Dror Bar-Natan — Handout Portfolio

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The Strongest Genuinely Computable Knot Invariant in 2024

Abstract. "Genuinely computable" means we have computed it for random knots with over 300 crossings. "Strongest" means it separates prime knots with up to 15 crossings better than the less-computable HOMFLY-PT and Khovanov homology taken together. And hey, it's also meaningful and fun.


van der Veen

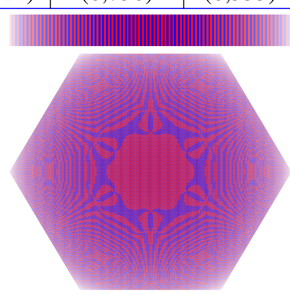
Continues Rozansky, Garoufalidis, Kricker, and Ohtsuki, joint with van der Veen.

Acknowledgement. This work was supported by NSERC grants RGPIN-2018-04350 and RGPIN-2025-06718 and by the Chu Family Foundation (NYC).

Strongest. Testing $\Theta = (\Delta, \theta)$ on prime knots up to mirrors and reversals, counting the number of distinct values (with deficits in parenthesis): (ρ_1: [Ro1, Ro2, Ro3, Ov, BV1])

	knots	(H, Kh)	(Δ, ρ_1)	$\Theta = (\Delta, \theta)$	together
reign		2005-22	2022-24	2024-	
xing ≤ 10	249	248 (1)	249 (0)	249 (0)	249 (0)
xing ≤ 11	801	771 (30)	787 (14)	798 (3)	798 (3)
xing ≤ 12	2,977	(214)	(95)	(19)	(18)
xing ≤ 13	12,965	(1,771)	(959)	(194)	(185)
xing ≤ 14	59,937	(10,788)	(6,253)	(1,118)	(1,062)
xing ≤ 15	313,230	(70,245)	(42,914)	(6,758)	(6,555)

Genuinely Computable. Here's Θ on a random 300 crossing knot (from [DHOEBL]). For almost every other invariant, that's science fiction.

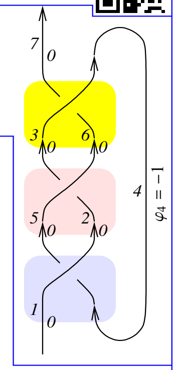


Fun. There's so much more to see in 2D pictures than in 1D ones! Yet almost nothing of the patterns you see we know how to prove. We'll have fun with that over the next few years. Would you join?

Meaningful. θ gives a genus bound (unproven yet with confidence). We hope (with reason) it says something about ribbon knots.

Conventions. T, T_1, and T_2 are indeterminates and $T_3 := T_1T_2$.

Preparation. Draw an n-crossing knot K as a diagram D as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \ldots, 2n+1\}$ and with rotation numbers φ_k.



Model T Traffic Rules. Cars always drive forward. When a car crosses over a sign-s bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0$. At the very end, cars fall off and disappear. On various edges *traffic counters* are placed. See also [Jo, LTW].

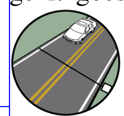


image credits: diamondtraffic.com


$p = 1 - T^s$

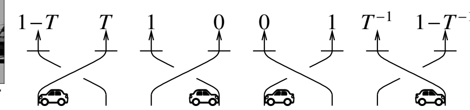
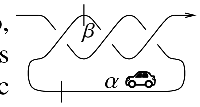


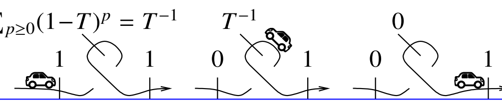

image credits: Dall-E

Definition. The *traffic function* $G = (g_{\alpha\beta})$ (also, the *Green function* or the *two-point function*) is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point). There are also model-T_ν traffic functions $G_\nu = (g_{\nu\alpha\beta})$ for $\nu = 1, 2, 3$.



Example.

$\sum_{p\geq 0}(1-T)^p = T^{-1}$



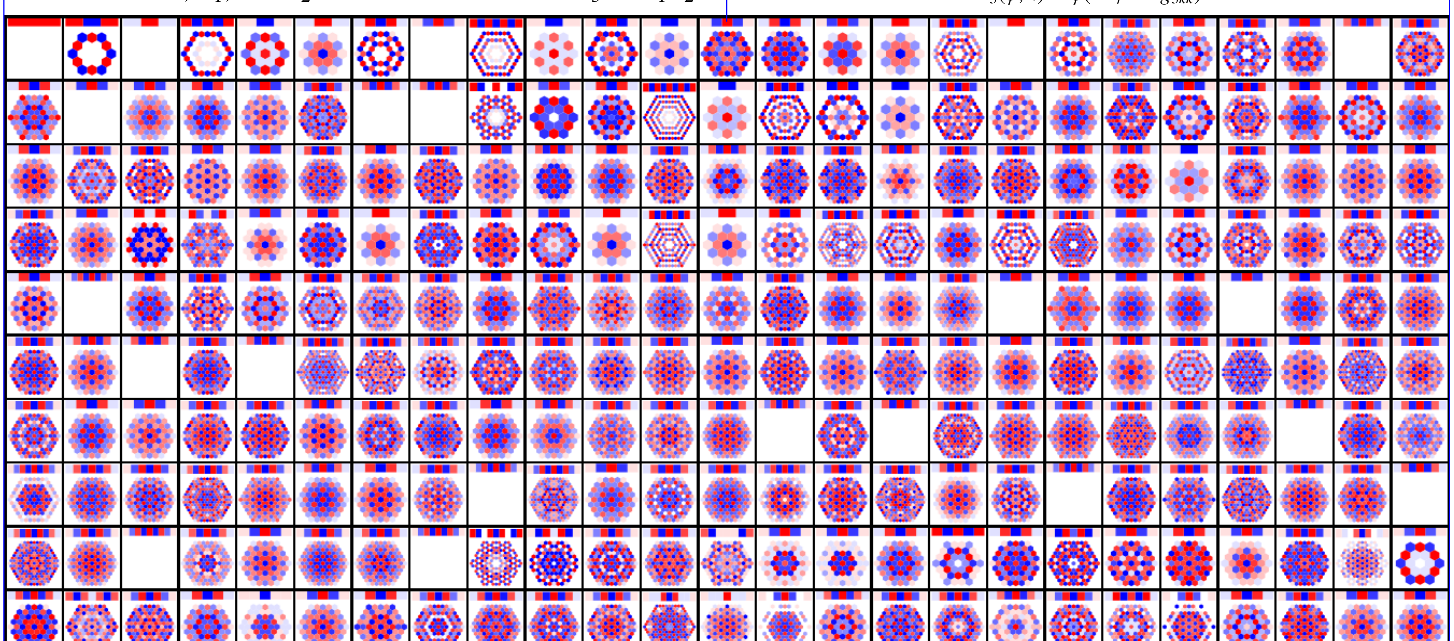
$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Don't Look.

$$\begin{aligned} F_1(c) = s\Big[&1/2 - g_{3ii} + T_2^s g_{1ii}g_{2ji} - T_2^s g_{3jj}g_{2ji} - (T_2^s-1)g_{3ii}g_{2ji} \\ &+(T_3^s-1)g_{2ji}g_{3ji} - g_{1ii}g_{2jj} + 2g_{3ii}g_{2jj} + g_{1ii}g_{3jj} - g_{2ii}g_{3jj}\Big] \\ &+ \frac{s}{T_2^s-1}\Big[(T_1^s-1)T_2^s\left(g_{3jj}g_{1ji} - g_{2jj}g_{1ji} + T_2^s g_{1ji}g_{2ji}\right) \\ &+ (T_3^s-1)\left(g_{3ji} - T_2^s g_{1ii}g_{3ji} + g_{2ij}g_{3ji} + (T_2^s-2)g_{2jj}g_{3ji}\right) \\ &-(T_1^s-1)(T_2^s+1)(T_3^s-1)g_{1ji}g_{3ji}\Big] \end{aligned}$$

$$F_2(c_0, c_1) = \frac{s_1(T_1^{s_0}-1)(T_3^{s_1}-1)g_{1j_1i_0}g_{3j_0i_1}}{T_2^{s_1}-1}\left(T_2^{s_0}g_{2i_1i_0} + g_{2j_1j_0} - T_2^{s_0}g_{2j_1i_0} - g_{2i_1j_0}\right)$$

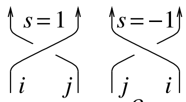
$$F_3(\varphi, k) = \varphi(-1/2 + g_{3kk})$$



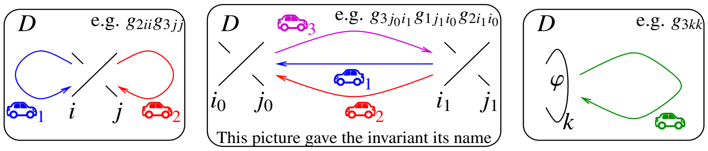
2025/06/04@23:23

Video and more at http://www.math.toronto.edu/~drorbn/Talks/PhuQuoc-2506.

Theorem [BV3]. With $c=(s,i,j)$, $c_0=(s_0,i_0,j_0)$, and $c_1=(s_1,i_1,j_1)$ denoting crossings, there is a quadratic $F_1(c)\in\mathbb{Q}(T_\nu)[g_{\nu\alpha\beta}:\alpha,\beta\in\{i,j\}]$, a cubic $F_2(c_0,c_1)\in\mathbb{Q}(T_\nu)[g_{\nu\alpha\beta}:\alpha,\beta\in\{i_0,j_0,i_1,j_1\}]$, and a linear $F_3(\varphi,k)$ such that θ is a knot invariant:

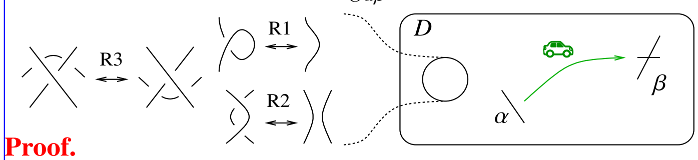


$$\theta(D) := \underbrace{\Delta_1\Delta_2\Delta_3}_{\substack{\text{normalization,}\\ \text{see later}}}\left(\sum_c F_1(c)+\sum_{c_0,c_1}F_2(c_0,c_1)+\sum_k F_3(\varphi_k,k)\right),$$

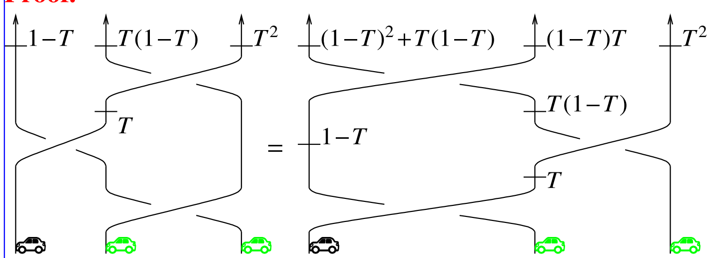


If these pictures remind you of Feynman diagrams, it's because they are Feynman diagrams [BN2].

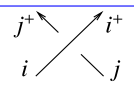
Lemma 1. The traffic function $g_{\alpha\beta}$ is a "relative invariant":



Proof.



Lemma 2. With $k^+ := k+1$, the "g-rules" hold near a crossing $c=(s,i,j)$:



$$g_{j\beta}=g_{j^+\beta}+\delta_{j\beta}\qquad g_{i\beta}=T^sg_{i^+\beta}+(1-T^s)g_{j^+\beta}+\delta_{i\beta}\qquad g_{2n^+,\beta}=\delta_{2n^+,\beta}$$
$$g_{\alpha i^+}=T^sg_{\alpha i}+\delta_{\alpha i^+}\qquad g_{\alpha j^+}=g_{\alpha j}+(1-T^s)g_{\alpha i}+\delta_{\alpha j^+}\qquad g_{\alpha,1}=\delta_{\alpha,1}$$

Corollary 1. G is easily computable, for $AG=I$ $(=GA)$, with A the $(2n+1)\times(2n+1)$ identity matrix with additional contributions:

$$c=(s,i,j)\mapsto\begin{array}{c|cc} A & \text{col } i^+ & \text{col } j^+\\ \hline \text{row } i & -T^s & T^s-1\\ \text{row } j & 0 & -1\end{array}$$

For the trefoil example, we have:

$$A=\begin{pmatrix}1&-T&0&0&T-1&0&0\\0&1&-1&0&0&0&0\\0&0&1&-T&0&0&T-1\\0&0&0&1&-1&0&0\\0&0&T-1&0&1&-T&0\\0&0&0&0&0&1&-1\\0&0&0&0&0&0&1\end{pmatrix},$$

$$G=\begin{pmatrix}1&T&1&T&1&T&1\\0&1&\frac{1}{T^2-T+1}&\frac{T}{T^2-T+1}&\frac{T}{T^2-T+1}&\frac{T^2}{T^2-T+1}&1\\0&0&\frac{1}{T^2-T+1}&\frac{T}{T^2-T+1}&\frac{T}{T^2-T+1}&\frac{T^2}{T^2-T+1}&1\\0&0&\frac{1-T}{T^2-T+1}&\frac{1}{T^2-T+1}&\frac{1}{T^2-T+1}&\frac{T}{T^2-T+1}&1\\0&0&\frac{1-T}{T^2-T+1}&-\frac{(T-1)T}{T^2-T+1}&\frac{1}{T^2-T+1}&\frac{T}{T^2-T+1}&1\\0&0&0&0&0&1&1\\0&0&0&0&0&0&1\end{pmatrix}$$

Note. The Alexander polynomial Δ is given by

$$\Delta=T^{(-\varphi-w)/2}\det(A),\qquad\text{with }\varphi=\textstyle\sum_k\varphi_k,\ w=\sum_c s.$$

We also set $\Delta_\nu:=\Delta(T_\nu)$ for $\nu=1,2,3$.

Questions, Conjectures, Expectations, Dreams.

Question 1. What's the relationship between Θ and the Garoufalidis-Kashaev invariants [GK, GL]?

Conjecture 2. On classical (non-virtual) knots, θ always has hexagonal (D_6) symmetry.

Conjecture 3. θ is the ϵ^1 contribution to the "solvable approximation" of the sl_3 universal invariant, obtained by running the quantization machinery on the double $\mathcal{D}(\mathfrak{b},b,\epsilon\delta)$, where $\mathfrak{b}$ is the Borel subalgebra of sl_3, b is the bracket of $\mathfrak{b}$, and δ the cobracket. See [BV2, BN1, Sch]

Conjecture 4. θ is equal to the "two-loop contribution to the Kontsevich Integral", as studied by Garoufalidis, Rozansky, Kricker, and in great detail by Ohtsuki [GR, Ro1, Ro2, Ro3, Kr, Oh].

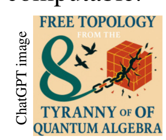
Fact 5. θ has a perturbed Gaussian integral formula, with integration carried out over over a space $6E$, consisting of 6 copies of the space of edges of a knot diagram D. See [BN2].

Conjecture 6. For any knot K, its genus $g(K)$ is bounded by the T_1-degree of θ: $2g(K)\geq\deg_{T_1}\theta(K)$.

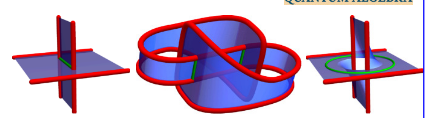
Conjecture 7. $\theta(K)$ has another perturbed Gaussian integral formula, with integration carried out over over the space $6H_1$, consisting of 6 copies of $H_1(\Sigma)$, where Σ is a Seifert surface for K.

Expectation 8. There are many further invariants like θ, given by Green function formulas and/or Gaussian integration formulas. One or two of them may be stronger than θ and as computable.

Dream 9. These invariants can be explained by something less foreign than semisimple Lie algebras.



Dream 10. With Conjecture 7 in mind, θ will have something to say about ribbon knots.



References.

[BN1] D. Bar-Natan, *Everything around sl^ϵ_{2+} is DoPeGDO. So what?,* talk in Da Nang, May 2019. Handout and video at ωεβ/DPG.

[BN2] —, *Knot Invariants from Finite Dimensional Integration,* talks in Beijing (July 2024, ωεβ/icbs24) and in Geneva (August 2024, ωεβ/ge24).

[BV1] —, R. van der Veen, *A Perturbed-Alexander Invariant,* Quantum Topology **15** (2024) 449–472, ωεβ/APAI.

[BV2] —, —, *Perturbed Gaussian Generating Functions for Universal Knot Invariants,* arXiv:2109.02057.

[BV3] —, —, *A Very Fast, Very Strong, Topologically Meaningful and Fun Knot Invariant,* in preparation, draft at ωεβ/Theta.

[DHOEBL] N. Dunfield, A. Hirani, M. Obeidin, A. Ehrenberg, S. Bhattacharyya, D. Lei, and others, *Random Knots: A Preliminary Report,* lecture notes at ωεβ/DHOEBL. Also a data file at ωεβ/DD.

[GK] S. Garoufalidis, R. Kashaev, *Multivariable Knot Polynomials from Braided Hopf Algebras with Automorphisms,* arXiv:2311.11528.

[GL] —, S. Y. Li, *Patterns of the V_2-polynomial of knots,* arXiv:2409.03557.

[GR] —, L. Rozansky, *The Loop Expansion of the Kontsevich Integral, the Null-Move, and S-Equivalence,* arXiv:math.GT/0003187.

[Jo] V. F. R. Jones, *Hecke Algebra Representations of Braid Groups and Link Polynomials,* Annals Math., **126** (1987) 335-388.

[Kr] A. Kricker, *The Lines of the Kontsevich Integral and Rozansky's Rationality Conjecture,* arXiv:math/0005284.

[LTW] X-S. Lin, F. Tian, Z. Wang, *Burau Representation and Random Walk on String Links,* Pac. J. Math., **182-2** (1998) 289–302, arXiv:q-alg/9605023.

[Oh] T. Ohtsuki, *On the 2–loop Polynomial of Knots,* Geom. Top. **11** (2007) 1357–1475.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, Aug. 2013, ωεβ/Ov.

[Ro1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

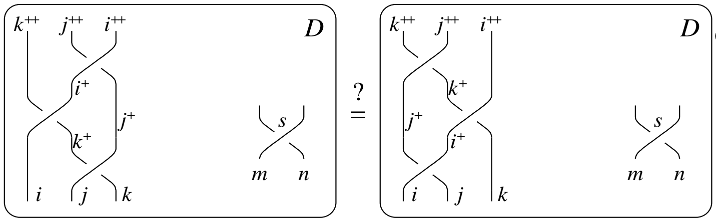
[Ro2] —, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] —, *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Sch] S. Schaveling, *Expansions of Quantum Group Invariants,* Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/PhuQuoc-2506`.

Corollary 2. Proving invariance is easy:



Invariance under R3

This is Theta.nb of http://drorbn.net/v25/ap.

```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

```
T3 = T1 T2;
```

```
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```

```
F1[{s_, i_, j_}] =
  CF[
   s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj -
       (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
       g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
       ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
          (T3^s - 1) g3ji
            (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji +
              (T2^s - 2) g2jj + g2ij)) / (T2^s - 1))];
```

```
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
  CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1
    ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
```

```
F3[φ_, k_] = -φ/2 + φ g3kk;
```

```
δi_,j_ := If[i === j, 1, 0];
gRs_,i_,j_ := {
   gν_jβ_ ⧴ gνj+β + δjβ,
   gν_iβ_ ⧴ Tν^s gνi+β + (1 - Tν^s) gνj+β + δiβ,
   gν_α_i+ ⧴ Tν^s gναi + δαi+,
   gν_α_j+ ⧴ gναj + (1 - Tν^s) gναi + δαj+
  }
```

```
DSum[Cs___] := Sum[F1[c], {c, {Cs}}] +
   Sum[F2[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
lhs = DSum[{1, j, k}, {1, i, k+}, {1, i+, j+},
     {s, m, n}] //. gR1,j,k ⋃ gR1,i,k+ ⋃ gR1,i+,j+;
rhs = DSum[{1, i, j}, {1, i+, k}, {1, j+, k+},
     {s, m, n}] //. gR1,i,j ⋃ gR1,i+,k ⋃ gR1,j+,k+;
Simplify[lhs == rhs]
```

```
True
```

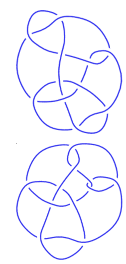
The Main Program

```
Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, θ},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} ⧴
     (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                         0    -1     ))];
   Δ = T^((-Total[φ] - Total[Cs[[All,1]]])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] :=
    Factor[ℰ /. gν_,α_,β_ ⧴ (G[[α, β]] /. T → Tν)];
   θ = ev[Sum_{k=1}^{n} F1[Cs[[k]]]];
   θ += ev[Sum_{k1=1}^{n} Sum_{k2=1}^{n} F2[Cs[[k1]], Cs[[k2]]]];
   θ += ev[Sum_{k=1}^{2 n} F3[φ[[k]], k]];
   Factor@
    {Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}];
```

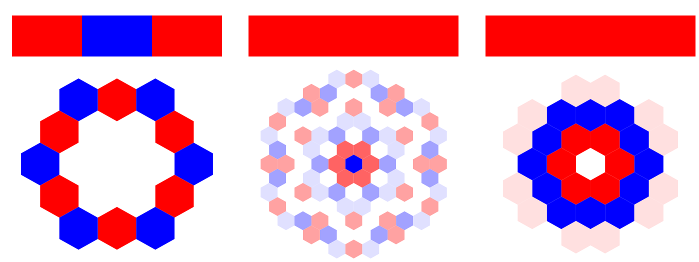
The Trefoil, Conway, and Kinoshita-Terasaka

```
Θ[Knot[3, 1]] // Expand
```

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2^2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}$$



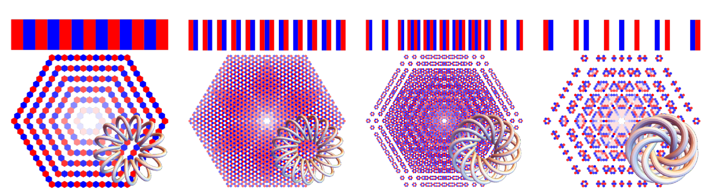
```
GraphicsRow[PolyPlot[Θ[Knot[#]]] & /@
  {"3_1", "K11n34", "K11n42"}]
```



(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

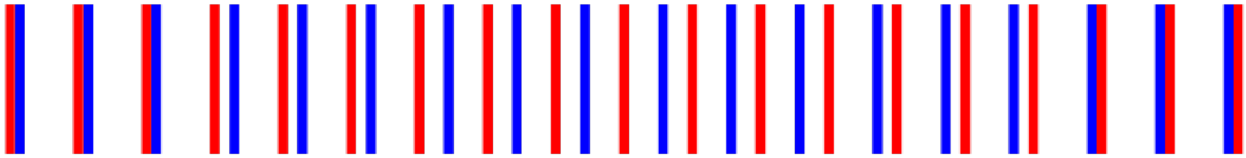
Some Torus Knots

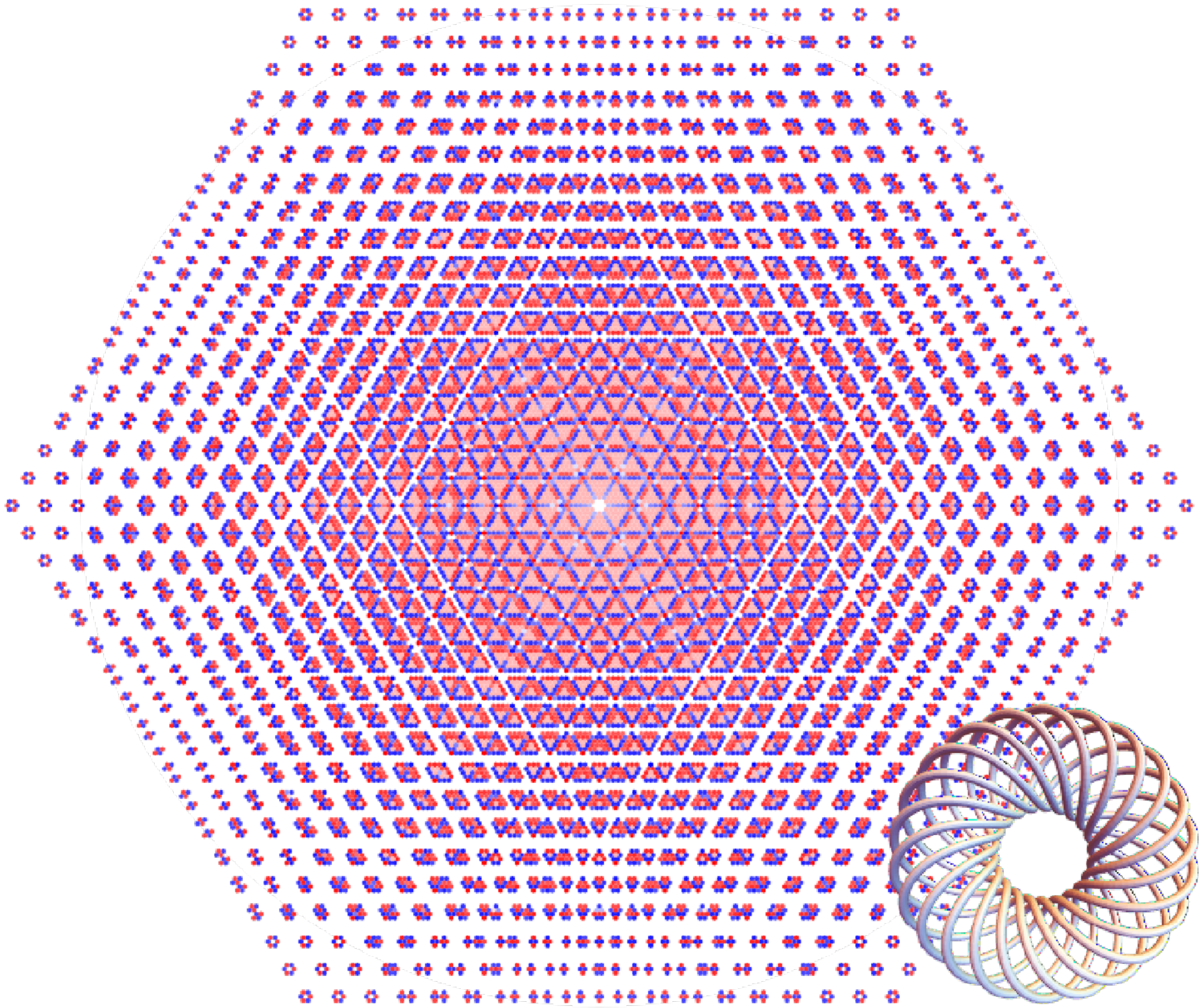
```
GraphicsRow[ImageCompose[
     PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
     TubePlot[TorusKnot @@ #, ImageSize → 240],
     {Right, Bottom}, {Right, Bottom}
   ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```



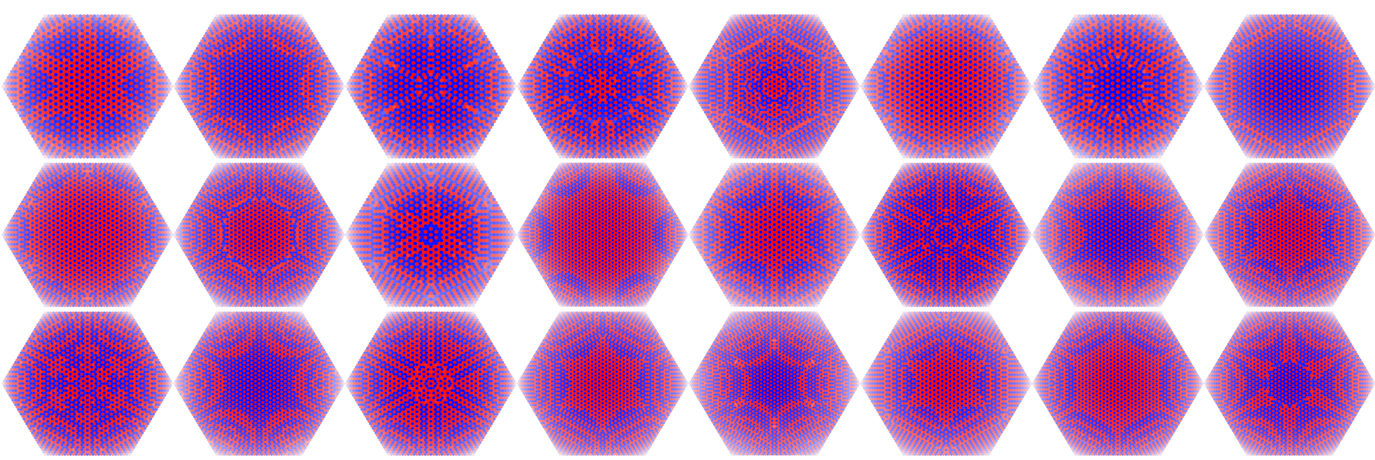
Video and more at http://www.math.toronto.edu/~drorbn/Talks/PhuQuoc-2506.

The 132-crossing torus knot $T_{22/7}$: (many more at ωεβ/TK)





Random knots from [DHOEBL], with 50-73 crossings: (many more at ωεβ/DK)



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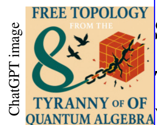
Dror Bar-Natan: Talks: Bonn-2505: ωεβ:=http://drorbn.net/bo25

Knot Invariants from Zero-Dimensional QFT

Abstract. For the purpose of today, an "I-Type Knot Invariant" is a knot invariant computed from a knot diagram by integrating the exponential of a *perturbed Gaussian* Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.


joint with R. van der Veen

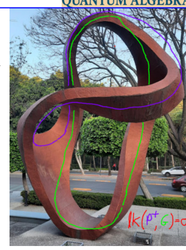
Q. Are there any such things? **A.** Yes.
Q. Are they any good? **A.** They are the strongest we know per CPU cycle, and are excellent in other ways too.
Q. Didn't Witten do that back in 1988 with path integrals?
A. No. His constructions are infinite dimensional and far from rigorous.
Q. But integrals belong in analysis!
A. Ours only use squeaky-clean algebra.



ChatGPT image

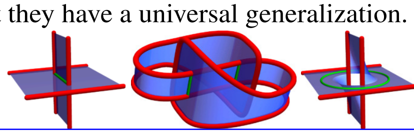
Dreams. Given a knot K with a Seifert surface Σ, we dream that there is a 0D Lagrangian $L_\Sigma\colon 6H_1(\Sigma;\mathbb{R})\to\mathbb{R}$ whose coefficients are (low degree) finite type invariants of graphs representing multiple homology classes, such that

$$Z=\oint_{6H_1(\Sigma;\mathbb{R})}\exp(L_\Sigma)$$

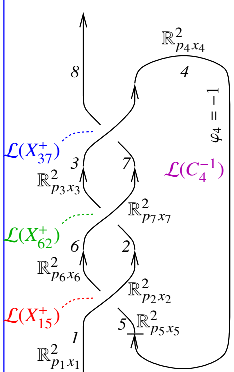


is the invariant θ presented below. We dream the formulas are simple and natural, and that they have a universal generalization.

Sweet. Even sweeter: May say something about ribbon knots!



The $sl_2^{/\epsilon^2}$ Example. With T an indeterminate and with $\epsilon^2=0$:



$$\longrightarrow Z=\oint_{\mathbb{R}^{14}_{p_ix_i}}\mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C_4^{-1})$$

measure on $\mathbb{R}$ is $(2\pi)^{-1/2}\cdot$standard

where $\mathcal{L}(X^s_{ij})=T^{s/2}e^{L(X^s_{ij})}$ and $\mathcal{L}(C^\varphi_i)=T^{\varphi/2}e^{L(C^\varphi_i)}$, and

$$L(X^s_{ij})=x_i(p_{i+1}-p_i)+x_j(p_{j+1}-p_j)+(T^s-1)x_i(p_{i+1}-p_{j+1})+\frac{\epsilon s}{2}\left(x_i(p_i-p_j)\begin{pmatrix}(T^s-1)x_ip_j\\+2(1-x_jp_j)\end{pmatrix}-1\right)$$

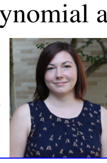
$$L(C^\varphi_i)=x_i(p_{i+1}-p_i)+\epsilon\varphi(1/2-x_ip_i)$$

So $Z=T\oint e^{L(\circledast)}dp_1\dots dp_7dx_1\dots dx_7$, where $L(\circledast)=$

$$\sum_{i=1}^7 x_i(p_{i+1}-p_i)+(T-1)(x_1(p_2-p_6)+x_6(p_7-p_3)+x_3(p_4-p_8))+\frac{\epsilon}{2}\begin{pmatrix}x_1(p_1-p_5)((T-1)x_1p_5+2(1-x_5p_5))-1\\+x_6(p_6-p_2)((T-1)x_6p_2+2(1-x_2p_2))-1\\+x_3(p_3-p_7)((T-1)x_3p_7+2(1-x_7p_7))-1\\+2x_4p_4-1\end{pmatrix},$$

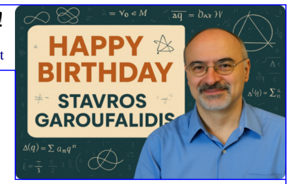
and so $Z=(T-1+T^{-1})^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right)=\Delta^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})\rho_1}{\Delta^2}\right)$. Here Δ is the Alexander polynomial and ρ_1 is the Rozansky-Overbay polynomial [R1, R2, R3, Ov, BV1, BV2].


Rozansky

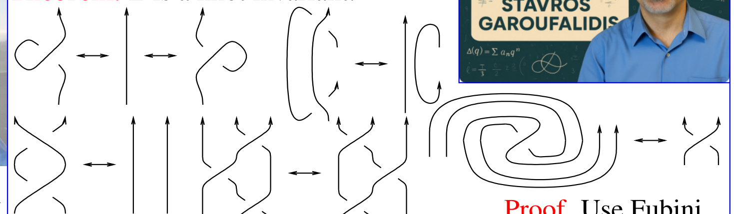

Overbay

Happy birthday, dear Stavros!

ChatGPT image, ωεβ/chat



Theorem. Z is a knot invariant.



Proof. Use Fubini.

(Alternative) Gaussian Integration.


Gauss

Goal. Compute (if convergent) $$\int_{\mathbb{R}^n}dx\exp\left(-\frac12 a^{ij}x_ix_j+V(x)\right).$$

Solution. Set $$\mathcal{Z}_\lambda(x):=\lambda^{-n/2}\int_{\mathbb{R}^n}dy\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right).$$

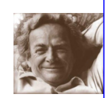
Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x)=(\det A)^{-1/2}\exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y=0$,

$$\frac12 g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x)$$
$$=\frac{\lambda^{-n/2}}{2}\int_{\mathbb{R}^n}dy\,g_{ij}(\partial_{x_i}-\partial_{y_i})(\partial_{x_j}-\partial_{y_j})\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$
$$=\frac{\lambda^{-n/2}}{2\lambda^2}\int_{\mathbb{R}^n}dy\left(g_{ij}a^{ii'}a^{jj'}y_{i'}y_{j'}-\lambda g_{ij}a^{ji}\right)\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$
$$=\frac{\lambda^{-n/2}}{2\lambda^2}\int_{\mathbb{R}^n}dy\left(a^{ij}y_iy_j-\lambda n\right)\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$
$$=\partial_\lambda\mathcal{Z}_\lambda(x).$$

Hence $$(*)\quad \partial_\lambda\mathcal{Z}_\lambda(x)=\frac12 g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x),$$

and therefore $$\mathcal{Z}_\lambda(x)=(\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x).$$

We've just witnessed the birth of "Feynman Diagrams".

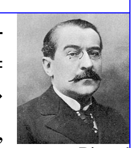

Feynman

Even better. With $Z_\lambda:=\log(\sqrt{\det A}\,\mathcal{Z}_\lambda)$, by a simple substitution into (*), we get the "Synthesis Equation":

$$Z_0=V,\quad \partial_\lambda Z_\lambda=\frac12\sum_{i,j=1}^n g_{ij}\left(\partial_{x_i,x_j}Z_\lambda+(\partial_{x_i}Z_\lambda)(\partial_{x_j}Z_\lambda)\right)=:F(Z_\lambda),$$

an ODE (in λ) whose solution is pure algebra.

Picard Iteration (used to prove the existence and uniqueness of solutions of ODEs). To solve $\partial_\lambda f_\lambda=F(f_\lambda)$ with a given f_0, start with f_0, iterate $f\mapsto f_0+\int_0^\lambda F(f_\lambda)d\lambda$, and seek a fixed point. In our cases, it is always reached after finitely many iterations!


Picard

Definition. $\oint$: The result of this process, ignoring the convergence of the actual integral.

Strong. The pair (Δ,ρ_1) attains 270,316 distinct values on the 313,230 prime knots with up to 15 crossings (a deficit of 42,914), whereas the pair (H = HOMFLYPT polynomial, Kh = Khovanov Homology) attains only 242,985 distinct values on the same knots (a deficit of 70,245). The pair (Δ,θ), discussed later, has a deficit of only 6,758, and the triple (Δ,θ,ρ_2), of only 6,341.
Yet better than (H,Kh) and other Reshetikhin-Turaev-Witten invariants and knot homologies, Δ, ρ_1, ρ_2 and θ can be computed in **polynomial time** (and hence, even for very large knots).
So ugly as the formulas may be (and θ's formulas are uglier), these invariants are possibly **the best we have!**

Acknowledgement. This work was supported by NSERC grants RGPIN-2018-04350 and RGPIN-2025-06718 and by the Chu Family Foundation (NYC).

2025/06/03@00:35

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505.

Implementation (see IType.nb of ωεβ/ap).

