega_1\omega_2 & X_1 & X_2 \\ \hline P_1 & A_1 & 0 \\ P_2 & 0 & A_2 \end{array} \qquad (2)\square$$

$$\begin{array}{c|ccc} \omega & x_i & x_j & \cdots \\ \hline p_i & \alpha & \beta & \theta \\ p_j & \gamma & \delta & \epsilon \\ \vdots & \phi & \psi & \Xi \end{array} \xrightarrow{hm_k^{ij}} \begin{array}{c|cc} (1+\gamma)\omega & x_k & \cdots \\ \hline p_k & 1+\beta-\frac{(1-\alpha)(1-\delta)}{1+\gamma} & \theta+\frac{(1-\alpha)\epsilon}{1+\gamma} \\ \vdots & \psi+\frac{(1-\delta)\phi}{1+\gamma} & \Xi-\frac{\phi\epsilon}{1+\gamma} \end{array} \qquad (3)$$

"Γ-calculus" relates via $A\leftrightarrow I-A^T$ and has slightly simpler formulas: $\omega\to(1-\beta)\omega$,

$$\begin{pmatrix}\alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi\end{pmatrix}\to\begin{pmatrix}\gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta} \\ \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{pmatrix}$$

Why Should You Categorify This? The simplest and fastest Alexander for tangles, easily generalizes to the multi-variable case, generalizes to v-tangles and w-tangles, generalizes to other Lie algebras. In fact, it's in almost any Lie algebra, and you don't even need to know what is $gl(1|1)$! But you'll have to deal with denominators and/or divisions!

Note. Example 1 ⇝ Example 2 via $\mathfrak{g}\hookrightarrow\mathfrak{h}(t)$ via $(y,b,a,x)\mapsto(-tp,t,px,x)$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LearningSeminarOnCategorification-2006/

The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Convention. For a finite set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$. $(p,x)^* = (\pi,\xi)$

The Generating Series $\mathcal{G}$: $\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \to \mathbb{Q}[\![\zeta_A, z_B]\!]$.

Claim. $L \in \mathrm{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B]) \xrightarrow[\mathcal{G}]{\sim} \mathbb{Q}[z_B][\![\zeta_A]\!] \ni \mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n\in\mathbb{N}^A} \frac{\zeta_A^n}{n!} L(z_A^n) = L\left(e^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{L} = {}_{\text{greek}}\mathcal{L}_{\text{latin}},$$

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{L}\right)_{\zeta_a=0} \quad \text{for } p \in \mathbb{Q}[z_A].$$

Claim. If $L \in \mathrm{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$, $M \in \mathrm{Hom}(\mathbb{Q}[z_B] \to \mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0}$.

Examples. • $\mathcal{G}(id\colon \mathbb{Q}[p,x]\to\mathbb{Q}[p,x]) = e^{\pi p+\xi x}$.
• Consider $R_{ij} \in (\mathfrak{h}_i\otimes\mathfrak{h}_j)[\![t]\!] \cong \mathrm{Hom}\left(\mathbb{Q}[] \to \mathbb{Q}[p_i,x_i,p_j,x_j]\right)[\![t]\!]$.
Then $\mathcal{G}(R_{ij}) = e^{(e^t-1)(p_i-p_j)x_j} = e^{(T-1)(p_i-p_j)x_j}$.

Heisenberg Algebras. Let $\mathfrak{h} = A\langle p,x\rangle/([p,x]=1)$, let $\mathbb{O}_i\colon \mathbb{Q}[p_i,x_i]\to\mathfrak{h}_i$ is the "p before x" PBW normal ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[p_i,x_i,p_j,x_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathfrak{h}_i\otimes\mathfrak{h}_j \xrightarrow{m_k^{ij}} \mathfrak{h}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[p_k,x_k].$$

Then $\mathcal{G}(hm_k^{ij}) = e^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}$.

Proof. Recall the "Weyl CCR" $e^{\xi x}e^{\pi p} = e^{-\xi\pi}e^{\pi p}e^{\xi x}$, and find

$$\begin{aligned}\mathcal{G}(hm_k^{ij}) &= e^{\pi_i p_i+\xi_i x_i+\pi_j p_j+\xi_j x_j}/\!\!/\mathbb{O}_i\otimes\mathbb{O}_j/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1}\\ &= e^{\pi_i p_i}e^{\xi_i x_i}e^{\pi_j p_j}e^{\xi_j x_j}/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1} = e^{\pi_i p_k}e^{\xi_i x_k}e^{\pi_j p_k}e^{\xi_j x_k}/\!\!/\mathbb{O}_k^{-1}\\ &= e^{-\xi_i\pi_j}e^{(\pi_i+\pi_j)p_k}e^{(\xi_i+\xi_j)x_k}/\!\!/\mathbb{O}_k^{-1} = e^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}.\end{aligned}$$

GDO := The category with objects finite sets and

$$\mathrm{mor}(A\to B) = \left\{\mathcal{L} = \omega e^{Q}\right\} \subset \mathbb{Q}[\![\zeta_A, z_B]\!],$$

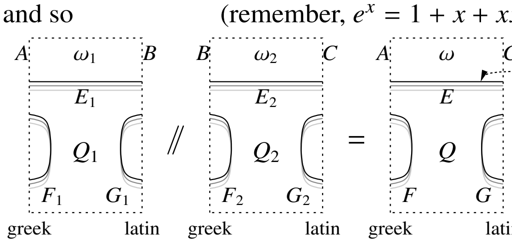
where: • ω is a scalar. • Q is a "small" quadratic in $\zeta_A\cup z_B$.
• Compositions: $\mathcal{L}/\!\!/\mathcal{M} := \left(\mathcal{L}|_{z_i\to\partial_{\zeta_i}}\mathcal{M}\right)_{\zeta_i=0}$.

Compositions. In $\mathrm{mor}(A\to B)$,


R. Feynman

$$Q = \sum_{i\in A, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij}z_i z_j,$$

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \dots$)



$$E_1E_2 + E_1F_2G_1E_2 + E_1F_2G_1F_2G_1E_2 + \dots = \sum_{r=0}^{\infty} E_1(F_2G_1)^r E_2$$

where • $E = E_1(I-F_2G_1)^{-1}E_2$ • $F = F_1 + E_1F_2(I-G_1F_2)^{-1}E_1^T$
• $G = G_2 + E_2^TG_1(I-F_2G_1)^{-1}E_2$ • $\omega = \omega_1\omega_2\det(I-F_2G_1)^{-1/2}$

Proof of Claim in Example 2. Let $\Phi_1 := e^{t(p_i-p_j)x_j}$ and $\Phi_2 := \mathbb{O}_{p_jx_j}\left(e^{(e^t-1)(p_i-p_j)x_j}\right) =: \mathbb{O}(\Psi)$. We show that $\Phi_1 = \Phi_2$ in $(\mathfrak{h}_i\otimes\mathfrak{h}_j)[\![t]\!]$ by showing that both solve the ODE $\partial_t\Phi = (p_i-p_j)x_j\Phi$ with $\Phi|_{t=0} = 1$. For Φ_1 this is trivial. $\Phi_2|_{t=0} = 1$ is trivial, and

$$\partial_t\Phi_2 = \mathbb{O}(\partial_t\Psi) = \mathbb{O}(e^t(p_i-p_j)x_j\Psi)$$

$$\begin{aligned}(p_i-p_j)x_j\Phi_2 &= (p_i-p_j)x_j\mathbb{O}(\Psi) = (p_i-p_j)\mathbb{O}(x_j\Psi - \partial_{p_j}\Psi)\\ &= \mathbb{O}\left((p_i-p_j)(x_j\Psi + (e^t-1)x_j\Psi)\right) = \mathbb{O}(e^t(p_i-p_j)x_j\Psi) \quad\square\end{aligned}$$

Implementation. Without, don't trust!

```
CF = ExpandNumerator@*ExpandDenominator@*PowerExpand@*Factor;
```

```
𝔼A1_→B1_[ω1_, Q1_] 𝔼A2_→B2_[ω2_, Q2_] ^:= 𝔼A1∪A2→B1∪B2[ω1 ω2, Q1 + Q2]
```

```
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I, M = Table},
  I = IdentityMatrix@Length@B1;
  E1 = M[∂i,j Q1, {i, A1}, {j, B1}]; E2 = M[∂i,j Q2, {i, A2}, {j, B2}];
  F1 = M[∂i,j Q1, {i, A1}, {j, A1}]; F2 = M[∂i,j Q2, {i, A2}, {j, A2}];
  G1 = M[∂i,j Q1, {i, B1}, {j, B1}]; G2 = M[∂i,j Q2, {i, B2}, {j, B2}];
  𝔼A1→B2[CF[ω1 ω2 Det[I - F2.G1]^(1/2)], CF@Plus[
     If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
     If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1ᵀ).A1],
     If[B2 === {}, 0, 1/2 B2.(G2 + E2ᵀ.G1.Inverse[I - F2.G1].E2).B2]]]]
```

```
A_ \ B_ := Complement[A, B];
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* =!= A2) :=
 𝔼A1∪(A2\B1*)→B1∪A2*[ω1, Q1 + Sum[ζ* ζ, {ζ, A2 \ B1*}]] //
  𝔼B1*∪A2→B2∪(B1\A2*)[ω2, Q2 + Sum[z* z, {z, B1 \ A2*}]]
```

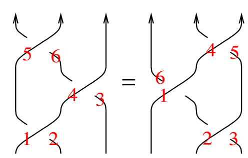
```
{p*, x*, π*, ξ*} = {π, ξ, p, x}; (u_i_)* := (u*)i;
l_List* := #* & /@ l;
```

```
Ri_,j_ := 𝔼{}→{pi,xi,pj,xj}[T^(-1/2), (1 - T) pj xj + (T - 1) pi xj];
R̄i_,j_ := 𝔼{}→{pi,xi,pj,xj}[T^(1/2), (1 - T^-1) pj xj + (T^-1 - 1) pi xj];
Ci_ := 𝔼{}→{pi,xi}[T^(-1/2), 0];
C̄i_ := 𝔼{}→{pi,xi}[T^(1/2), 0];
```

```
hmi_,j_→k_ := 𝔼{πi,ξi,πj,ξj}→{pk,xk}[1, -ξi πj + (πi + πj) pk + (ξi + ξj) xk]
```

```
𝔼{}→vs_[ωi_, Q_]h := Module[{ps, xs, M},
  ps = Cases[vs, p_]; xs = Cases[vs, x_];
  M = Table[ωi, 1 + Length@ps, 1 + Length@xs];
  M[[2 ;;, 2 ;;]] = Table[CF[∂i,j Q], {i, ps}, {j, xs}];
  M[[2 ;;, 1]] = ps; M[[1, 2 ;;]] = xs;
  MatrixForm[M]h]
```

Proof of Reidemeister 3.




```
(R1,2 R4,3 R5,6 // hm1,4→1 hm2,5→2 hm3,6→3) ==
 (R2,3 R1,6 R4,5 // hm1,4→1 hm2,5→2 hm3,6→3)
```

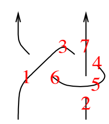
True $\square$

The "First Tangle".

```
Factor /@
 (z = R1,6 C̄3 R̄7,4 R̄5,2 // hm1,3→1 // hm1,4→1 // hm1,5→1 // hm1,6→1 // hm2,7→2)
```

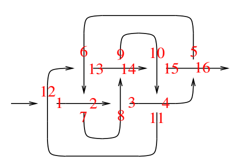
$$\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}\left[\frac{-1+2T}{T}, \frac{(-1+T)(p_1-p_2)(Tx_1-x_2)}{-1+2T}\right]$$

```
zh
```



$$\begin{pmatrix} \frac{-1+2T}{T} & x_1 & x_2 \\ p_1 & \frac{-T+T^2}{-1+2T} & \frac{1-T}{-1+2T} \\ p_2 & \frac{T-T^2}{-1+2T} & \frac{-1+T}{-1+2T}\end{pmatrix}_h$$