```
Once[<< KnotTheory`; << Rot.m];
```

Loading KnotTheory` version
of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/AP/Talks/Bonn-2505
to compute rotation numbers.

```
CF[ω_. ℰ_𝔼] := CF[ω] × CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs, ps, c},
  vs = Cases[ℰ, (x | p | ξ | π | g)__, ∞] ⋃ {ϵ};
  Total[CoefficientRules[Expand[ℰ], vs] /.
    (ps_ → c_) :→ Factor[c] (Times @@ vs^ps)]];
```

Integration using Picard iteration. The core is in yellow and hacks are in pink.

```
𝔼 /: 𝔼[A_] × 𝔼[B_] := 𝔼[A + B];
$π = Identity; (* The Wisdom Projection *)
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) :=
  Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, DDZ,
    FZ, a, b},
   n = Length@vs; L0 = L /. ϵ → 0;
   Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /.
      (p | x)__ → 0, {a, n}, {b, n}];
   If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
   Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
   FixedPoint[(DZ = Table[∂_v Z, {v, vs}];
      DDZ = Table[∂_u DZ, {u, vs}];
      FZ = Sum[G⟦a, b⟧ (DDZ⟦a, b⟧ + DZ⟦a⟧ × DZ⟦b⟧),
        {a, n}, {b, n}] / 2;
      Z = CF[Z0 + ∫_0^λ $π[FZ] ⅆλ]) &, Z];
   PowerExpand@Factor[ω Δ^(-1/2)] ×
    𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

```
∫𝔼[-μ x^2 / 2 + ⅈ ξ x] ⅆ{x}
```

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

```
FofG = ∫𝔼[-μ (x - a)^2 / 2 + ⅈ ξ x] ⅆ{x}
```

$$\frac{\mathbb{E}\left[\frac{i\,(2 a \mu + i \xi)\,\xi}{2\mu}\right]}{\sqrt{\mu}}$$

```
∫FofG 𝔼[-ⅈ ξ x] ⅆ{ξ}
```

$$\mathbb{E}\left[-\frac{1}{2}\,(a-x)^2\,\mu\right]$$

So we've tested and nearly proven the Fourier inversion formula!

Joseph Fourier

```
L = -1/2 {x1, x2}.(a b; b c).{x1, x2} + {ξ1, ξ2}.{x1, x2};
Z12 = ∫𝔼[L] ⅆ{x1, x2}
```

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\,(-b^2+a c)}+\frac{b\,\xi_1\,\xi_2}{b^2-a c}+\frac{a\,\xi_2^2}{2\,(-b^2+a c)}\right]}{\sqrt{-b^2+a c}}$$

Guido Fubini

```
{Z1 = ∫𝔼[L] ⅆ{x1}, Z12 == ∫Z1 ⅆ{x2}}
```

$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2+a c)\,x_2^2}{2a}-\frac{b\,x_2\,\xi_1}{a}+\frac{\xi_1^2}{2a}+x_2\,\xi_2\right]}{\sqrt{a}},\ \text{True}\right\}$$

```
$π = Normal[# + O[ϵ]^13] &; ∫𝔼[-ϕ^2 / 2 + ϵ ϕ^3 / 6] ⅆ{ϕ}
```

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24}+\frac{5\,\epsilon^4}{16}+\frac{1105\,\epsilon^6}{1152}+\frac{565\,\epsilon^8}{128}+\frac{82825\,\epsilon^{10}}{3072}+\frac{19675\,\epsilon^{12}}{96}\right]$$

From https://oeis.org/A226260:

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A226260 Numerators of mass formula for connected vacuum graphs on 2n nodes for a phi^3 field theory.

1, 5, 5, 1105, 565, 82825, 19675, 1282031525, 80727925, 1683480621875, 13209845125, 2239646759308375, 19739117098375, 6320791709083309375, 32468078556378125, 3836267676884504575l875, 281365778405032973125, 2824650747089425586152484375, 776632157034116712734375 (list; graph; refs; listen; history; text; internal format)

The Right-Handed Trefoil.

```
K = Mirror@Knot[3, 1]; Features[K]
```

Features[7, $C_4[-1]$ $X_{1,5}[1]$ $X_{3,7}[1]$ $X_{6,2}[1]$]

```
ℒ[X_{i_,j_}[s_]] := T^(s/2) 𝔼[
   x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) +
    (T^s - 1) x_i (p_{i+1} - p_{j+1}) +
    (ϵ s / 2) ×
     (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)]
ℒ[C_i_[φ_]] := T^(φ/2) 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧]
vs[K_] :=
 Join @@ Table[{p_i, x_i}, {i, Features[K]⟦1⟧}]
{vs[K], ℒ[K]}
```

$$\Big\{\{p_1, x_1, p_2, x_2, p_3, x_3, p_4, x_4, p_5, x_5, p_6, x_6, p_7, x_7\},$$
$$T\,\mathbb{E}\Big[-2\epsilon - p_1 x_1 + \epsilon p_1 x_1 + T p_2 x_1 - \epsilon p_5 x_1 + (1-T)\,p_6 x_1 + \frac{1}{2}(-1+T)\,\epsilon p_1 p_5 x_1^2 + \frac{1}{2}(1-T)\,\epsilon p_5^2 x_1^2 - p_2 x_2 + p_3 x_2 - p_3 x_3 +$$
$$\epsilon p_3 x_3 + T p_4 x_3 - \epsilon p_7 x_3 + (1-T)\,p_8 x_3 + \frac{1}{2}(-1+T)\,\epsilon p_3 p_7 x_3^2 + \frac{1}{2}(1-T)\,\epsilon p_7^2 x_3^2 - p_4 x_4 + \epsilon p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 -$$
$$\epsilon p_1 p_5 x_1 x_5 + \epsilon p_5^2 x_1 x_5 - \epsilon p_2 x_6 + (1-T)\,p_3 x_6 - p_6 x_6 + \epsilon p_6 x_6 + T p_7 x_6 + \epsilon p_2^2 x_2 x_6 - \epsilon p_2 p_6 x_2 x_6 + \frac{1}{2}(1-T)\,\epsilon p_2^2 x_6^2 +$$
$$\frac{1}{2}(-1+T)\,\epsilon p_2 p_6 x_6^2 - p_7 x_7 + p_8 x_7 - \epsilon p_3 p_7 x_3 x_7 + \epsilon p_7^2 x_3 x_7\Big]\Big\}$$

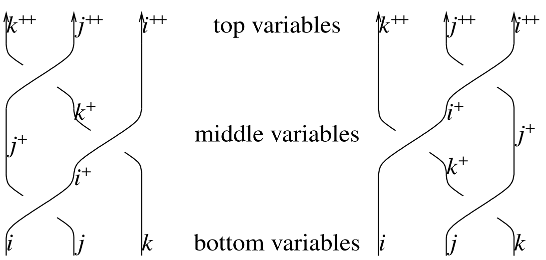
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505.

☺ $\$\pi = \text{Normal}\left[\# + O[\epsilon]^2\right]\ \&;\ \int \mathcal{L}[K]\, d\, \text{vs}[K]$

💻 $$-\frac{i\, T\, \mathbb{E}\left[-\frac{(-1+T)^2\,(1+T^2)\,\epsilon}{(1-T+T^2)^2}\right]}{1-T+T^2}$$

A faster program to compute ρ_1, and more stories about it, are at [BV2].

Invariance Under Reidemeister 3.




Reidemeister

☺ $\text{lhs} = \int (\mathcal{L}\ /@\ (X_{i,j}[1]\, X_{i+1,k}[1]\, X_{j+1,k+1}[1]))$
$\quad d\{p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$
$\text{rhs} = \int (\mathcal{L}\ /@\ (X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1]))$
$\quad d\{x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$
$\text{lhs} === \text{rhs}$

💻 False

Invariance Under Reidemeister 3, Take 2.

☺ $\text{lhs} = \int (\mathcal{L}\ /@\ (X_{i,j}[1]\, X_{i+1,k}[1]\, X_{j+1,k+1}[1]))$
$\quad d\{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$
$\text{rhs} = \int (\mathcal{L}\ /@\ (X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1]))$
$\quad d\{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\};$
$\text{lhs} === \text{rhs}$

💻 True

☺ lhs

💻 Degenerate Q!

Invariance Under Reidemeister 3, Take 3.

☺ $\text{lhs} = \int (\mathbb{E}[i\,\pi_i\, p_i + i\,\pi_j\, p_j + i\,\pi_k\, p_k] \times \mathcal{L}\ /@\ (X_{i,j}[1]\, X_{i+1,k}[1]\, X_{j+1,k+1}[1]))$
$\quad d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$
$\text{rhs} = \int (\mathbb{E}[i\,\pi_i\, p_i + i\,\pi_j\, p_j + i\,\pi_k\, p_k] \times \mathcal{L}\ /@\ (X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1]))$
$\quad d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$
$\text{lhs} == \text{rhs}$

💻 True

☺ lhs

💻 $$T^{3/2}\,\mathbb{E}\Big[-\frac{3\epsilon}{2} + i\,T^2 p_{2+i}\pi_i - i\,(-1+T)\,T\,p_{2+j}\pi_i + i\,T^2\epsilon\, p_{2+j}\pi_i - i\,(-1+T)\,p_{2+k}\pi_i + i\,T\,\epsilon\, p_{2+k}\pi_i - \frac{1}{2}(-1+T)\,T^3\epsilon\, p_{2+i}\,p_{2+j}\,\pi_i^2 + \frac{1}{2}(-1+T)\,T^3\epsilon\, p_{2+j}^2\pi_i^2 - \frac{1}{2}(-1+T)\,T^2\epsilon\, p_{2+i}\,p_{2+k}\,\pi_i^2 + \frac{1}{2}(-1+T)^2\,T\,\epsilon\, p_{2+j}\,p_{2+k}\,\pi_i^2 + \frac{1}{2}(-1+T)\,T\,\epsilon\, p_{2+k}^2\pi_i^2 + i\,T\,p_{2+j}\pi_j - i\,T\,\epsilon\, p_{2+j}\pi_j - i\,(-1+T)\,p_{2+k}\pi_j + i\,(-1+2T)\,\epsilon\, p_{2+k}\pi_j + T^3\epsilon\, p_{2+i}\,p_{2+j}\,\pi_i\pi_j - T^3\epsilon\, p_{2+j}^2\pi_i\pi_j - (-1+T)\,T^2\epsilon\, p_{2+i}\,p_{2+k}\,\pi_i\pi_j + (-1+T)^2\,T\,\epsilon\, p_{2+j}\,p_{2+k}\,\pi_i\pi_j + (-1+T)\,T\,\epsilon\, p_{2+k}^2\pi_i\pi_j - \frac{1}{2}(-1+T)\,T\,\epsilon\, p_{2+j}\,p_{2+k}\,\pi_j^2 + \frac{1}{2}(-1+T)\,T\,\epsilon\, p_{2+k}^2\pi_j^2 + i\,p_{2+k}\pi_k - 2\,i\,\epsilon\, p_{2+k}\pi_k + T^2\epsilon\, p_{2+i}\,p_{2+k}\,\pi_i\pi_k - (-1+T)\,T\,\epsilon\, p_{2+j}\,p_{2+k}\,\pi_i\pi_k - T\,\epsilon\, p_{2+k}^2\pi_i\pi_k + T\,\epsilon\, p_{2+j}\,p_{2+k}\,\pi_j\pi_k - T\,\epsilon\, p_{2+k}^2\pi_j\pi_k\Big]$$

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at ωεβ/ap.

There's more! To get sl_2 invariants mod ϵ^3, add the following to $L(X^+_{ij})$, $L(X^-_{ij})$, and $L(C^\varphi_i)$, respectively (and see More.nb at ωεβ/ap for the verifications):

☺ $\epsilon^2\, r_2[1, i, j]$

💻 $$\frac{1}{12}\epsilon^2\big(-6 p_i x_i + 6 p_j x_i - 3(-1+3T)\,p_i p_j x_i^2 + 3(-1+3T)\,p_j^2 x_i^2 + 4(-1+T)\,p_i^2 p_j x_i^3 - 2(-1+T)(5+T)\,p_i p_j^2 x_i^3 + 2(-1+T)(3+T)\,p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j - 6 p_i^2 p_j x_i^2 x_j + 6(2+T)\,p_i p_j^2 x_i^2 x_j - 6(1+T)\,p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2\big)$$

☺ $\epsilon^2\, r_2[-1, i, j]$

💻 $$\frac{1}{12T^2}\epsilon^2\big(-6T^2 p_i x_i + 6T^2 p_j x_i + 3(-3+T)\,T\,p_i p_j x_i^2 - 3(-3+T)\,T\,p_j^2 x_i^2 - 4(-1+T)\,T\,p_i^2 p_j x_i^3 + 2(-1+T)(1+5T)\,p_i p_j^2 x_i^3 - 2(-1+T)(1+3T)\,p_j^3 x_i^3 + 18T^2 p_i p_j x_i x_j - 18T^2 p_j^2 x_i x_j - 6T^2 p_i^2 p_j x_i^2 x_j + 6T(1+2T)\,p_i p_j^2 x_i^2 x_j - 6T(1+T)\,p_j^3 x_i^2 x_j - 6T^2 p_i p_j^2 x_i x_j^2 + 6T^2 p_j^3 x_i x_j^2\big)$$

☺ $\epsilon^2\, \gamma_2[\varphi, i]$

💻 $$-\frac{1}{2}\epsilon^2\varphi^2 p_i x_i$$

Even more! • The sl_2 formulas mod ϵ^4 are in the last page of the handout of [BN3].

• Using [GPV] we can show that every finite type invariant is I-Type.

• Probably, ⟨Reshetikhin-Turaev⟩ ⊂ ⟨I-Type⟩ efficiently.

• Possibly, ⟨Rozansky Polynomials⟩ ⊂ ⟨I-Type⟩ efficiently.

• Knot signatures are I-Type, at least mod 8.

• We already have some work on sl_3, and it leads to the strongest genuinely-computable knot invariant presently known.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505.

The $sl_3^{/\epsilon^2}$ Example [BV3]. Here we have two formal variables T_1 and T_2, we set $T_3 := T_1T_2$, we integrate over 6 variables for each edge: p_{1i}, p_{2i}, p_{3i}, x_{1i}, x_{2i}, and x_{3i}.

```
T3 = T1 T2;  i_+ := i + 1;
$π =
  (CF@Normal[# + O[ϵ]^2] /.
      {π_{is__} :→ B^-1 π_{is}, x_{is__} :→ B^-1 x_{is}, p_{is__} :→ B p_{is}} /.
    ϵ B^b_ /; b < 0 → 0 /. B → 1) &;
```

```
vs_{i_} := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
𝓕[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
ℒ[K_] := CF[ℒ /@ Features[K][[2]]];
vs[K_] :=
 Union @@ Table[{vs_i}, {i, Features[K][[1]]}]
```

The Lagrangian.

```
ℒ[X_{i_,j_}[s_]] := T3^s 𝔼[CF@Plus[
    Σ_{ν=1}^3 (x_{νi} (p_{νi+} - p_{νi}) + x_{νj} (p_{νj+} - p_{νj}) + (T_ν^s - 1) x_{νi} (p_{νi+} - p_{νj+})),
    (T1^s - 1) p_{3j} x_{1i} (T2^s x_{2i} - x_{2j}),
    ϵ s (T3^s - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T2^s - 1),
    ϵ s (1/2 + T2^s p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} -
       (T2^s - 1) p_{2j} p_{3i} x_{2i} x_{3i} + (T3^s - 1) p_{2j} p_{3j} x_{2i} x_{3i} +
       2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} -
       T2^s p_{2j} p_{3j} x_{2i} x_{3j} +
       ((T1^s - 1) p_{1j} x_{1i} (T2^{2s} p_{2j} x_{2i} - T2^s p_{2j} x_{2j} -
            (T2^s + 1) (T3^s - 1) p_{3j} x_{3i} + T2^s p_{3j} x_{3j}) +
          (T3^s - 1) p_{3j} x_{3i} (1 - T2^s p_{1i} x_{1i} + p_{2i} x_{2j} + (T2^s - 2) p_{2j} x_{2j})) /
       (T2^s - 1))]]
```

```
ℒ[C_{i_}[φ_]] := T3^φ 𝔼[Σ_{ν=1}^3 x_{νi} (p_{νi+} - p_{νi}) + ϵ φ (p_{3i} x_{3i} - 1/2)]
```

Reidemeister 3.

```
Short[
 lhs = ∫ 𝓕[i, j, k] × ℒ /@ (X_{i,j}[1] X_{i+,k}[1] X_{j+,k+}[1])
    d{vs_i, vs_j, vs_k, vs_{i+}, vs_{j+}, vs_{k+}}]
```

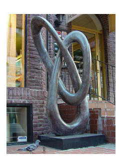
$T_1^3\, T_2^3$

$\mathbb{E}\left[\frac{3\,\epsilon}{2} + T_1^2\, p_{1,2+i}\, \pi_{1,i} - (-1 + T_1)\, T_1\, p_{1,2+j}\, \pi_{1,i} + \ll 150 \gg\right]$

```
rhs = ∫ 𝓕[i, j, k] × ℒ /@ (X_{j,k}[1] X_{i,k+}[1] X_{i+,j+}[1])
    d{vs_i, vs_j, vs_k, vs_{i+}, vs_{j+}, vs_{k+}};
lhs == rhs
```

True

The Trefoil.



```
K = Knot[3, 1];  ∫ ℒ[K] d vs[K]
```

$$-\Big((i\, T_1^2\, T_2^2\ \mathbb{E}\Big[-\Big(\big(\epsilon\, (1 - T_1 + T_1^2 - T_2 - T_1^3\, T_2 + T_2^2 + T_1^4\, T_2^2 - T_1\, T_2^3 - T_1^4\, T_2^3 + T_1^2\, T_2^4 - T_1^3\, T_2^4 + T_1^4\, T_2^4)\big) \Big/ \big((1 - T_1 + T_1^2)\, (1 - T_2 + T_2^2)\, (1 - T_1\, T_2 + T_1^2\, T_2^2)\big)\Big)\Big]\Big) \Big/ \Big((1 - T_1 + T_1^2)\, (1 - T_2 + T_2^2)\, (1 - T_1\, T_2 + T_1^2\, T_2^2)\Big)\Big)$$

A faster program, in which the Feynman diagrams are "pre-computed" (see theta.nb at ωεβ/ap):

```
F1[{s_, i_, j_}] := CF[
  s (1/2 - g_{3ii} + T2^s g_{1ii} g_{2ji} - g_{1ii} g_{2jj} - (T2^s - 1) g_{2ji} g_{3ii} +
     2 g_{2jj} g_{3ii} - (1 - T3^s) g_{2ji} g_{3ji} - g_{2ii} g_{3jj} - T2^s g_{2ji} g_{3jj} +
     g_{1ii} g_{3jj} +
     ((T1^s - 1) g_{1ji} (T2^{2s} g_{2ji} - T2^s g_{2jj} + T2^s g_{3jj}) +
        (T3^s - 1) g_{3ji} (1 - T2^s g_{1ii} - (T1^s - 1) (T2^s + 1) g_{1ji} +
           (T2^s - 2) g_{2jj} + g_{2ij})) / (T2^s - 1))]
```

```
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
 CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g_{1,j1,i0} g_{3,j0,i1}
   ((T2^s0 g_{2,i1,i0} - g_{2,i1,j0}) - (T2^s0 g_{2,j1,i0} - g_{2,j1,j0}))]
```

```
F3[φ_, k_] = φ g_{3kk} - φ/2;
```

We call the invariant computed θ:

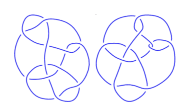
```
Θ[K_] := Θ[K] = Module[{X, φ, n, A, Δ, G, ev, θ},
   {X, φ} = Rot[K]; n = Length[X];
   A = IdentityMatrix[2 n + 1];
   Cases[X, {s_, i_, j_} :→
     (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                        0    -1    ))];
   Δ = T^((-Total[φ]-Total[X[[All,1]]])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] := Factor[ℰ /. g_{ν_,α_,β_} :→ (G[[α, β]] /. T → T_ν)];
   θ = ev[Σ_{k=1}^n F1[X[[k]]]];
   θ += ev[Σ_{k1=1}^n Σ_{k2=1}^n F2[X[[k1]], X[[k2]]]];
   θ += ev[Σ_{k=1}^{2 n} F3[φ[[k]], k]];
   Factor@{Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}
  ];
```

Some Knots.

```
Expand[Θ[Knot[3, 1]]]
```

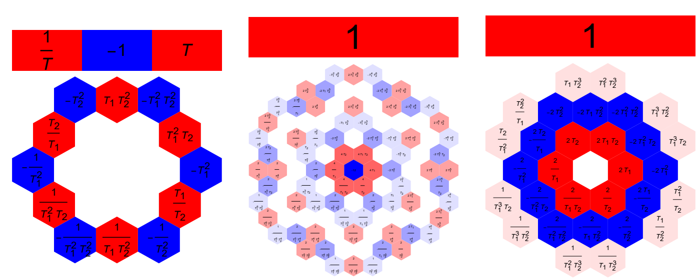
$$\left\{-1 + \frac{1}{T} + T,\ -\frac{1}{T_1^2} - T_1^2 - \frac{1}{T_2^2} - \frac{1}{T_1^2\, T_2^2} + \frac{1}{T_1\, T_2^2} + \frac{1}{T_1^2\, T_2} + \frac{T_1}{T_2} + \frac{T_2}{T_1} + T_1^2\, T_2 - T_2^2 + T_1\, T_2^2 - T_1^2\, T_2^2\right\}$$

```
(* PolyPlot suppressed *)
```



K11n34 K11n42

```
GraphicsRow[PolyPlot[Θ[Knot[#]],
    Labeled → True] &
  /@ {"3_1", "K11n34", "K11n42"}]
```

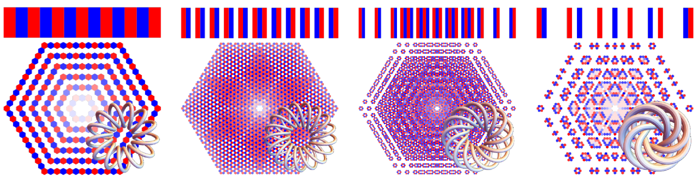


So θ detects knot mutation and separates the Conway knot K11n34 from the Kinoshita-Terasaka knot K11n42!

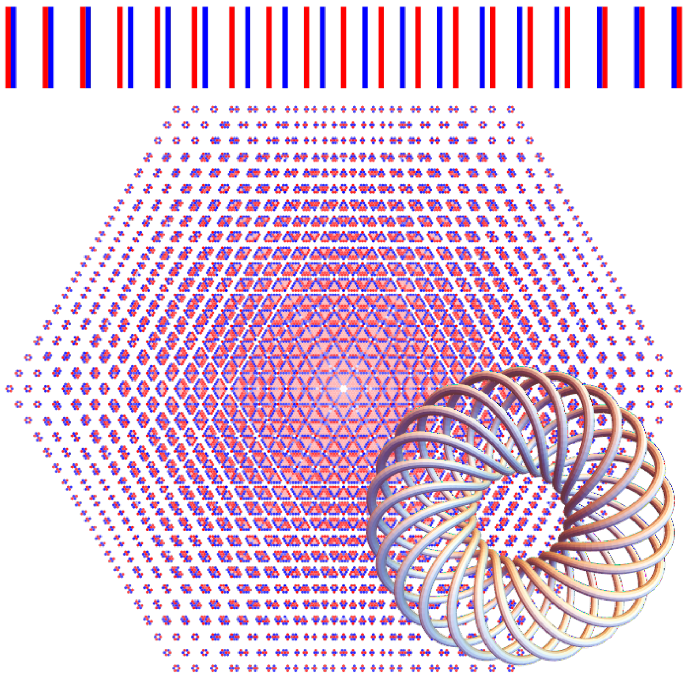
Conway Kinoshita Terasaka

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505`.