The knot 8_{17}.




```
z = R̄12,1 R̄27 R̄83 R̄4,11 R16,5 R6,13 R14,9 R10,15;
Table[z = z // hm1k→1, {k, 2, 16}] // Last
```

$$\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T^3}, 0\right]$$

Proof of Theorem 3, (3).

```
{(γ1 = 𝔼{}→{p1,x1,p2,x2,p3,x3}[ω, {p1, p2, p3}.(α β θ; γ δ ϵ; ϕ ψ Ξ).{x1, x2, x3}])h,
 (γ1 // hm1,2→0)h}
```

$$\left\{\begin{pmatrix}\omega & x_1 & x_2 & x_3\\ p_1 & \alpha & \beta & \theta\\ p_2 & \gamma & \delta & \epsilon\\ p_3 & \phi & \psi & \Xi\end{pmatrix}_h, \begin{pmatrix}\omega+\gamma\omega & x_0 & x_3\\ p_0 & \frac{\alpha+\beta+\gamma+\beta\gamma+\delta-\alpha\delta}{1+\gamma} & \frac{\epsilon-\alpha\epsilon+\theta+\gamma\theta}{1+\gamma}\\ p_3 & \frac{\phi-\delta\phi+\psi+\gamma\psi}{1+\gamma} & \frac{\Xi+\gamma\Xi-\epsilon\phi}{1+\gamma}\end{pmatrix}_h\right\}$$

$\square$

References. On ωεβ=http://drorbn.net/cat20

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LearningSeminarOnCategorification-2006/

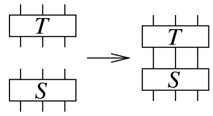
Dror Bar-Natan: Talks: Toronto-1912: ωεβ:=http://drorbn.net/to19/

Geography vs. Identity

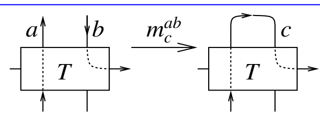
Thanks for inviting me to the *Topology* session!

Abstract. Which is better, an emphasis on where things happen or on who are the participants? I can't tell; there are advantages and disadvantages either way. Yet much of quantum topology seems to be heavily and unfairly biased in favour of geography.

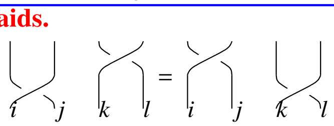
Geographers care for placement; for them, braids and tangles have ends at some distinguished points, hence they form categories whose objects are the placements of these points. For them, the basic operation is a binary "stacking of tangles". They are lead to monoidal categories, braided monoidal categories, representation theory, and much or most of we call "quantum topology".

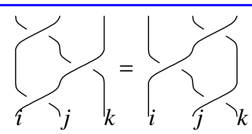


Identiters believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation m^{ab}_c, and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See ωεβ/reg, ωεβ/kbh.



Braids.





Geography: (better topology!)

$$GB := \langle \gamma_i \rangle \bigg/ \left(\begin{matrix} \gamma_i\gamma_k = \gamma_k\gamma_i \text{ when } |i-k|>1 \\ \gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1} \end{matrix} \right) = B.$$

Identity: (captures quantum algebra!)

$$IB := \langle \sigma_{ij} \rangle \bigg/ \left(\begin{matrix} \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \text{ when } |\{i,j,k,l\}|=4 \\ \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \text{ when } |\{i,j,k\}|=3 \end{matrix} \right) = PvB.$$

Theorem. Let $S = \{\tau\}$ be the symmetric group. Then vB is both

$$PvB \rtimes S \cong B * S \Big/ \big(\gamma_i\tau = \tau\gamma_j \text{ when } \tau i = j,\ \tau(i+1) = (j+1)\big)$$

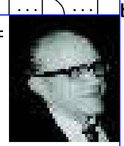
(and so PvB is "bigger" then B, and hence quantum algebra doesn't see topology very well).

Proof. Going left, $\gamma_i \mapsto \sigma_{i,i+1}(i\ i+1)$. Going right, if $i<j$ map $\sigma_{ij} \mapsto (j{-}1\ j{-}2\ \dots\ i)\gamma_{j-1}(i\ i{+}1\ \dots\ j)$ and if $i>j$ use $\sigma_{ij} \mapsto (j\ j{+}1\ \dots\ i)\gamma_j(i\ i{-}1\ \dots\ j{+}1)$.

vB views of σ_{ij}:

The Burau Representation of PvB_n acts on $R^n := \mathbb{Z}[t^{\pm 1}]^n = R\langle v_1, \dots, v_n \rangle$ by

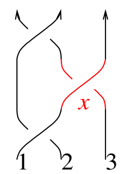
$$\sigma_{ij}v_k = v_k + \delta_{kj}(t-1)(v_j - v_i).$$



Werner Burau

```
δ /: δi_,j_ := If[i == j, 1, 0];                    ωεβ/code
Bi_,j_[ℰ_] := ℰ /. vk_ :> vk + δk,j (t - 1) (vj - vi) // Expand
(bas3 = {v1, v2, v3}) // B1,2
{v1, v1 - t v1 + t v2, v3}
bas3 // B1,2 // B1,3 // B2,3
{v1, v1 - t v1 + t v2, v1 - t v1 + t v2 - t^2 v2 + t^2 v3}
bas3 // B2,3 // B1,3 // B1,2
{v1, v1 - t v1 + t v2, v1 - t v1 + t v2 - t^2 v2 + t^2 v3}
```

S_n acts on R^n by permuting the v_i so the Burau representation extends to vB_n and restricts to B_n. With this, γ_i maps $v_i \mapsto v_{i+1}$, $v_{i+1} \mapsto tv_i + (1-t)v_{i+1}$, and otherwise $v_k \mapsto v_k$.



Geography view:

so x is γ_2.

Identity view: At x strand 1 crosses strand 3, so x is σ_{13}.

The Gold Standard is set by the "Γ-calculus" Alexander formulas (ωεβ/mac). An S-component tangle T has

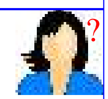
$$\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\} \text{ with } R_S := \mathbb{Z}(\{T_a : a \in S\}):$$

$$\left({}_a\nearrow_b,\ {}_b\nwarrow_a \right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1 - T_a^{\pm 1} \\ b & 0 & T_a^{\pm 1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[T_a, T_b \to T_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array} \right)$$

The Gassner Representation of PvB_n acts on $V = R^n := \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^n = R\langle v_1, \dots, v_n \rangle$ by

$$\sigma_{ij}v_k = v_k + \delta_{kj}(t_i - 1)(v_j - v_i).$$



?

Betty Jane Gassner deserves to be more famous

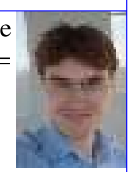
```
Gi_,j_[ℰ_] := ℰ /. vk_ :> vk + δk,j (ti - 1) (vj - vi) // Expand
(bas3 // G1,2 // G1,3 // G2,3) == (bas3 // G2,3 // G1,3 // G1,2)
True
```

S_n acts on R^n by permuting the v_i *and* the t_i, so the Gassner representation extends to vB_n and then restricts to B_n as a $\mathbb{Z}$-linear ∞-dimensional representation. It then descends to PB_n as a finite-rank R-linear representation, with lengthy non-local formulas.

Geographers: Gassner is an obscure partial extension of Burau.

Identiters: Burau is a trivial silly reduction of Gassner.

The Turbo-Gassner Representation. With the same R and V, TG acts on $V \oplus (R^n \otimes V) \oplus (S^2V \otimes V^*) = R\langle v_k, v_{lk}, u_iu_jw_k \rangle$ by



With Roland van der Veen

```
TGi_,j_[ℰ_] := ℰ /. {
    vk_ :> vk + δk,j ((ti - 1) (vj - vi) + vi,j - vi,i) +
      δk,i (uj - ui) ui wj,
    vl_,k_ :> vl,k + (ti - 1) ×
      (δk,j (vl,j - vl,i) + (δl,i - δl,j ti^-1 tj)
        (uk + δk,j (ti - 1) (uj - ui)) ui wj),
    uk_ :> uk + δk,j (ti - 1) (uj - ui),
    wk_ :> wk + (δk,j - δk,i) (ti^-1 - 1) wj} // Expand
bas3 = {v1, v2, v3, v1,1, v1,2, v1,3, v2,1, v2,2, v2,3, v3,1,
   v3,2, v3,3, u1^2 w1, u1^2 w2, u1^2 w3, u1 u2 w1, u1 u2 w2, u1 u2 w3,
   u1 u3 w1, u1 u3 w2, u1 u3 w3, u2^2 w1, u2^2 w2, u2^2 w3, u2 u3 w1,
   u2 u3 w2, u2 u3 w3, u3^2 w1, u3^2 w2, u3^2 w3};
(bas3 // TG1,2 // TG1,3 // TG2,3) == (bas3 // TG2,3 // TG1,3 // TG1,2)
True
```

Gassner motifs

Adjoint-Gassner

Like Gassner, TG is also a representation of PB_n.

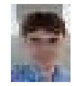
I have no idea where it belongs!

My talk tomorrow, at the *chord diagrams everywhere* session: More Dror: ωεβ/talks

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/

Some Feynman Diagrams in Pure Algebra

With Roland van der Veen



Thanks for allowing me in Columbia U!
ωεβ:=http://drorbn.net/co19/
Slides w/ no handout/URL should be banned!

Abstract. I will explain how the computation of compositions of maps of a certain natural class, from one polynomial ring into another, naturally leads to a certain composition operation of quadratics and to Feynman diagrams. I will also explain, with very little detail, how this is used in the construction of some very well-behaved poly-time computable knot polynomials.

The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Gentle Agreement. Everything converges!

Convention. For a finite set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$. $(y,b,a,x)^* = (\eta,\beta,\alpha,\xi)$

The Generating Series $\mathcal{G}$: $\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \to \mathbb{Q}[\![\zeta_A, z_B]\!]$.

Claim. $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \xrightarrow[\mathcal{G}]{\sim} \mathbb{Q}[z_B][\![\zeta_A]\!] \ni \mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}L(z_A^n) = L\left(\mathbb{e}^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{L} = {}_{\text{greek}}\mathcal{L}_{\text{latin}},$$

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{L}\right)_{\zeta_a=0} \quad \text{for } p\in\mathbb{Q}[z_A].$$

Claim. If $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $M\in\operatorname{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0}$.

Basic Examples. 1. $\mathcal{G}(id\colon \mathbb{Q}[y,a,x]\to\mathbb{Q}[y,a,x]) = \mathbb{e}^{\eta y+\alpha a+\xi x}$.