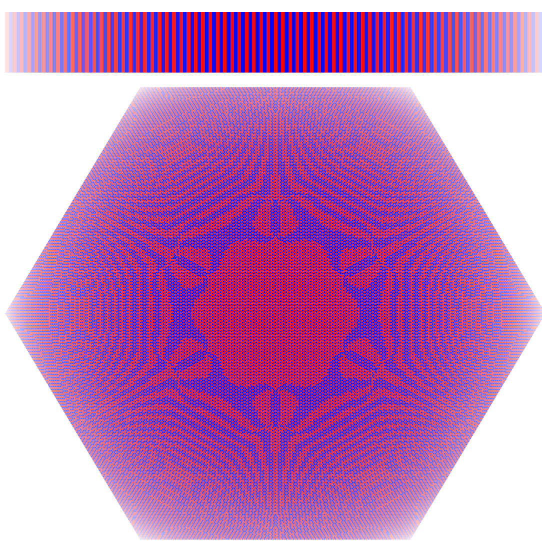
☺
```
GraphicsRow[ImageCompose[
    PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
    TubePlot[TorusKnot @@ #, ImageSize → 240],
    {Right, Bottom}, {Right, Bottom}
  ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```



The torus knot $T_{22/7}$:

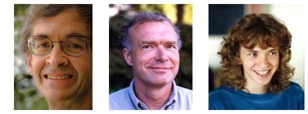


Next, a random 300 crossing knot from [DHOEBL] (more at ωεβ/DK):

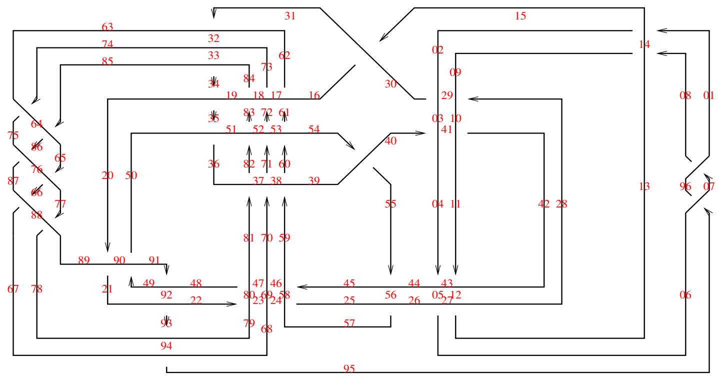


Unproven Fact. For any knot K, twice its genus $g(K)$ bounds the T_1 degree of θ: $\deg_{T_1}\theta(K) \leq 2g(K)$.

The 48-crossing Gompf-Scharlemann-Thompson GST_{48} knot [GST] is significant because it may be a counterexample to the slice-ribbon conjecture:



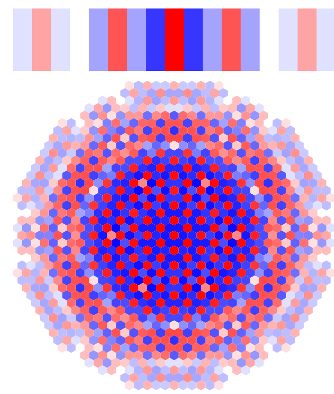
Gompf Scharlemann Thompson



☺ GST_{48} = EPD$[X_{14,1}, \overline{X}_{2,29}, X_{3,40}, X_{43,4}, \overline{X}_{26,5}, X_{6,95}, X_{96,7}, X_{13,8}, \overline{X}_{9,28}, X_{10,41}, X_{42,11}, \overline{X}_{27,12}, X_{30,15}, \overline{X}_{16,61}, \overline{X}_{17,72}, \overline{X}_{18,83}, X_{19,34}, \overline{X}_{89,20}, \overline{X}_{21,92}, \overline{X}_{79,22}, \overline{X}_{68,23}, \overline{X}_{57,24}, \overline{X}_{25,56}, X_{62,31}, X_{73,32}, X_{84,33}, \overline{X}_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, \overline{X}_{39,54}, X_{44,55}, X_{58,45}, X_{69,46}, X_{80,47}, X_{48,91}, X_{90,49}, X_{51,82}, X_{52,71}, X_{53,60}, \overline{X}_{63,74}, \overline{X}_{64,85}, \overline{X}_{76,65}, \overline{X}_{87,66}, \overline{X}_{67,94}, \overline{X}_{75,86}, \overline{X}_{88,77}, \overline{X}_{78,93}]$;

AbsoluteTiming[PolyPlot[Θ_{48} = Θ@GST_{48}, ImageSize → Small]]

{17.1103,



}

☺ {Exponent[Θ_{48}[[1]], T], Floor[Exponent[Θ_{48}[[2]], T_2] / 2]}

{8, 10}

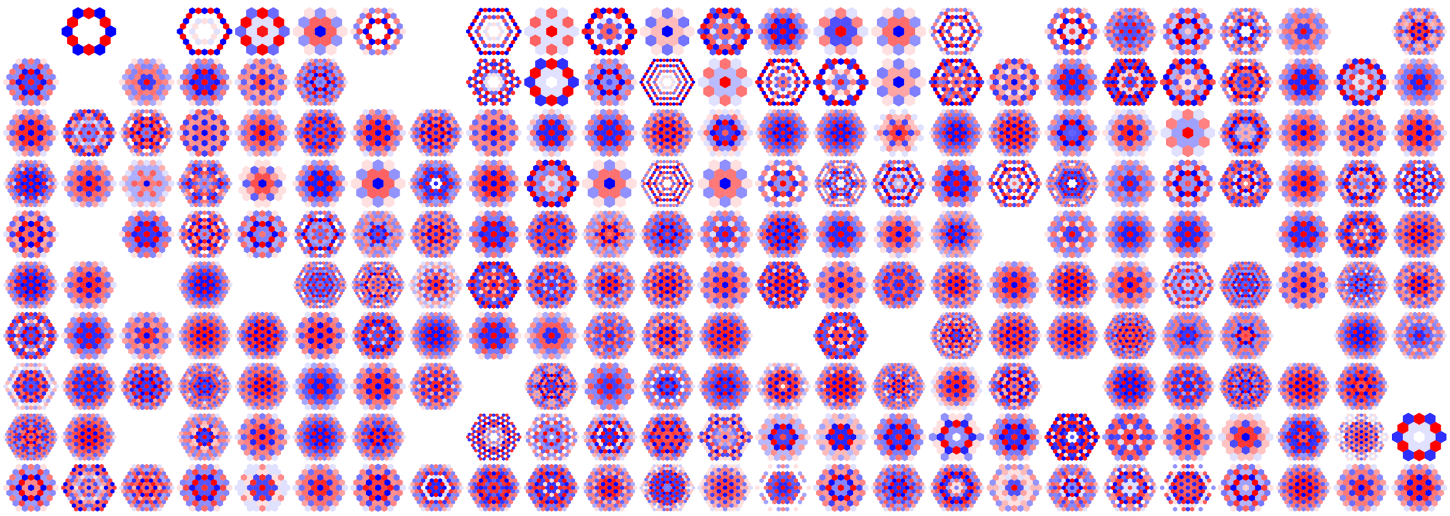
So Θ knows things about GST_{48} that Δ doesn't!

Ohtsuki Garoufalidis Rozansky Kricker Schaveling Kashaev

Prior Art. θ is probably equal to the "2-loop polynomial" studied by Ohtsuki [Oh2] (at greater difficulty, with harder computations), continuing B-N, Garoufalidis, Rozansky, Kricker, and Schaveling [BNG, GR, R1, R2, R3, Kr, Sch]. θ is related, but probably not equivalent, to the invariant studied by Garoufalidis–Kashaev [GK].

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505.

The Rolfsen Table of Knots.



Where is it coming from? The most honest answer is “we don’t know” (and that’s good!). The second most, “undetermined coefficients for an ansatz that made sense”. The ansatz comes from the following principles / earlier work:

Morphisms have generating functions. Indeed, there is an isomorphism

$$\mathcal{G}\colon \operatorname{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j]) \to \mathbb{Q}[y_j][\![\xi_i]\!],$$

and by PBW, many relevant spaces are polynomial rings, though only as vector spaces.

Composition is integration. Indeed, if $f \in \operatorname{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j])$ and $g \in \operatorname{Hom}(\mathbb{Q}[y_j], \mathbb{Q}[z_k])$, then

$$\mathcal{G}(g \circ f) = \int \mathbb{e}^{-y\cdot\eta} f g \, dy \, d\eta$$

Use universal invariants. These take values in a universal enveloping algebra (perhaps quantized), and thus they are expressible as long compositions of generating functions. See [La, Oh1].

“Solvable approximation” $\rightsquigarrow$ perturbed Gaussians. Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{h}$ be its Cartan subalgebra, and let $\mathfrak{b}^u$ and $\mathfrak{b}^l$ be its upper and lower Borel subalgebras. Then $\mathfrak{b}^u$ has a bracket β, and as the dual of $\mathfrak{b}^l$ it also has a cobracket δ, and in fact, $\mathfrak{g} \oplus \mathfrak{h} \equiv \operatorname{Double}(\mathfrak{b}^u, \beta, \delta)$. Let $\mathfrak{g}^+_\epsilon := \operatorname{Double}(\mathfrak{b}^u, \beta, \epsilon\delta)$ (mod ϵ^{d+1} it is solvable for any d). Then by [BV3, BN1] (in the case of $\mathfrak{g} = sl_2$) all the interesting tensors of $\mathcal{U}(\mathfrak{g}^+_\epsilon)$ (quantized or not) are perturbed Gaussian with perturbation parameter ϵ with with understood bounds on the degrees of the perturbations.

References.

[BN1] D. Bar-Natan, *Everything around sl^ϵ_{2+} is DoPeGDO. So what?,* talk given in “Quantum Topology and Hyperbolic Geometry Conference”, Da Nang, Vietnam, May 2019. Handout and video at ωεβ/DPG.

[BN2] D. Bar-Natan, *Algebraic Knot Theory,* talk given in Sydney, September 2019. Handout and video at ωεβ/AKT.

[BN3] D. Bar-Natan, *Cars, Interchanges, Traffic Counters, and some Pretty Darned Good Knot Invariants,* talk given in “Using Quantum Invariants to do Interesting Topology”, Oaxaca, Mexico, October 2022. Handout and video at ωεβ/Cars.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BV1] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* Proc. Amer. Math. Soc. **147** (2019) 377–397, arXiv:1708.04853.

[BV2] D. Bar-Natan and R. van der Veen, *A Perturbed-Alexander Invariant,* Quantum Topology **15** (2024) 449–472, ωεβ/APAI.

[BV3] D. Bar-Natan and R. van der Veen, *A Very Fast, Very Strong, Topologically Meaningful and Fun Knot Invariant,* in preparation, draft at ωεβ/Theta.

[BV3] D. Bar-Natan and R. van der Veen, *Perturbed Gaussian Generating Functions for Universal Knot Invariants,* arXiv:2109.02057.

[BG] J. Becerra Garrido, *Universal Quantum Knot Invariants,* Ph.D. thesis, University of Groningen, ωεβ/BG.

[DHOEBL] N. Dunfield, A. Hirani, M. Obeidin, A. Ehrenberg, S. Bhattacharya, D. Lei, and others, *Random Knots: A Preliminary Report,* lecture notes at ωεβ/DHOEBL. Also a data file at ωεβ/DD.

[GK] S. Garoufalidis and R. Kashaev, *Multivariable Knot Polynomials from Braided Hopf Algebras with Automorphisms,* arXiv:2311.11528.

[GR] S. Garoufalidis and L. Rozansky, *The Loop Expansion of the Kontsevich Integral, the Null-Move, and S-Equivalence,* arXiv:math.GT/0003187.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[Kr] A. Kricker, *The Lines of the Kontsevich Integral and Rozansky’s Rationality Conjecture,* arXiv:math/0005284.

[La] R. J. Lawrence, *Universal Link Invariants using Quantum Groups,* Proc. XVII Int. Conf. on Diff. Geom. Methods in Theor. Phys., Chester, England, August 1988. World Scientific (1989) 55–63.

[LV] D. López Neumann and R. van der Veen, *Genus Bounds from Unrolled Quantum Groups at Roots of Unity,* arXiv:2312.02070.

[Oh1] T. Ohtsuki, *Quantum Invariants,* Series on Knots and Everything **29**, World Scientific 2002.

[Oh2] T. Ohtsuki, *On the 2–Loop Polynomial of Knots,* Geom. Topol. **11-3** (2007) 1357–1475.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, August 2013, ωεβ/Ov.

[R1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten’s Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[R2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[R3] L. Rozansky, *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Sch] S. Schaveling, *Expansions of Quantum Group Invariants,* Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Thanks for bearing with me!

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Bonn-2505`.

Dror Bar-Natan: Talks: Pitzer-250308: ωεβ:=http://drorbn.net/pi25

A Seifert Dream

Thanks for inviting me to Pitzer College!

Abstract. Given a knot K with a Seifert surface Σ, I dream that the well-known Seifert linking form Q, a quadratic form on $H_1(\Sigma)$, has plenty docile local perturbations P_ϵ such that the formal Gaussian integrals of $\exp(Q+P_\epsilon)$ are invariants of K.
In my talk I will explain what the above means, why this dream is oh so sweet, and why it is in fact closer to a plan than to a delusion. Joint with Roland van der Veen.

The Seifert-Alexander Formula. With $P, Q \in H_1(\Sigma)$,

$$Q(P,G) = T^{1/2}lk(P^+,G) - T^{-1/2}lk(P,G^+)$$

$$\Delta(K) = \det(Q)$$

$$\int_{2H_1(\Sigma)} dp\,dx \exp Q(p,x) \doteq \det(Q)^{-1}$$

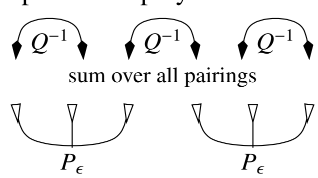
(where $\doteq$ means "ignoring silly factors").

From Mexico City, tariffs exempt

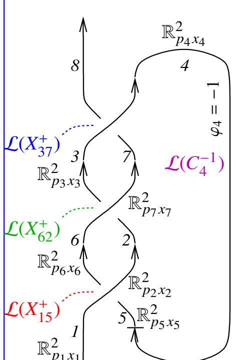
Perturbed Gaussian Integration. We say that $P_\epsilon \in \epsilon\mathbb{Q}[x_1,\dots x_n][\![\epsilon]\!]$ is M-docile (for some $M\colon \mathbb{N}\to\mathbb{N}$) if for every monomial m in P_ϵ we have $\deg_{x_1,\dots,x_n}(m) \leq M(\deg_\epsilon(m))$.

Theorem (Feynman). If Q is a quadratic in $x_1,\dots,x_n$ and P_ϵ is docile, set $Z_\epsilon = \int_{\mathbb{R}^n} dx_1\cdots x_n \exp(Q+P_\epsilon)$. Then every coefficient in the ϵ-expansion of Z_ϵ is computable in polynomial time in n. in fact,

$$\Delta^{1/2}Z_\epsilon \doteq \left\langle \exp Q^{-1}(\partial_{x_i}), \exp P_\epsilon \right\rangle =$$



$\theta(T,1)$ is like that! With $\epsilon^2=0$,



$$\longrightarrow Z \doteq \oint_{2E=\mathbb{R}^{14}_{p_ix_i}} \mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C^{-1}_4)$$

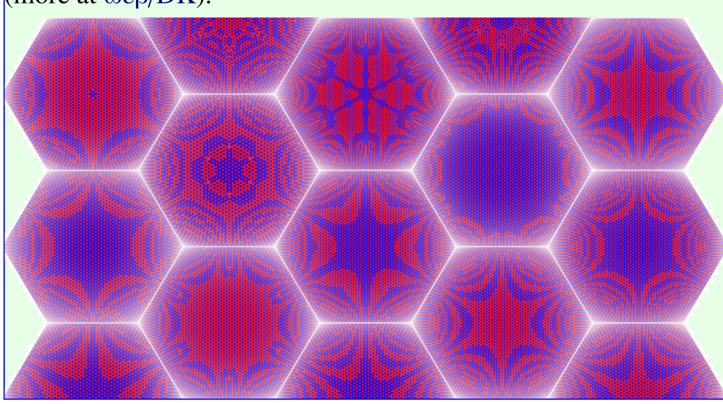
where $\mathcal{L}(X^s_{ij}) \doteq \mathbb{e}^{L(X^s_{ij})}$, $\mathcal{L}(C^\varphi_i) \doteq \mathbb{e}^{L(C^\varphi_i)}$,

$$L(X^s_{ij}) = x_i(p_{i+1}-p_i) + x_j(p_{j+1}-p_j) + (T^s-1)x_i(p_{i+1}-p_{j+1}) + \frac{\epsilon s}{2}\left(x_i(p_i-p_j)\begin{pmatrix}(T^s-1)x_ip_j\\ +2(1-x_jp_j)\end{pmatrix}-1\right)$$

$$L(C^\varphi_i) = x_i(p_{i+1}-p_i) + \epsilon\varphi(1/2 - x_ip_i)$$

$\theta(T_1,T_2)$ is likewise, with harder formulas and integration over $6E$.

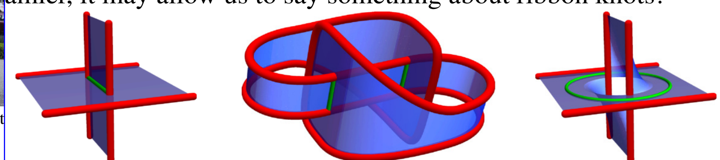
Right. The 132-crossing torus knot $T_{22/7}$ (more at ωεβ/TK).
Below. Random knots from [DHOEBL], with 101-115 crossings (more at ωεβ/DK).



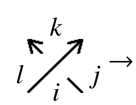
Dream. There is a similar perturbed Gaussian integral formula for θ, but with integration over $6H_1(\Sigma)$. The quadratic Q will be the same as in the Seifert-Alexander formula (but repeated 3 times, for each T_ν). The perturbation P_ϵ will be given by low-degree finite type invariants of curves on Σ (possibly also dependent on the intersection points of such curves, or on other information coming from Σ).

Evidence. Experimentally (yet undeniably), $\deg\theta$ is bounded by the genus of Σ. How else could such a genus bound arise? Further very strong evidence comes from the conjectural (yet undeniable) understanding of θ as the two-loop contribution to the Kontsevich integral [Oh] and/or as the "solvable approximation" of the universal sl_3 invariant [BN1, BV2].

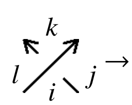
Why so sweet? It will allow us to prove the aforementioned genus bound and likely, the hexagonal symmetry. Sweeter and dreamier, it may allow us to say something about ribbon knots!



What's "local"? How will we compute? The Będlewo Alexander formula: Let F be the faces of a knot diagram. Make an $F\times F$ matrix A by adding for each crossing contributions

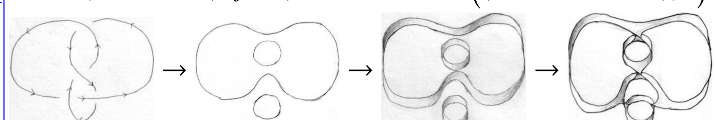


$$\to \begin{pmatrix}-1&-1&2&0\\0&0&0&0\\0&1&-1&0\\1&0&-1&0\end{pmatrix}$$



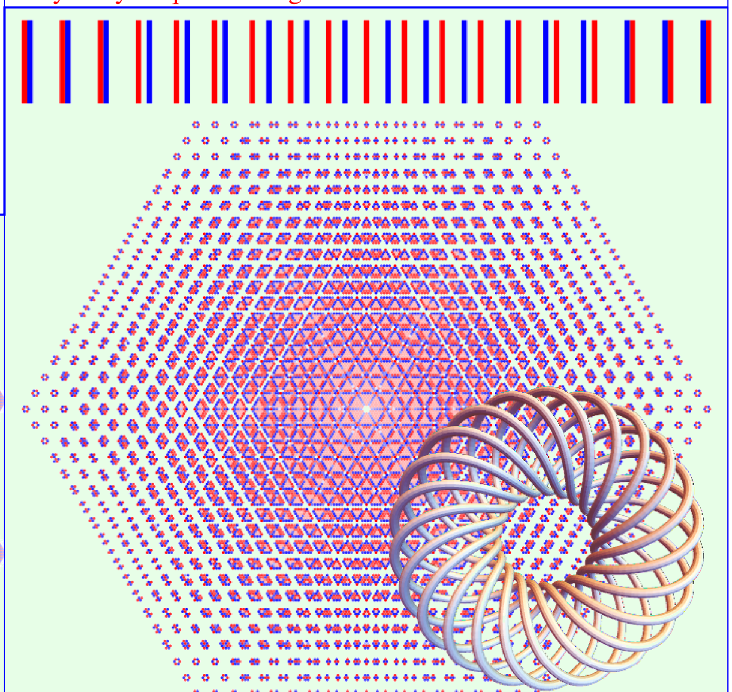
$$\to \begin{pmatrix}1&-1&0&0\\0&0&0&0\\-2&1&1&0\\1&0&-1&0\end{pmatrix}$$

at rows / columns (i,j,k,l). Then $\Delta = \det'\left((T^{1/2}A - T^{-1/2}A)/2\right)$.



(the Seifert algorithm by Emily Redelmeier)

Expect the like for θ! Expect more like θ! Topology first! Resist the tyranny of quantum algebra!



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Video and more at http://www.math.toronto.edu/~drorbn/Talks/Pitzer-250308.

The Strongest Genuinely Computable Knot Invariant ~~Since~~ In 2024

The First International On-line Knot Theory Congress
February 1-5, 2025

Dror Bar-Natan

Abstract. "Genuinely computable" means we have computed it for random knots with over 300 crossings. "Strongest" means it separates prime knots with up to 15 crossings better than the less-computable HOMFLY-PT and Khovanov homology taken together. And hey, it's also meaningful and fun.
Continues Rozansky, Garoufalidis, Kricker, and Ohtsuki, joint with van der Veen.

These slides and the code within are online at ωεβ:=http://drorbn.net/ktc25

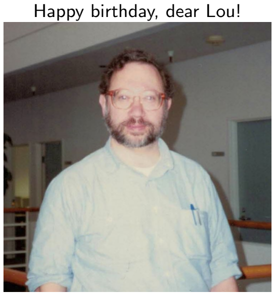
(I wish all speakers were making their slides available **before** / **for** their talks).

(I'll post the video there too)

A paper-in-progress is at ωεβ/Theta.

If you can, please turn your video on!

Happy birthday, dear Lou!



Lou Kauffman at MSRI, March 1991

Acknowledgement.

This work was supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

The Strongest Genuinely Computable Knot Invariant ~~Since~~ In 2024

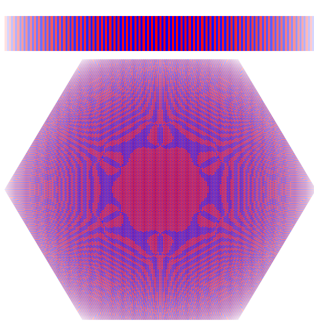
Strongest? Genuinely Computable?

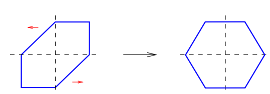
Strongest.

Testing $\Theta = (\Delta, \theta)$ on prime knots up to mirrors and reversals, counting the number of distinct values (with deficits in parenthesis): (ρ_1: [Ro1, Ro2, Ro3, Ov, BV1])

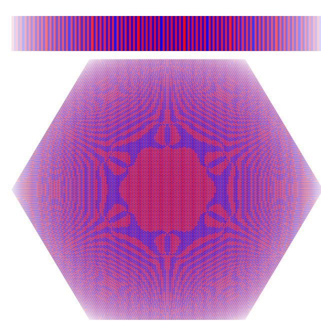
	knots	(H, Kh)	(Δ, ρ_1)	$\Theta = (\Delta, \theta)$	(Δ, θ, ρ_2)	all together
reign		2005-22	2022-24	2024	2025-	
xing ≤ 10	249	248 (1)	249 (0)	249 (0)	249(0)	249 (0)
xing ≤ 11	801	771 (30)	787 (14)	798 (3)	798 (3)	798 (3)
xing ≤ 12	2,977	(214)	(95)	(19)	(10)	(10)
xing ≤ 13	12,965	(1,771)	(959)	(194)	(169)	(169)
xing ≤ 14	59,937	(10,788)	(6,253)	(1,118)	(982)	(981)
xing ≤ 15	313,230	(70,245)	(42,914)	(6,758)	(6,341)	(6,337)

Genuinely Computable. Here's Θ on a random 300 crossing knot (from [DHOEBL]). For almost every other knot invariant, that's science fiction.
Gukov: Should take 300 years if Moore's law persists.
Us: A few hours on a laptop, 0 GPUs.



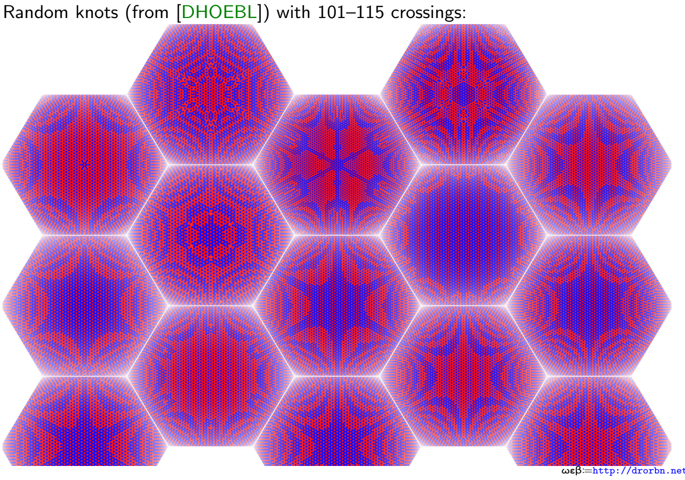


Fun. There's so much more to see in 2D pictures than in 1D ones! Yet almost nothing of the patterns you see we know how to prove. We'll have fun with that over the next few years. Would you join?



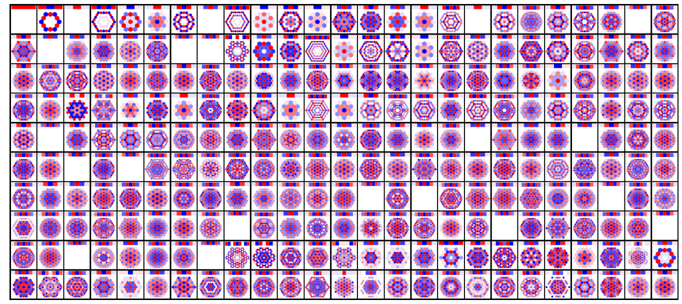
Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.

Random knots (from [DHOEBL]) with 101–115 crossings:



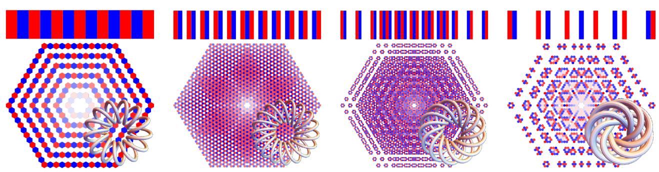
ωεβ:=http://drorbn.net/ktc25

The Rolfsen Table:

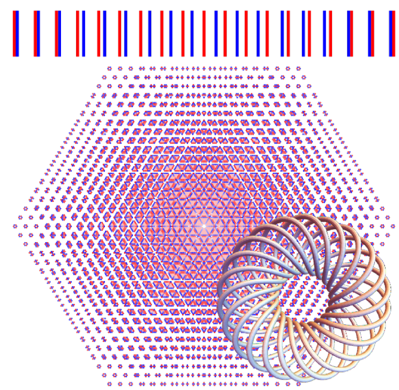


ωεβ:=http://drorbn.net/ktc25

The torus knots $TK_{13/2}$, $TK_{17/3}$, $TK_{13/5}$, and $TK_{7/6}$:



ωεβ:=http://drorbn.net/ktc25



The torus knot $TK_{22/7}$:

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Meaningful.

θ gives a genus bound (unproven yet with confidence). We hope (with reason) it says something about ribbon knots.

ωεβ:=http://drorbn.net/ktc25

Convention.

T, T_1, and T_2 are indeterminates and $T_3 := T_1T_2$.

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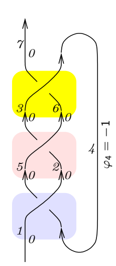
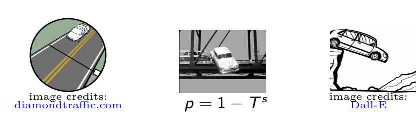
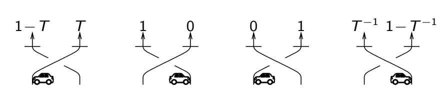
Preparation. Draw an n-crossing knot K as a diagram D as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \ldots, 2n+1\}$ and with rotation numbers φ_k.



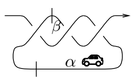
ωεβ:=http://drorbn.net/ktc25



Model T Traffic Rules. Cars always drive forward. When a car crosses over a sign-s bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0$. At the very end, cars fall off and disappear. On various edges traffic counters are placed. See also [Jo, LTW].



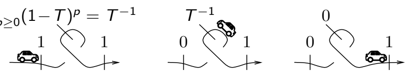
Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.



Definition. The traffic function $G=(g_{\alpha\beta})$ (also, the Green function or the two-point function) is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha=\beta$, the counter is after the injection point). There are also model-T_ν traffic functions $G_\nu=(g_{\nu\alpha\beta})$ for $\nu=1,2,3$.

Example.

$\sum_{p\geq 0}(1-T)^p=T^{-1}$



$$G=\begin{pmatrix}1 & T^{-1} & 1\\ 0 & T^{-1} & 1\\ 0 & 0 & 1\end{pmatrix}$$

Given crossings $c=(s,i,j)$, $c_0=(s_0,i_0,j_0)$, and $c_1=(s_1,i_1,j_1)$, let

$$\begin{aligned}F_1(c)=s\,[&1/2-g_{3ii}+T_2^s g_{1ii}g_{2ji}-T_2^s g_{3jj}g_{2ji}-(T_2^s-1)g_{3ii}g_{2ji}\\ &+(T_3^s-1)g_{2ji}g_{3ji}-g_{1ii}g_{2jj}+2g_{3ii}g_{2jj}+g_{1ii}g_{3jj}-g_{2ii}g_{3jj}]\\ &+\frac{s}{T_2^s-1}\,[(T_1^s-1)\,T_2^s\,(g_{3jj}g_{1ji}-g_{2jj}g_{1ji}+T_2^s g_{1ji}g_{2ji})\\ &+(T_3^s-1)\,(g_{3ji}-T_2^s g_{1ii}g_{3ji}+g_{2ij}g_{3ji}+(T_2^s-2)g_{2jj}g_{3ji})\\ &-(T_1^s-1)(T_2^s+1)(T_3^s-1)g_{1ji}g_{3ji}]\end{aligned}$$

$$F_2(c_0,c_1)=\frac{s_1(T_1^{s_0}-1)(T_3^{s_1}-1)g_{1j_1i_0}g_{3j_0i_1}}{T_2^{s_1}-1}\,(T_2^{s_0}g_{2i_1i_0}+g_{2j_1j_0}-T_2^{s_0}g_{2j_1i_0}-g_{2i_1j_0})$$

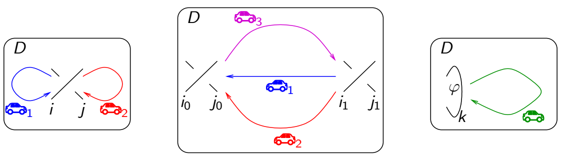
$$F_3(\varphi_k,k)=\varphi_k(g_{3kk}-1/2)$$

(Computers don't care!)

Main Theorem.

The following is a knot invariant: (the Δ_ν are normalizations discussed later)

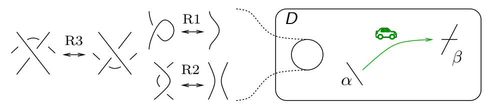
$$\theta(D):=\Delta_1\Delta_2\Delta_3\left(\sum_c F_1(c)+\sum_{c_0,c_1}F_2(c_0,c_1)+\sum_k F_3(\varphi_k,k)\right).$$



If these pictures remind you of Feynman diagrams, it's because they are Feynman diagrams [BN2].

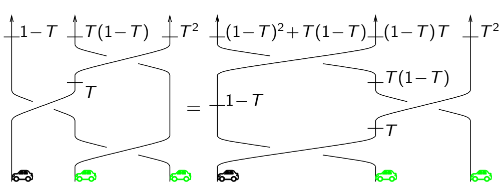
Lemma 1.

The traffic function $g_{\alpha\beta}$ is a "relative invariant":

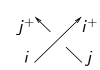


(There is some small print for R1 and R2 which change the numbering of the edges and sometimes collapse a pair of edges into one)

Proof.



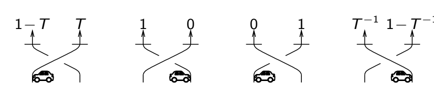
Lemma 2.



With $k^+:=k+1$, the "g-rules" hold near a crossing $c=(s,i,j)$:

$$g_{j\beta}=g_{j^+\beta}+\delta_{j\beta}\quad g_{i\beta}=T^s g_{i^+\beta}+(1-T^s)g_{j^+\beta}+\delta_{i\beta}\quad g_{2n^+,\beta}=\delta_{2n^+,\beta}$$

$$g_{\alpha i^+}=T^s g_{\alpha i}+\delta_{\alpha i^+}\quad g_{\alpha j^+}=g_{\alpha j}+(1-T^s)g_{\alpha i}+\delta_{\alpha j^+}\quad g_{\alpha,1}=\delta_{\alpha,1}$$



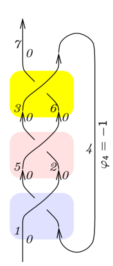
Corollary 1.

G is easily computable, for $AG=I$ $(=GA)$, with A the $(2n+1)\times(2n+1)$ identity matrix with additional contributions:

$$c=(s,i,j)\mapsto\begin{array}{c|cc}A & \text{col } i^+ & \text{col } j^+\\ \hline \text{row } i & -T^s & T^s-1\\ \text{row } j & 0 & -1\end{array}$$

For the trefoil example, we have:

$$A=\begin{pmatrix}1 & -T & 0 & 0 & T-1 & 0 & 0\\ 0 & 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -T & 0 & 0 & T-1\\ 0 & 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & T-1 & 0 & 1 & -T & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$



And so,

$$G=\begin{pmatrix}1 & T & 1 & T & 1 & T & 1\\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1\\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1\\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1\\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1\\ 0 & 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.

Note.

The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2}\det(A),$$

with

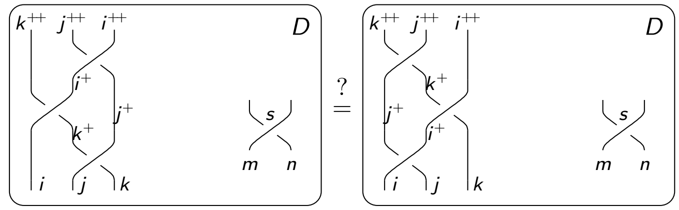
$$\varphi = \sum_k \varphi_k, \qquad w = \sum_c s.$$

We also set $\Delta_\nu := \Delta(T_\nu)$ for $\nu = 1, 2, 3$. This defines and explains the normalization factors in the Main Theorem.

ωεβ:=http://drorbn.net/ktc25

Corollary 2.

Proving invariance is easy:



ωεβ:=http://drorbn.net/ktc25

Invariance under R3

This is Theta.nb of http://drorbn.net/ktc25/ap.

```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/ktc25/ap to compute rotation numbers.
Loading PolyPlot.m from
  http://drorbn.net/ktc25/ap to plot 2-variable polynomials.
```

```
T₃ = T₁ T₂;
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```

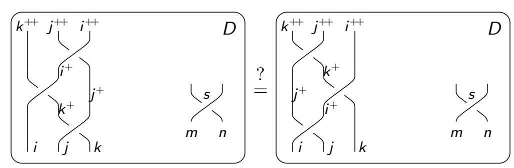
ωεβ:=http://drorbn.net/ktc25

```
F₁[{s_, i_, j_}] =
  CF[
   s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj - (T2^s - 1) g2ji g3ii + 2 g2jj g3ii -
      (1 - T3^s) g2ji g3ji - g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
      ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
         (T3^s - 1) g3ji (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji + (T2^s - 2) g2jj + g2ij)) /
       (T2^s - 1))];
F₂[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
 CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1
   ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
F₃[φ_, k_] = -φ/2 + φ g3kk;
```

ωεβ:=http://drorbn.net/ktc25

```
δ_{i_,j_} := If[i === j, 1, 0];
gR_{s_,i_,j_} := {
   g_{ν_ jβ_} :→ g_{ν j⁺ β} + δ_{jβ}, g_{ν_ iβ_} :→ T_ν^s g_{ν i⁺ β} + (1 - T_ν^s) g_{ν j⁺ β} + δ_{iβ},
   g_{ν_ α_ i⁺} :→ T_ν^s g_{ναi} + δ_{αi⁺}, g_{ν_ α_ j⁺} :→ g_{ναj} + (1 - T_ν^s) g_{ναi} + δ_{αj⁺}
  }
```

ωεβ:=http://drorbn.net/ktc25




```
DSum[Cs___] := Sum[F₁[c], {c, {Cs}}] +
   Sum[F₂[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
lhs = DSum[{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}, {s, m, n}] //.
   gR_{1,j,k} ∪ gR_{1,i,k⁺} ∪ gR_{1,i⁺,j⁺};
rhs = DSum[{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}, {s, m, n}] //.
   gR_{1,i,j} ∪ gR_{1,i⁺,k} ∪ gR_{1,j⁺,k⁺};
Simplify[lhs == rhs]
```

```
True
```

ωεβ:=http://drorbn.net/ktc25

The Main Program

```
Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, θ},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :→ (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                                             0      -1   ))];
   Δ = T^((-Total[φ] - Total[Cs[[All,1]]])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] := Factor[ℰ /. g_{ν_,α_,β_} :→ (G[[α, β]] /. T → T_ν)];
   θ = ev[Sum_{k=1}^{n} F₁[Cs[[k]]]];
   θ += ev[Sum_{k1=1}^{n} Sum_{k2=1}^{n} F₂[Cs[[k1]], Cs[[k2]]]];

   θ += ev[Sum_{k=1}^{2 n} F₃[φ[[k]], k]];
   Factor@{Δ, (Δ /. T → T₁) (Δ /. T → T₂) (Δ /. T → T₃) θ}];
```

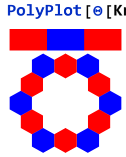
ωεβ:=http://drorbn.net/ktc25

The Trefoil Knot

```
Θ[Knot[3, 1]] // Expand
```

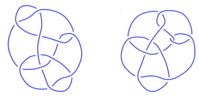
$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2T_2^2}+\frac{1}{T_1T_2^2}+\frac{1}{T_1^2T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2T_2-T_2^2+T_1T_2^2-T_1^2T_2^2\right\}$$

```
PolyPlot[Θ[Knot[3, 1]], ImageSize → Tiny]
```

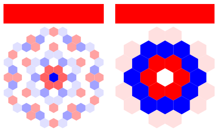


Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.

The Conway and Kinoshita-Terasaka Knots



```
GraphicsRow[PolyPlot[Θ[Knot[#]], ImageSize → Tiny] & /@
  {"K11n34", "K11n42"}]
```

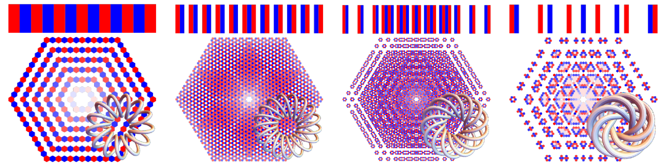


(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

ωεβ:=http://drorbn.net/ktc25

The Torus Knots $TK_{13/2}$, $TK_{17/3}$, $TK_{13,5}$, and $TK_{7,6}$

```
GraphicsRow[ImageCompose[
    PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
    TubePlot[TorusKnot @@ #, ImageSize → 240],
    {Right, Bottom}, {Right, Bottom}
  ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```



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Questions, Conjectures, Expectations, Dreams.

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Question 1.

What's the relationship between Θ and the Garoufalidis-Kashaev invariants [GK, GL]?

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Conjecture 2.

On classical (non-virtual) knots, θ always has hexagonal (D_6) symmetry.

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Conjecture 3.

θ is the ϵ^1 contribution to the "solvable approximation" of the sl_3 universal invariant, obtained by running the quantization machinery on the double $\mathcal{D}(\mathfrak{b}, b, \epsilon\delta)$, where $\mathfrak{b}$ is the Borel subalgebra of sl_3, b is the bracket of $\mathfrak{b}$, and δ the cobracket. See [BV2, BN1, Sch]

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Conjecture 4.

θ is equal to the "two-loop contribution to the Kontsevich Integral", as studied by Garoufalidis, Rozansky, Kricker, and in great detail by Ohtsuki [GR, Ro1, Ro2, Ro3, Kr, Oh].

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Fact 5. θ has a perturbed Gaussian integral formula, with integration carried out over a space $6E$, consisting of 6 copies of the space of edges of a knot diagram D. See [BN2].

Conjecture 6. For any knot K, its genus $g(K)$ is bounded by the T_1-degree of θ: $2g(K) \geq \deg_{T_1} \theta(K)$.

Conjecture 7. $\theta(K)$ has another perturbed Gaussian integral formula, with integration carried out over over the space $6H_1$, consisting of 6 copies of $H_1(\Sigma)$, where Σ is a Seifert surface for K.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.

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Question 8.

Is there a direct quantum field theory derivation of θ? Perhaps using the ϵ-expansion (at constant k!) of Chern-Simons-Witten theory with gauge group $\mathfrak{g}_+^\epsilon := \mathcal{D}(\mathfrak{b}, b, \epsilon\delta)$ with some Seifert-surface-dependent gauge fixing?

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Expectation 9.

There are many further invariants like θ, given by Green function formulas and/or Gaussian integration formulas. One or two of them may be stronger than θ and as computable.

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Dream 10.

These invariants can be explained by something less foreign than semisimple Lie algebras.

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Dream 11.

θ will have something to say about ribbon knots.

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Thank You!

ωεβ:=http://drorbn.net/ktc25

References.

[BN1] D. Bar-Natan, Everything around sl_{2+}^ϵ is DoPeGDO. So what?, talk in Da Nang, May 2019. Handout and video at ωεβ/DPG.

[BN2] —, Knot Invariants from Finite Dimensional Integration, talks in Beijing (July 2024, ωεβ/icbs24) and in Geneva (August 2024, ωεβ/ge24).

[BV1] —, R. van der Veen, A Perturbed-Alexander Invariant, Quantum Topology **15** (2024) 449–472, ωεβ/APAI.

[BV2] —, —, Perturbed Gaussian Generating Functions for Universal Knot Invariants, arXiv: 2109.02057.

[DHOEBL] N. Dunfield, A. Hirani, M. Obeidin, A. Ehrenberg, S. Bhattacharyya, D. Lei, and others, Random Knots: A Preliminary Report, lecture notes at ωεβ/DHOEBL. Also a data file at ωεβ/DD.

[GK] S. Garoufalidis, R. Kashaev, Multivariable Knot Polynomials from Braided Hopf Algebras with Automorphisms, arXiv:2311.11528.

[GL] —, S. Y. Li, Patterns of the V_2-polynomial of knots, arXiv:2409.03557.

[GR] —, L. Rozansky, The Loop Expansion of the Kontsevich Integral, the Null-Move, and S-Equivalence, arXiv:math.GT/0003187.

ωεβ:=http://drorbn.net/ktc25

[Jo] V. F. R. Jones, Hecke Algebra Representations of Braid Groups and Link Polynomials, Annals Math., **126** (1987) 335-388.

[Kr] A. Kricker, The Lines of the Kontsevich Integral and Rozansky's Rationality Conjecture, arXiv:math/0005284.

[LTW] X-S. Lin, F. Tian, Z. Wang, Burau Representation and Random Walk on String Links, Pac. J. Math., **182-2** (1998) 289–302, arXiv:q-alg/9605023.

[Oh] T. Ohtsuki, On the 2–loop Polynomial of Knots, Geom. Top. **11** (2007) 1357–1475.

[Ov] A. Overbay, Perturbative Expansion of the Colored Jones Polynomial, Ph.D. thesis, University of North Carolina, Aug. 2013, ωεβ/Ov.

[Ro1] L. Rozansky, A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I, Comm. Math. Phys. **175-2** (1996) 275–296, arXiv: hep-th/9401061.

[Ro2] —, The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial, Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] —, A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture, arXiv: math/0201139.

ωεβ:=http://drorbn.net/ktc25

[Sch] S. Schaveling, Expansions of Quantum Group Invariants, Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.

Shifted Partial Quadratics, their Pushforwards, and Signature Invariants for Tangles

Abstract. Following a general discussion of the computation of zombians of unfinished columbaria (with examples), I will tell you about my recent joint work w/ Jessica Liu on what we feel is the "textbook" extension of knot signatures to tangles, which for unknown reasons, is not in any of the textbooks that we know.

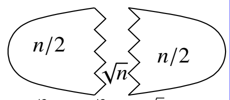

Jessica Liu

Columbaria in an East Sydney Cemetery Jacobian, Hamiltonian, Zombian

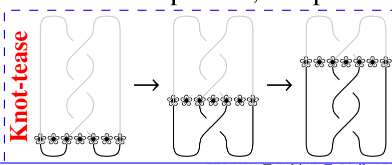
Prior Art on signatures for tangles / braids. Gambaudo and Ghys [GG], Cimasoni and Conway [CC], Conway [Co], Merz [Me]. All define signatures of tangles / braids by first closing them to links and then work hard to derive composition properties.

Why Tangles? • Faster!

- Conceptually clearer proofs of invariance (and of skein relations).
- Often fun and consequential:
  - The Jones Polynomial ⇝ The Temperley-Lieb Algebra.
  - Khovanov Homology ⇝ "Unfinished complexes", complexes in a category.
  - The Kontsevich Integral ⇝ Associators.
  - HFK ⇝ OMG, type D, type A, $\mathcal{A}_\infty$, ...



$2^{n/2}+2^{n/2}+2^{\sqrt{n}} \ll 2^n$



Computing Zombians of Unfinished Columbaria.

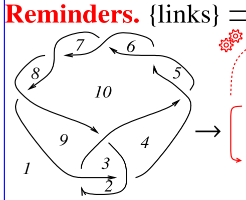
- Must be no slower than for finished ones.
- Future zombies must be able to complete the computation.
- Future zombies must not even know the size of the task that today's zombies were facing.
- We must be able to extend to ZPUCs, Zombie Processed Unfinished Columbaria!

Example / Exercise. Compute the determinant of a $1,000 \times 1,000$ matrix in which 50 entries are not yet given.

Columbarium near Assen

Homework / Research Projects. • What with ZPUCs? • Use this to get an Alexander tangle invariant.

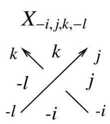
Reminders. $\{\text{links}\} \rightrightarrows \{\text{matrices / quadratic forms}\} \xrightarrow[\sigma]{\text{signature}} \mathbb{Z}$:



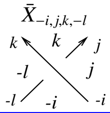
$\to$ $A=\bar{A}^T$ $\to$ $\begin{array}{c}1 \quad \cdots \quad 10\end{array}$ $\begin{pmatrix} 1 & & & & \\ & \ddots\ \sigma_+ & & & \\ & & -1 & & \\ & & & \ddots\ \sigma_- & \\ & & & & 0\ \ddots \end{pmatrix}$ $\to \sigma = \sigma_+ - \sigma_- + s$

With $|\omega|=1$, $t=1-\omega$, $r=t+\bar{t}$, $v=\mathrm{Re}(\omega)$, and $u=\mathrm{Re}(\omega^{1/2})$:

Tristram-Levine *(TL)* — Kashaev *(Kas)*

$X_{-i,j,k,-l}$ 

$A \mathrel{+}= \begin{array}{c} \begin{pmatrix} -r & -t & 2t & \bar{t} \\ -\bar{t} & 0 & \bar{t} & 0 \\ 2\bar{t} & t & -r & -\bar{t} \\ t & 0 & -t & 0 \end{pmatrix} \end{array}\begin{array}{c} i \\ j \\ k \\ l \end{array}$, $s=0$ $\qquad$ $A \mathrel{+}= \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}\begin{array}{c} i \\ j \\ k \\ l \end{array}$, $s \mathrel{-}= 1$

$\bar{X}_{-i,j,k,-l}$ 

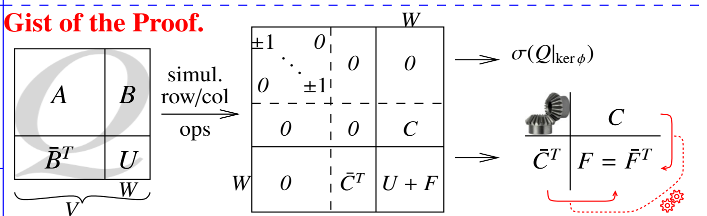
$A \mathrel{+}= \begin{pmatrix} r & -t & -2\bar{t} & \bar{t} \\ -\bar{t} & 0 & \bar{t} & 0 \\ -2t & t & r & -\bar{t} \\ t & 0 & -t & 0 \end{pmatrix}\begin{array}{c} i \\ j \\ k \\ l \end{array}$, $s=0$ $\qquad$ $A \mathrel{-}= \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}\begin{array}{c} i \\ j \\ k \\ l \end{array}$, $s \mathrel{+}= 1$

~~**Kashaev's Conjecture**~~ [Ka] For links, $\sigma_{Kas} = 2\sigma_{TL}$.
Liu's Theorem [Li].

A *Partial Quadratic (PQ)* on V is a quadratic Q defined only on a subspace $\mathcal{D}_Q \subset V$. We add PQs with $\mathcal{D}_{Q_1+Q_2} := \mathcal{D}_{Q_1} \cap \mathcal{D}_{Q_2}$. Given a linear $\psi\colon V \to W$ and a PQ Q on W, there is an obvious *pullback* ψ^*Q, a PQ on V.

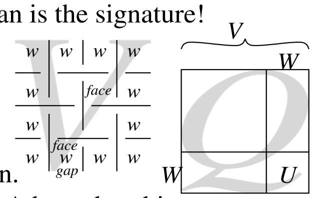
Theorem 1. Given a linear $\phi\colon V \to W$ and a PQ Q on V, there is a unique *pushforward* PQ ϕ_*Q on W such that for every *PQ* U on W, $\sigma_V(Q+\phi^*U) = \sigma_{\ker\phi}(Q|_{\ker\phi}) + \sigma_W(U+\phi_*Q)$.
(If you must, $\mathcal{D}(\phi_*Q) = \phi(\mathrm{ann}_Q(\mathcal{D}(Q) \cap \ker\phi))$ and $(\phi_*Q)(w) = Q(v)$, where v is s.t. $\phi(v)=w$ and $Q(v, \mathrm{rad}\, Q|_{\ker\phi}) = 0$).

Gist of the Proof.



... and the quadratic $F =: \phi_*Q$ is well-defined only on $D := \ker C$.

Exactly what we want, if the Zombian is the signature!
V: The full space of *faces*.
W: The boundary, made of *gaps*.
Q: The known parts.
U: The part yet unknown.
$\sigma_V(Q+\phi^*(U))$: The overall Zombian.
$\sigma(Q|_{\ker\phi})$: An internal bit. $U+\phi_*Q$: A boundary bit.
And so our ZPUC is the pair $S = (\sigma(Q|_{\ker\phi}), \phi_*Q)$.



A *Shifted Partial Quadratic (SPQ)* on V is a pair $S = (s \in \mathbb{Z}, Q$ a PQ on $V)$. addition also adds the shifts, pullbacks keep the shifts, yet $\phi_*S := (s + \sigma_{\ker\phi}(Q|_{\ker\phi}), \phi_*Q)$ and $\sigma(S) := s + \sigma(Q)$.

Theorem 1' (*Reciprocity*). Given $\phi\colon V \to W$, for SPQs S on V and U on W we have $\sigma_V(S+\phi^*U) = \sigma_W(U+\phi_*S)$ (and this characterizes ϕ_*S). Note. ψ^* is additive but ϕ_* is not.

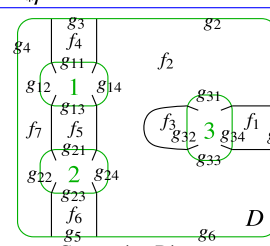
Theorem 2. ψ^* and ϕ_* are functorial.

Theorem 3. "The pullback of a pushforward scene is a pushforward scene": If, on the right, β and δ are arbitrary, $Y = \mathrm{EQ}(\beta,\gamma) = V \oplus_Z W = \{(v,w)\colon \beta v = \gamma w\}$ and μ and ν are the obvious projections, then $\gamma^*\beta_* = \nu_*\mu^*$.

$$\begin{array}{ccc} Y & \xrightarrow{\nu} & W \\ {\scriptstyle\mu}\downarrow & \nearrow & \downarrow{\scriptstyle\gamma} \\ V & \xrightarrow[\beta]{} & Z \end{array}$$

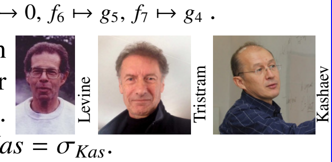
Definition. $\mathcal{S}\left(\begin{array}{c} g_2 \\ g_3 \quad\quad g_1 \\ \ldots \end{array}\right) := \left\{\begin{array}{l}\text{SPQ } S \\ \text{on } \langle g_i \rangle\end{array}\right\}$.

Theorem 4. $\{\mathcal{S}(\text{cyclic sets})\}$ is a planar algebra, with compositions $\mathcal{S}(D)((S_i)) := \phi_*^D(\psi_D^*(\bigoplus_i S_i))$, where $\psi_D\colon \langle f_i \rangle \to \langle g_{\alpha i} \rangle$ maps every face of D to the sum of the input gaps adjacent to it and $\phi^D\colon \langle f_i \rangle \to \langle g_i \rangle$ maps every face to the sum of the output gaps adjacent to it. So for our D, $\psi_D\colon$ $f_1 \mapsto g_{34}$, $f_2 \mapsto g_{31}+g_{14}+g_{24}+g_{33}$, $f_3 \mapsto g_{32}$, $f_4 \mapsto g_{11}$, $f_5 \mapsto g_{13}+g_{21}$, $f_6 \mapsto g_{23}$, $f_7 \mapsto g_{12}+g_{22}$ and ϕ^D: $f_1 \mapsto g_1$, $f_2 \mapsto g_2+g_6$, $f_3 \mapsto 0$, $f_4 \mapsto g_3$, $f_5 \mapsto 0$, $f_6 \mapsto g_5$, $f_7 \mapsto g_4$.



Connection Diagram

Theorem 5. *TL* and *Kas*, defined on X and $\bar{X}$ as before, extend to planar algebra morphisms $\{\text{tangles}\} \to \{\mathcal{S}\}$. Restricted to links, $TL = \sigma_{TL}$ and $Kas = \sigma_{Kas}$.



Video: http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201. Handout: http://www.math.toronto.edu/~drorbn/Talks/USC-240205.

Implementation (sources: http://drorbn.net/icerm23/ap). I like it most when the implementation matches the math perfectly. We failed here.

```
Once[<< KnotTheory`];
```

```
Loading KnotTheory` version
  of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

Utilities. The step function, algebraic numbers, canonical forms.

```
θ[x_] /; NumericQ[x] := UnitStep[x]
```

```
ω2[v_][p_] := Module[{q = Expand[p], n, c},
   If[q === 0, 0,
    c = Coefficient[q, ω, n = Exponent[q, ω]];
    c v^n + ω2[v][q - c (ω + ω^-1)^n]]];
```

```
sign[ℰ_] := Module[{n, d, v, p, rs, e, k},
  {n, d} = NumeratorDenominator[ℰ];
  {n, d} /= ω^(Exponent[n,ω]/2+Exponent[n,ω,Min]/2);
  p = Factor[ω2[v]@n * ω2[v]@d /. v → 4 u^2 - 2];
  rs = Solve[p == 0, u, Reals];
  If[rs === {}, Sign[p /. u → 0],
   rs = Union@(u /. rs);
   Sign[(-1)^(e=Exponent[p,u]) Coefficient[p, u, e]] + Sum[
     k = 0;
     While[(d = RootReduce[∂_{u,++k} p /. u → r]) == 0];
     If[EvenQ[k], 0, 2 Sign[d]] * θ[u - r],
     {r, rs}]
  ]
 ]
```

```
SetAttributes[B, Orderless];
CF[b_B] := RotateLeft[#, First@Ordering[#] - 1] & /@
  DeleteCases[b, {}]
```

```
CF[ℰ_] := Module[{γs = Union@Cases[ℰ, γ_ | γ̄_, ∞]},
  Total[CoefficientRules[ℰ, γs] /.
    (ps_ → c_) :→ Factor[c] × Times @@ γs^ps] ]
```

```
CF[{}] = {};
CF[𝒞_List] :=
 Module[{γs = Union@Cases[𝒞, γ_, ∞], γ},
  CF /@ DeleteCases[0][
    RowReduce[Table[∂_γ r, {r, 𝒞}, {γ, γs}]].γs] ]
```

```
(ℰ_)* := ℰ /. {γ̄ → γ, γ → γ̄, ω → ω^-1, c_Complex :→ c*};
r_Rule⁺ := {r, r*}
```

```
RulesOf[γi_ + rest_.] := (γi → -rest)⁺;
CF[PQ[𝒞_, q_]] := Module[{n𝒞 = CF[𝒞]},
  PQ[n𝒞, CF[q /. Union @@ RulesOf /@ n𝒞]] ]
```

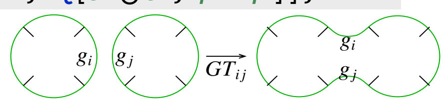
```
CF[Σ_b_[σ_, pq_]] := Σ_CF[b][σ, CF[pq]]
```

Pretty-Printing.

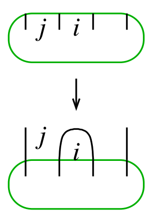
```
Format[Σ_b_B[σ_, PQ[𝒞_, q_]]] := Module[{γs},
  γs = γ_# & /@ Join @@ b;
  Column[{TraditionalForm@σ,
    TableForm[Join[
      Prepend[""] /@ Table[TraditionalForm[∂_c r],
        {r, 𝒞}, {c, γs}],
      {Prepend[""][
        Join @@
         (b /. {l_, m___, r_} :→
            {DisplayForm@RowBox[{"(", l}],
             m, DisplayForm@RowBox[{r, ")"}]}) /.
         i_Integer :→ γi ]},
      MapThread[Prepend,
       {Table[TraditionalForm[∂_{r,c} q], {r, γs*},
         {c, γs}], γs*}]
     ], TableAlignments → Center]
   }, Center] ];
```

The Face-Centric Core.

```
Σ_b1_[σ1_, PQ[𝒞1_, q1_]] ⊕ Σ_b2_[σ2_, PQ[𝒞2_, q2_]] ^:=
  CF@Σ_Join[b1,b2][σ1 + σ2, PQ[𝒞1 ⋃ 𝒞2, q1 + q2]];
```

GT for Gap Touch: 


```
GT_i_,j_@Σ_B[{li___,i_,ri___},{lj___,j_,rj___},bs___][σ_,
   PQ[𝒞_, q_]] :=
  CF@Σ_B[{ri,li,j,rj,lj,i},bs][σ, PQ[𝒞 ⋃ {γi - γj}, q]]
```



cor·don (kôr′dn)

n.

1. A line of people, military posts, or ships stationed around an area to enclose or guard it: *a police cordon.*
2. A rope, line, tape, or similar border stretched around an area, usually by the police, indicating that access is restricted.

$$\begin{array}{c} s \\ i \\ \\ \end{array}\begin{pmatrix} 0 & \phi & C_{\text{rest}} \\ \bar\phi^T & \lambda & \theta \\ \bar C^T_{\text{rest}} & \bar\theta^T & A_{\text{rest}} \end{pmatrix} \to \begin{cases} \exists p\ \phi_p \neq 0 & \text{use } \phi_p \text{ to kill its row and column, drop a } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ summand} \\ \phi = 0, \lambda \neq 0 & \text{use } \lambda \text{ to kill } \theta\text{, let } s \mathrel{+}= \operatorname{sign}(\lambda) \\ \phi = 0, \lambda = 0 & \text{append } \theta \text{ to } C_{\text{rest}}. \end{cases}$$

```
Cordon_i_@Σ_B[{li___,i_,ri___},bs___][σ_, PQ[𝒞_, q_]] :=
 Module[{ϕ = ∂_γi 𝒞, λ = ∂_{γ̄i,γi} q, nσ = σ, n𝒞, nq, p},
  {p} = FirstPosition[(# =!= 0) & /@ ϕ, True, {0}];
  {n𝒞, nq} = Which[
    p > 0, {𝒞, q} /. (γi → -𝒞[[p]] / ϕ[[p]])⁺ /. (γi → 0)⁺,
    λ =!= 0, (nσ += sign[λ];
      {𝒞, q /. (γi → -(∂_γ̄i q) / λ)⁺ /. (γi → 0)⁺}),
    λ === 0, {𝒞 ⋃ {∂_γ̄i q}, q /. (γi → 0)⁺}];
  CF@Σ_B[Most@{ri,li},bs][nσ,
    PQ[n𝒞, nq] /. (γ_Last@{ri,li} → γ_First@{ri,li})⁺] ]
```

Video: http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201. Handout: http://www.math.toronto.edu/~drorbn/Talks/USC-240205.

Strand Operations. c for contract, mc for magnetic contract:

```
c_{i_,j_}@t : Σ_B[{li___,i_,ri___},{___,j_,___},___][__] :=
  t // GT_{j,First@{ri,li}} // Cordon_j
```

```
c_{i_,j_}@t : Σ_B[{___,i_,j_,___},___][__] := Cordon_j@t
c_{i_,j_}@t : Σ_B[{j_,___,i_},___][__] := Cordon_j@t
c_{i_,j_}@t : Σ_B[{___,j_,i_,___},___][__] := Cordon_i@t
c_{i_,j_}@t : Σ_B[{i_,___,j_},___][__] := Cordon_i@t
```

```
mc[ℰ_] := ℰ //.
  t : Σ_B[{___,i_,___},{___,j_,___},___][__] |
      Σ_B[{___,i_,j_,___},___][__] | Σ_B[{j_,___,i_},___][__] /;
    i + j == 0 :> c_{i,j}@t
```

The Crossings (and empty strands).

```
Kas@P_{i_,j_} := CF@Σ_B[{i,j}][0, PQ[{}, 0]];
TL@P_{i_,j_} := CF@Σ_B[{i,j}][0, PQ[{}, 0]]
```

```
Kas[x : X[i_, j_, k_, l_]] :=
  Kas@If[PositiveQ[x], X_{-i,j,k,-l}, X̄_{-j,k,l,-i}];
Kas[(x : X | X̄)_{fs__}] := Module[{v = 2 u^2 - 1, p, γs, m},
  γs = γ_# & /@ {fs}; p = (x === X);
  m = If[p, {{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}},
           -{{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}}];
  CF@Σ_B[{fs}][If[p, -1, 1], PQ[{}, γs*.m.γs]]]
```

$$m = \mathrm{If}\left[p,\ \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix},\ -\begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}\right];$$

```
TL[x : X[i_, j_, k_, l_]] :=
  TL@If[PositiveQ[x], X_{-i,j,k,-l}, X̄_{-j,k,l,-i}];
TL[(x : X | X̄)_{fs__}] := Module[{t = 1 - ω, r, γs, m},
  r = t + t*; γs = γ_# & /@ {fs};
  m = If[x === X,
        {{-r, -t, 2t, t*}, {-t*, 0, t*, 0}, {2t*, t, -r, -t*}, {t, 0, -t, 0}},
        {{r, -t, -2t*, t*}, {-t*, 0, t*, 0}, {-2t, t, r, -t*}, {t, 0, -t, 0}}];
  CF@Σ_B[{fs}][0, PQ[{}, γs*.m.γs]]]
```

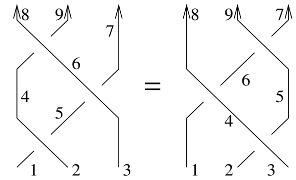
$$m = \mathrm{If}\left[x === X,\ \begin{pmatrix} -r & -t & 2t & t^* \\ -t^* & 0 & t^* & 0 \\ 2t^* & t & -r & -t^* \\ t & 0 & -t & 0 \end{pmatrix},\ \begin{pmatrix} r & -t & -2t^* & t^* \\ -t^* & 0 & t^* & 0 \\ -2t & t & r & -t^* \\ t & 0 & -t & 0 \end{pmatrix}\right];$$

Evaluation on Tangles and Knots.

```
Kas[K_] := Fold[mc[#1 ⊕ #2] &, Σ_B[][0, PQ[{}, 0]],
    List @@ (Kas /@ PD@K)];
KasSig[K_] := Expand[Kas[K]〚1〛 / 2]
```

```
TL[K_] :=
  Fold[mc[#1 ⊕ #2] &, Σ_B[][0, PQ[{}, 0]],
    List @@ (TL /@ PD@K)] /.
   θ[c_ + u] /; Abs[c] ≥ 1 :> θ[c];
TLSig[K_] := TL[K]〚1〛
```

Reidemeister 3.




```
R3L = PD[X_{-2,5,4,-1}, X_{-3,7,6,-5},
    X_{-6,9,8,-4}];
R3R = PD[X_{-3,5,4,-2}, X_{-4,6,8,-1},
    X_{-5,7,9,-6}];
{TL@R3L == TL@R3R, Kas@R3L == Kas@R3R}
```

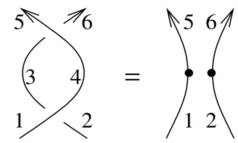
{True, True}

```
Kas@R3L
```

$$2\,\theta\left(u-\tfrac{1}{2}\right) - 2\,\theta\left(u+\tfrac{1}{2}\right) - 2$$

$$\begin{array}{c|cccccc}
 & (\gamma_{-3} & \gamma_7 & \gamma_9 & \gamma_8 & \gamma_{-1} & \gamma_{-2}) \\
\bar\gamma_{-3} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} \\
\bar\gamma_{7} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2(2u^2-1)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} \\
\bar\gamma_{9} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} \\
\bar\gamma_{8} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} \\
\bar\gamma_{-1} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2(2u^2-1)}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} \\
\bar\gamma_{-2} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & -\frac{2u}{(2u-1)(2u+1)} & -\frac{1}{(2u-1)(2u+1)} & \frac{u(4u^2-3)}{(2u-1)(2u+1)} & \frac{2u^2(4u^2-3)}{(2u-1)(2u+1)}
\end{array}$$

Reidemeister 2.




```
TL@PD[X_{-2,4,3,-1}, X̄_{-4,6,5,-3}]
```

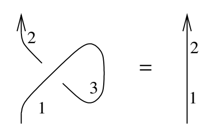
$$0$$

$$\begin{array}{c|cccc}
 & 1 & 0 & -1 & 0 \\
 & (\gamma_{-2} & \gamma_6 & \gamma_5 & \gamma_{-1}) \\
\bar\gamma_{-2} & 0 & 0 & 0 & 0 \\
\bar\gamma_{6} & 0 & 0 & 0 & 0 \\
\bar\gamma_{5} & 0 & 0 & 0 & 0 \\
\bar\gamma_{-1} & 0 & 0 & 0 & 0
\end{array}$$

```
{TL@PD[X_{-2,4,3,-1}, X̄_{-4,6,5,-3}] == GT_{5,-2}@TL@PD[P_{-1,5}, P_{-2,6}],
 Kas@PD[X_{-2,4,3,-1}, X̄_{-4,6,5,-3}] == GT_{5,-2}@Kas@PD[P_{-1,5}, P_{-2,6}]}
```

{True, True}

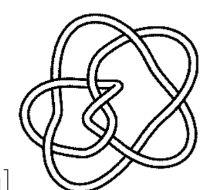
Reidemeister 1.




```
{TL@PD[X_{-3,3,2,-1}] == TL@P_{-1,2},
 Kas@PD[X_{-3,3,2,-1}] == Kas@P_{-1,2}}
```

{True, True}

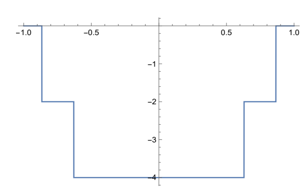
A Knot.