2. The standard commutative product m_k^{ij} of polynomials is given by $z_i, z_j \to z_k$. Hence $\mathcal{G}(m_k^{ij}) = m_k^{ij}(\mathbb{e}^{\zeta_i z_i+\zeta_j z_j}) = \mathbb{e}^{(\zeta_i+\zeta_j)z_k}$.

$$\begin{array}{ccc} \mathbb{Q}[z]_i\otimes\mathbb{Q}[z]_j & \xrightarrow{m_k^{ij}} & \mathbb{Q}[z]_k \\ \| & & \| \\ \mathbb{Q}[z_i,z_j] & \xrightarrow{m_k^{ij}} & \mathbb{Q}[z_k] \end{array}$$

3. The standard co-commutative coproduct Δ_{jk}^i of polynomials is given by $z_i\to z_j+z_k$. Hence $\mathcal{G}(\Delta_{jk}^i) = \Delta_{jk}^i(\mathbb{e}^{\zeta_i z_i}) = \mathbb{e}^{\zeta_i(z_j+z_k)}$.

$$\begin{array}{ccc} \mathbb{Q}[z]_i & \xrightarrow{\Delta_{jk}^i} & \mathbb{Q}[z]_j\otimes\mathbb{Q}[z]_k \\ \| & & \| \\ \mathbb{Q}[z_i] & \xrightarrow{\Delta_{jk}^i} & \mathbb{Q}[z_j,z_k] \end{array}$$

Heisenberg Algebras. Let $\mathbb{H} = \langle x,y\rangle/[x,y] = \hbar$ (with $\hbar$ a scalar), let $\mathbb{O}_i\colon \mathbb{Q}[x_i,y_i]\to\mathbb{H}_i$ is the "x before y" PBW ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[x_i,y_i,x_j,y_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathbb{H}_i\otimes\mathbb{H}_j \xrightarrow{m_k^{ij}} \mathbb{H}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[x_k,y_k].$$

Then $\mathcal{G}(hm_k^{ij}) = \mathbb{e}^{\Lambda_\hbar}$, where $\Lambda_\hbar = -\hbar\eta_i\xi_j + (\xi_i+\xi_j)x_k + (\eta_i+\eta_j)y_k$.

Proof 1. Recall the "Weyl form of the CCR" $\mathbb{e}^{\eta y}\mathbb{e}^{\xi x} = \mathbb{e}^{-\hbar\eta\xi}\mathbb{e}^{\xi x}\mathbb{e}^{\eta y}$, and compute

$$\begin{aligned}\mathcal{G}(hm_k^{ij}) &= \mathbb{e}^{\xi_i x_i+\eta_i y_i+\xi_j x_j+\eta_j y_j}/\!/\mathbb{O}_i\otimes\mathbb{O}_j/\!/m_k^{ij}/\!/\mathbb{O}_k^{-1} \\ &= \mathbb{e}^{\xi_i x_i}\mathbb{e}^{\eta_i y_i}\mathbb{e}^{\xi_j x_j}\mathbb{e}^{\eta_j y_j}/\!/m_k^{ij}/\!/\mathbb{O}_k^{-1} = \mathbb{e}^{\xi_i x_k}\mathbb{e}^{\eta_i y_k}\mathbb{e}^{\xi_j x_k}\mathbb{e}^{\eta_j y_k}/\!/\mathbb{O}_k^{-1} \\ &= \mathbb{e}^{-\hbar\eta_i\xi_j}\mathbb{e}^{(\xi_i+\xi_j)x_k}\mathbb{e}^{(\eta_i+\eta_j)y_k}/\!/\mathbb{O}_k^{-1} = \mathbb{e}^{\Lambda_\hbar}.\end{aligned}$$

Proof 2. We compute in a faithful 3D representation ρ of $\mathbb{H}$: (ωεβ/hm)

```
{x̂ = (0 1 0
      0 0 0
      0 0 0), ŷ = (0 0 0
                   0 0 ħ
                   0 0 0), ĉ = (0 0 1
                                0 0 0
                                0 0 0)};
{x̂.ŷ - ŷ.x̂ == ħ ĉ, x̂.ĉ == ĉ.x̂, ŷ.ĉ == ĉ.ŷ}
{True, True, True}
Λ = -ħ ηi ξj ck + (ξi + ξj) xk + (ηi + ηj) yk;
Simplify@With[{𝔼 = MatrixExp},
  𝔼[x̂ ξi].𝔼[ŷ ηi].𝔼[x̂ ξj].𝔼[ŷ ηj] ==
   𝔼[x̂ ∂xk Λ].𝔼[ŷ ∂yk Λ].𝔼[ĉ ∂ck Λ]]
True
```

A Real DoPeGDO Example (DoPeGDO:=Docile Perturbed Gaussian Differential Operators). Let $sl_{2+}^\epsilon := L\langle y,b,a,x\rangle$ subject to $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $[x,y]=\epsilon a+b$. So $t:=\epsilon a-b$ is central and if $\exists\epsilon^{-1}$, $sl_{2+}^\epsilon \cong sl_2\oplus\langle t\rangle$. Let $CU := \mathcal{U}(sl_{2+}^\epsilon)$, and let cm_k^{ij} be the composition below, where $\mathbb{O}_i\colon \mathbb{Q}[y_i,b_i,a_i,x_i]\to CU_i$ be the PBW ordering map in the order $ybax$:

$$\begin{array}{ccc} CU_i\otimes CU_j & \xrightarrow{m_k^{ij}} & CU_k \\ \uparrow{\scriptstyle\mathbb{O}_{i,j}} & & \uparrow{\scriptstyle\mathbb{O}_k} \\ \mathbb{Q}[y_i,b_i,a_i,x_i,y_j,b_j,a_j,x_j] & \xrightarrow{cm_k^{ij}} & \mathbb{Q}[y_k,b_k,a_k,x_k] \end{array}$$

Claim. Let (all brawn and no brains)

$$\Lambda = \left(\eta_i + \frac{e^{-\alpha_i-\epsilon\beta_i}\eta_j}{1+\epsilon\eta_j\xi_i}\right)y_k + \left(\beta_i+\beta_j+\frac{\log\left(1+\epsilon\eta_j\xi_i\right)}{\epsilon}\right)b_k + \left(\alpha_i+\alpha_j+\log\left(1+\epsilon\eta_j\xi_i\right)\right)a_k + \left(\frac{e^{-\alpha_j-\epsilon\beta_j}\xi_i}{1+\epsilon\eta_j\xi_i}+\xi_j\right)x_k$$

Then $\mathbb{e}^{\eta_i y_i+\beta_i b_i+\alpha_i a_i+\xi_i x_i+\eta_j y_j+\beta_j b_j+\alpha_j a_j+\xi_j x_j}/\!/\mathbb{O}_{i,j}/\!/cm_k^{ij} = \mathbb{e}^\Lambda/\!/\mathbb{O}_k$, and hence $\mathcal{G}(cm_k^{ij}) = \mathbb{e}^\Lambda$.

Proof. We compute in a faithful 2D representation ρ of CU: (ωεβ/sl2)

```
{ŷ = (0 0
      ϵ 0), b̂ = (0  0
                 0 -ϵ), â = (1 0
                             0 0), x̂ = (0 1
                                        0 0)};
{â.x̂ - x̂.â == x̂, â.ŷ - ŷ.â == -ŷ, b̂.ŷ - ŷ.b̂ == -ϵ ŷ,
 b̂.x̂ - x̂.b̂ == ϵ x̂, x̂.ŷ - ŷ.x̂ == b̂ + ϵ â}
{True, True, True, True, True}
Simplify@With[{𝔼 = MatrixExp},
  𝔼[ηi ŷ].𝔼[βi b̂].𝔼[αi â].𝔼[ξi x̂].𝔼[ηj ŷ].𝔼[βj b̂].
   𝔼[αj â].𝔼[ξj x̂] == 𝔼[ŷ ∂yk Λ].𝔼[b̂ ∂bk Λ].𝔼[â ∂ak Λ].
   𝔼[x̂ ∂xk Λ]]
True
Series[Λ, {ϵ, 0, 2}]
```

$$\begin{aligned}&\left(a_k(\alpha_i+\alpha_j) + y_k(\eta_i + e^{-\alpha_i}\eta_j) + b_k(\beta_i+\beta_j+\eta_j\xi_i) + x_k(e^{-\alpha_j}\xi_i+\xi_j)\right) + \\ &\left(a_k\eta_j\xi_i - \frac{1}{2}b_k\eta_j^2\xi_i^2 - e^{-\alpha_i}y_k\eta_j(\beta_i+\eta_j\xi_i) - e^{-\alpha_j}x_k\xi_i(\beta_j+\eta_j\xi_i)\right)\epsilon + \\ &\left(-\frac{1}{2}a_k\eta_j^2\xi_i^2 + \frac{1}{3}b_k\eta_j^3\xi_i^3 + \frac{1}{2}e^{-\alpha_i}y_k\eta_j\left(\beta_i^2+2\beta_i\eta_j\xi_i+2\eta_j^2\xi_i^2\right) + \frac{1}{2}e^{-\alpha_j}x_k\xi_i\left(\beta_j^2+2\beta_j\eta_j\xi_i+2\eta_j^2\xi_i^2\right)\right)\epsilon^2 + O[\epsilon]^3\end{aligned}$$

Note 1. If the lower half of the alphabet (a,b,α,β) is regarded as constants, then $\Lambda = C+Q+\sum_{k\geq1}\epsilon^k P^{(k)}$ is a docile perturbed Gaussian relative to the upper half of the alphabet (x,y,ξ,η): C is a scalar, Q is a quadratic, and $\deg P^{(k)}\leq 2k+2$.

Note 2. $\mathrm{wt}(x,y,\xi,\eta;a,b,\alpha,\beta;\epsilon) = (1,1,1,1;2,0,0,2;-2)$.

Quadratic Casimirs. If $t\in\mathfrak{g}\otimes\mathfrak{g}$ is the quadratic Casimir of a semi-simple Lie algebra $\mathfrak{g}$, then $\mathbb{e}^t$, regarded by PBW as an element of $\mathcal{S}^{\otimes2} = \operatorname{Hom}\left(\mathcal{S}(\mathfrak{g})^{\otimes0}\to\mathcal{S}(\mathfrak{g})^{\otimes2}\right)$, has a latin-latin dominant Gaussian factor. Likewise for R-matrices.

(Baby) **DoPeGDO** := The category with objects finite sets[†1] and

$$\operatorname{mor}(A\to B) = \{\mathcal{L} = \omega\exp(Q+P)\} \subset \mathbb{Q}[\![\zeta_A, z_B, \epsilon]\!],$$

where: • ω is a scalar.[†2] • Q is a "small" ϵ-free quadratic in $\zeta_A\cup z_B$.[†3] • P is a "docile perturbation": $P=\sum_{k\geq1}\epsilon^k P^{(k)}$, where $\deg P^{(k)}\leq 2k+2$.[†4] • Compositions:[†6] $\mathcal{L}/\!/\mathcal{M} := \left(\mathcal{L}|_{z_i\to\partial_{\zeta_i}}\mathcal{M}\right)_{\zeta_i=0}$.

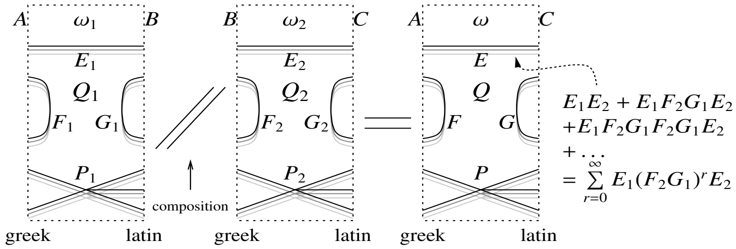
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Columbia-191125/

So What? If V is a representation, then $V^{\otimes n}$ explodes as a function of n, while in **DoPeGDO** up to a fixed power of ϵ, the ranks of $\mathrm{mor}(A \to B)$ grow polynomially as a function of $|A|$ and $|B|$.

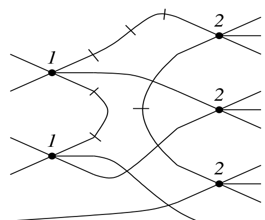
Compositions. In $\mathrm{mor}(A \to B)$,

$$Q = \sum_{i\in A, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j,$$

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \ldots$)



where • $E = E_1(I - F_2G_1)^{-1}E_2$.
• $F = F_1 + E_1F_2(I - G_1F_2)^{-1}E_1^T$.
• $G = G_2 + E_2^T G_1(I - F_2G_1)^{-1}E_2$.
• $\omega = \omega_1\omega_2 \det(I - F_2G_1)^{-1}$.
• P is computed as the solution of a messy PDE or using "connected Feynman diagrams" (yet we're still in pure algebra!). Docility is preserved.



DoPeGDO Footnotes. Each variable has a "weight"$\in \{0, 1, 2\}$, and always $\mathrm{wt}\, z_i + \mathrm{wt}\,\zeta_i = 2$.

†1. Really, "weight-graded finite sets" $A = A_0 \sqcup A_1 \sqcup A_2$.

†2. Really, a power series in the weight-0 variables†5.

†3. The weight of Q must be 2, so it decomposes as $Q = Q_{20}+Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series†5.

†4. Setting $\mathrm{wt}\,\epsilon = -2$, the weight of P is ≤ 2 (so the powers of the weight-0 variables are not constrained)†5.

†5. In the knot-theoretic case, all weight-0 power series are rational functions of bounded degree in the exponentials of the weight-0 variables.

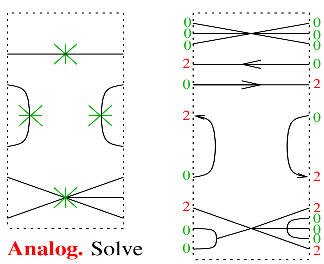
†6. There's also an obvious product

$$\mathrm{mor}(A_1 \to B_1)\times \mathrm{mor}(A_2 \to B_2) \to \mathrm{mor}(A_1\sqcup A_2 \to B_1 \sqcup B_2).$$

Full DoPeGDO. Compute compositions in two phases:

• A 1-1 phase over the ring of power series in the weight-0 variables, in which the weight-2 variables are spectators.

• A (slightly modified) 2-0 phase over $\mathbb{Q}$, in which the weight-1 variables are spectators.



Analog. Solve $Ax = a,\ B(x)y = b$

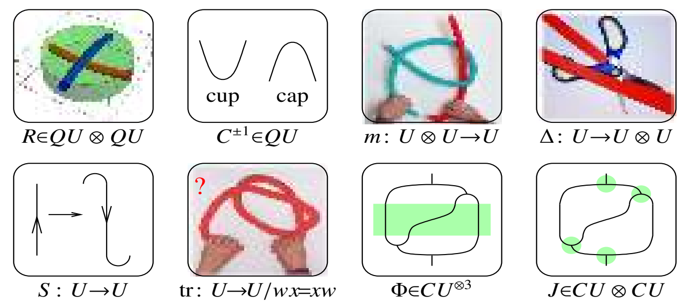
Questions. • Are there QFT precedents for "two-step Gaussian integration"?

• In QFT, one saves even more by considering "one-particle-irreducible" diagrams and "effective actions". Does this mean anything here?

• Understanding $\mathrm{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$ seems like a good cause. Can you find other applications for the technology here?

$QU = \mathcal{U}_\hbar(sl^\epsilon_{2+}) = A\langle y, b, a, x\rangle[\![\hbar]\!]$ with $[a, x] = x$, $[b, y] = -\epsilon y$, $[a, b] = 0$, $[a, y] = -y$, $[b, x] = \epsilon x$, and $xy-qyx = (1-AB)/\hbar$, where $q = e^{\hbar\epsilon}$, $A = e^{-\hbar\epsilon a}$, and $B = e^{-\hbar b}$. Also $\Delta(y, b, a, x) = (y_1 + B_1y_2, b_1 + b_2, a_1 + a_2, x_1 + A_1x_2)$, $S(y, b, a, x) = (-B^{-1}y, -b, -a, -A^{-1}x)$, and $R = \sum \hbar^{j+k}y^kb^j \otimes a^jx^k/j![k]_q!$.

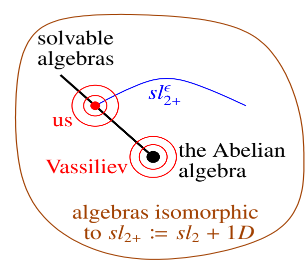
Theorem. Everything of value regrading $U = CU$ and/or its quantization $U = QU$ is **DoPeGDO**:

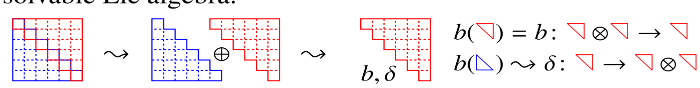


also Cartan's θ, the $\mathbb{D}$equantizator, and more, and all of their compositions.

Solvable Approximation. In sl_n, half is enough! Indeed $sl_n \oplus \mathfrak{a}_{n-1} = \mathcal{D}(◹, b, \delta)$. Now define $sl^\epsilon_{n+} := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon◺$, and $[◹, ◺] = ◺ + \epsilon◹$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1} = 0$ always yields a solvable Lie algebra.

4D Metrized Lie Algebras





Conclusion. There are lots of poly-time-computable well-behaved near-Alexander knot invariants: • They extend to tangles with appropriate multiplicative behaviour. • They have cabling and strand reversal formulas. ωεβ/akt

The invariant for $sl^\epsilon_{2+}/(\epsilon^2 = 0)$ (prior art: ωεβ/Ov) attains 2,883 distinct values on the 2,978 prime knots with ≤ 12 crossings. HOMFLY-PT and Khovanov homology together attain only 2,786 distinct values.

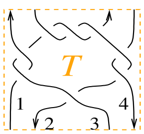
knot diag	n^t_k / $(\rho'_1)^+$ / $(\rho'_2)^+$	Alexander's ω^+	genus / ribbon; unknotting # / amphi?
	0^a_1 / 0 / 0	1	0 / ✔; 0 / ✔
	3^a_1 / T / $3T^3-12T^2+26T-38$	$T-1$	1 / ✘; 1 / ✘
	4^a_1 / 0 / $T^4-3T^3-15T^2+74T-110$	$3-T$	1 / ✘; 1 / ✔
	5^a_1 / $2T^3+3T$ / $5T^7-20T^6+55T^5-120T^4+217T^3-338T^2+450T-510$	T^2-T+1	2 / ✘; 2 / ✘
	5^a_2 / $5T-4$ / $-10T^4+120T^3-487T^2+1054T-1362$	$2T-3$	1 / ✘; 1 / ✘
	6^a_1 / $T-4$ / $14T^4-16T^3-293T^2+1098T-1598$	$5-2T$	1 / ✔; 1 / ✘
	6^a_2 / T^3-4T^2+4T-4 / $3T^8-21T^7+49T^6+15T^5-433T^4+1543T^3-3431T^2+5482T-6410$	$-T^2+3T-3$	2 / ✘; 1 / ✘
	6^a_3 / 0 / $4T^8-33T^7+121T^6-203T^5-111T^4+1499T^3-4210T^2+7186T-8510$	T^2-3T+5	2 / ✘; 1 / ✔
	7^a_1 / $3T^5+5T^3+6T$ / $7T^{11}-28T^{10}+77T^9-168T^8+322T^7-560T^6+891T^5-1310T^4+1777T^3-2238T^2+2604T-2772$	T^3-T^2+T-1	3 / ✘; 3 / ✘

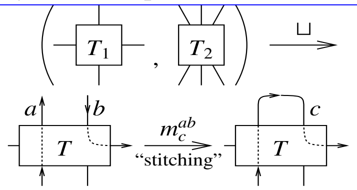
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Columbia-191125/

Algebraic Knot Theory

Abstract. This will be a very "light" talk: I will explain why about 13 years ago, in order to have a say on some problems in knot theory, I've set out to find tangle invariants with some nice compositional properties. In other talks in Sydney (ωεβ/talks) I have explained / will explain how such invariants were found - though they are yet to be explored and utilized.

(v-)Tangles.

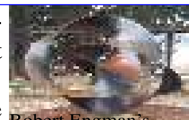




(meta-associativity: $m^{ab}_c /\!\!/ m^{xc}_y = m^{bc}_x /\!\!/ m^{ax}_y$) (tangles are generated by ⤯ and ⤰)

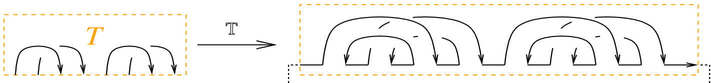
Strand doubling: $a \xrightarrow{\Delta^a_{bc}}$ b, c. Strand reversal: $a \xrightarrow{S_a}$ a

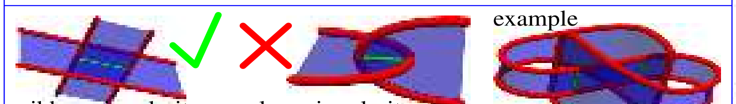
Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.



Robert Engman's "The Loop"

Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via





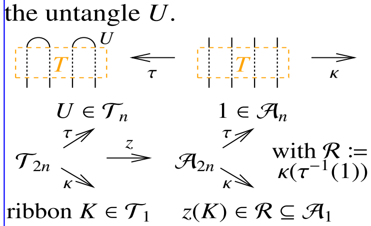
a ribbon singularity a clasp singularity

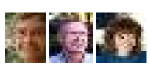
A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

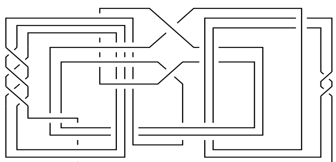
Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

Theorem. K is ribbon iff it is κT for a tangle T for which τT is the untangle U.

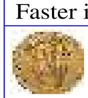




Gompf, Scharlemann, Thompson [GST]



Faster is better, leaner is meaner!



The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{T_a\colon a \in S\})$:

$$\left({}_a\!\nearrow_b, {}_b\!\nwarrow_a\right) \to \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[T_a,T_b\to T_c]{m^{ab}_c} \left(\begin{array}{c|cc}(1-\beta)\omega & c & S\\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta}\\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

Strand Doubling and Reversal.

$$\begin{array}{c|cc}\omega & a & S\\ \hline a & \alpha & \theta\\ S & \phi & \Xi\end{array} \xrightarrow[\substack{\mu=T_a-1\\ \nu=\alpha-\sigma_a\\ T_a\to T_bT_c}]{q\Delta^a_{bc}} \left(\begin{array}{c|ccc}\omega & b & c & S\\ \hline b & (\sigma_a-\alpha T_a-\nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu\\ c & (T_c-1)\nu/\mu & (\alpha-\sigma_aT_a-\nu T_c)/\mu & (T_c-1)\theta/\mu\\ S & \phi & \phi & \Xi\end{array}\right)$$

$dS^a \downarrow T_a \to T_a^{-1}$

$$\left(\begin{array}{c|cc}\alpha\omega/\sigma_a & a & S\\ \hline a & 1/\alpha & \theta/\alpha\\ S & -\phi/\alpha & (\alpha\Xi-\phi\theta)/\alpha\end{array}\right)$$

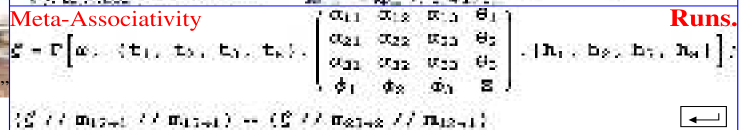
Where σ assigns to every $a \in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left({}_a\!\nearrow_b, {}_b\!\nwarrow_a\right) = (a \to 1, b \to t_a^{\pm1})$, $\sigma(T_1 \sqcup T_2) = \sigma(T_1) \sqcup \sigma(T_2)$, and $\sigma /\!\!/ m^{ab}_c = (\sigma \setminus \{a,b\}) \cup (c \to \sigma_a\sigma_b)|_{t_a,t_b\to t_c}$.

Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

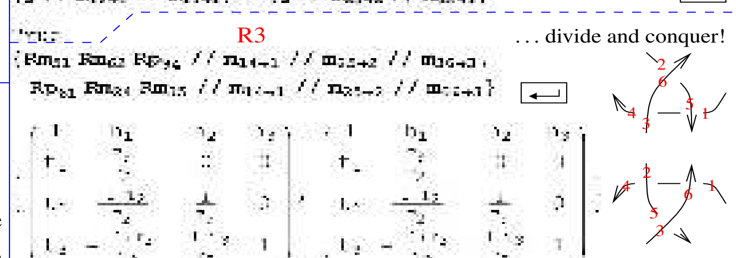
Implementation key idea: ωεβ/AlexDemo

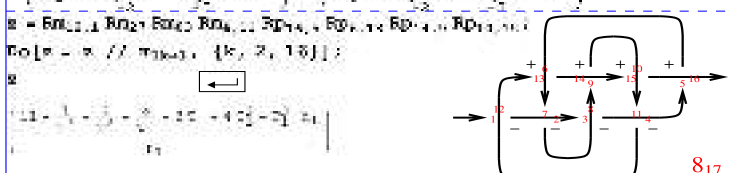
$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

Meta-Associativity **Runs.**



R3 ... divide and conquer!

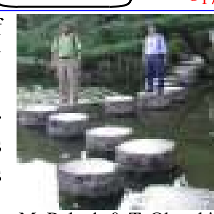




Fact. Γ is better viewed as an invariant of a certain class of 2D knotted objects in $\mathbb{R}^4$ [BND, BN].

Fact. Γ is the "0-loop" part of an invariant that generalizes to "n-loops" (1D tangles only, see further talks and future publications with van der Veen).

Speculation. Stepping stones to categorification?



M. Polyak & T. Ohtsuki @ Heian Shrine, Kyoto

Ask me about geography vs. identity!

References.

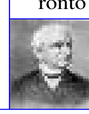
[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: w-Knots and the Alexander Polynomial,* Alg. and Geom. Top. **16-2** (2016) 1063–1133, arXiv:1405.1956, ωεβ/WKO1.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[Vo] H. Vo, *Alexander Invariants of Tangles via Expansions,* University of Toronto Ph.D. thesis, ωεβ/Vo.



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Macquarie-191016/

Everything around sl_{2+}^{ϵ} is **DoPeGDO**. So what?

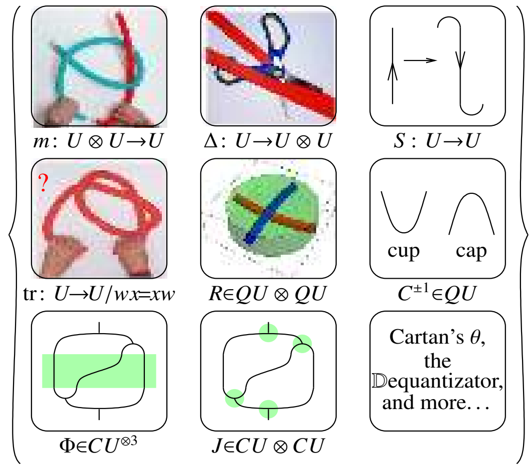
Thanks for inviting me to UCLA!
ωεβ:=http://drorbn.net/la19/
More at ωεβ/talks

Continues Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen [BV].

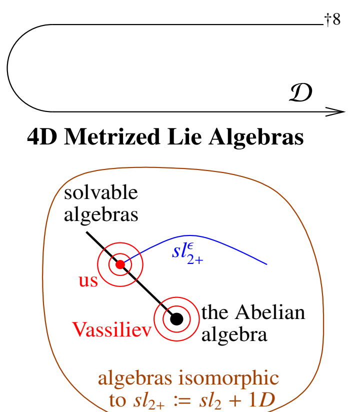
Abstract. I'll explain what "everything around" means: classical and quantum m, Δ, S, tr, R, C, and θ, as well as P, Φ, J, $\mathbb{D}$, and more, and all of their compositions. What **DoPeGDO** means: the category of Docile Perturbed Gaussian Differential Operators. And what sl_{2+}^{ϵ} means: a solvable approximation of the semi-simple Lie algebra sl_2.

Knot theorists should rejoice because all this leads to very powerful and well-behaved poly-time-computable knot invariants. Quantum algebraists should rejoice because it's a realistic playground for testing complicated equations and theories.

Conventions. 1. For a set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$.†1 2. Everything converges!



Less Abstract



DoPeGDO := The category with objects finite sets†2 and mor$(A\to B)$:

$$\{\mathcal{F} = \omega\exp(Q+P)\}\subset \mathbb{Q}[\![\zeta_A, z_B, \epsilon]\!]$$

Where: • ω is a scalar.†3 • Q is a "small" ϵ-free quadratic in $\zeta_A\cup z_B$.†4 • P is a "docile perturbation": $P=\sum_{k\geq 1}\epsilon^k P^{(k)}$, where $\deg P^{(k)}\leq 2k+2$.†5 • Compositions:†6

$$\mathcal{F}/\!\!/\mathcal{G} = \mathcal{G}\circ\mathcal{F} := \left(\mathcal{G}|_{\zeta_i\to\partial_{z_i}}\mathcal{F}\right)_{z_i=0} = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

Cool! $(V^*)^{\otimes\Sigma}\otimes V^{\otimes S}$ explodes; the ranks of quadratics and bounded-degree polynomials grow slowly!†7 Representation theory is over-rated!

Cool! How often do you see a computational toolbox so successful?

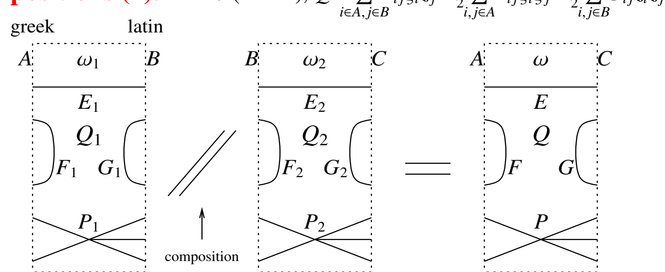
Our Algebras. Let $sl_{2+}^{\epsilon} := L\langle y,b,a,x\rangle$ subject to $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $[x,y]=\epsilon a+b$. So $t:=\epsilon a-b$ is central and if $\exists\epsilon^{-1}$, $sl_{2+}^{\epsilon}/\langle t\rangle\cong sl_2$. ωεβ/oa
U is either $CU=\mathcal{U}(sl_{2+}^{\epsilon})[\![\hbar]\!]$ or $QU=\mathcal{U}_\hbar(sl_{2+}^{\epsilon}) = A\langle y,b,a,x\rangle[\![\hbar]\!]$ with $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $xy-qyx=(1-AB)/\hbar$, where $q=\mathbb{e}^{\hbar\epsilon}$, $A=\mathbb{e}^{-\hbar\epsilon a}$, and $B=\mathbb{e}^{-\hbar b}$. Set also $T=A^{-1}B=\mathbb{e}^{\hbar t}$.

The Quantum Leap. Also decree that in QU,

$$\Delta(y,b,a,x) = (y_1+B_1y_2, b_1+b_2, a_1+a_2, x_1+A_1x_2),$$
$$S(y,b,a,x) = (-B^{-1}y, -b, -a, -A^{-1}x),$$

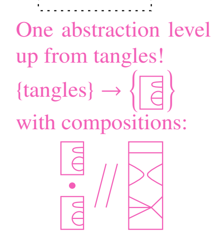
and $R=\sum\hbar^{j+k}y^kb^j\otimes a^jx^k/j![k]_q!$.

Compositions (1). In mor$(A\to B)$, $Q=\sum_{i\in A,j\in B}E_{ij}\zeta_iz_j+\frac12\sum_{i,j\in A}F_{ij}\zeta_i\zeta_j+\frac12\sum_{i,j\in B}G_{ij}z_iz_j$



Where • $E=E_1(I-F_2G_1)^{-1}E_2$.
• $F=F_1+E_1F_2(I-G_1F_2)^{-1}E_1^T$.
• $G=G_2+E_2^TG_1(I-F_2G_1)^{-1}E_2$.
• $\omega=\omega_1\omega_2\det(I-F_2G_1)^{-1}$.
• P is computed using "connected Feynman diagrams" or as the solution of a messy PDE (yet we're still in algebra!).

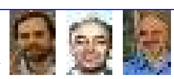
Alexander lives here



Mid-Talk Debts. • What is this good for in quantum algebra?
• In knot theory?
• How does the "inclusion" $\mathcal{D}$: Hom$(U^{\otimes\Sigma}\to U^{\otimes S})\rightsquigarrow$ **DoPeGDO** work?
• Proofs that everything around sl_{2+}^{ϵ} really is **DoPeGDO**.
• Relations with prior art.
• The rest of the "compositions" story.

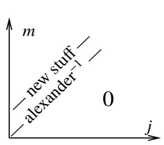
DoPeGDO Footnotes. †1. Each variable has a "weight"$\in\{0,1,2\}$, and always wt z_i + wt $\zeta_i=2$.
†2. Really, "weight-graded finite sets" $A=A_0\sqcup A_1\sqcup A_2$.
†3. Really, a power series in the weight-0 variables†9.
†4. The weight of Q must be 2, so it decomposes as $Q=Q_{20}+Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series†9.
†5. Setting wt $\epsilon=-2$, the weight of P is ≤ 2 (so the powers of the weight-0 variables are not constrained†9).
†6. There's also an obvious product
mor$(A_1\to B_1)\times$ mor$(A_2\to B_2)\to$ mor$(A_1\sqcup A_2\to B_1\sqcup B_2)$.
†7. That is, if the weight-0 variables are ignored. Otherwise more care is needed yet the conclusion remains.
†8. Hom$(U^{\otimes\Sigma}\to U^{\otimes S})\rightsquigarrow$ mor$(\{\eta_i,\beta_i,\tau_i,\alpha_i,\xi_i\}_{i\in\Sigma}\to\{y_i,b_i,t_i,a_i,x_i\}_{i\in S})$, where wt$(\eta_i,\xi_i,y_i,x_i)=1$ and wt$(\beta_i,\tau_i,\alpha_i;b_i,t_i,a_i)=(2,2,0;0,0,2)$.
†9. For tangle invariants the wt-0 power series are always rational functions in the exponentials of the wt-0 variables (for knots: just one variable), with degrees bounded linearly by the crossing number.

Theorem ([BG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing



$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0}a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$

Video and more: http://www.math.toronto.edu/~drorbn/Talks/CRM-1907, http://www.math.toronto.edu/~drorbn/Talks/UCLA-191101.

Computation without Representation

Thanks for inviting me to Ohio!

ωεβ:=http://drorbn.net/o19/

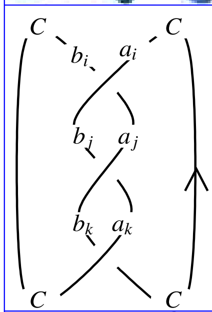
Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at [BV] and at ωεβ/talks.

Abstract. A major part of "quantum topology" is the definition and computation of various knot invariants by carrying out computations in quantum groups. Traditionally these computations are carried out "in a representation", but this is very slow: one has to use tensor powers of these representations, and the dimensions of powers grow exponentially fast.

In my talk, I will describe a direct method for carrying out such computations without having to choose a representation and explain why in many ways the results are better and faster. The two key points we use are a technique for composing infinite-order "perturbed Gaussian" differential operators, and the little-known fact that every semi-simple Lie algebra can be approximated by solvable Lie algebras, where computations are easier.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know. (experimental analysis @ωεβ/kiw)

Knotted Candies ωεβ/kc



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R=\sum a_i\otimes b_i\in U\otimes U\quad\text{and}\quad C\in U,$$

form

$$Z=\sum_{i,j,k} Ca_ib_ja_kC^2b_ia_jb_kC.$$

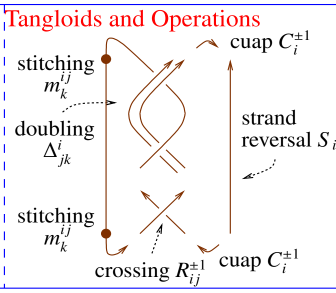
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

A Knot Theory Portfolio.