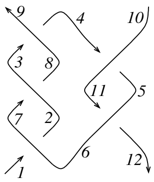
```
f = TLSig[Knot[8, 5]]
```

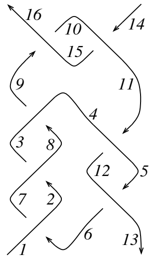
$$2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right] - 2\,\theta\left[\frac{\sqrt{3}}{2}+u\right] - 2\,\theta\left[u - (-0.630\ldots)\right] + 2\,\theta\left[u - 0.630\ldots\right]$$

```
Plot[f, {u, -1, 1}]
```



The Conway-Kinoshita-Terasaka Tangles.





```
T1 = PD[X̄-6,2,7,-1, X̄-2,8,3,-7,
    X̄-8,4,9,-3, X-11,6,12,-5,
    X-4,11,5,-10];
T2 = PD[X̄-6,2,7,-1, X̄-2,8,3,-7,
    X-8,4,9,-3, X̄-12,6,13,-5,
    X̄-4,12,5,-11, X̄-10,15,11,-14, X̄-15,10,16,-9];
```

```
Column@{TL[T1], Kas[T1]}
```

$$-2\,\theta\left(u-\tfrac{\sqrt{3}}{2}\right)+2\,\theta\left(u+\tfrac{\sqrt{3}}{2}\right)-1$$

$$\begin{array}{c|cccc}
 & \gamma_{-10} & \gamma_{9} & \gamma_{-1} & \gamma_{12} \\
\hline
\bar\gamma_{-10} & 0 & 1-\omega & 0 & \omega-1 \\
\bar\gamma_{9} & \frac{\omega-1}{\omega} & \frac{2\omega}{\omega^2-\omega+1} & -\frac{\omega-1}{\omega} & -\frac{2\omega}{\omega^2-\omega+1} \\
\bar\gamma_{-1} & 0 & \omega-1 & 0 & 1-\omega \\
\bar\gamma_{12} & -\frac{\omega-1}{\omega} & -\frac{2\omega}{\omega^2-\omega+1} & \frac{\omega-1}{\omega} & \frac{2\omega}{\omega^2-\omega+1}
\end{array}$$

$$-2\,\theta\left(u-\tfrac{\sqrt{3}}{2}\right)+2\,\theta\left(u+\tfrac{\sqrt{3}}{2}\right)-1$$

$$\begin{array}{c|cccc}
 & \gamma_{-10} & \gamma_{9} & \gamma_{-1} & \gamma_{12} \\
\hline
\bar\gamma_{-10} & 2(u-1)(u+1)(4u^2-3) & 0 & -2(u-1)(u+1)(4u^2-3) & 0 \\
\bar\gamma_{9} & 0 & \frac{1}{2(4u^2-3)} & 0 & -\frac{1}{2(4u^2-3)} \\
\bar\gamma_{-1} & -2(u-1)(u+1)(4u^2-3) & 0 & 2(u-1)(u+1)(4u^2-3) & 0 \\
\bar\gamma_{12} & 0 & -\frac{1}{2(4u^2-3)} & 0 & \frac{1}{2(4u^2-3)}
\end{array}$$

```
Column@{TL[T2], Kas[T2]}
```

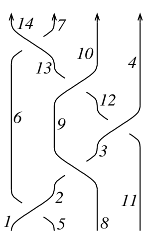
$$0$$

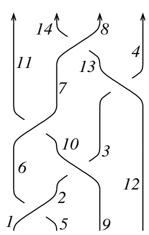
$$\begin{array}{c|cccc}
 & \gamma_{-14} & \gamma_{16} & \gamma_{-1} & \gamma_{13} \\
\hline
\bar\gamma_{-14} & 0 & 1-\omega & 0 & \omega-1 \\
\bar\gamma_{16} & \frac{\omega-1}{\omega} & -\frac{2(\omega-1)^2\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1} & -\frac{\omega-1}{\omega} & \frac{2(\omega-1)^2\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1} \\
\bar\gamma_{-1} & 0 & \omega-1 & 0 & 1-\omega \\
\bar\gamma_{13} & -\frac{\omega-1}{\omega} & \frac{2(\omega-1)^2\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1} & \frac{\omega-1}{\omega} & -\frac{2(\omega-1)^2\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1}
\end{array}$$

$$1$$

$$\begin{array}{c|cccc}
 & \gamma_{-14} & \gamma_{16} & \gamma_{-1} & \gamma_{13} \\
\hline
\bar\gamma_{-14} & \frac12(-16u^4+28u^2-13) & 0 & \frac12(16u^4-28u^2+13) & 0 \\
\bar\gamma_{16} & 0 & -\frac{2(u-1)(u+1)}{16u^4-28u^2+13} & 0 & \frac{2(u-1)(u+1)}{16u^4-28u^2+13} \\
\bar\gamma_{-1} & \frac12(16u^4-28u^2+13) & 0 & \frac12(-16u^4+28u^2-13) & 0 \\
\bar\gamma_{13} & 0 & \frac{2(u-1)(u+1)}{16u^4-28u^2+13} & 0 & -\frac{2(u-1)(u+1)}{16u^4-28u^2+13}
\end{array}$$

Examples with non-trivial co-dimension.





```
B1 = PD[X̄-5,2,6,-1, X̄-8,3,9,-2,
    X-11,4,12,-3, X-12,10,13,-9,
    X̄-13,7,14,-6];
B2 = PD[X-5,2,6,-1, X̄-9,3,10,-2,
    X-10,7,11,-6, X̄-12,4,13,-3, X-13,8,14,-7];
```

```
Column@{TL[B1], Kas[B1]}
```

$$0$$

$$\begin{array}{c|cccccccc}
 & 1 & 0 & -1 & 0 & \frac{1}{\omega} & 0 & -\frac{1}{\omega} & 0 \\
 & 0 & 0 & 0 & -1 & \frac{1}{\omega} & 0 & -\frac{1}{\omega} & 1 \\
 & \gamma_{-11} & \gamma_{4} & \gamma_{10} & \gamma_{7} & \gamma_{14} & \gamma_{-1} & \gamma_{-5} & \gamma_{-8} \\
\hline
\bar\gamma_{-11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{4} & 0 & 0 & 0 & 0 & \frac{\omega-1}{\omega^2} & 0 & -\frac{\omega-1}{\omega^2} & 0 \\
\bar\gamma_{10} & 0 & 0 & 0 & 0 & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 \\
\bar\gamma_{7} & 0 & 0 & 0 & 0 & \frac{(\omega-1)^2}{\omega^2} & 0 & -\frac{(\omega-1)^2}{\omega^2} & 0 \\
\bar\gamma_{14} & 0 & -((\omega-1)\omega) & \omega-1 & (\omega-1)^2 & 0 & -\frac{\omega-1}{\omega} & \frac{\omega-1}{\omega} & 0 \\
\bar\gamma_{-1} & 0 & 0 & 0 & 0 & \omega-1 & 0 & 1-\omega & 0 \\
\bar\gamma_{-5} & 0 & (\omega-1)\omega & 1-\omega & -(\omega-1)^2 & 1-\omega & \frac{\omega-1}{\omega} & \frac{(\omega-1)^2}{\omega} & 0 \\
\bar\gamma_{-8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

$$0$$

$$\begin{array}{c|cccccccc}
 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\
 & \gamma_{-11} & \gamma_{4} & \gamma_{10} & \gamma_{7} & \gamma_{14} & \gamma_{-1} & \gamma_{-5} & \gamma_{-8} \\
\hline
\bar\gamma_{-11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{4} & 0 & 0 & 0 & -1 & -u & 0 & u & 1 \\
\bar\gamma_{10} & 0 & 0 & 0 & -u & 1-2u^2 & 0 & 2u^2-1 & u \\
\bar\gamma_{7} & 0 & -1 & -u & 2u^2-3 & -u & -1 & 0 & 1 \\
\bar\gamma_{14} & 0 & -u & 1-2u^2 & -u & -1 & -u & -2(u-1)(u+1) & u \\
\bar\gamma_{-1} & 0 & 0 & 0 & -1 & -u & 0 & u & 1 \\
\bar\gamma_{-5} & 0 & u & 2u^2-1 & 0 & -2(u-1)(u+1) & u & 4u^2-3 & 0 \\
\bar\gamma_{-8} & 0 & 1 & u & 1 & u & 1 & 0 & 1-2u^2
\end{array}$$

```
Column@{TL[B2], Kas[B2]}
```

$$0$$

$$\begin{array}{c|cccccccc}
 & \gamma_{-12} & \gamma_{4} & \gamma_{8} & \gamma_{14} & \gamma_{11} & \gamma_{-1} & \gamma_{-5} & \gamma_{-9} \\
\hline
\bar\gamma_{-12} & \frac{(\omega-1)^2}{\omega} & \omega-1 & -2(\omega-1) & \frac{2(\omega-1)^2}{\omega} & \frac{2(\omega-1)}{\omega^2} & 0 & -\frac{2(\omega-1)}{\omega^2} & -\frac{(\omega-1)(2\omega-3)}{\omega} \\
\bar\gamma_{4} & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{8} & \frac{2(\omega-1)}{\omega} & 1-\omega & \frac{(\omega-1)^2}{\omega} & -\frac{(\omega-1)(2\omega-3)}{\omega} & -\frac{2(\omega-1)}{\omega^2} & 0 & \frac{2(\omega-1)}{\omega^2} & \frac{2(\omega-2)(\omega-1)}{\omega} \\
\bar\gamma_{14} & \frac{2(\omega-1)^2}{\omega} & 0 & -\frac{(\omega-1)(3\omega-2)}{\omega} & \frac{3(\omega-1)^2}{\omega} & -\frac{(\omega-2)(\omega-1)}{\omega^2} & 0 & -\frac{2(\omega-1)}{\omega^2} & -\frac{2(\omega-2)(\omega-1)}{\omega} \\
\bar\gamma_{11} & -2(\omega-1)\omega & 0 & 2(\omega-1)\omega & -((\omega-1)(2\omega-1)) & \frac{(\omega-1)^2}{\omega} & -\frac{\omega-1}{\omega} & \frac{2(\omega-1)}{\omega} & 2(\omega-1)^2 \\
\bar\gamma_{-1} & 0 & 0 & 0 & 0 & \omega-1 & 0 & 1-\omega & 0 \\
\bar\gamma_{-5} & 2(\omega-1)\omega & 0 & -2(\omega-1)\omega & 2(\omega-1)\omega & -2(\omega-1) & \frac{\omega-1}{\omega} & \frac{(\omega-1)^2}{\omega} & -((\omega-1)(2\omega-1)) \\
\bar\gamma_{-9} & -\frac{(\omega-1)(3\omega-2)}{\omega} & 0 & \frac{2(\omega-1)(2\omega-1)}{\omega} & -\frac{2(\omega-1)(2\omega-1)}{\omega} & \frac{2(\omega-1)^2}{\omega^2} & 0 & -\frac{(\omega-2)(\omega-1)}{\omega^2} & \frac{3(\omega-1)^2}{\omega}
\end{array}$$

$$2\,\theta\left(u-\tfrac{\sqrt{3}}{2}\right)-2\,\theta\left(u+\tfrac{\sqrt{3}}{2}\right)$$

$$\begin{array}{c|cccccccc}
 & 1 & \frac{1}{2u} & 0 & -\frac{1}{2u} & -1 & -\frac{1}{2u} & 0 & \frac{1}{2u} \\
 & \gamma_{-12} & \gamma_{4} & \gamma_{8} & \gamma_{14} & \gamma_{11} & \gamma_{-1} & \gamma_{-5} & \gamma_{-9} \\
\hline
\bar\gamma_{-12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{4} & 0 & -\frac{(2u-1)(2u+1)(2u^2-1)}{4u^2(4u^2-3)} & -\frac{2u^2-1}{2u} & \frac{1}{4u^2(4u^2-3)} & 0 & -\frac{(2u-1)(2u+1)}{4u^2(4u^2-3)} & -\frac{1}{2u(4u^2-3)} & \frac{8u^4-6u^2-1}{4u^2(4u^2-3)} \\
\bar\gamma_{8} & 0 & -\frac{2u^2-1}{2u} & -2(u-1)(u+1) & \frac{2u^2-1}{2u} & 0 & -\frac{1}{2u} & 0 & \frac{1}{2u} \\
\bar\gamma_{14} & 0 & \frac{1}{4u^2(4u^2-3)} & \frac{2u^2-1}{2u} & \frac{(2u^2-1)(16u^4-16u^2+1)}{4u^2(4u^2-3)} & 0 & -\frac{8u^4-10u^2+1}{4u^2(4u^2-3)} & \frac{1}{2u(4u^2-3)} & \frac{1}{4u^2(4u^2-3)} \\
\bar\gamma_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\gamma_{-1} & 0 & -\frac{(2u-1)(2u+1)}{4u^2(4u^2-3)} & -\frac{1}{2u} & -\frac{8u^4-10u^2+1}{4u^2(4u^2-3)} & 0 & \frac{8u^4-10u^2-1}{4u^2(4u^2-3)} & \frac{8u^4-10u^2+1}{2u(4u^2-3)} & \frac{16u^4-16u^2+1}{4u^2(4u^2-3)} \\
\bar\gamma_{-5} & 0 & -\frac{1}{2u(4u^2-3)} & 0 & \frac{1}{2u(4u^2-3)} & 0 & \frac{8u^4-10u^2+1}{2u(4u^2-3)} & \frac{2(u-1)(u+1)(2u-1)(2u+1)}{4u^2-3} & \frac{8u^4-6u^2-1}{2u(4u^2-3)} \\
\bar\gamma_{-9} & 0 & \frac{8u^4-6u^2-1}{4u^2(4u^2-3)} & \frac{1}{2u} & \frac{1}{4u^2(4u^2-3)} & 0 & \frac{16u^4-16u^2+1}{4u^2(4u^2-3)} & \frac{8u^4-6u^2-1}{2u(4u^2-3)} & -\frac{32u^6-64u^4+30u^2+1}{4u^2(4u^2-3)}
\end{array}$$

$$\begin{pmatrix}A & B\\ C & U\end{pmatrix} \xrightarrow{\det(A)} \begin{pmatrix}I & A^{-1}B\\ C & U\end{pmatrix} \xrightarrow{1} \begin{pmatrix}I & A^{-1}B\\ 0 & U-CA^{-1}B\end{pmatrix},$$
so $\det\begin{pmatrix}A & B\\ C & U\end{pmatrix}=\det(A)\det(U-CA^{-1}B)$.

Roughly, $\det(A)$ is "det on ker", $-CA^{-1}B$ is "a pushforward of $\begin{pmatrix}A & B\\ C & U\end{pmatrix}$". (what if $\nexists A^{-1}$?)

Questions. 1. Does this have a topological meaning? 2. Is there a version of the Kashaev Conjecture for tangles? 3. Find all solutions of R123 in our "algebra". 4. Braids and the Burau representation. 5. Recover the work in "Prior Art". 6. Are there any concordance properties? 7. What is the "SPQ group"? 8. The jumping points of signatures are the roots of the Alexander polynomial. Does this generalize to tangles? 9. Which of the three Cordon cases is the most common? 10. Are there interesting examples of tangles for which rels is non-trivial? 11. Is the pq part determined by Γ-calculus? 12. Is the pq part determined by finite type invariants? 13. Does it work with closed components / links? 14. Strand-doubling formulas? 15. A multivariable version? 16. Mutation invariance? 17. Ribbon knots? 18. Are there "face-virtual knots"? 19. Does the pushforward story extend to ranks? To formal Gaussian measures? To super Gaussian measures?

References.

[CC] D. Cimasoni, A. Conway, *Colored Tangles and Signatures,* Math. Proc. Camb. Phil. Soc. **164** (2018) 493–530, arXiv:1507.07818.

[Co] A. Conway, *The Levine-Tristram Signature: A Survey,* arXiv:1903.04477.

[GG] J-M. Gambaudo, É. Ghys, *Braids and Signatures,* Bull. Soc. Math. France **133-4** (2005) 541–579.

[Ka] R. Kashaev, *On Symmetric Matrices Associated with Oriented Link Diagrams,* in *Topology and Geometry, A Collection of Essays Dedicated to Vladimir G. Turaev,* EMS Press 2021, arXiv:1801.04632.

[Li] J. Liu, *A Proof of the Kashaev Signature Conjecture,* arXiv:2311.01923.

[Me] A. Merz, *An Extension of a Theorem by Cimasoni and Conway,* arXiv:2104.02993.

Acknowledgement. This work was partially supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

Video: http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201. Handout: http://www.math.toronto.edu/~drorbn/Talks/USC-240205.

Some Rigor. (Exercises hints and partial solutions at end)

Exercise 1. Show that if two SPQ's S_1 and S_2 on V satisfy $\sigma(S_1+U)=\sigma(S_2+U)$ for every quadratic U on V, then they have the same shifts and the same domains.

Exercise 2. Show that if two full quadratics Q_1 and Q_2 satisfy $\sigma(Q_1+U)=\sigma(Q_2+U)$ for every U, then $Q_1=Q_2$.

Proof of Theorem 1'. Fix W and consider triples $(V,S,\phi\colon V\to W)$ where $S=(s,D,Q)$ is an SPQ on V. Say that two triples are "push-equivalent", $(V_1,S_1,\phi_1)\sim(V_2,S_2,\phi_2)$ if for every quadratic U on W,

$$\sigma_{V_1}(S_1+\phi_1^*U)=\sigma_{V_2}(S_2+\phi_2^*U).$$

Given our (V,S,ϕ), we need to show:
1. There is an SPQ S' on W such that $(V,S,\phi)\sim(W,S',I)$.
2. If $(W,S',I)\sim(W,S'',I)$ then $S'=S''$.

Property 2 is easy (Exercises 1, 2). Property 1 follows from the following three claims, each of which is easy.

Claim 1. If $v\in\ker\phi\cap D(S)$, and $\lambda:=Q(v,v)\neq 0$, then $(V,S,\phi)\sim$

$$\left(V/\langle v\rangle,\left(s+\operatorname{sign}(\lambda),D(S)/\langle v\rangle,Q-\lambda^{-1}Q(-,v)\otimes Q(v,-)\right),\phi/\langle v\rangle\right).$$

So wlog $Q|_{\ker\phi}=0$ (meaning, $Q|_{\ker\phi\otimes\ker\phi}=0$). □

Claim 2. If $Q|_{\ker\phi}=0$ and $v\in\ker\phi\cap D(S)$, let $V'=\ker Q(v,-)$ and then $(V,S,\phi)\sim(V',S|_{V'},\phi|_{V'})$ so wlog $Q|_{V\otimes\ker\phi+\ker\phi\otimes V}=0$. □

Claim 3. If $Q|_{V\otimes\ker\phi+\ker\phi\otimes V}=0$ then $S=\phi^*S'$ for some SPQ S' on $\operatorname{im}\phi$ and then $(V,S,\phi)\sim(W,S',I)$. □ □

Proof of Theorem 2. The functoriality of pullbacks needs no proof. Now assume $V_0\xrightarrow{\alpha}V_1\xrightarrow{\beta}V_2$ and that S is an SPQ on V_0. Then for every SPQ U on V_2 we have, using reciprocity three times, that $\sigma(\beta_*\alpha_*S+U)=\sigma(\alpha_*S+\beta^*U)=\sigma(S+\alpha^*\beta^*U)=\sigma(S+(\beta\alpha)^*U)=\sigma((\beta\alpha)_*S+U)$. Hence $\beta_*\alpha_*S=(\beta\alpha)_*S$. □

Definition. A commutative square as on the right is called *admissible* if $\gamma^*\beta_*=\nu_*\mu^*$.

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma"] \\ V \arrow[r, "\beta"'] \arrow[ur, dashed] & Z \end{tikzcd}$$

Lemma 1. If $V=W=Y=Z$ and $\beta=\gamma=\mu=\nu=I$, the square is admissible. □

Lemma 2. The following are equivalent:
1. A square as above is admissible.
2. The *Pairing Condition* holds. Namely, if S_1 is an SPQ on V (write $S_1\vdash V$) and $S_2\vdash W$, then $\sigma(\mu^*S_1+\nu^*S_2)=\sigma(\beta_*S_1+\gamma_*S_2)$.
3. The square is mirror admissible: $\beta^*\gamma_*=\mu_*\nu^*$.

$$\begin{tikzcd} & Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma"] & \dashv S_2 \\ S_1 \vdash & V \arrow[r, "\beta"'] & Z & \end{tikzcd}
\qquad
\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma"] \arrow[dl, dashed] \\ V \arrow[r, "\beta"'] & Z \end{tikzcd}$$

Proof. Using Exercises 1 and 2 below, and then using reciprocity on both sides, we have $\forall S_1\,\gamma^*\beta_*S_1=\nu_*\mu^*S_1\Leftrightarrow\forall S_1\forall S_2\,\sigma(\gamma^*\beta_*S_1+S_2)=\sigma(\nu_*\mu^*S_1+S_2)\Leftrightarrow\forall S_1\forall S_2\,\sigma(\beta_*S_1+\gamma_*S_2)=\sigma(\mu^*S_1+\nu^*S_2)$, and thus $1\Leftrightarrow 2$. But the condition in 2 is symmetric under $\beta\leftrightarrow\gamma$, $\mu\leftrightarrow\nu$, so also $2\Leftrightarrow 3$. □

Lemma 3. If the first diagram below is admissible, then so is the second. □

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma"] \\ V \arrow[r, "\beta"'] \arrow[ur, dashed] & Z \end{tikzcd}
\qquad
\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma\oplus 0"] \\ V \arrow[r, "\beta\oplus 0"'] \arrow[ur, dashed] & Z\oplus F \end{tikzcd}$$

Lemma 4. A pushforward by an inclusion is the do nothing operation (though note that the pushforward via an inclusion of a fully defined quadratic retains its domain of definition, which now may become partial). □

Lemma 5. For any linear $\phi\colon V\to W$, the diagram on the right is admissible, where ι denotes the inclusion maps.

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[d, "\phi"'] & V\oplus C \arrow[d, "\phi\oplus I"] \\ W \arrow[r, "\iota"'] \arrow[ur, dashed] & W\oplus C \end{tikzcd}$$

Proof. Follows easily from Lemma 4. □

Definition. If S is an SPQ with domain D and quadratic Q, the radical of S is the radical of Q considered as a fully-defined quadratic on D. Namely, $\operatorname{rad}S:=\{u\in D\colon\forall v\in D,\ Q(u,v)=0\}$.

Lemma 6. Always, $\phi(\operatorname{rad}S)\subset\operatorname{rad}\phi_*S$.

Proof. Pick $w\in\phi(\operatorname{rad}S)$ and repeat the proof of Theorem 1' but now considering quadruples (V,S,ϕ,v), where (V,S,ϕ) are as before and $v\in\operatorname{rad}S$ satisfies $\phi(v)=w$. Clearly our initial triple (V,S,ϕ) can be extended to such a quadruple, and it is easy to repeat the steps of the proof of Theorem 1' extending everything to such quadruples. □

We have to acknowledge that our proof of Lemma 6 is ugly. We wish we had a cleaner one.

Exercise 3. Show that if two SPQ's S_1 and S_2 on $V\oplus A$ satisfy $A\subset\operatorname{rad}S_i$ and $\sigma(S_1+\pi^*U)=\sigma(S_2+\pi^*U)$ for every quadratic U on V, where $\pi\colon V\oplus A\to V$ is the projection, then $S_1=S_2$.

Exercise 4. Show that if $\phi\colon V\to W$ is surjective and Q is a quadratic on W, then $\sigma(Q)=\sigma(\phi^*Q)$.

Exercise 5. Show that always, $\phi_*\phi^*S=S|_{\operatorname{im}\phi}$.

Lemma 7. For any linear $\phi\colon V\to W$, the diagram on the right is admissible, where $\phi^+:=\phi\oplus I$ and α and β denote the projection maps.

$$\begin{tikzcd} V\oplus C \arrow[r, "\phi^+"] \arrow[d, "\alpha"'] & W\oplus C \arrow[d, "\beta"] \\ V \arrow[r, "\phi"'] \arrow[ur, dashed] & W \end{tikzcd}$$

Proof. Let S be an SPQ on V. Clearly $C\subset\beta^*\phi_*S$. Also, $C\subset\operatorname{rad}\alpha^*S$ so by Lemma 6, $C=\phi^+(C)\subset\phi^+(\operatorname{rad}\alpha^*S)\subset\operatorname{rad}\phi^+_*\alpha^*S$. Hence using Exercise 3, it is enough to show that $\sigma(\phi^+_*\alpha^*S+\beta^*U)=\sigma(\beta^*\phi_*S+\beta^*U)$ for every U on W. Indeed, $\sigma(\phi^+_*\alpha^*S+\beta^*U)\overset{(1)}{=}\sigma(\beta_*\phi^+_*\alpha^*S+U)\overset{(2)}{=}\sigma(\phi_*\alpha_*\alpha^*S+U)\overset{(3)}{=}\sigma(\phi_*S+U)\overset{(4)}{=}\sigma(\beta^*(\phi_*S+U))\overset{(5)}{=}\sigma(\beta^*\phi_*S+\beta^*U)$, using (1) reciprocity, (2) the commutativity of the diagram and the functoriality of pushing, (3) Exercise 5, (4) Exercise 4, and (5) the additivity of pullbacks. □

Lemma 8. If the first diagram below is admissible, then so are the other two.

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\mu"'] & W \arrow[d, "\gamma"] \\ V \arrow[r, "\beta"'] \arrow[ur, dashed] & Z \end{tikzcd}
\qquad
\begin{tikzcd} Y\oplus E \arrow[r, "\nu\oplus 0"] \arrow[d, "\mu\oplus I"'] & W \arrow[d, "\gamma"] \\ V\oplus E \arrow[r, "\beta\oplus 0"'] \arrow[ur, dashed] & Z \end{tikzcd}
\qquad
\begin{tikzcd} Y \arrow[r, "\nu\oplus 0"] \arrow[d, "\mu"'] & W\oplus F \arrow[d, "\gamma\oplus I"] \\ V \arrow[r, "\beta\oplus 0"'] \arrow[ur, dashed] & Z\oplus F \end{tikzcd}$$

Proof. In the diagram

$$\begin{tikzcd} Y\oplus E \arrow[r, "\pi"] \arrow[d, "\mu\oplus I"'] & Y \arrow[r, "\nu"] \arrow[d, "\mu"] & W \arrow[r, "\iota"] \arrow[d, "\gamma"] & W\oplus F \arrow[d, "\gamma\oplus I"] \\ V\oplus E \arrow[r, "\pi"'] \arrow[ur, dashed] & V \arrow[r, "\beta"'] \arrow[ur, dashed] & Z \arrow[r, "\iota"'] \arrow[ur, dashed] & Z\oplus F, \end{tikzcd}$$

with π marking projections and ι inclusions, the left square is admissible by Lemma 7, the middle square by assumption, and the right square by Lemma 5. Along with the functoriality of pushforwards this shows the admissibility of both the left and the right 1×2 subrectangles, and these are the diagrams we wanted. □

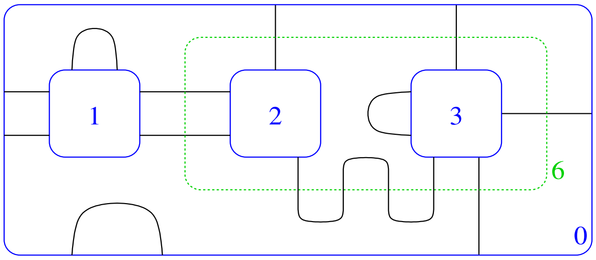
Proof of Theorem 3. Decompose $Z=A\oplus B\oplus C\oplus D$, where $A=\operatorname{im}\beta\cap\operatorname{im}\gamma$, $\operatorname{im}\beta=A\oplus B$, and $\operatorname{im}\gamma=A\oplus C$. Write $V\simeq A\oplus B\oplus E$ with $\beta=I$ on $A\oplus B$ yet $\beta=0$ on E, and write $W\simeq A\oplus C\oplus F$ with $\gamma=I$ on $A\oplus C$ yet $\gamma=0$ on F. Then $Y=V\oplus_Z W\simeq A\oplus E\oplus F$ and our square is as shown on the right, with all maps equal to I on like-named summands and equal to 0 on non-like-named summands. But this diagram is admissible: build it up using Lemma 1 for the A's, and then Lemma 8 for E and C, and then again Lemma 8 along with the mirror property of Lemma 2 for B and F, and then Lemma 3 for D. □

$$\begin{tikzcd} A\oplus E\oplus F \arrow[r] \arrow[d] & A\oplus C\oplus F \arrow[d] \\ A\oplus B\oplus E \arrow[r] \arrow[ur, dashed] & A\oplus B\oplus C\oplus D \end{tikzcd}$$

To prove Theorem 4, given three[1] SPQ's S_1, S_2, and S_3, we need to show that planar-multiplying them in two steps, first using a planar connection diagram D_I (I for Inner) to yield $S_6=\mathcal{S}(D_I)(S_2,S_3)$ and then using a second planar connection diagram D_O (O for Outer) to yield $\mathcal{S}(D_O)(S_1,S_6)$, gives the same answer as multiplying them all at once using the composition planar connection diagram $D_B=D_O\circ_6 D_I$ (B for Big) to yield $\mathcal{S}(D_B)(S_1,S_2,S_3)$.[2] An example should help:

[1] Truly, we need the same for any number of input SPQ's that are divided into two groups, "multiply in the first step" and "multiply in the second step". But there's no added difficulty here, only an added notational complexity.

[2] Aren't we sassy? We picked "6" for the name of the product of "2" and "3".



In this example, if you ignore the dotted green line (marked "6"), you see the planar connection diagram D_B, which has three inputs (1,2,3) and a single output, the cycle 0. If you only look inside the green line, you see D_I, with inputs 2 and 3 and an output cycle 6. If you ignore the inside of 6 you see D_O, with inputs 1 and 6 and output cycle 0.

Let F_B (Big Faces) denote the vector space whose basis are the faces of D_B, let F_I (Inner Faces) be the space of faces of D_I, and let F_O (Outer Faces) be the space of faces of D_O. Let G_1, G_2, G_3, G_6, and G_0 be the spaces of gaps (edges) along the cycles 1,2,3,6, and 0, respectively. Let $\psi := \psi_{D_B}$ and $\phi := \phi^{D_B}$ be the maps defining $\mathcal{S}(D_B)$ and let $\gamma := \psi_{D_O}$ and $\delta := \phi^{D_O}$ be the maps defining $\mathcal{S}(D_O)$. Further, let $\alpha := \psi_{D_I}\colon F_I \to G_2 \oplus G_3$ and $\beta := \phi^{D_I}\colon F_I \to G_6$ be the maps defining $\mathcal{S}(D_I)$, and let $\alpha_+ := I \oplus \alpha$ and $\beta^+ := I \oplus \beta$ be the extensions of α and β by an identity on an extra factor of G_1, so that $\beta^+_*\alpha^*_+ = I_{G_1} \oplus \mathcal{S}(D_I)$. Let μ map any big face to the sum of G_1 gaps around it, plus the sum of the inner faces it contains. Let ν map any big face to the sum of the outer faces it contains. It is easy to see that the master diagram (MD) shown on the right, made of all of these spaces and maps, is commutative.

$$(MD)\qquad \begin{tikzcd} & & G_0 \\ & F_B \arrow[ur, "\phi"] \arrow[r, "\nu"] \arrow[dl, "\psi"'] \arrow[d, "\mu"] & F_O \arrow[u, "\delta"'] \arrow[d, "\gamma"] \\ G_1 \oplus G_2 \oplus G_3 & G_1 \oplus F_I \arrow[l, "\alpha_+"] \arrow[r, "\beta^+"'] & G_1 \oplus G_6 \end{tikzcd}$$

Claim. The bottom right square of (MD) is an equalizer square, namely $F_B \simeq EQ(\beta^+, \gamma)$. Hence $\nu_*\mu^* = \gamma^*\beta^+_*$.

$$\begin{tikzcd} F_B \arrow[r, "\nu"] \arrow[d, "\mu"] & F_O \arrow[d, "\gamma"] \\ G_1 \oplus F_I \arrow[r, "\beta^+"'] & G_1 \oplus G_6 \end{tikzcd}$$

Proof. A big face (an element of F_B) is a sum of outer faces f_o and a sum of inner faces f_i, and it has a boundary g_1 on input cycle 1, such that the boundary of the outer pieces f_o is equal to the boundary of the inner pieces f_i plus g_1. That matches perfectly with the definition of the equalizer: $EQ(\beta^+, \gamma) = \{(g_1, f_i, f_o)\colon \beta^+(g_1, f_i) = \gamma(f_o)\} = \{(g_1, f_i, f_o)\colon \gamma(f_o) = (g_1, \beta(f_i))\}$. □

Proof of Theorem 4. With notation as above, with the example above (which is general enough), and with the claim above, and also using functoriality, we have $\mathcal{S}(D_B) = \phi_*\psi^* = \delta_*\nu_*\mu^*\alpha^*_+ = \delta_*\gamma^*\beta^+_*\alpha^*_+ = \mathcal{S}(D_O) \circ (I_{G_1} \oplus \mathcal{S}(D_I))$, as required. □

Proof of Theorem 5. We need to verify the Reidemeister moves and that was done in the computational section, and the statement about the restriction to links, which is easy: simply assemble an n-crossing knot using an n-input planar connection diagram, and the formulas clearly match. □

Further Homework.

Exercise 6. By taking $U = 0$ in the reciprocity statement, prove that always $\sigma(\phi_*S) = \sigma(S)$. But that seems wrong, if $\phi = 0$. What saves the day?

Exercise 7. By taking $S = 0$ in the reciprocity statement, frove that always $\sigma(\phi^*U) = \sigma(U)$. But wait, this is nonsense! What went wrong?

Exercise 8. Given $\phi\colon V \to W$ and a subspace $D \subset V$, show that there is a unique subspace $\phi_*D \subset W$ such that for every quadratic Q on W, $\sigma(\phi^*Q|_D) = \sigma(Q|_{\phi_*D})$.

Exercise 9. When are diagrams as on the right equalizer diagrams? What then do we learn from Theorem 3?