- Has operations $\sqcup$, m^{ij}_k, Δ^i_{jk}, S_i.
- All tangloids are generated by $R^{\pm1}$ and $C^{\pm1}$ (so "easy" to produce invariants).
- Makes some knot properties ("genus", "ribbon") become "definable".

Tangloids and Operations



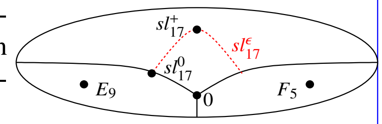
A "Quantum Group" Portfolio consists of a vector space U along with maps (and some axioms...)

$$\begin{tikzcd}
\mathbb{Q}=\hat{U}^{\otimes\emptyset} \arrow[r, "C_i"] & \hat{U}^{\otimes\{i\}} \arrow[loop above, "S_i"] \arrow[r, shift left, "\Delta^i_{jk}"] & \hat{U}^{\otimes\{j,k\}} \arrow[l, shift left, "m^{jk}_i"] & \mathbb{Q}=\hat{U}^{\otimes\emptyset} \arrow[l, "R_{jk}"'] \\
\hat{\mathcal{S}}(\emptyset) \arrow[u, "\mathbb{O}_\emptyset"] \arrow[r, "C_i"] & \hat{\mathcal{S}}(B_i) \arrow[u, "\mathbb{O}_{y_ix_i\dots}"] \arrow[loop below, "S_i"] \arrow[r, shift left, "\Delta^i_{jk}"] & \hat{\mathcal{S}}(B_j,B_k) \arrow[u, "\mathbb{O}_{y_jx_j\dots\otimes y_kx_k\dots}"] \arrow[l, shift left, "m^{jk}_i"] & \hat{\mathcal{S}}(\emptyset) \arrow[u, "\mathbb{O}_\emptyset"] \arrow[l, "R_{jk}"']
\end{tikzcd}$$

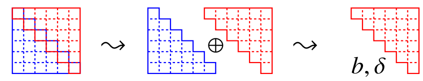
$\sqcup\to\otimes$ (top row), $\sqcup\to\cdot$ (bottom row)

Our Way. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes2}\otimes V$ is on the right. We care about $sl^k_{17}:=sl^\epsilon_{17}/(\epsilon^{k+1}=0)$.



Solvable Approximation. In gl_n, half is enough! Indeed $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(\nabla,b,\delta)$:



Now define $gl^\epsilon_n:=\mathcal{D}(\nabla,b,\epsilon\delta)$. Schematically, this is $[\nabla,\nabla]=\nabla$, $[\triangle,\triangle]=\epsilon\triangle$, and $[\nabla,\triangle]=\triangle+\epsilon\nabla$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1}=0$ always yields a solvable Lie algebra.

CU and QU. Starting from sl_2, get $CU_\epsilon=\langle y,a,x,t\rangle/([t,-]=0,\ [a,y]=-y,\ [a,x]=x,\ [x,y]=2\epsilon a-t)$. Quantize using standard tools (I'm sorry) and get $QU_\epsilon=\langle y,a,x,t\rangle/([t,-]=0,\ [a,y]=-y,\ [a,x]=x,\ xy-\mathbb{e}^{\hbar\epsilon}yx=(1-T\mathbb{e}^{-2\hbar\epsilon a})/\hbar)$.

PBW Bases. The U's we care about always have "Poincaré-Birkhoff-Witt" bases; there is some finite set $B=\{y,x,\dots\}$ of "generators" and isomorphisms $\mathbb{O}_{y,x,\dots}\colon\hat{\mathcal{S}}(B)\to U$ defined by "ordering monomials" to some fixed $y,x,\dots$ order. The quantum group portfolio now becomes a "symmetric algebra" portfolio, or a "power series" portfolio.

Operations are Objects.

★ $B^*:=\{z_i^*=\zeta_i\colon\ z_i\in B\}$,

$$\langle z_i^m,\zeta_i^n\rangle=\delta_{mn}n!,$$

$$\left\langle\prod z_i^{m_i},\prod\zeta_i^{n_i}\right\rangle=\prod\delta_{m_in_i}n_i!,$$

in general, for $f\in\mathcal{S}(z_i)$ and $g\in\mathcal{S}(\zeta_i)$,

$$\langle f,g\rangle=f(\partial_{\zeta_i})g\big|_{\zeta_i=0}=g(\partial_{z_i})f\big|_{z_i=0}.$$

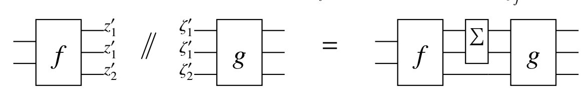
$$f\in\mathrm{Hom}_{\mathbb{Q}}(S(B)\to S(B'))\ \|\ S(B)^*\otimes S(B')\ \|\star\ S(B^*)\otimes S(B')\ \|\ S(B^*\sqcup B')\ \|\ \tilde{f}\in\mathbb{Q}[\zeta_i,z_i']$$

The Composition Law. If

$$\mathcal{S}(B)\xrightarrow[\tilde{f}\in\mathbb{Q}[\![\zeta_i,z'_j]\!]]{f}\mathcal{S}(B')\xrightarrow[\tilde{g}\in\mathbb{Q}[\![\zeta'_j,z''_k]\!]]{g}\mathcal{S}(B'')$$

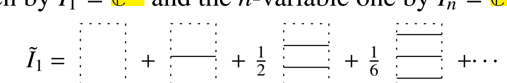
then

$$\widetilde{(f/\!\!/g)}=\widetilde{(g\circ f)}=\left(\tilde{g}\big|_{\zeta'_j\to\partial_{z'_j}}\tilde{f}\right)_{z'_j=0}=\left(\tilde{f}\big|_{z'_j\to\partial_{\zeta'_j}}\tilde{g}\right)_{\zeta'_j=0}:$$



Examples

1. The 1-variable identity map $I\colon\mathcal{S}(z)\to\mathcal{S}(z)$ is given by $\tilde{I}_1=\mathbb{e}^{z\zeta}$ and the n-variable one by $\tilde{I}_n=\mathbb{e}^{z_1\zeta_1+\dots+z_n\zeta_n}$:



2. The "archetypal multiplication map $m^{ij}_k\colon\mathcal{S}(z_i,z_j)\to\mathcal{S}(z_k)$" has $\tilde{m}=\mathbb{e}^{z_k(\zeta_i+\zeta_j)}$.
3. The "archetypal coproduct $\Delta^i_{jk}\colon\mathcal{S}(z_i)\to\mathcal{S}(z_j,z_k)$", given by $z_i\to z_j+z_k$ or $\Delta z=z\otimes1+1\otimes z$, has $\tilde{\Delta}=\mathbb{e}^{(z_j+z_k)\zeta_i}$.
4. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1x_2}\in\mathcal{U}_q\otimes\mathcal{U}_q$. The "baby R-matrix" is $\tilde{R}=\mathbb{e}^{\hbar yx}\in\mathcal{S}(y,x)$.
5. The "Weyl form of the canonical commutation relations" states that if $[y,x]=tI$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y}=\mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. So with

$$\begin{tikzcd}\mathcal{S}(y,x) \arrow[loop left, "SW_{xy}"] \arrow[r, shift left, "\mathbb{O}_{xy}"] \arrow[r, shift right, "\mathbb{O}_{yx}"'] & \mathcal{U}(y,x)\end{tikzcd}$$

we have $\widetilde{SW}_{xy}=\mathbb{e}^{\eta y+\xi x-\eta\xi t}$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ohio-1901

Do Not Turn Over Until Instructed

Dror Bar-Natan: Talks: MAASeaway-1810:

Thanks for inviting me to the fall 2018 MAA Seaway Section meeting!

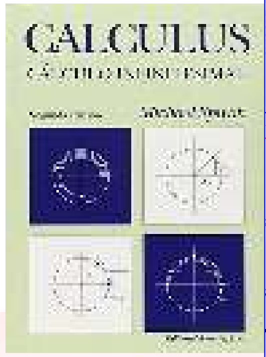
My Favourite First-Year Analysis Theorem

Handout, video, links at ωεβ:=http://drorbn.net/maa18/

Abstract. Whatever it may be, it should say something useful and exciting and it should not be *about* rigour, yet it should *demand* rigour. You can't guess. You probably think it the dreariest. You are wrong.

Contents Ⓢ

Several excerpts here are from Spivak's "Calculus" Ⓢ. I believe they fall under "fair use".



Prologue
1 Basic Properties of Numbers 3
2 Numbers of Various Sorts 21
Foundations
3 Functions 39
4 Graphs 56
5 Limits 90
6 Continuous Functions 113
7 Three Hard Theorems 120
8 Least Upper Bounds 142
Derivatives and Integrals
9 Derivatives 147
10 Differentiation 166
11 Significance of the Derivative 185
12 Inverse Functions 227
13 Integrals 250
14 The Fundamental Theorem of Calculus 282
15 The Trigonometric Functions 300
∗16 π is Irrational 321
∗17 Planetary Motion 327
18 The Logarithm and Exponential Functions 336
19 Integration in Elementary Terms 359
Infinite Sequences and Infinite Series
20 Approximation by Polynomial Functions 405 . . .

6 Continuous Functions.

for every $\varepsilon > 0$ there is $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Ⓢ

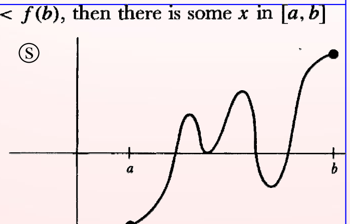
If f and g are continuous at a, then

(1) $f + g$ is continuous at a,

(2) $f \cdot g$ is continuous at a.

7 Three Hard Theorems.

If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some x in $[a, b]$ such that $f(x) = 0$. Ⓢ



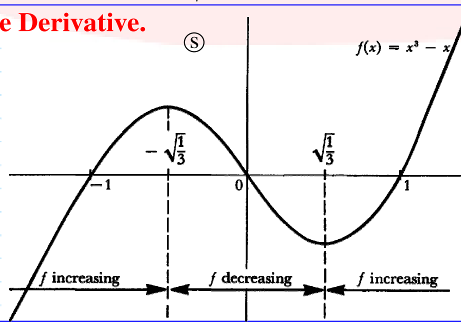
11 Significance of the Derivative. Ⓢ

$y = x^3 - x$

$y' = 3x^2 - 1$

$= (\sqrt{3}x + 1)(\sqrt{3}x - 1)$

$= \begin{cases} > 0 & x > \sqrt{1/3} \\ < 0 & -\sqrt{1/3} < x < \sqrt{1/3} \\ > 0 & x < -\sqrt{1/3} \end{cases}$



14 The Fundamental Theorem of Calculus.

If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a).$$

Ⓢ

∗16 π is Irrational.