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ V \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d] & W \arrow[d] \\ V \arrow[r] & 0 \end{tikzcd}$$

Exercise 10. There are 11 types or irreducible commutative squares:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}, \begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}, \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 1 \arrow[r] & 0 \end{tikzcd}, \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 1 \end{tikzcd}, \begin{tikzcd} 1 \arrow[r, "1"'] \arrow[d] & 1 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}, \begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "1"'] \\ 0 \arrow[r] & 1 \end{tikzcd}, \begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "1"'] \\ 0 \arrow[r] & 1 \end{tikzcd}, \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 1 \arrow[r, "1"] & 1 \end{tikzcd},$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "1"'] \\ 1 \arrow[r, "1"] & 1 \end{tikzcd}, \begin{tikzcd} 1 \arrow[r, "1"] \arrow[d, "1"] & 1 \arrow[d] \\ 1 \arrow[r] & 0 \end{tikzcd}, \text{ and } \begin{tikzcd} 1 \arrow[r, "1"] \arrow[d, "1"] & 1 \arrow[d, "1"'] \\ 1 \arrow[r, "1"] & 1 \end{tikzcd}.$$

Show that pushing commutes with pulling for all but four of them. Compare with the statement of Theorem 3.

Exercise 11. Prove that a square is admissible iff it is an equalizer square, with an additional direct summand A added to the Y term, and with the maps μ and ν extended by 0 on A.

Exercise 12. Prove that the direct sum of two admissible squares is admissible. *Warning:* Harder than it seems! Not all quadratics on $V_1 \oplus V_2$ are direct sums of quadratics on V_1 and on V_2.

Exercise 13. Given a quadratic Q on a space V, let π be the projection $V \to V/\operatorname{rad}(Q)$ and show that $\pi_*Q = Q/\operatorname{rad}(Q)$, with the obvious definition for the latter.

Exercise 14. Show that for any partial quadratic Q on a space W there exists a space A and a fully-defined quadratic F on $W \oplus A$ such that $\pi_*F = Q$, where $\pi\colon W \oplus A \to W$ is the projection (these are not unique). Furthermore, if $\phi\colon V \to W$, then $\phi^*Q = \pi_*\phi^*_+F$, where $\phi_+ = \phi \oplus I\colon V \oplus A \to W \oplus A$ and π also denotes the projection $V \oplus A \to V$.

Solutions / Hints.

Hint for 1. On a vector in the domain of one but not the other, take an outrageous value for U, that will raise or lower the signature.

Hint for 2. WLOG, Q_1 is diagonal and $Q_1 = 0$.

Hint for 5. It's enough to test that against U with $\mathcal{D}(U) = \operatorname{im}\phi$.

Hint for 6. The "shift" part of 0_*S is $\sigma(S)$.

Hint for 7. ϕ_*S isn't 0, it's the *partial* quadratic "0 on $\operatorname{im}\phi$" (and indeed, $\sigma(\phi^*U) = \sigma(U)$ if ϕ is surjective).

Hint for 10. The exceptions are $\begin{smallmatrix}10\\00\end{smallmatrix}$, $\begin{smallmatrix}00\\01\end{smallmatrix}$, $\begin{smallmatrix}10\\11\end{smallmatrix}$, and $\begin{smallmatrix}11\\01\end{smallmatrix}$.

Hint for 12. Use Exercise 11.

Dror Bar-Natan: Talks: Tokyo-230911: Thanks for inviting me to UTokyo!

Rooting the BKT for FTI

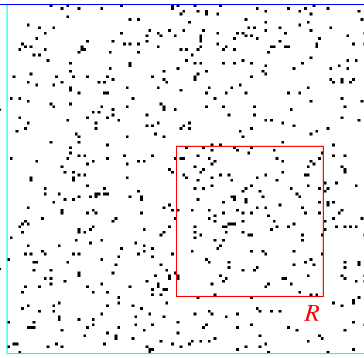
$\omega\varepsilon\beta$:=http://drorbn.net/tok2309

Abstract. Following joint work with Itai Bar-Natan, Iva Halacheva, and Nancy Scherich, I will show that the Best Known Time (BKT) to compute a typical Finite Type Invariant (FTI) of type d on a typical knot with n crossings is roughly equal to $n^{d/2}$, which is roughly the square root of what I believe was the standard belief before, namely about n^d.

Conventions. • $\underline{n} := \{1, 2, \ldots, n\}$. • For complexity estimates we ignore constant and logarithmic terms: $n^3 \sim 2023d!(\log n)^d n^3$.

A Key Preliminary. Let $Q \subset \underline{n}^l$ be an enumerated subset, with $1 \ll q = |Q| \ll n^l$. In time $\sim q$ we can set up a lookup table of size $\sim q$ so that we will be able to compute $|Q \cap R|$ in time ~ 1, for any rectangle $R \subset \underline{n}^l$.

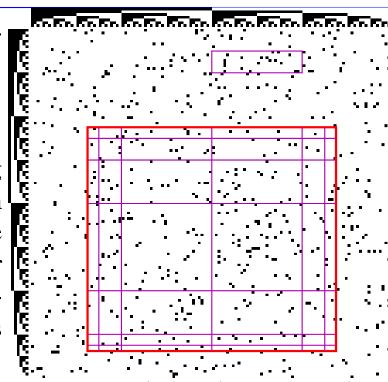
Fails. • Count after R is presented. • Make a lookup table of $|Q \cap R|$ counts for all R's.



Unfail. Make a restricted lookup table of the form

$$\left\{ \underset{\text{dyadic}}{R} \to \underset{>0}{|Q \cap R|} \right\}.$$

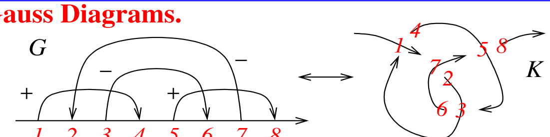
• Make the table by running through $x \in Q$, and for each one increment by 1 only the entries for dyadic $R \ni x$ (or create such an entry, if it didn't exist already). This takes $q \cdot (\log_2 n)^l \sim q$ ops.



• Entries for empty dyadic R's are not needed and not created.
• Using standard sorting techniques, access takes $\log_2 q \sim 1$ ops.
• A general R is a union of at most $(2\log_2 n)^l \sim 1$ dyadic ones, so counting $|Q \cap R|$ takes ~ 1 ops.

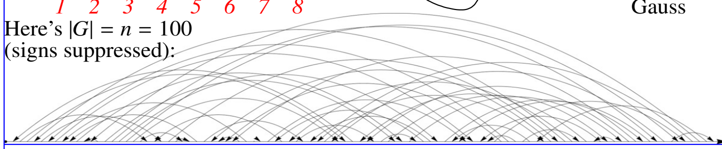
Generalization. Without changing the conclusion, replace counts $|Q \cap R|$ with summations $\sum_R \theta$, where $\theta\colon \underline{n}^l \to V$ is supported on a sparse Q, takes values in a vector space V with $\dim V \sim 1$, and in some basis, all of its coefficients are "easy".

Gauss Diagrams.



Gauss

Here's $|G| = n = 100$ (signs suppressed):

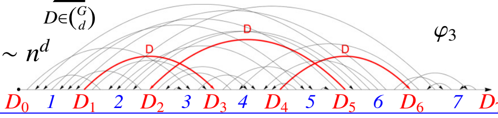


Definitions. Let $\mathcal{G} := \mathbb{Q}\langle\text{Gauss Diagrams}\rangle$, with $\mathcal{G}_d$ / $\mathcal{G}_{\le d}$ the diagrams with exactly / at most d arrows. Let $\varphi_d\colon \mathcal{G} \to \mathcal{G}_d$ be

$$\varphi_d\colon G \mapsto \sum_{D \subset G,\ |D| = d} D = \sum_{D \in \binom{G}{d}} D,$$

and let $\varphi_{\le d} = \sum_{e \le d} \varphi_e$.

Naively, it takes $\binom{n}{d} \sim n^d$ ops to compute φ_d.



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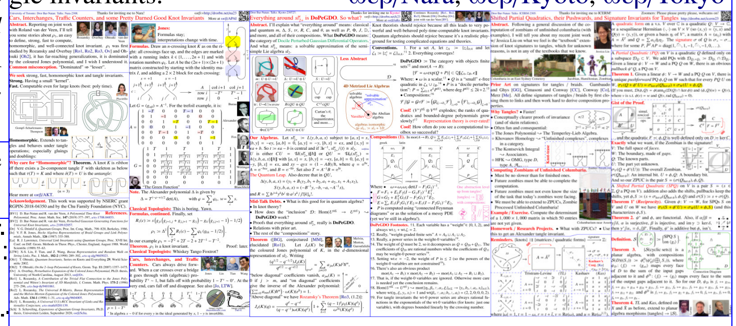
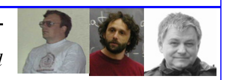
My Primary Interest. Strong, fast, homomorphic knot and tangle invariants. $\omega\varepsilon\beta$/Nara, $\omega\varepsilon\beta$/Kyoto, $\omega\varepsilon\beta$/Tokyo



The [GPV] Theorem. A knot invariant is finite type of type d iff it is of the form $\omega \circ \varphi_{\le d}$ for some $\omega \in \mathcal{G}^*_{\le d}$.

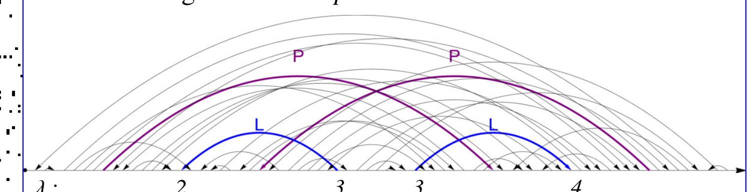


Goussarov-Polyak-Viro

• $\Leftarrow$ is easy; $\Rightarrow$ is hard and IMHO not well understood.
• $\varphi_{\le d}$ is not an invariants and not every ω gives an invariant!
• The theory of finite type invariants is very rich. Many knot invariants factor through finite type invariants, and it is possible that they separate knots.
• We need a fast algorithm to compute $\varphi_{\le d}$!

Our Main Theorem. On an n-arrow Gauss diagram, φ_d can be computed in time $\sim n^{\lceil d/2 \rceil}$.

Proof. With $d = p + l$ (p for "put", l for "lookup"), pick p arrows and look up in how many ways the remaining l can be placed in between the legs of the first p:



To reconstruct $D = P\#_\lambda L$ from P and L we need a non-decreasing "placement function" $\lambda\colon \underline{2l} \to \underline{2p+1}$.

$$\varphi_d(G) = \sum_{D \in \binom{G}{d}} D = \binom{d}{p}^{-1} \sum_{P \in \binom{G}{p}} \sum_{\substack{\text{non-decreasing} \\ \lambda\colon \underline{2l} \to \underline{2p+1}}} \sum_{\substack{L \in \binom{G}{l} \\ L_i \in (P_{\lambda(i)-1}, P_{\lambda(i)})}} P\#_\lambda L$$

Define $\theta_G\colon \underline{2n}^{2l} \to \mathcal{G}_l$ by

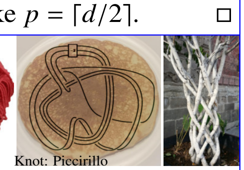
$$(L_1, \ldots, L_{2l}) \mapsto \begin{cases} L & \text{if } (L_1, \ldots, L_{2l}) \text{ are the ends of some } L \subset G \\ 0 & \text{otherwise} \end{cases}$$

and now
$$\varphi_d(G) = \binom{d}{p}^{-1} \sum_{P \in \binom{G}{p}} \sum_{\substack{\text{non-decreasing} \\ \lambda\colon \underline{2l} \to \underline{2p+1}}} P\#_\lambda \left(\sum_{\prod_i (P_{\lambda(i)-1}, P_{\lambda(i)})} \theta_G \right)$$

can be computed in time $\sim n^p + n^l$. Now take $p = \lceil d/2 \rceil$. □

Question ([BBHS], $\omega\varepsilon\beta$/Fields). For computations, planar projections are better than braids (as likely $l \sim n^{3/2}$). But are yarn balls better than planar projections (here likely $n \sim L^{4/3}$)?

Length L



n crossings   length l

References.

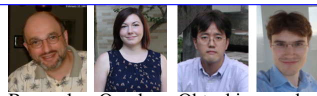
[BBHS] D. Bar-Natan, I. Bar-Natan, I. Halacheva, and N. Scherich, *Yarn Ball Knots and Faster Computations,* J. of Appl. and Comp. Topology (to appear), arXiv:2108.10923.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Tokyo-230911/

Cars, Interchanges, Traffic Counters, and some Pretty Darned Good Knot Invariants

More at ωεβ/APAI

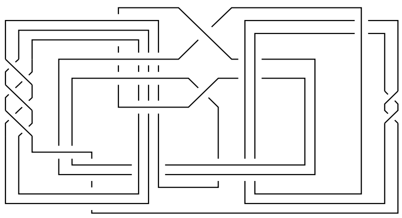


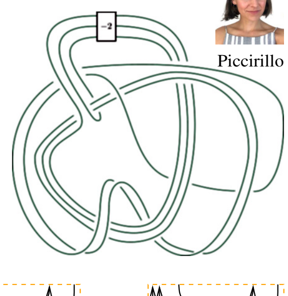
Rozansky Overbay Ohtsuki van der Veen

Abstract. Reporting on joint work with Roland van der Veen, I'll tell you some stories about ρ_1, an easy to define, strong, fast to compute, homomorphic, and well-connected knot invariant. ρ_1 was first studied by Rozansky and Overbay [Ro1, Ro2, Ro3, Ov] and Ohtsuki [Oh2], it has far-reaching generalizations, it is elementary and dominated by the coloured Jones polynomial, and I wish I understood it.

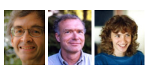
Common misconception. Dominated, elementary ⇒ lesser.

We seek strong, fast, homomorphic knot and tangle invariants.
Strong. Having a small "kernel".
Fast. Computable even for large knots (best: poly time).



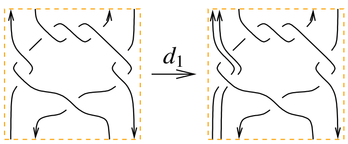


Piccirillo

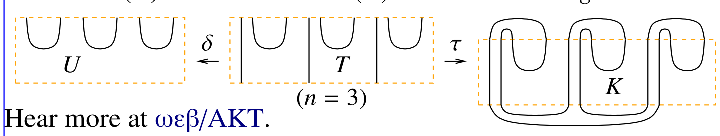


Gompf–Scharlemann–Thompson

Homomorphic. Extends to tangles and behaves under tangle operations; especially gluings and doublings:

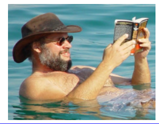


Why care for "Homomorphic"? Theorem. A knot K is *ribbon* iff there exists a $2n$-component tangle T with skeleton as below such that $\tau(T) = K$ and where $\delta(T) = U$ is the *untangle*:



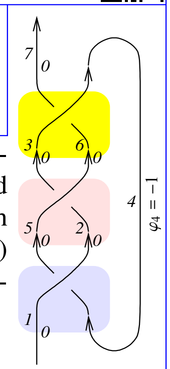
Hear more at ωεβ/AKT.

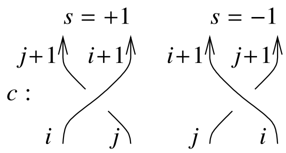
Acknowledgement. This work was supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).



Jones: Formulas stay; interpretations change with time.

Formulas. Draw an n-crossing knot K as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \dots, 2n+1\}$ and with rotation numbers φ_k. Let A be the $(2n+1)\times(2n+1)$ matrix constructed by starting with the identity matrix I, and adding a 2×2 block for each crossing:



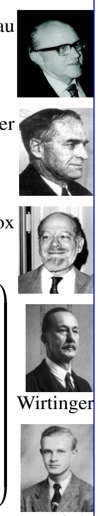


A	col $i+1$	col $j+1$
row i	$-T^s$	$T^s - 1$
row j	0	-1

Let $G = (g_{\alpha\beta}) = A^{-1}$. For the trefoil example, it is:

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Burau Alexander Fox Wirtinger Blanchfield

"The Green Function"

Note. The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2}\det(A), \qquad \text{with } \varphi = \sum_k \varphi_k,\ w = \sum_c s.$$

Classical Topologists: This is boring. Yawn.

Formulas, continued. Finally, set

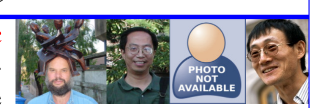
$$R_1(c) := s\left(g_{ji}\left(g_{j+1,j} + g_{j,j+1} - g_{ij}\right) - g_{ii}\left(g_{j,j+1} - 1\right) - 1/2\right)$$

$$\rho_1 := \Delta^2\left(\sum_c R_1(c) - \sum_k \varphi_k\left(g_{kk} - 1/2\right)\right).$$

In our example $\rho_1 = -T^2 + 2T - 2 + 2T^{-1} - T^{-2}$.

Theorem. ρ_1 is a knot invariant. Proof: later.

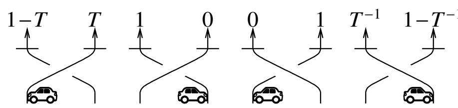
Classical Topologists: Whiskey Tango Foxtrot?



Jones Lin Tian Wang

Cars, Interchanges, and Traffic Counters. Cars always drive forward. When a car crosses over a bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0^*$. At the very end, cars fall off and disappear. See also [Jo, LTW].

$p = 1 - T^s$



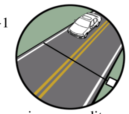


image credits: diamondtraffic.com

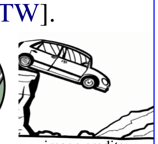


image credits: Dall-E

* In algebra $x \sim 0$ if for every y in the ideal generated by x, $1-y$ is invertible.

References.

[BV1] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* Proc. Amer. Math. Soc. **147** (2019) 377–397, arXiv:1708.04853.

[BV2] D. Bar-Natan and R. van der Veen, *Perturbed Gaussian Generating Functions for Universal Knot Invariants,* arXiv:2109.02057.

[Dr] V. G. Drinfel'd, *Quantum Groups,* Proc. Int. Cong. Math., 798–820, Berkeley, 1986.

[Jo] V. F. R. Jones, *Hecke Algebra Representations of Braid Groups and Link Polynomials,* Annals Math., **126** (1987) 335-388.

[La] R. J. Lawrence, *Universal Link Invariants using Quantum Groups,* Proc. XVII Int. Conf. on Diff. Geom. Methods in Theor. Phys., Chester, England, August 1988. World Scientific (1989) 55–63.

[LTW] X-S. Lin, F. Tian, and Z. Wang, *Burau Representation and Random Walk on String Links,* Pac. J. Math., **182-2** (1998) 289–302, arXiv:q-alg/9605023.

[Oh1] T. Ohtsuki, *Quantum Invariants,* Series on Knots and Everything **29**, World Scientific 2002.

[Oh2] T. Ohtsuki, *On the 2–loop Polynomial of Knots,* Geom. Top. **11** (2007) 1357–1475.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, August 2013, ωεβ/Ov.

[Ro1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Sch] S. Schaveling, *Expansions of Quantum Group Invariants,* Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Video: http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210. Handout: http://www.math.toronto.edu/~drorbn/Talks/Nara-2308.

Preliminaries

This is Rho.nb of http://drorbn.net/oa22/ap.

```
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version
  of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/la22/ap
  to compute rotation numbers.
```

The Program

$$R_1[s\_, i\_, j\_] := s\,(g_{ji}\,(g_{j^+,j} + g_{j,j^+} - g_{ij}) - g_{ii}\,(g_{j,j^+} - 1) - 1/2);$$

$$Z[K\_] := \text{Module}\Big[\{Cs, \varphi, n, A, s, i, j, k, \Delta, G, \rho1\},$$
$$\{Cs, \varphi\} = \text{Rot}[K];\ n = \text{Length}[Cs];$$
$$A = \text{IdentityMatrix}[2n+1];$$
$$\text{Cases}\Big[Cs, \{s\_, i\_, j\_\} \mapsto \Big(A[[\{i, j\}, \{i+1, j+1\}]] \mathrel{+}= \begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}\Big)\Big];$$
$$\Delta = T^{(-\text{Total}[\varphi] - \text{Total}[Cs[[\text{All},1]]])/2}\,\text{Det}[A];$$
$$G = \text{Inverse}[A];$$
$$\rho1 = \sum_{k=1}^{n} R_1 \mathbin{@@} Cs[[k]] - \sum_{k=1}^{2n} \varphi[[k]]\,(g_{kk} - 1/2);$$
$$\text{Factor@}\{\Delta, \Delta^2 \rho1\ /.\ \alpha\_^+ \mapsto \alpha + 1\ /.\ g_{\alpha\_,\beta\_} \mapsto G[[\alpha, \beta]]\}\Big];$$

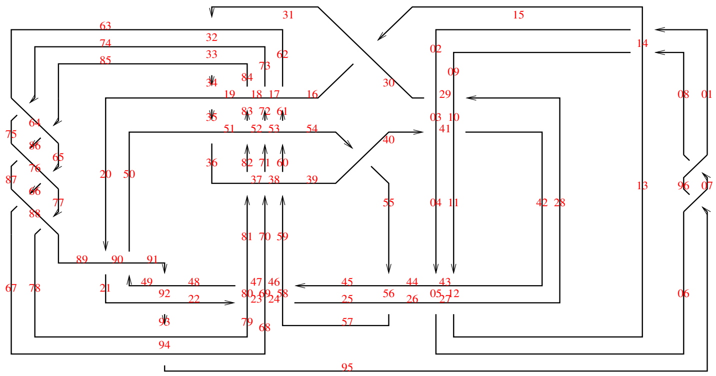
The First Few Knots

```
TableForm[Table[Join[{K[[1]]_K[[2]]}, Z[K]],
  {K, AllKnots[{3, 6}]}], TableAlignments → Center]
```

3_1	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2\,(1+T^2)}{T^2}$
4_1	$-\frac{1-3T+T^2}{T}$	0
5_1	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2\,(1+T^2)\,(2+T^2+2T^4)}{T^4}$
5_2	$\frac{2-3T+2T^2}{T}$	$\frac{(-1+T)^2\,(5-4T+5T^2)}{T^2}$
6_1	$-\frac{(-2+T)\,(-1+2T)}{T}$	$\frac{(-1+T)^2\,(1-4T+T^2)}{T^2}$
6_2	$-\frac{1-3T+3T^2-3T^3+T^4}{T^2}$	$\frac{(-1+T)^2\,(1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{T^4}$
6_3	$\frac{1-3T+5T^2-3T^3+T^4}{T^2}$	0

$p = 1 - T^s$

Fast!



Timing@

$$Z\Big[GST48 = \text{EPD}\Big[X_{14,1}, \overline{X}_{2,29}, X_{3,40}, X_{43,4}, \overline{X}_{26,5}, X_{6,95}, X_{96,7}, X_{13,8}, \overline{X}_{9,28}, X_{10,41}, X_{42,11}, \overline{X}_{27,12}, X_{30,15}, \overline{X}_{16,61}, \overline{X}_{17,72}, \overline{X}_{18,83}, X_{19,34}, \overline{X}_{89,20}, \overline{X}_{21,92}, \overline{X}_{79,22}, \overline{X}_{68,23}, \overline{X}_{57,24}, \overline{X}_{25,56}, X_{62,31}, X_{73,32}, X_{84,33}, \overline{X}_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, \overline{X}_{39,54}, X_{44,55}, X_{58,45}, X_{69,46}, X_{80,47}, X_{48,91}, X_{90,49}, X_{51,82}, X_{52,71}, X_{53,60}, \overline{X}_{63,74}, \overline{X}_{64,85}, \overline{X}_{76,65}, \overline{X}_{87,66}, \overline{X}_{67,94}, \overline{X}_{75,86}, \overline{X}_{88,77}, \overline{X}_{78,93}\Big]\Big]$$

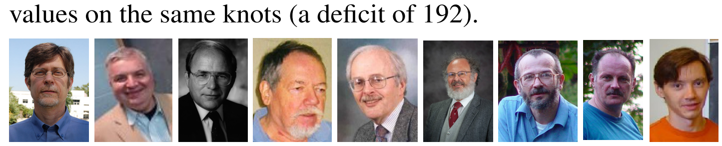
$$\Big\{170.313, \Big\{-\frac{1}{T^8}\,(-1+2T-T^2-T^3+2T^4-T^5+T^8)\,(-1+T^3-2T^4+T^5+T^6-2T^7+T^8),\ \frac{1}{T^{16}}(-1+T)^2\,(5-18T+33T^2-32T^3+2T^4+42T^5-62T^6-8T^7+166T^8-242T^9+108T^{10}+132T^{11}-226T^{12}+148T^{13}-11T^{14}-36T^{15}-11T^{16}+148T^{17}-226T^{18}+132T^{19}+108T^{20}-242T^{21}+166T^{22}-8T^{23}-62T^{24}+42T^{25}+2T^{26}-32T^{27}+33T^{28}-18T^{29}+5T^{30})\Big\}\Big\}$$

Strong!

```
{NumberOfKnots[{3, 12}],
 Length@
  Union@Table[Z[K], {K, AllKnots[{3, 12}]}],
 Length@
  Union@Table[{HOMFLYPT[K], Kh[K]},
    {K, AllKnots[{3, 12}]}]}
```

{2977, 2882, 2785}

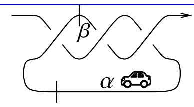
So the pair (Δ, ρ_1) attains 2,882 distinct values on the 2,977 prime knots with up to 12 crossings (a deficit of 95), whereas the pair (HOMFLYPT, Khovanov Homology) attains only 2,785 distinct values on the same knots (a deficit of 192).



Hoste Ocneanu Millett Freyd Lickorish Yetter Przytycki Traczyk Khovanov

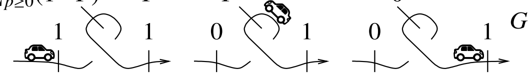
Video: http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210. Handout: http://www.math.toronto.edu/~drorbn/Talks/Nara-2308.

Theorem. The Green function $g_{\alpha\beta}$ is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point).



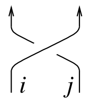
Example.

$\sum_{p\geq 0}(1-T)^p = T^{-1}$



$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof. Near a crossing c with sign s, incoming upper edge i and incoming lower edge j, both sides satisfy the *g-rules*:



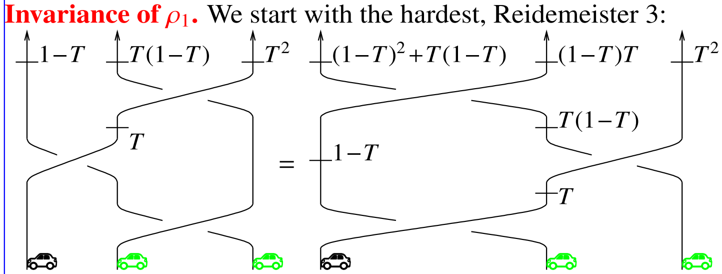
$$g_{i\beta} = \delta_{i\beta} + T^s g_{i+1,\beta} + (1-T^s) g_{j+1,\beta}, \qquad g_{j\beta} = \delta_{j\beta} + g_{j+1,\beta},$$

and always, $g_{\alpha,2n+1} = 1$: use common sense and $AG = I$ $(= GA)$.

Bonus. Near c, both sides satisfy the further *g-rules*:

$$g_{\alpha i} = T^{-s}(g_{\alpha,i+1} - \delta_{\alpha,i+1}), \qquad g_{\alpha j} = g_{\alpha,j+1} - (1-T^s) g_{\alpha i} - \delta_{\alpha,j+1}.$$

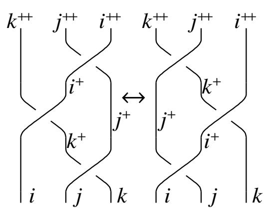
Invariance of ρ_1. We start with the hardest, Reidemeister 3:



⇒ Overall traffic patterns are unaffected by Reid3!

⇒ Green's $g_{\alpha\beta}$ is unchanged by Reid3, provided the cars injection site α and the traffic counters β are away.

⇒ Only the contribution from the R_1 terms within the Reid3 move matters, and using g-rules the relevant $g_{\alpha\beta}$'s can be pushed outside of the Reid3 area:

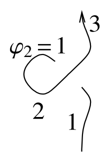



```
δi_,j_ := If[i === j, 1, 0];
gRules s_,i_,j_ :=
 {g iβ_ :→ δ iβ + T^s g i+,β + (1 - T^s) g j+,β, g jβ_ :→ δ jβ + g j+,β,
  g α_,i :→ T^-s (g α,i+ - δ α,i+),
  g α_j :→ g α,j+ - (1 - T^s) g αi - δ α,j+}
lhs = R1[1, j, k] + R1[1, i, k+] + R1[1, i+, j+] //.
   gRules 1,j,k ⋃ gRules 1,i,k+ ⋃ gRules 1,i+,j+;
rhs = R1[1, i, j] + R1[1, i+, k] + R1[1, j+, k+] //.
   gRules 1,i,j ⋃ gRules 1,i+,k ⋃ gRules 1,j+,k+;
Simplify[lhs == rhs]
True
```

Next comes Reid1, where we use results from an earlier example:

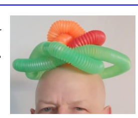
```
R1[1, 2, 1] - 1 (g22 - 1/2) /. g α_,β_ :→ (1 T^-1 1; 0 T^-1 1; 0 0 1)[[α, β]]
```

$$\frac{1}{T^2} - \frac{1}{T} - \frac{-1 + \frac{1}{T}}{T} = 0$$



Invariance under the other moves is proven similarly.

Wearing my Topology hat the formula for R_1, and even the idea to look for R_1, remain a complete mystery to me.



Wearing my Quantum Algebra hat, I spy a Heisenberg algebra $\mathbb{H} = A\langle p, x\rangle/([p,x] = 1)$:

$$\text{cars} \leftrightarrow p \qquad \text{traffic counters} \leftrightarrow x$$

Where did it come from? Consider $\mathfrak{g}_\epsilon := sl_{2+}^\epsilon := L\langle y, b, a, x\rangle$ with relations

$$[b,x] = \epsilon x, \quad [b,y] = -\epsilon y, \quad [b,a] = 0,$$
$$[a,x] = x, \quad [a,y] = -y, \quad [x,y] = b + \epsilon a.$$

At invertible ϵ, it is isomorphic to sl_2 plus a central factor, and it can be quantized à la Drinfel'd [Dr] much like sl_2 to get an algebra $QU = A\langle y, b, a, x\rangle$ subject to (with $q = e^{\hbar\epsilon}$):

$$[b,a] = 0, \quad [b,x] = \epsilon x, \quad [b,y] = -\epsilon y,$$
$$[a,x] = x, \quad [a,y] = -y, \quad xy - qyx = \frac{1 - e^{-\hbar(b+\epsilon a)}}{\hbar}.$$

Now QU has an R-matrix solving Yang-Baxter (meaning Reid3),

$$R = \sum_{m,n\geq 0} \frac{y^n b^m \otimes (\hbar a)^m (\hbar x)^n}{m![n]_q!}, \qquad ([n]_q! \text{ is a "quantum factorial"})$$

and so it has an associated "universal quantum invariant" à la Lawrence and Ohtsuki [La, Oh1], $Z_\epsilon(K) \in QU$.

Now $QU \cong \mathcal{U}(\mathfrak{g}_\epsilon)$ (only as algebras!) and $\mathcal{U}(\mathfrak{g}_\epsilon)$ represents into $\mathbb{H}$ via

$$y \to -tp - \epsilon \cdot xp^2, \quad b \to t + \epsilon \cdot xp, \quad a \to xp, \quad x \to x,$$

(abstractly, $\mathfrak{g}_\epsilon$ acts on its Verma module

$$\mathcal{U}(\mathfrak{g}_\epsilon)/(\mathcal{U}(\mathfrak{g}_\epsilon)\langle y, a, b - \epsilon a - t\rangle) \cong \mathbb{Q}[x]$$

by differential operators, namely via $\mathbb{H}$), so R can be pushed to $\mathcal{R} \in \mathbb{H} \otimes \mathbb{H}$.

Everything still makes sense at $\epsilon = 0$ and can be expanded near $\epsilon = 0$ resulting with $\mathcal{R} = \mathcal{R}_0(1 + \epsilon\mathcal{R}_1 + \cdots)$, with $\mathcal{R}_0 = e^{t(xp\otimes 1 - x\otimes p)}$ and $\mathcal{R}_1$ a quartic polynomial in p and x. So p's and x's get created along K and need to be pushed around to a standard location ("normal ordering"). This is done using

$$(p \otimes 1)\mathcal{R}_0 = \mathcal{R}_0(T(p\otimes 1) + (1-T)(1\otimes p)),$$
$$(1\otimes p)\mathcal{R}_0 = \mathcal{R}_0(1\otimes p),$$

and when the dust settles, we get our formulas for ρ_1. But QU is a quasi-triangular Hopf algebra, and hence **ρ_1 is homomorphic**. Read more at [BV1, BV2] and hear more at ωεβ/SolvApp, ωεβ/Dogma, ωεβ/DoPeGDO, ωεβ/FDA, ωεβ/AQDW.

Schaveling

Also, we can (and know how to) look at higher powers of ϵ and we can (and more or less know how to) replace sl_2 by arbitrary semi-simple Lie algebra (e.g., [Sch]). So **ρ_1 is not alone!**

These constructions are very similar to Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] and hence to the "loop expansion" of the Kontsevich integral and the coloured Jones polynomial [Oh2].