20 Approximation by Polynomial Functions.

Suppose that f is a function for which

$$f'(a), \ldots, f^{(n)}(a)$$

all exist. Let

$$a_k = \frac{f^{(k)}(a)}{k!}, \quad 0 \le k \le n,$$

and define

$$P_{n,a}(x) = a_0 + a_1(x - a) + \cdots + a_n(x - a)^n.$$

Then

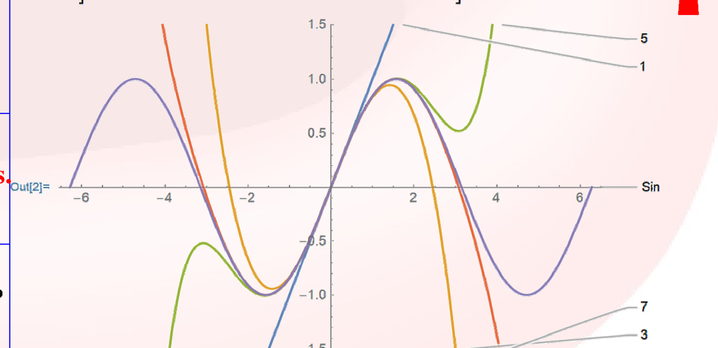
$$\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x - a)^n} = 0.$$

Ⓢ

For example for $f(x) = \sin(x)$ at $a = 0$, $f^{(k)} = \sin, \cos, -\sin, -\cos, \sin, \ldots$, so

$$a_k = \begin{cases} \frac{(-1)^{(k-1)/2}}{k!} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

```
In[1]:= a_k_ := { (-1)^((k-1)/2) / k!   OddQ[k]
                  0                      EvenQ[k] ;
        Plot[Evaluate@Append[
            Table[Labeled[Sum_{k=0}^{n} a_k x^k, n], {n, {1, 3, 5, 7}}],
            Labeled[Sin[x], Sin]
          ], {x, -2π, 2π}, PlotRange → {-1.5, 1.5}]
```

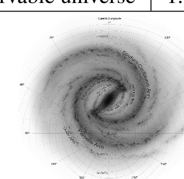


Out[2]=

```
In[3]:= Column@Table[k → N[a_k 157^k], {k, {0, 3, 9, 13, 29, 35, 157, 223, 457}}]

Out[3]= 0 → 0.
        3 → -644982.
        9 → 1.59711 × 10^14
        13 → 5.65477 × 10^18
        29 → 5.42689 × 10^32
        35 → -6.95433 × 10^36
        157 → 4.86366 × 10^66
        223 → -1.94045 × 10^61
        457 → 4.87404 × 10^-16
```

Some sizes (in multiples of the diameter of a Hydrogen atom:

A red blood cell	1.56×10^5
The CN Tower	1.11×10^{13}
The rings of Saturn	5.6×10^{18}
The Milky Way galaxy	1.89×10^{31}
The observable universe	1.76×10^{37}



```
In[4]:= {N[Sum_{k=0}^{457} a_k 157^k], Sum_{k=0}^{457} N[a_k 157^k]}

Out[4]= {-0.0795485, 5.10624 × 10^50}

In[5]:= N@Sin[157]

Out[5]= -0.0795485
```

Do Not Turn Over Until Instructed

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MAASeaway-1810/

Solvable Approximations of the Quantum sl_2 Portfolio

See also [BV]  With Roland van der Veen ωεβ:=http://drorbn.net/mm18/ W.I.P. Warning  Brute Warning

Our Main Theorem (loosely stated). Everything that matters in the quantum sl_2 portfolio can be continuously expressed in terms of docile perturbed Gaussians using solvable approximations. ○

Our Main Points.

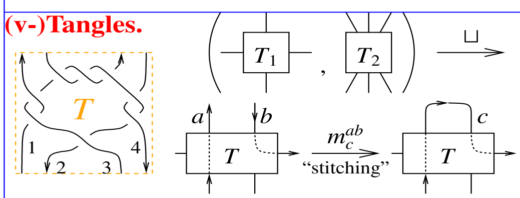
- What's the "quantum sl_2 portfolio"?
- What in it "matters" and why? (the most important question)
- What's "solvable approximation"? What's "continuously"?
- What are "docile perturbed Gaussians"?
- Why do they matter? (2^{nd} most important)
- How proven? (docile)
- How implemented? (sacred; the work of unsung heroes)
- Some context and background.
- What's next?

The quantum sl_2 Portfolio includes a classical universal enveloping algebra CU, its quantization QU, their tensor powers $CU^{\otimes S}$ and $QU^{\otimes S}$ with the "tensor operations" $\otimes$, their products m_k^{ij}, coproducts Δ^i_{jk} and antipodes S_i, their Cartan automophisms $C\theta\colon CU \to CU$ and $Q\theta\colon QU \to QU$, the "dequantizators" $A\mathbb{D}\colon QU \to CU$ and $S\mathbb{D}\colon QU \to CU$, and most importantly, the R-matrix R and the Drinfel'd element s. All this in any PBW basis, and change of basis maps are included.

$$R, s \in \{QU^{\otimes S}\} \xrightarrow{A\mathbb{D},\, S\mathbb{D}} \{CU^{\otimes S}\}$$

(each side with a loop labelled $\otimes, m_k^{ij}, \Delta^i_{jk}, S_i, \theta$)

(v-)Tangles.

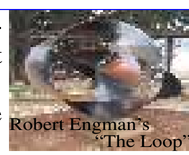


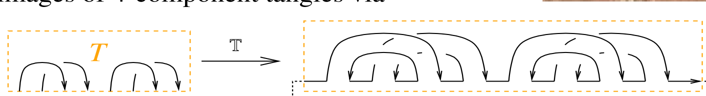
(meta-associativity: $m^{ab}_x /\!\!/ m^{xc}_y = m^{bc}_x /\!\!/ m^{ax}_y$) (tangles are generated by ⤯ and ⤰)

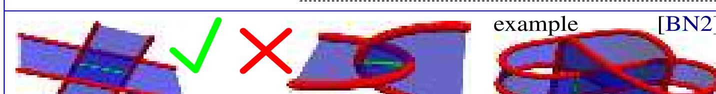
Strand doubling: $a \to$ Δ^a_{bc} b, c Strand reversal: a S_a a

Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.

Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via


Robert Engman's "The Loop"



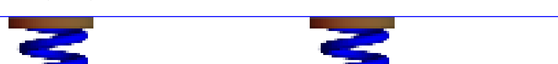


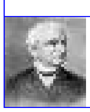
a ribbon singularity a clasp singularity

A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

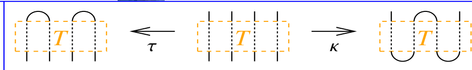
Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

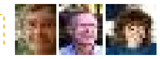




"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified) www.katlas.org 



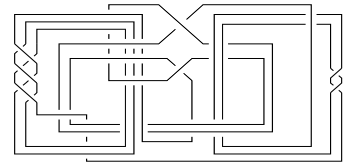

Gompf, Scharlemann, Thompson [GST]

$U \in \mathcal{T}_n$ $\quad 1 \in \mathcal{A}_n$

$\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n}$ with $\mathcal{R} := \kappa(\tau^{-1}(1))$

ribbon $K \in \mathcal{T}_1$ $\quad z(K) \in \mathcal{R} \subseteq \mathcal{A}_1$

Faster is better, leaner is meaner!




The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\}$ with $R_S := \mathbb{Z}(\{t_a : a \in S\})$:

$$\left({}_a\!\nearrow_b, {}_b\!\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm 1} \\ b & 0 & t_a^{\pm 1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a, t_b \to t_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}\right)$$

(Roland: "add to A the product of column b and row a, divide by $(1 - A_{ab})$, delete column b and row a".)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow[\substack{\mu = T_a - 1 \\ \nu = \alpha - \sigma_a \\ T_a \mapsto T_b T_c}]{q\Delta^a_{bc}} \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu \\ c & (T_c-1)\nu/\mu & (\alpha - \sigma_a T_a - \nu T_c)/\mu & (T_c-1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array}\right)$$

$dS^a \Big\downarrow T_a \to T_a^{-1}$

$$\left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha \end{array}\right)$$

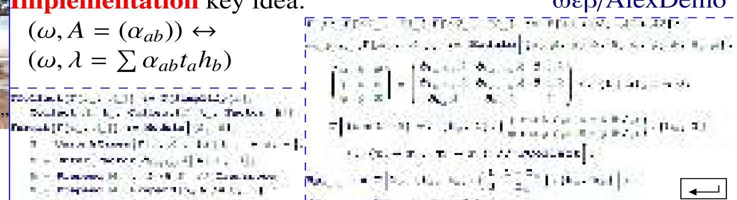
Where σ assigns to every $a \in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left({}_a\!\nearrow_b, {}_b\!\nwarrow_a\right) = (a \to 1, b \to t_a^{\pm 1})$, $\sigma(T_1 \sqcup T_2) = \sigma(T_1) \sqcup \sigma(T_2)$, and $\sigma /\!\!/ m^{ab}_c = (\sigma \setminus \{a, b\}) \cup (c \to \sigma_a\sigma_b)|_{t_a, t_b \to t_c}$.

Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more). 

Implementation key idea: ωεβ/AlexDemo

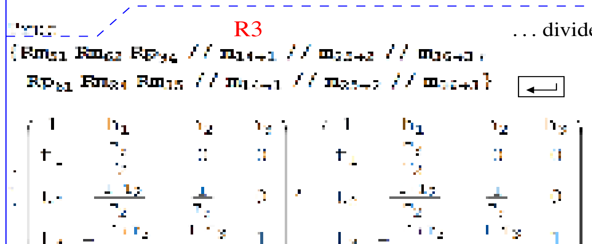
$(\omega, A = (\alpha_{ab})) \leftrightarrow$

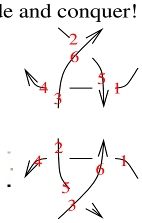
$(\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

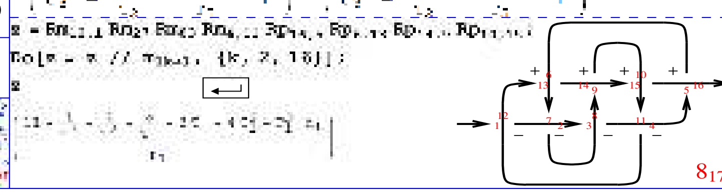


Meta-Associativity **Runs.**

R3 . . . divide and conquer!







Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/

The Dogma is Wrong

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. Preliminary writeup [BV1], fuller writeup [BV2]. More at ωεβ/talks.

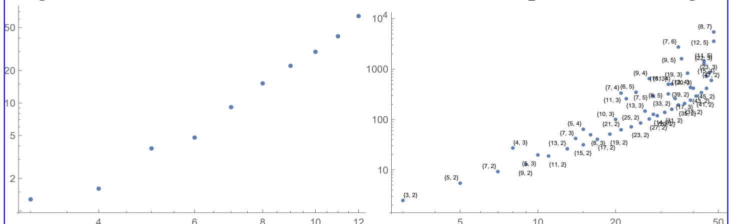
Happy Birthday Anton!

ωεβ:=http://drorbn.net/ld17/

Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: "quantize and use representation theory". We present an alternative and better procedure: "centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra". While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

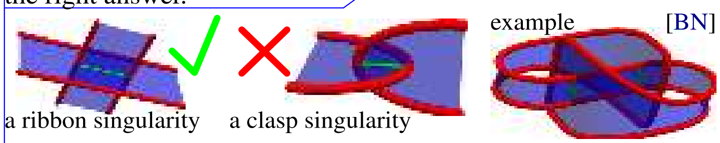
Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:

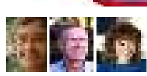


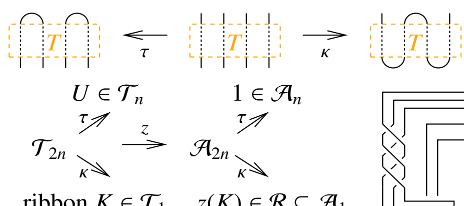
Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

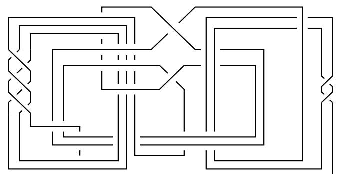
Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg \rho_1^+ \leq 2g - 1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Ribbon Knots.









[Vo]: Works for Alexander!  with $\mathcal{R} := \kappa(\tau^{-1}(1))$

$A^+ = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$

$\rho_1^+ = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 + 108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$

Faster is better, leaner is meaner!

Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of $poly$ in $S(\oplus_i \mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to $specs$. E.g.,

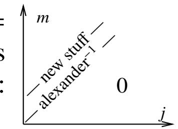
$$\mathbb{O}\left(a_1^3 y_1 a_2 e^{y_3} x_3^9 \mid x_3 a_1 \otimes y_1 y_3 a_2\right) = x^9 a^3 \otimes y e^y a \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

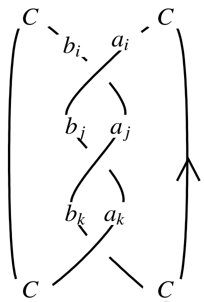
$$\left.\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K) d^j \hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot \omega(K)(e^\hbar) = 1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1})\omega(K)(q^d)} \left(1 + \sum_{k=1}^\infty \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i \otimes b_i \in U \otimes U \quad \text{and} \quad C \in U,$$

form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

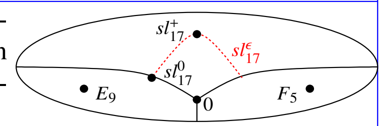
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

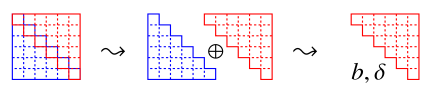
The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

$$m^{ij}_k \circlearrowright \{\mathcal{F}_S\} \xrightarrow{\mathbb{E}} \{U^{\otimes S}\} \circlearrowleft m^{ij}_k$$

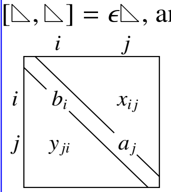
The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^\epsilon/(\epsilon^{k+1} = 0)$.



Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon ◺$, and $[◹, ◺] = ◺ + \epsilon ◹$. In detail, it is



$[x_{ij}, x_{kl}] = \delta_{jk}x_{il} - \delta_{li}x_{kj}$ $\quad [y_{ij}, y_{kl}] = \epsilon\delta_{jk}y_{il} - \epsilon\delta_{li}y_{kj}$

$[x_{ij}, y_{kl}] = \delta_{jk}(\epsilon\delta_{j<k}x_{il} + \delta_{il}(b_i + \epsilon a_i)/2 + \delta_{i>l}y_{il})$
$\qquad -\delta_{li}(\epsilon\delta_{k<j}x_{kj} + \delta_{kj}(b_j + \epsilon a_j)/2 + \delta_{k>j}y_{kj})$

$[a_i, x_{jk}] = (\delta_{ij} - \delta_{ik})x_{jk}$ $\quad [b_i, x_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})x_{jk}$

$[a_i, y_{jk}] = (\delta_{ij} - \delta_{ik})y_{jk}$ $\quad [b_i, y_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})y_{jk}$

The Main sl_2 Theorem. Let $\mathfrak{g}^\epsilon = \langle t, y, a, x\rangle/([t, \cdot] = 0, [a, x] = x, [a, y] = -y, [x, y] = t - 2\epsilon a)$ and let $\mathfrak{g}_k = \mathfrak{g}^\epsilon/(\epsilon^{k+1} = 0)$. The $\mathfrak{g}_k$-invariant of any S-component tangle K can be written in the form $Z(K) = \mathbb{O}\left(\omega \mathbb{e}^{L+Q+P} : \bigotimes_{i\in S} y_i a_i x_i\right)$, where ω is a scalar (a rational function in the variables t_i and their exponentials $T_i := \mathbb{e}^{t_i}$), where $L = \sum l_{ij} t_i a_j$ is a quadratic in t_i and a_j with integer coefficients l_{ij}, where $Q = \sum q_{ij} y_i x_j$ is a quadratic in the variables y_i and x_j with scalar coefficients q_{ij}, and where P is a polynomial in $\{\epsilon, y_i, a_i, x_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\{y_i, \sqrt{a_i}, x_i\}$. Furthermore, after setting $t_i = t$ and $T_i = T$ for all i, the invariant $Z(K)$ is poly-time computable.

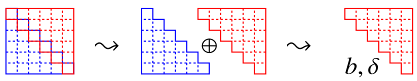
Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1708/

What else can you do with solvable approximations?

Thanks for the invitation!

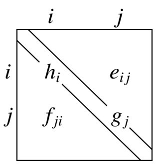
Abstract. Recently, Roland van der Veen and myself found that there are sequences of solvable Lie algebras "converging" to any given semi-simple Lie algebra (such as sl_2 or sl_3 or $E8$). Certain computations are much easier in solvable Lie algebras; in particular, using solvable approximations we can compute in polynomial time certain projections (originally discussed by Rozansky) of the knot invariants arising from the Chern-Simons-Witten topological quantum field theory. This provides us with the first strong knot invariants that are computable for truly large knots.
But sl_2 and sl_3 and similar algebras occur in physics (and in mathematics) in many other places, beyond the Chern-Simons-Witten theory. Do solvable approximations have further applications?

Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b\colon ◹ \otimes ◹ \to ◹$
$b(◺) \rightsquigarrow \delta\colon ◹ \to ◹ \otimes ◹$

Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon◺$, and $[◹, ◺] = ◺ + \epsilon◹$. In detail, it is



$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj} \qquad [f_{ij}, f_{kl}] = \epsilon\delta_{jk}f_{il} - \epsilon\delta_{li}f_{kj}$$
$$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon\delta_{j<k}e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l}f_{il}) - \delta_{li}(\epsilon\delta_{k<j}e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j}f_{kj})$$
$$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik})e_{jk} \qquad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})e_{jk}$$
$$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik})f_{jk} \qquad [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})f_{jk}$$

Solvable Approximation. At $\epsilon = 1$ and modulo $h = g$, the above is just gl_n. By rescaling at $\epsilon \neq 0$, gl_n^ϵ is independent of ϵ. We let gl_n^k be gl_n^ϵ regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1} = 0$. It is the "$k$-smidgen solvable approximation" of gl_n!
Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_3 := [\mathfrak{g}_2, \mathfrak{g}_2]$, …, $\mathfrak{g}_d = 0$. Equivalently, if it is a subalgebra of some large-size ◹ algebra.
Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

Why are "solvable algebras" any good? Contrary to common beliefs, computations in semi-simple Lie algebras are just awful:

```
MatrixExp[( a b
            c d )] // FullSimplify // MatrixForm
```
Enter

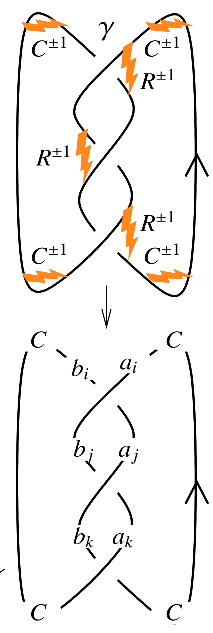
Yet in solvable algebras, exponentiation is fine and even BCH, $z = \log(e^x e^y)$, is bearable:

```
MatrixExp[( a b
            0 c )] // MatrixForm
```

```
MatrixExp[( a1 b1
            0  c1 )].MatrixExp[( a2 b2
                                 0  c2 )] //
   MatrixLog // PowerExpand // Simplify //
  MatrixForm
```
Enter

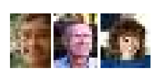
Question. What else can you do with solvable approximation? Chern-Simons-Witten theory is often "solved" using ideas from conformal field theory and using quantization of various moduli spaces. Does it make sense to use solvable approximation there too? Elsewhere in physics? Elsewhere in mathematics?
See Also. Talks at George Washington University [ωεβ/gwu], Indiana [ωεβ/ind], and Les Diablerets [ωεβ/ld], and a University of Toronto "Algebraic Knot Theory" class [ωεβ/akt].

Chern-Simons-Witten. Given a knot $\gamma(t)$ in $\mathbb{R}^3$ and a metrized Lie algebra $\mathfrak{g}$, set $Z(\gamma) :=$
$$\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\, e^{ik\,cs(A)} PExp_\gamma(A),$$
where $cs(A) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(AdA + \frac{2}{3}A^3\right)$ and
$$PExp_\gamma(A) := \prod_0^1 \exp(\gamma^* A) \in \mathcal{U} = \hat{\mathcal{U}}(\mathfrak{g}),$$
and $\mathcal{U}(\mathfrak{g}) := \langle\text{words in } \mathfrak{g}\rangle/(xy - yx = [x, y])$.
In a favourable gauge, one may hope that this computation will localize near the crossings and the bends, and all will depend on just two quantities,
$$R = \sum a_i \otimes b_i \in \mathcal{U} \otimes \mathcal{U} \quad \text{and} \quad C \in \mathcal{U}.$$
This was never done formally, yet R and C can be "guessed" and all "quantum knot invariants" arise in this way. So for the trefoil,
$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

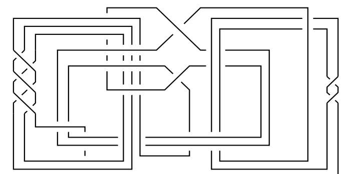


But Z lives in $\mathcal{U}$, a complicated space. How do you extract information out of it?
Solution 1, Representation Theory. Choose a finite dimensional representation ρ of $\mathfrak{g}$ in some vector space V. By luck and the wisdom of Drinfel'd and Jimbo, $\rho(R) \in V^* \otimes V^* \otimes V \otimes V$ and $\rho(C) \in V^* \otimes V$ are computable, so Z is computable too. But in exponential time!

Ribbon=Slice?



Gompf, Scharlemann, Thompson



Solution 2, Solvable Approximation. Work directly in $\hat{\mathcal{U}}(\mathfrak{g}_k)$, where $\mathfrak{g}_k = sl_2^k$ (or a similar algebra); everything is expressible using low-degree polynomials in a small number of variables, hence everything is poly-time computable!

Example 0. Take $\mathfrak{g}_0 = sl_2^0 = \mathbb{Q}\langle h, e, l, f\rangle$, with h central and $[f, l] = f$, $[e, l] = -e$, $[e, f] = h$. In it, using normal orderings,
$$R = \mathbb{O}\left(\exp\left(hl + \frac{e^h - 1}{h}ef\right) \mid e \otimes lf\right), \quad \text{and,}$$
$$\mathbb{O}\left(e^{\delta ef} \mid fe\right) = \mathbb{O}\left(\nu e^{\nu\delta ef} \mid ef\right) \quad \text{with } \nu = (1 + h\delta)^{-1}.$$

Example 1. Take $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$ and $\mathfrak{g}_1 = sl_2^1 = R\langle h, e, l, f\rangle$, with h central and $[f, l] = f$, $[e, l] = -e$, $[e, f] = h - 2\epsilon l$. In it,
$$\mathbb{O}\left(e^{\delta ef} \mid fe\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\delta\Lambda/2)e^{\nu\delta ef} \mid elf\right), \quad \text{where } \Lambda \text{ is}$$
$$4\nu^3\delta^2e^2f^2 + 3\nu^3\delta^3he^2f^2 + 8\nu^2\delta ef + 4\nu^2\delta^2hef + 4\nu\delta elf - 2\nu\delta h + 4l.$$
Fact. Setting $h_i = h$ (for all i) and $t = e^h$, the $\mathfrak{g}_1$ invariant of any tangle T can be written in the form
$$Z_{\mathfrak{g}_1}(T) = \mathbb{O}\left(\omega^{-1}e^{hL + \omega^{-1}Q}(1 + \epsilon\omega^{-4}P) \mid \bigotimes_i e_i l_i f_i\right),$$
where L is linear, Q quadratic, and P quartic in the $\{e_i, l_i, f_i\}$ with ω and all coefficients polynomials in t. Furthermore, everything is poly-time computable.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/McGill-1702/