If this all reads like **insanity** to you, it should (and you haven't seen half of it). Simple things should have simple explanations. Hence, **Homework.** Explain ρ_1 with no reference to quantum voodoo and find it a topology home (large enough to house generalizations!). Make explicit the homomorphic properties of ρ_1. Use them to do topology!

P.S. As a friend of Δ, ρ_1 gives a genus bound, sometimes better than Δ's. How much further does this friendship extend?

Video: http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210. Handout: http://www.math.toronto.edu/~drorbn/Talks/Nara-2308.

A Small-Print Page on ρ_d, $d > 1$.

Definition. $\langle f(z_i), h(\zeta_i)\rangle_{\{z_i\}} := f(\partial_{\zeta_i})h\big|_{\zeta_i=0}$, so $\langle p^2x^2, e^{g\pi\xi}\rangle = 2g^2$.

Baby Theorem. There exist (non unique) power series $r^{\pm}(p_1,p_2,x_1,x_2) = \sum_d \epsilon^d r_d^{\pm}(p_1,p_2,x_1,x_2) \in \mathbb{Q}[T^{\pm1},p_1,p_2,x_1,x_2][\![\epsilon]\!]$ with $\deg r_d^{\pm} \le 2d+2$ ("docile") such that the power series $Z^b = \sum \rho_d^b \epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i,p_j,x_i,x_j)\right), \exp\left(\sum_{\alpha,\beta} g_{\alpha\beta}\pi_\alpha\xi_\beta\right)\right\rangle_{\{p_\alpha,x_\beta\}}$$

is a bnot invariant. Beyond the once-and-for-all computation of $g_{\alpha\beta}$ (a matrix inversion), Z^b is computable in $O(n^d)$ operations in the ring $\mathbb{Q}[T^{\pm1}]$.

(Bnots are knot diagrams modulo the braid-like Reidemeister moves, but not the cyclic ones).

Theorem. There also exist docile power series $\gamma^\varphi(\bar p,\bar x) = \sum_d \epsilon^d\gamma_d^\varphi \in \mathbb{Q}[T^{\pm1},\bar p,\bar x][\![\epsilon]\!]$ such that the power series $Z = \sum \rho_d\epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i,p_j,x_i,x_j) + \sum_k \gamma^{\varphi_k}(\bar p_k,\bar x_k)\right),\right.$$

$$\left.\exp\left(\sum_{\alpha,\beta} g_{\alpha\beta}(\pi_\alpha+\bar\pi_\alpha)(\xi_\beta+\bar\xi_\beta) + \sum_\alpha \pi_\alpha\bar\xi_\alpha\right)\right\rangle_{\{p_\alpha,\bar p_\alpha,x_\beta,\bar x_\beta\}}$$

is a knot invariant, as easily computable as Z^b.

Implementation. Data, then program (with output using the Conway variable $z = \sqrt{T} - 1/\sqrt{T}$), and then a demo. See `Rho.nb` of ωεβ/ap.

```
V@γ_{1,φ_}[k_] = φ (1/2 - p̄_k x̄_k); V@γ_{2,φ_}[k_] = -φ^2 p̄_k x̄_k / 2;
V@γ_{3,φ_}[k_] := -φ^3 p̄_k x̄_k / 6
```

```
V@r_{1,s_}[i_, j_] :=
 s (-1 + 2 p_i x_i - 2 p_j x_i + (-1 + T^s) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2
```

```
V@r_{2,1}[i_, j_] :=
 (-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 -
   2 (-1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (-1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j -
   18 p_j^2 x_i x_j - 6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j -
   6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) / 12
```

```
V@r_{2,-1}[i_, j_] :=
 (-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 -
   4 (-1 + T) T p_i^2 p_j x_i^3 + 2 (-1 + T) (1 + 5 T) p_i p_j^2 x_i^3 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^3 +
   18 T^2 p_i p_j x_i x_j - 18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j -
   6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) / (12 T^2)
```

```
V@r_{3,1}[i_, j_] :=
 (4 p_i x_i - 4 p_j x_i + 2 (5 + 7 T) p_i p_j x_i^2 - 2 (5 + 7 T) p_j^2 x_i^2 - 4 (-5 + 6 T) p_i^2 p_j x_i^3 +
   4 (-16 + 17 T + 2 T^2) p_i p_j^2 x_i^3 - 4 (-11 + 11 T + 2 T^2) p_j^3 x_i^3 + 3 (-1 + T) p_i^3 p_j x_i^4 -
   3 (-1 + T) (4 + 3 T) p_i^2 p_j^2 x_i^4 + (-1 + T) (13 + 22 T + T^2) p_i p_j^3 x_i^4 -
   (-1 + T) (4 + 13 T + T^2) p_j^4 x_i^4 - 28 p_i p_j x_i x_j + 28 p_j^2 x_i x_j + 36 p_i^2 p_j x_i^2 x_j -
   12 (9 + 2 T) p_i p_j^2 x_i^2 x_j + 24 (3 + T) p_j^3 x_i^2 x_j - 4 p_i^3 p_j x_i^3 x_j + 28 T p_i^2 p_j^2 x_i^3 x_j -
   4 (-6 + 17 T + T^2) p_i p_j^3 x_i^3 x_j + 4 (-5 + 10 T + T^2) p_j^4 x_i^3 x_j + 24 p_i p_j^2 x_i x_j^2 -
   24 p_j^3 x_i x_j^2 - 24 p_i^2 p_j^2 x_i^2 x_j^2 + 6 (10 + T) p_i p_j^3 x_i^2 x_j^2 - 6 (6 + T) p_j^4 x_i^2 x_j^2 -
   4 p_i p_j^3 x_i x_j^3 + 4 p_j^4 x_i x_j^3) / 24
```

```
V@r_{3,-1}[i_, j_] :=
 (-4 T^3 p_i x_i + 4 T^3 p_j x_i - 2 T^2 (7 + 5 T) p_i p_j x_i^2 + 2 T^2 (7 + 5 T) p_j^2 x_i^2 -
   4 T^2 (-6 + 5 T) p_i^2 p_j x_i^3 + 4 T (-2 - 17 T + 16 T^2) p_i p_j^2 x_i^3 -
   4 T (-2 - 11 T + 11 T^2) p_j^3 x_i^3 + 3 (-1 + T) T^2 p_i^3 p_j x_i^4 - 3 (-1 + T) T (3 + 4 T) p_i^2 p_j^2 x_i^4 +
   (-1 + T) (1 + 22 T + 13 T^2) p_i p_j^3 x_i^4 - (-1 + T) (1 + 13 T + 4 T^2) p_j^4 x_i^4 +
   28 T^3 p_i p_j x_i x_j - 28 T^3 p_j^2 x_i x_j - 36 T^3 p_i^2 p_j x_i^2 x_j + 12 T^2 (2 + 9 T) p_i p_j^2 x_i^2 x_j -
   24 T^2 (1 + 3 T) p_j^3 x_i^2 x_j + 4 T^3 p_i^3 p_j x_i^3 x_j - 28 T^2 p_i^2 p_j^2 x_i^3 x_j -
   4 T (-1 - 17 T + 6 T^2) p_i p_j^3 x_i^3 x_j + 4 T (-1 - 10 T + 5 T^2) p_j^4 x_i^3 x_j -
   24 T^3 p_i p_j^2 x_i x_j^2 + 24 T^3 p_j^3 x_i x_j^2 + 24 T^3 p_i^2 p_j^2 x_i^2 x_j^2 - 6 T^2 (1 + 10 T) p_i p_j^3 x_i^2 x_j^2 +
   6 T^2 (1 + 6 T) p_j^4 x_i^2 x_j^2 + 4 T^3 p_i p_j^3 x_i x_j^3 - 4 T^3 p_j^4 x_i x_j^3) / (24 T^3)
```

```
{p*, x*, p̄*, x̄*} = {π, ξ, π̄, ξ̄}; (z_{i__})* := (z*)_i;
Zip_{{}}[𝓔_] := 𝓔;
Zip_{{z_,zs___}}[𝓔_] :=
 (Collect[𝓔 // Zip_{{zs}}, z] /. f_. z^{d_.} :> (D[f, {z*, d}])) /. z* → 0
```

```
gPair[fs_, w_] :=
 gPair[fs, w] =
  Collect[Zip_{Join@@Table[{p_α, p̄_α, x_α, x̄_α}, {α, w}]}[
    (Times @@ (V /@ fs))
     Exp[Sum[g_{α,β} (π_α + π̄_α) (ξ_β + ξ̄_β), {α, w}, {β, w}] - Sum[ξ̄_α π_α, {α, w}]]],
   g__, Factor]
```

```
T2z[p_] := Module[{q = Expand[p], n, c},
  If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
   c z^{2 n} + T2z[q - c (T^{1/2} - T^{-1/2})^{2 n}]]];
```

```
Z_{d_}[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, d1, Z1, Z2, Z3},
  {Cs, φ} = Rot[K]; n = Length[Cs]; A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                                               0     -1     ))];
  {Δ, G} = Factor@{T^{(-Total[φ] - Total[Cs[[All,1]]])/2} Det@A, Inverse@A};
  Z1 =
   Exp[Total[Cases[Cs, {s_, i_, j_} :> Sum[ϵ^{d1} r_{d1,s}[i, j], {d1, d}]]] +
     Sum[ϵ^{d1} γ_{d1,φ[[k]]}[k], {k, 2 n}, {d1, d}] /. γ_{_,0}[_] → 0];
  Z2 = Expand[F[{}, {}] × Normal@Series[Z1, {ϵ, 0, d}]] //.
    F[fs_, {es___}] × (f : (r | γ)_{ps__}[is__])^{p_.} :>
     F[Join[fs, Table[f, p]], DeleteDuplicates@{es, is}];
  Z3 = Expand[Z2 /. F[fs_, es_] :> Expand[gPair[
        Replace[fs, Thread[es → Range@Length@es], {2}], Length@es
       ] /. g_{α_,β_} :> G[[es[[α]], es[[β]]]]] ];
  Collect[{Δ, Z3 /. ϵ^{p_.} → p! Δ^{2 p} ϵ^p}, ϵ, T2z] ];
```

```
Z_2[GST48] (* takes a few minutes *)
```

$\{1 - 4z^2 - 61z^4 - 207z^6 - 296z^8 - 210z^{10} - 77z^{12} - 14z^{14} - z^{16},$

$1 + (38z^2 + 255z^4 + 1696z^6 + 16\,281z^8 + 86\,952z^{10} + 259\,994z^{12} + 487\,372z^{14} + 615\,066z^{16} + 543\,148z^{18} + 341\,714z^{20} + 153\,722z^{22} + 48\,983z^{24} + 10\,776z^{26} + 1554z^{28} + 132z^{30} + 5z^{32})\,\epsilon +$

$(-8 - 484z^2 + 9709z^4 + 165\,952z^6 + 1\,590\,491z^8 + 16\,256\,508z^{10} + 115\,341\,797z^{12} + 432\,685\,748z^{14} + 395\,838\,354z^{16} - 4\,017\,557\,792z^{18} - 23\,300\,064\,167z^{20} - 70\,082\,264\,972z^{22} - 142\,572\,271\,191z^{24} - 209\,475\,503\,700z^{26} - 221\,616\,295\,209z^{28} - 151\,502\,648\,428z^{30} - 23\,700\,199\,243z^{32} + 99\,462\,146\,328z^{34} + 164\,920\,463\,074z^{36} + 162\,550\,825\,432z^{38} + 119\,164\,552\,296z^{40} + 69\,153\,062\,608z^{42} + 32\,547\,596\,611z^{44} + 12\,541\,195\,448z^{46} + 3\,961\,384\,155z^{48} + 1\,021\,219\,696z^{50} + 212\,773\,106z^{52} + 35\,264\,208z^{54} + 4\,537\,548z^{56} + 436\,600z^{58} + 29\,536z^{60} + 1252z^{62} + 25z^{64})\,\epsilon^2\}$

```
TableForm[Table[Join[{K[[1]]_{K[[2]]}}, Z_3[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center] (* takes a few minutes *)
```

3_1	$1+z^2$	$1+(2z^2+z^4)\epsilon+(2-4z^2+3z^4+4z^6+z^8)\epsilon^2+(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12})\epsilon^3$
4_1	$1-z^2$	$1+(-2+2z^4)\epsilon^2$
5_1	$1+3z^2+z^4$	$1+(10z^2+21z^4+12z^6+2z^8)\epsilon+(6-28z^2+33z^4+364z^6+655z^8+536z^{10}+227z^{12}+48z^{14}+4z^{16})\epsilon^2+(-60+970z^2+645z^4-3380z^6-3280z^8+7470z^{10}+19\,475z^{12}+20\,536z^{14}+12\,564z^{16}+4774z^{18}+1109z^{20}+144z^{22}+8z^{24})\epsilon^3$
5_2	$1+2z^2$	$1+(6z^2+5z^4)\epsilon+(4-20z^2+43z^4+64z^6+26z^8)\epsilon^2+(-36+498z^2-883z^4+100z^6+816z^8+556z^{10}+146z^{12})\epsilon^3$
6_1	$1-2z^2$	$1+(-2z^2+z^4)\epsilon+(-4+4z^2+25z^4-8z^6+2z^8)\epsilon^2+(12+154z^2-223z^4-608z^6+100z^8-52z^{10}+10z^{12})\epsilon^3$
6_2	$1-z^2-z^4$	$1+(-2z^2-3z^4+2z^6+z^8)\epsilon+(-2-4z^2+29z^4+28z^6+42z^8-8z^{10}-2z^{12}+4z^{14}+z^{16})\epsilon^2+(12+166z^2+155z^4-194z^6-2453z^8-1622z^{10}-1967z^{12}-258z^{14}+49z^{16}-30z^{18}+z^{20}+6z^{22}+z^{24})\epsilon^3$
6_3	$1+z^2+z^4$	$1+(2+8z^2-16z^6-24z^8-16z^{10}-2z^{12})\epsilon^2$

Video: http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210. Handout: http://www.math.toronto.edu/~drorbn/Talks/Nara-2308.

Computing the Zombian of an Unfinished Columbarium

Confession. It's about 50% of what I do.

Apology. It's a 20 minutes talk. Necessarily, it will be superficial.

Abstract. The zombies need to compute a quantity, the zombian, that pertains to some structure — say, a columbarium. But unfortunately (for them), a part of that structure will only be known in the future. What can they compute today with the parts they already have to hasten tomorrow's computation?
That's a common quest, and I will illustrate it with a few examples from knot theory and with two examples about matrices — determinants and signatures. I will also mention two of my dreams (perhaps delusions): that one day I will be able to reproduce, and extend, the Rolfsen table of knots using code of the highest level of beauty.

Columbaria in an East Sydney Cemetery

Image: Freepik.com

Jacobian, Hamiltonian, Zombian

Zombies: Freepik.com

Computing Zombians of Unfinished Columbaria.

- Future zombies must be able to complete the computation.
- Must be no slower than for finished ones.
- Future zombies must not even know the size of the task that today's zombies were facing.
- We must be able to extend to ZPUCs, Zombie Processed Unfinished Columbaria!

Columbarium near Assen

Exercise 1. Compute the sum of 1,000 numbers, the last 50 of which are still unknown.

Exercise 2. Compute the determinant of a $1,000 \times 1,000$ matrix in which 50 entries are not yet given.

Example 3. Same, for signatures of matrices / quadratic forms.

A quadratic form on a v.s. V over $\mathbb{C}$ is a quadratic $Q\colon V \to \mathbb{C}$, or a sesquilinear Hermitian $\langle\cdot,\cdot\rangle$ on $V \times V$ (so $\langle x, y\rangle = \overline{\langle y, x\rangle}$ and $Q(y) = \langle y, y\rangle$), or given a basis η_i of V^*, a matrix $A = (a_{ij})$ with $A = \bar{A}^T$ and $Q = \sum a_{ij}\bar{\eta}_i\eta_j$. The signature σ of Q is $\sigma_+ - \sigma_-$, where for some P, $\bar{P}^T A P = \operatorname{diag}(1, \overset{\sigma_+}{\cdots}, 1, -1, \overset{\sigma_-}{\cdots}, -1, 0, \ldots)$.

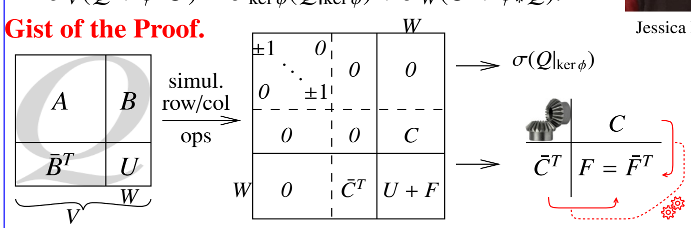
A *Partial Quadratic (PQ)* on V is a quadratic Q defined only on a subspace $\mathcal{D}_Q \subset V$. We add PQs with $\mathcal{D}_{Q_1+Q_2} := \mathcal{D}_{Q_1} \cap \mathcal{D}_{Q_2}$. Given a linear $\psi\colon V \to W$ and a PQ Q on W, there is an obvious *pullback* ψ^*Q, a PQ on V.

Theorem 1 (with Jessica Liu). Given a linear $\phi\colon V \to W$ and a PQ Q on V, there is a unique *pushforward* PQ ϕ_*Q on W such that for every *PQ* U on W,

$$\sigma_V(Q + \phi^*U) = \sigma_{\ker\phi}(Q|_{\ker\phi}) + \sigma_W(U + \phi_*Q).$$

Jessica Liu

Gist of the Proof.



…and the quadratic $F =: \phi_*Q$ is well-defined only on $D := \ker C$.

(more at ωεβ/icerm.)

Acknowledgement. This work was partially supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC).

Knots and Tangles.

"Nautical Knots"

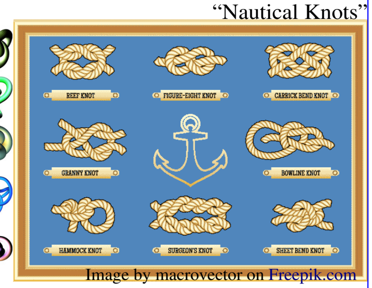
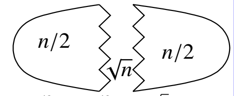


Image by macrovector on Freepik.com

Why Tangles? • As common as knots!

- Faster computations!
- Conceptually clearer proofs of invariance (and of skein relations).
- Often fun and consequential:

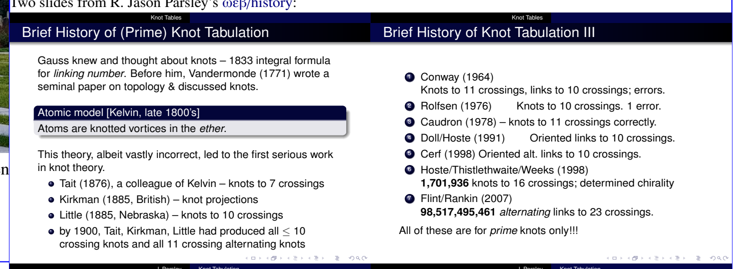


$2^{n/2} + 2^{n/2} + 2^{\sqrt{n}} \ll 2^n$

- ∘ The Alexander polynomial ⇝ Zombian = det.
- ∘ Knot signatures ⇝ Pushforwards of quadratic forms.
- ∘ The Jones Polynomial ⇝ The Temperley-Lieb Algebra.
- ∘ Khovanov Homology ⇝ "Unfinished complexes", complexes in a category.
- ∘ The Kontsevich Integral ⇝ Drinfel'd Associators. …

One more story is left to tell, of knot tabulation.

Two slides from R. Jason Parsley's ωεβ/history:



Knot Tables

Brief History of (Prime) Knot Tabulation

Gauss knew and thought about knots – 1833 integral formula for *linking number*. Before him, Vandermonde (1771) wrote a seminal paper on topology & discussed knots.

Atomic model [Kelvin, late 1800's]

Atoms are knotted vortices in the *ether*.

This theory, albeit vastly incorrect, led to the first serious work in knot theory.

- Tait (1876), a colleague of Kelvin – knots to 7 crossings
- Kirkman (1885, British) – knot projections
- Little (1885, Nebraska) – knots to 10 crossings
- by 1900, Tait, Kirkman, Little had produced all ≤ 10 crossing knots and all 11 crossing alternating knots

J. Parsley Knot Tabulation

Knot Tables

Brief History of Knot Tabulation III

1. Conway (1964) Knots to 11 crossings, links to 10 crossings; errors.
2. Rolfsen (1976) Knots to 10 crossings. 1 error.
3. Caudron (1978) – knots to 11 crossings correctly.
4. Doll/Hoste (1991) Oriented links to 10 crossings.
5. Cerf (1998) Oriented alt. links to 10 crossings.
6. Hoste/Thistlethwaite/Weeks (1998) **1,701,936** knots to 16 crossings; determined chirality
7. Flint/Rankin (2007) **98,517,495,461** *alternating* links to 23 crossings.

All of these are for *prime* knots only!!!

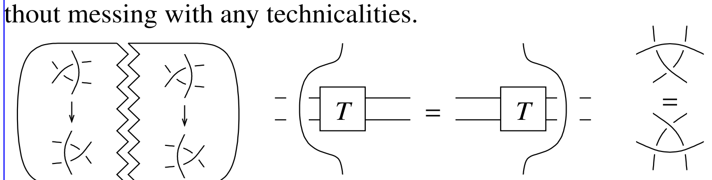
J. Parsley Knot Tabulation

There's also Burton's tabulation to 19 crossings ωεβ/Burton, and Khesin's K250, arXiv:1705.10319.

Embarrassment 1 (personal). I don't know how to reproduce the Rolfsen table of knots! Many others can, yet I still take it on faith, contradicting one of the tenets of our practice, "thou shalt not use what thou canst not prove".
It's harder than it seems! Producing all knot diagrams is a mess, identifying all available Reidemeister moves is a mess, and you sometimes have to go up in crossing number before you can go down again.

Embarrassment 2 (communal). There isn't anywhere a tabulation of tangles! When you want to test your new discoveries, where do you go?

Dream. Conquer both embarrassments at once. Reproduce the Rolfsen table, and extend it to tangles, using code of the highest level of beauty. The algorithm should be so clear and simple that anyone should be able to easily implement it in an afternoon without messing with any technicalities.



We don't even need to look at all knot diagrams!

The dreaded slide moves, which go up in crossing number, are parametrized by tangles!

R-moves are tangle equalities!

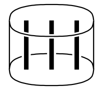
$\begin{pmatrix} A & B \\ C & U \end{pmatrix} \xrightarrow{\det(A)} \begin{pmatrix} I & A^{-1}B \\ C & U \end{pmatrix} \xrightarrow{1} \begin{pmatrix} I & A^{-1}B \\ 0 & U - CA^{-1}B \end{pmatrix}$, so $\det\begin{pmatrix} A & B \\ C & U \end{pmatrix} = \det(A)\det(U - CA^{-1}B)$.

Roughly, $\det(A)$ is "det on ker", $-CA^{-1}B$ is "a pushforward of $\begin{pmatrix} A & B \\ C & U \end{pmatrix}$". (what if $\nexists A^{-1}$?)

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ottawa-2306/

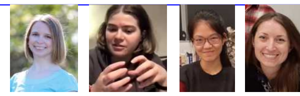
Tangles in a Pole Dance Studio: A Reading of Massuyeau, Alekseev, and Naef

Preliminary Definitions. Fix $p \in \mathbb{N}$ and $\mathbb{F} = \mathbb{Q}/\mathbb{C}$. Let $D_p := D^2\backslash(p \text{ pts})$, and let the Pole Dance Studio be $PDS_p := D_p \times I$.



PDS_3

Abstract. I will report on joint work with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich. Little of what we do is original, and much of it is simply a reading of Massuyeau [Ma] and Alekseev and Naef [AN1].



Dancso Hogan Liu Scherich

We study the pole-strand and strand-strand double filtration on the space of tangles in a pole dance studio (a punctured disk cross an interval), the corresponding homomorphic expansions, and a strand-only HOMFLY-PT relation. When the strands are transparent or nearly transparent to each other we recover and perhaps simplify substantial parts of the work of the above-mentioned authors on expansions for the Goldman-Turaev Lie bi-algebra.

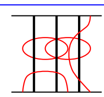
Jessica, Nancy, Tamara, Zsuzsi, & Dror in PDS_4

$\Longrightarrow$ Expansions W: $FG\langle X_i\rangle \to FA\langle x_i\rangle$; Magnus: $X_i \mapsto 1 + x_i$, $X_i^{-1} \mapsto 1 - x_i + x_i^2 - \ldots$; Exponential: $X_i^{\pm 1} \mapsto e^{\pm x_i}$.

Definitions. Let $\pi := FG\langle X_1, \ldots, X_p\rangle$ be the free group (of deformation classes of based curves in D_p), $\bar\pi$ be the framed free group (deformation classes of based immersed curves), $|\pi|$ and $|\bar\pi|$ denote $\mathbb{F}$-linear combinations of cyclic words ($|x_i w| = |w x_i|$, unbased curves), $A := FA\langle x_1, \ldots, x_p\rangle$ be the free associative algebra, and let $|A| := A/(x_i w = w x_i)$ denote cyclic algebra words.

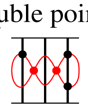
Theorem 1 (Goldman, Turaev, Massuyeau, Alekseev, Kawazumi, Kuno, Naef). $|\bar\pi|$ and $|A|$ are Lie bialgebras, and there is a "homomorphic expansion" $W\colon |\bar\pi| \to |A|$: a morphism of Lie bialgebras with $W(|X_i|) = 1 + |x_i| + \ldots$.

Further Definitions. • $\mathcal{K} = \mathcal{K}_0 = \mathcal{K}_0^0 = \mathcal{K}(S) := \mathbb{F}\langle\text{framed tangles in } PDS_p\rangle$.



• $\mathcal{K}_t^s$:=(the image via $\times \to \overcrossing - \undercrossing$ of tangles in PDS_p that have t double points, of which s are strand-strand).

E.g.,

$$\mathcal{K}_5^2(\bigcirc) = \Big\langle$$  $$\Big\rangle / . \quad \times \to \overcrossing - \undercrossing$$

• $\mathcal{K}^{/s} := \mathcal{K}/\mathcal{K}^s$. Most important, $\mathcal{K}^{/1}(\bigcirc) = |\bar\pi|$, and there is $P\colon \mathcal{K}(\bigcirc) \to |\bar\pi|$.

• $\mathcal{A} := \prod \mathcal{K}_t/\mathcal{K}_{t+1}$, $\quad \mathcal{A}^s := \prod \mathcal{K}_t^s/\mathcal{K}_{t+1}^s \subset \mathcal{A}$, $\quad \mathcal{A}^{/s} := \mathcal{A}/\mathcal{A}^s$.

Fact 1. The Kontsevich Integral is an "expansion" $Z\colon \mathcal{K} \to \mathcal{A}$, compatible with several noteworthy structures.

Fact 2 (Le-Murakami, [LM1]). Z satisfies the strand-strand HOMFLY-PT relations: It descends to $Z_H\colon \mathcal{K}_H \to \mathcal{A}_H$, where

$$\mathcal{K}_H := \mathcal{K}\Big/\!\!\Big(\overcrossing - \undercrossing = (e^{\hbar/2} - e^{-\hbar/2})\cdot)(\Big)$$

$$\mathcal{A}_H := \mathcal{A}/(\ \longleftrightarrow = \hbar \asymp \ \text{ or } \ \longleftrightarrow = \hbar \asymp\ )$$

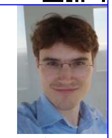
and $\deg \hbar = (1, 1)$.

Proof of Fact 2. $Z(\overcrossing) - Z(\undercrossing) = \times \cdot \left(e^{\hbar/2} - e^{-\hbar/2}\right)$

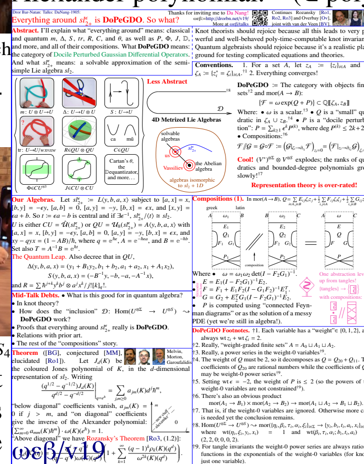
$$= \times \cdot \left(e^{\hbar\times/2} - e^{-\hbar\times/2}\right) = \left(e^{\hbar/2} - e^{-\hbar/2}\right))(. \qquad \square$$

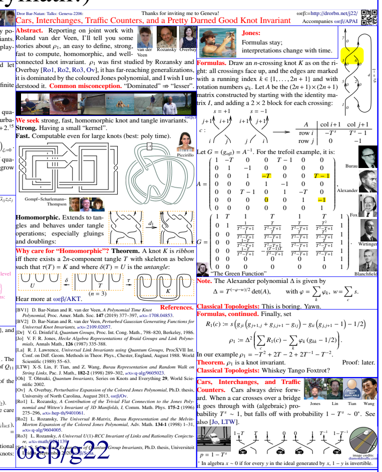
Le, Murakami

Other Passions. With Roland van der Veen, I use "solvable approximation" and "Perturbed Gaussian Differential Operators" to unveil simple, strong, fast to compute, and topologically meaningful knot invariants near the Alexander polynomial. ($\subset$ polymath!)



van der Veen





Key 1. $W\colon |\bar\pi| \to |A|$ is $Z_H^{/1}\colon \mathcal{K}_H^{/1}(\bigcirc) \to \mathcal{A}_H^{/1}(\bigcirc)$.

Key 2 (Schematic). Suppose $\lambda_0, \lambda_1\colon |\bar\pi| \to \mathcal{K}(\bigcirc)$ are two ways of lifting plane curves into knots in PDS_p (namely, $P \circ \lambda_i = I$). Then for $\gamma \in |\bar\pi|$,

Lemma 1. "Division by $\hbar$" is well-defined.

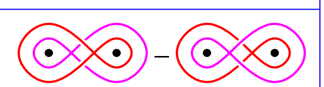
$$\eta(\gamma) := (\lambda_0(\gamma) - \lambda_1(\gamma))/\hbar \in \mathcal{K}_H^{/1}(\bigcirc\bigcirc) = |\bar\pi| \otimes |\bar\pi|$$

and we get an operation η on plane curves. If Kontsevich likes λ_0 and λ_1 (namely if there are λ_i^a with $Z^{/2}(\lambda_i(\gamma)) = \lambda_i^a(W(\gamma))$), then η will have a compatible algebraic companion η^a:

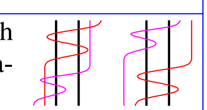
$$\eta^a(\alpha) := (\lambda_0^a(\alpha) - \lambda_1^a(\alpha))/\hbar \in \mathcal{A}_H^{/1}(\bigcirc\bigcirc) = |A| \otimes |A|.$$

For indeed, in $\mathcal{A}_H^{/2}$ we have $\hbar W(\eta(\gamma)) = \hbar Z(\eta(\gamma)) = Z(\lambda_0(\gamma)) - Z(\lambda_1(\gamma)) = \lambda_0^a(W(\gamma)) - \lambda_1^a(W(\gamma)) = \hbar\eta^a(W(\gamma))$.

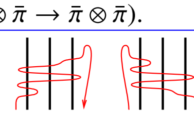
Example 1. With $\gamma_1, \gamma_2 \in |\pi|$ (or $|\bar\pi|$) set $\lambda_0(\gamma_1, \gamma_2) = \tilde\gamma_1 \cdot \tilde\gamma_2$ and $\lambda_1(\gamma_1, \gamma_2) = \tilde\gamma_2 \cdot \tilde\gamma_1$ where $\tilde\gamma_i$ are arbitrary lifts of γ_i. Then η_1 is the Goldman bracket! Note that here λ_0 and λ_1 are not well-defined, yet η_1 is.



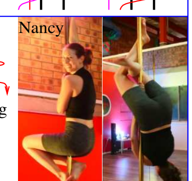
Example 2. With $\gamma_1, \gamma_2 \in \pi$ (or $\bar\pi$) and with λ_0, λ_1 as on the right, we get the "double bracket" $\eta_2\colon \pi \otimes \pi \to \pi \otimes \pi$ (or $\bar\pi \otimes \bar\pi \to \bar\pi \otimes \bar\pi$).



Example 3. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending realization as a bottom tangle and $\lambda_1(\gamma)$ its descending realization as a bottom tangle, we get $\eta_3\colon \bar\pi \to \bar\pi \otimes |\bar\pi|$. Closing the first component and anti-symmetrizing, this is the Turaev cobracket.

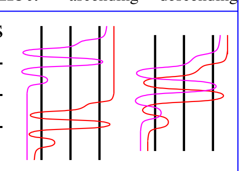


ascending descending



ascending descending

Example 4 [Ma]. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending outer double and $\lambda_1(\gamma)$ its ascending inner double we get $\eta_4\colon \bar\pi \to \bar\pi \otimes \bar\pi$. After some massaging, it too becomes the Turaev cobracket.

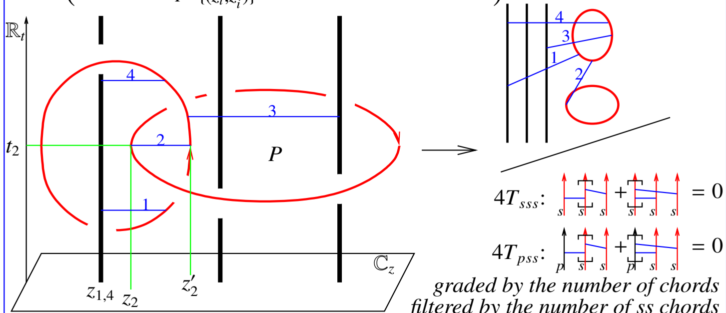


The rest is essentially **Exercises:** **1.** Lemma 1? **2.** $\mathcal{A}$? **3.** Fact 2? **4.** $\mathcal{A}^{/1}$? Especially, $\mathcal{A}^{/1}(\bigcirc) \cong |A|$! **5.** Explain why Kontsevich likes our λ's. **6.** Figure out η_i^a, $i = 1, \ldots, 4$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-2208/

Kontsevich in a Pole Dance Studio. (w/o poles? See [Ko, BN])

$$Z=\left(\sum_{m=0}^{\infty}\frac{1}{(2\pi i)^m}\sum_{\substack{t_1<\dots<t_m\\ P=\{(z_i,z_i')\}}}(-1)^{\#P_\downarrow}D_P\bigwedge_{i=1}^{m}\frac{dz_i-dz_i'}{z_i-z_i'}\right)^{\sim}\in\mathcal{A}$$



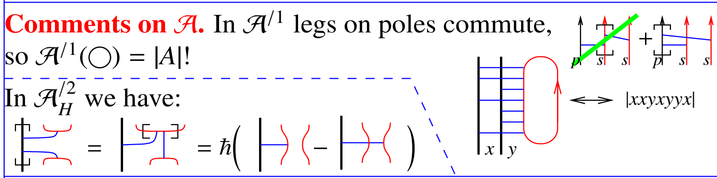
graded by the number of chords
filtered by the number of ss chords

Comments on the Kontsevich Integral.

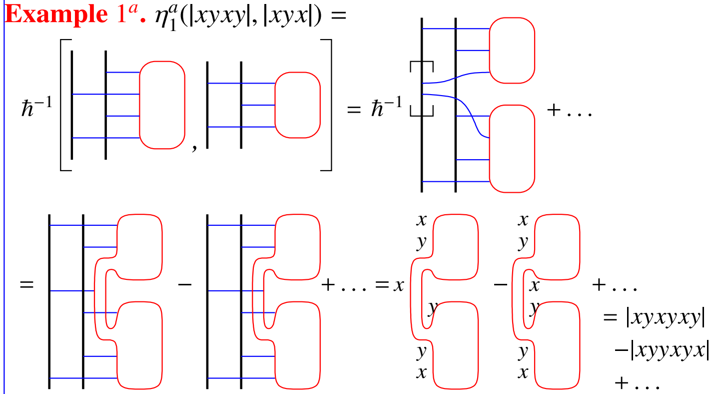
1. In the tangle case, the endpoints are fixed at top and bottom.
2. The $(\cdots)^\sim$ means "a correction is needed near the caps and the cups" (for the framed version, see [LM2, Da]).
3. There are never pp chords, and no $4T_{pps}$ and $4T_{ppp}$ relations.
4. Z is an "expansion".
5. Z respects the ss filtration and so descends to $Z^{/s}\colon \mathcal{K}^{/s}\to\mathcal{A}^{/s}$.

Comments on $\mathcal{A}$. In $\mathcal{A}^{/1}$ legs on poles commute, so $\mathcal{A}^{/1}(\bigcirc)=|A|$!

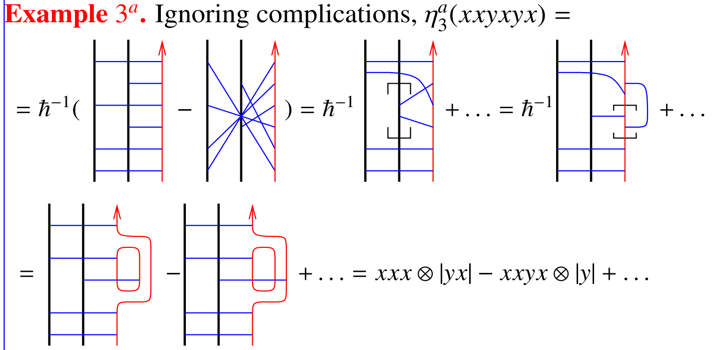
In $\mathcal{A}^{/2}_H$ we have:



Example 1^a. $\eta_1^a(|xyxy|,|xyx|)=$



Example 3^a. Ignoring complications, $\eta_3^a(xxyxyx)=$



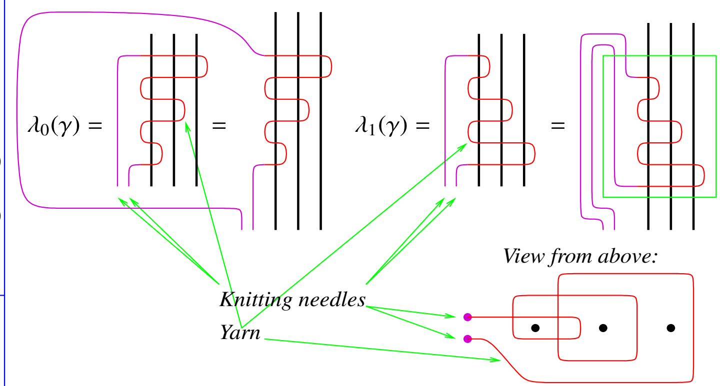
Proof of Lemma 1. We partially prove Theorem 2 instead:

Theorem 2. $\mathrm{gr}^\bullet\,\mathcal{K}_H\cong\mathbb{F}[\![\hbar]\!]\otimes(\mathcal{K}^{/1})_0$.

Proof mod $\hbar^2$. The map $\leftarrow$ is obvious. To go $\rightarrow$, map $\mathcal{K}_H\to\mathbb{F}[\![\hbar]\!]\otimes\mathcal{K}^{/1}$ using $\mapsto + \frac{\hbar}{2}$ and $\mapsto - \frac{\hbar}{2}$ and apply the functor $\mathrm{gr}^\bullet$.

Unignoring the Complications. We need λ_0 and λ_1 such that:

1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping all self-intersections from ascending to descending.
2. Up to conjugation, $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip.
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma))=W(\gamma)$.



Homework

1. Is there more than Examples 1–4?
2. Derive the bialgebra axioms from this perspective.
3. What more do we get if we don't mod out by HOMFLY-PT?
4. What more do we get if we allow more than one strand-strand interaction?
5. In this language, recover Kashiwara-Vergne [AKKN1, AKKN2].
6. How is all this related to w-knots?
7. Do the same with associators. Use that to derive formulas for solutions of Kashiwara-Vergne.
8. What's the relationship with the Habiro-Massuyeau invariants of links in handlebodies [HM] (different filtration!).
9. Pole dance on other surfaces!
10. Explore the action of the mapping class group.

Kashiwara Vergne

Acknowledgement. This work was partially supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC). I also wish to thanks A. Alekseev, F. Naef, and M. Ren for listening to an earlier version and catching some bugs, and Dhanya S. for the dance studio photos. And of course, thanks for listening!

References

[AKKN1] A. Alekseev, N. Kawazumi, Y. Kuno, & F. Naef, *The Goldman-Turaev Lie Bialgebra in Genus Zero and the Kashiwara-Vergne Problem,* Adv. Math. **326** (2018) 1–53, arXiv:1703.05813.

[AKKN2] A. Alekseev, N. Kawazumi, Y. Kuno, & F. Naef, *Goldman-Turaev formality implies Kashiwara-Vergne,* Quant. Topol. **11-4** (2020) 657-–689, arXiv:1812.01159.

[AN1] A. Alekseev & F. Naef, *Goldman-Turaev Formality from the Knizhnik-Zamolodchikov Connection,* Comp. Rend. Math. **355-11** (2017) 1138–1147, arXiv:1708.03119.

[BN] D. Bar-Natan, *On the Vassiliev Knot Invariants,* Top. **34** (1995) 423–472.

[Da] Z. Dancso, *On the Kontsevich Integral for Knotted Trivalent Graphs,* Alg. Geom. Topol. **10** (2010) 1317–1365, arXiv:0811.4615.

[HM] K. Habiro & G. Massuyeau, *The Kontsevich Integral for Bottom Tangles in Handlebodies,* Quant. Topol. **12-4** (2021) 593–703, arXiv:1702.00830.

[Ko] M. Kontsevich, *Vassiliev's Knot Invariants,* Adv. in Sov. Math. **16(2)** (1993) 137–150.

[LM1] T. Q. T. Le & J. Murakami, *Kontsevich's Integral for the HOMFLY Polynomial and Relations Between Values of Multiple Zeta Functions,* Top. and its Appl. **62-2** (1995) 193–206.

[LM2] T. Q. T. Le & J. Murakami, *The Universal Vassiliev-Kontsevich Invariant for Framed Oriented Links,* Comp. Math. **102-1** (1996) 41–64, arXiv:hep-th/9401016.

[Ma] G. Massuyeau, *Formal Descriptions of Turaev's Loop Operations,* Quant. Topol. **9-1** (2018) 39-–117, arXiv:1511.03974.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-2208/

Kashaev's Signature Conjecture

CMS Winter 2021 Meeting, December 4, 2021

Dror Bar-Natan with Sina Abbasi

Agenda. Show and tell with signatures.

Abstract. I will display side by side two nearly identical computer programs whose inputs are knots and whose outputs seem to always be the same. I'll then admit, very reluctantly, that I don't know how to prove that these outputs are always the same. One program I wrote mostly in Bedlewo, Poland, in the summer of 2003 and as of recently I understand why it computes the Levine-Tristram signature of a knot. The other is based on the 2018 preprint *On Symmetric Matrices Associated with Oriented Link Diagrams* by Rinat Kashaev (arXiv:1801.04632), where he conjectures that a certain simple algorithm also computes that same signature.

If you can, please turn your video on! (And mic, whenever needed).

These slides and all the code within are available at http://drorbn.net/cms21.

(I'll post the video there too)


```
Bed[K_, ω_] :=
 Module[{t, r, XingsByArmpits, bends, faces, p, A, is},
  t = 1 - ω; r = t + t*;
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X+[-i, j, k, -l], X-[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p[a,-d] p[b,-a] p[c,-b] p[d,-c];
  faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z];
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X+,
     ( -r   -t  2t   t* )   (  r   -t  -2t*  t* )
     ( -t*   0   t*  0  ) , ( -t*   0   t*   0  ) ],
     ( 2t*   t  -r  -t* )   ( -2t   t   r   -t* )
     (  t    0  -t   0  )   (  t    0  -t    0  )
   {x, XingsByArmpits}];
  MatrixSignature[A]];
```

```
Kas[K_, ω_] :=
 Module[{u, v, XingsByArmpits, bends, faces, p, A, is},
  u = Re[ω^(1/2)]; v = Re[ω];
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X+[-i, j, k, -l], X-[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p[a,-d] p[b,-a] p[c,-b] p[d,-c];
  faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z];
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X+,
     ( v u 1 u )    ( v u 1 u )
     ( u 1 u 1 ) , -( u 1 u 1 ) ],
     ( 1 u v u )    ( 1 u v u )
     ( u 1 u 1 )    ( u 1 u 1 )
   {x, XingsByArmpits}];
  (MatrixSignature[A] - Writhe[K]) / 2];
```


Why am I showing you 💀code💀?

- I love code — it's fun!
- Believe it or not, it is more expressive than math-talk (though I'll do the math-talk as well, to confirm with prevailing norms).
- It is directly verifiable. Once it is up and running, you'll never ask yourself "did he misplace a sign somewhere"?

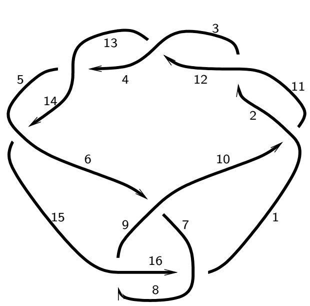

```
Bed[K_, ω_] :=
 Module[{t, r, XingsByArmpits, bends, faces, p, A, is},
  t = 1 - ω; r = t + t*;
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X+[-i, j, k, -l], X-[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p[a,-d] p[b,-a] p[c,-b] p[d,-c];
  faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z];
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X+,
     ( -r   -t  2t   t* )   (  r   -t  -2t*  t* )
     ( -t*   0   t*  0  ) , ( -t*   0   t*   0  ) ],
     ( 2t*   t  -r  -t* )   ( -2t   t   r   -t* )
     (  t    0  -t   0  )   (  t    0  -t    0  )
   {x, XingsByArmpits}];
  MatrixSignature[A]];
```

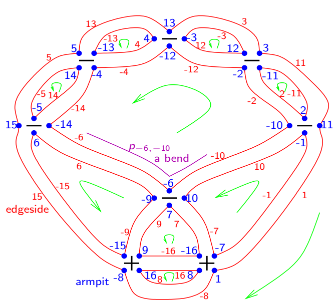
```
Kas[K_, ω_] :=
 Module[{u, v, XingsByArmpits, bends, faces, p, A, is},
  u = Re[ω^(1/2)]; v = Re[ω];
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X+[-i, j, k, -l], X-[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p[a,-d] p[b,-a] p[c,-b] p[d,-c];
  faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z];
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X+,
     ( v u 1 u )    ( v u 1 u )
     ( u 1 u 1 ) , -( u 1 u 1 ) ],
     ( 1 u v u )    ( 1 u v u )
     ( u 1 u 1 )    ( u 1 u 1 )
   {x, XingsByArmpits}];
  (MatrixSignature[A] - Writhe[K]) / 2];
```


Verification.

```
Once[<< KnotTheory`]
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
MatrixSignature[A_] :=
  Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-12 &]]];
Writhe[K_] := Sum[If[PositiveQ[x], 1, -1], {x, List @@ PD@K}];
Sum[ω = e^(i RandomReal[{0,2π}]); Bed[K, ω] == Kas[K, ω], {10},
  {K, AllKnots[{3, 10}]}]
KnotTheory: Loading precomputed data in PD4Knots`.
2490 True
```


Label everything!

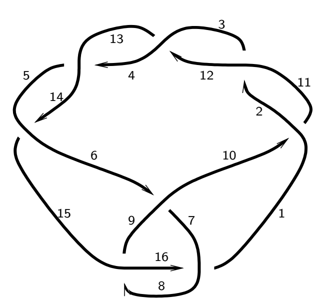




$PD[X[10,1,11,2], X[2,11,3,12], \ldots]$ $\{X_-[-1,11,2,-10], X_-[-11,3,12,-2], \ldots\}$

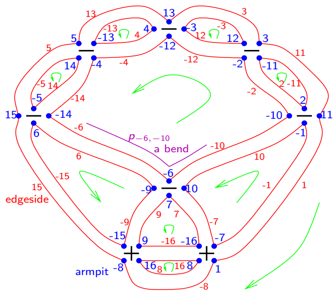
Lets run our code line by line. . .

```
PD[8₂] = PD[X[10, 1, 11, 2],
   X[2, 11, 3, 12], X[12, 3, 13, 4],
   X[4, 13, 5, 14], X[14, 5, 15, 6],
   X[8, 16, 9, 15], X[16, 8, 1, 7],
   X[6, 9, 7, 10]];
K = 8₂;
```



```
XingsByArmpits =
  List @@ PD[K] /.
   x : X[i_, j_, k_, l_] :>
    If[PositiveQ[x], X+[-i, j, k, -l],
     X-[-j, k, l, -i]]
```

$\{X_-[-1, 11, 2, -10], X_-[-11, 3, 12, -2], X_-[-3, 13, 4, -12], X_-[-13, 5, 14, -4], X_-[-5, 15, 6, -14], X_+[-8, 16, 9, -15], X_+[-16, 8, 1, -7], X_-[-9, 7, 10, -6]\}$

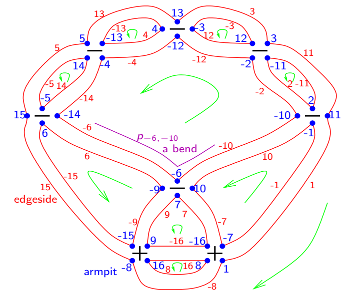



```
bends = Times @@ XingsByArmpits /.
   _[X][a_, b_, c_, d_] :>
    p[a,-d] p[b,-a] p[c,-b] p[d,-c]
```

$p_{-16,7}\, p_{-15,-9}\, p_{-14,-6}\, p_{-13,4}\, p_{-12,-4}\, p_{-11,2}\, p_{-10,-2}\, p_{-9,6}\, p_{-8,15}\, p_{-7,-1}\, p_{-6,-10}\, p_{-5,14}\, p_{-4,-14}\, p_{-3,12}\, p_{-2,-12}\, p_{-1,10}\, p_{1,-8}\, p_{2,-11}\, p_{3,11}\, p_{4,-13}\, p_{5,13}\, p_{6,-15}\, p_{7,9}\, p_{8,16}\, p_{9,-16}\, p_{10,-7}\, p_{11,1}\, p_{12,-3}\, p_{13,3}\, p_{14,-5}\, p_{15,5}\, p_{16,8}$

```
faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z]
```

$p_{-13,4,-13}\, p_{-11,2,-11}\, p_{-5,14,-5}\, p_{-3,12,-3}\, p_{8,16,8}\, p_{6,-15,-9,6}\, p_{9,-16,7,9}\, p_{10,-7,-1,10}\, p_{-10,-2,-12,-4,-14,-6,-10}\, p_{1,-8,15,5,13,3,11,1}$




```
A = Table[0, Length@faces, Length@faces];
A // MatrixForm
```

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$


```
Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
 A[[is, is]] += If[Head[x] === X+,
```

$$\boxed{\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}}, -\boxed{\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}}],$$

```
 {x, XingsByArmpits}];
```



```
x = XingsByArmpits[[1]]
```

$X_-[-1, 11, 2, -10]$

```
faces
```

$p_{-13,4,-13}\, p_{-11,2,-11}\, p_{-5,14,-5}\, p_{-3,12,-3}\, p_{8,16,8}\, p_{6,-15,-9,6}\, p_{9,-16,7,9}\, p_{10,-7,-1,10}\, p_{-10,-2,-12,-4,-14,-6,-10}\, p_{1,-8,15,5,13,3,11,1}$

```
is = Position[faces, #][[1, 1]] & /@ List @@ x
```

{8, 10, 2, 9}


```
A[[is, is]] += If[Head[x] === X+,
```

$$\boxed{\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}}, -\boxed{\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}}];$$

```
A // MatrixForm
```

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&-v&0&0&0&0&0&-1&-u&-u\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&-v&-u&-u\\ 0&-u&0&0&0&0&0&-u&-1&-1\\ 0&-u&0&0&0&0&0&-u&-1&-1 \end{pmatrix}$$

Recall, $is = \{8, 10, 2, 9\}$


```
Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
 A[[is, is]] += If[Head[x] === X+,
```

$$\boxed{\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}}, -\boxed{\begin{pmatrix} v&u&1&u\\ u&1&u&1\\ 1&u&v&u\\ u&1&u&1 \end{pmatrix}}],$$

```
 {x, Rest@XingsByArmpits}]
```

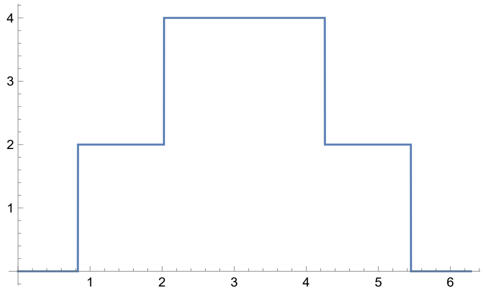


```
A // MatrixForm
```

$$\begin{pmatrix} -2v&0&-1&-1&0&0&0&0&-2u&-2u\\ 0&-2v&0&-1&0&0&0&-1&-2u&-2u\\ -1&0&-2v&0&0&-1&0&0&-2u&-2u\\ -1&-1&0&-2v&0&0&0&0&-2u&-2u\\ 0&0&0&0&2&1&2u&1&0&2u\\ 0&0&-1&0&1&1-2v&0&-1&-2u&0\\ 0&0&0&0&2u&0&-1+2v&0&-1&2\\ 0&-1&0&0&1&-1&0&1-2v&-2u&0\\ -2u&-2u&-2u&-2u&0&-2u&-1&-2u&-6&-5\\ -2u&-2u&-2u&-2u&2u&0&2&0&-5&-5+2v \end{pmatrix}$$

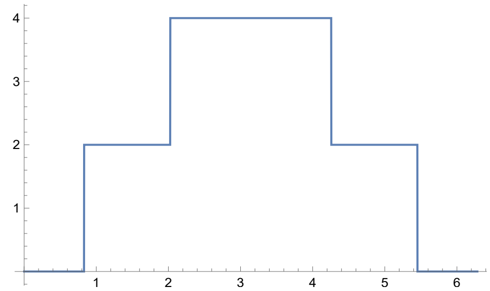
Video and more at http://www.math.toronto.edu/~drorbn/Talks/CMS-2112/

```
Plot[ω = e^(i t); u = Re[ω^(1/2)]; v = Re[ω];
 (MatrixSignature[A] - Writhe[K]) / 2,
 {t, 0, 2π}]
```



http://drorbn.net/cms21

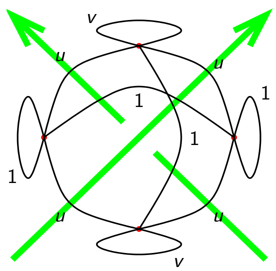
```
Plot[Bed[Knot[8, 2], e^(i t)], {t, 0, 2π}]
```

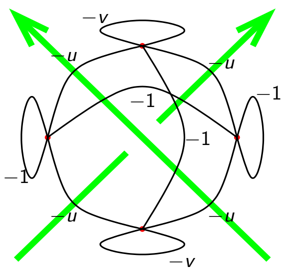


http://drorbn.net/cms21

Kashaev for Mathematicians.

For a knot K and a complex unit ω set $u = \Re(\omega^{1/2})$, $v = \Re(\omega)$, make an $F \times F$ matrix A with contributions



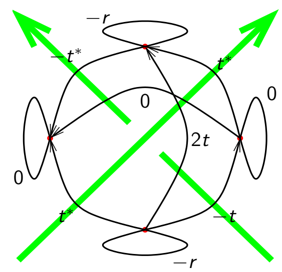


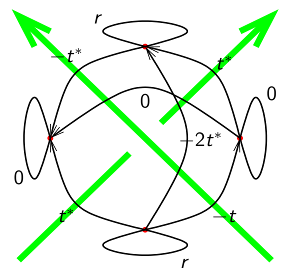
and output $\frac{1}{2}(\sigma(A) - w(K))$.

http://drorbn.net/cms21

Bedlewo for Mathematicians.

For a knot K and a complex unit ω set $t = 1 - \omega$, $r = 2\Re(t)$, make an $F \times F$ matrix A with contributions





(conjugate if going against the flow) and output $\sigma(A)$.

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Why are they equal?

I dunno, yet note that

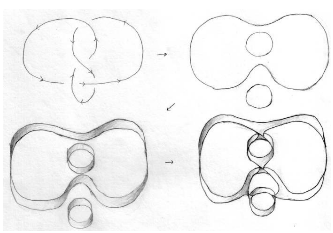
- Kashaev is over the $\mathbb{R}$eals, Bedlewo is over the $\mathbb{C}$omplex numbers.
- There's a factor of 2 between them, and a shift.

. . . so it's not merely a matrix manipulation.

http://drorbn.net/cms21

Theorem. The Bedlewo program computes the Levine-Tristram signature of K at ω.

(Easy) **Proof.** Levine and Tristram tell us to look at $\sigma((1-\omega)L + (1-\omega^*)L^T)$, where L is the linking matrix for a Seifert surface S for K: $L_{ij} = \mathrm{lk}(\gamma_i, \gamma_j^+)$ where γ_i run over a basis of $H_1(S)$ and γ_i^+ is the pushout of γ_i. But signatures don't change if you run over and over-determined basis, and the faces make such and over-determined basis whose linking numbers are controlled by the crossings. The rest is details.



Art by Emily Redelmeier

http://drorbn.net/cms21

Thank You!

Video and more at http://www.math.toronto.edu/~drorbn/Talks/CMS-2112/

Warning. The second formula on page (−2) "**Conclusion**" is silly-wrong. A fix will be posted here soon: some of the numbers written in this handout are a bit off, yet the qualitative results remain exactly the same (namely, for finite type, 3D seems to beat 2D, with the same algorithms).

Yarn-Ball Knots

[K-OS] on October 21, 2021

Dror Bar-Natan with Itai Bar-Natan, Iva Halacheva, and Nancy Scherich

Agenda. A modest light conversation on how knots should be measured.

Abstract. Let there be scones! Our view of knot theory is biased in favour of pancakes.
Technically, if K is a 3D knot that fits in volume V (assuming fixed-width yarn), then its projection to 2D will have about $V^{4/3}$ crossings. You'd expect genuinely 3D quantities associated with K to be computable straight from a 3D presentation of K. Yet we can hardly ever circumvent this $V^{4/3} \gg V$ "projection fee". Exceptions include linking numbers (as we shall prove), the hyperbolic volume, and likely finite type invariants (as we shall discuss in detail). But knot polynomials and knot homologies seem to always pay the fee. Can we exempt them?

More at http://drorbn.net/kos21

Thanks for inviting me to speak at [K-OS]!

Most important: http://drorbn.net/kos21

See also arXiv:2108.10923.

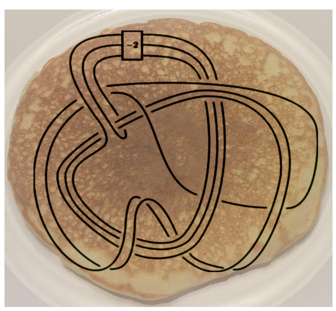
If you can, please turn your video on! (And mic, whenever needed).

A recurring question in knot theory is "do we have a 3D understanding of our invariant?"

- See Witten and the Jones polynomial.
- See Khovanov homology.

I'll talk about my perspective on the matter. . .

We often think of knots as planar diagrams. 3-dimensionally, they are embedded in "pancakes".



Knot by Lisa Piccirillo, pancake by DBN

But real life knots are 3D!

A Yarn Ball

'Connector' by Alexandra Griess and Jorel Heid (Hamburg, Germany). Image from www.waterfrontbia.com/ice-breakers-2019-presented-by-ports/.

The difference matters when

- We make statements about "random knots".
- We figure out computational complexity.

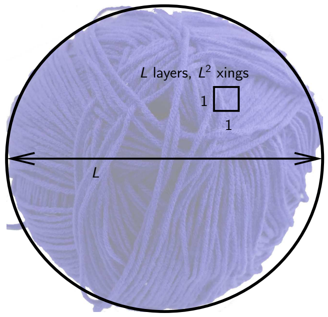
Let's try to make it quantitative. . .

Video and more at http://www.math.toronto.edu/~drorbn/Talks/KOS-211021/

$V \sim L^3$

$n = \text{xing number} \sim L^2L^2 = L^4 = V^{4/3}$

("$\sim$" means "equal up to constant terms and log terms")

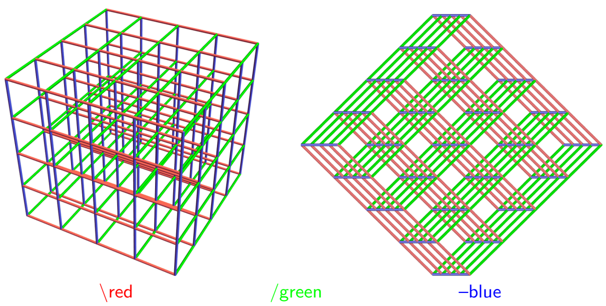


Conversation Starter 1. A knot invariant ζ is said to be Computationally 3D, or C3D, if

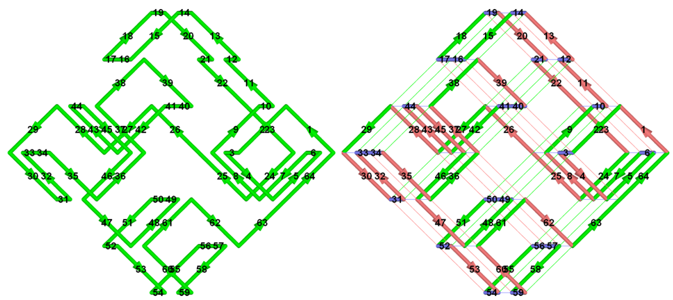
$$C_\zeta(3D, V) \ll C_\zeta(2D, V^{4/3}).$$

This isn't a rigorous definition! It is time- and naïveté-dependent! But there's room for less-stringent rigour in mathematics, and on a philosophical level, our definition means something.

Theorem 1. Let lk denote the linking number of a 2-component link. Then $C_{lk}(2D, n) \sim n$ while $C_{lk}(3D, V) \sim V$, so lk is C3D!
Proof. WLOG, we are looking at a link in a grid, which we project as on the right:

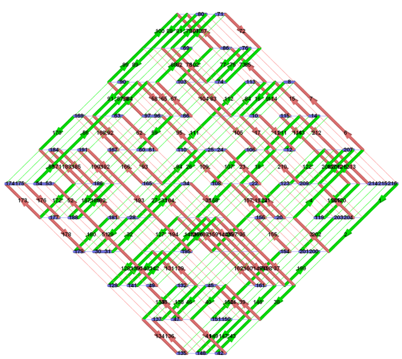


Here's what it look like, in the case of a knot:



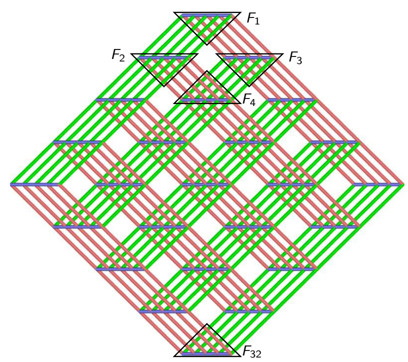
And here's a bigger knot.

This may look like a lot of information, but if V is big, it's less than the information in a planar diagram, and it is easily computable.



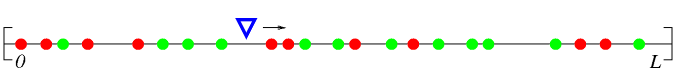
There are $2L^2$ triangular "crossings fields" F_k in such a projection.

WLOG, in each F_k all over strands and all under strands are oriented in the same way and all green edges belong to one component and all red edges to the other.

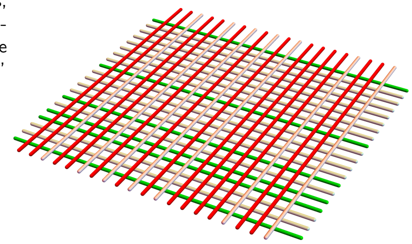


So $2L^2$ times we have to solve the problem "given two sets R and G of integers in $[0, L]$, how many pairs $\{(r, g) \in R \times G : r < g\}$ are there?". This takes time $\sim L$ (see below), so the overall computation takes time $\sim L^3$.

Below. Start with $rb = cf = 0$ ("reds before" and "cases found") and slide $\triangledown$ from left to right, incrementing rb by one each time you cross a $\bullet$ and incrementing cf by rb each time you cross a $\bullet$:



In general, with our limited tools, speedup arises because appropriately projected 3D knots have many uniform "red over green" regions:



Video and more at http://www.math.toronto.edu/~drorbn/Talks/KOS-211021/

Great Embarrassment 1. I don't know if any of the Alexander, Jones, HOMFLY-PT, and Kauffman polynomials is C3D. I don't know if any Reshetikhin-Turaev invariant is C3D. I don't know if any knot homology is C3D.

Or maybe it's a cause for optimism — there's still something very basic we don't know about (say) the Jones polynomial. Can we understand it well enough 3-dimensionally to compute it well? If not, why not?

Conversation Starter 2. Similarly, if η is a stingy quantity (a quantity we expect to be small for small knots), we will say that η has Savings in 3D, or "has S3D" if $M_\eta(3D, V) \ll M_\eta(2D, V^{4/3})$.

Example (R. van der Veen, D. Thurston, private communications). The hyperbolic volume has *S3D*.

Great Embarrassment 2. I don't know if the genus of a knot has S3D! In other words, even if a knot is given in a 3-dimensional, the best way I know to find a Seifert surface for it is to first project it to 2D, at a great cost.

Next we argue that most finite type invariants are probably C3D...

(What a weak statement!)

Theorem FT2D. If ζ is a finite type invariant of type d then $C_\zeta(2D, n)$ is at most $\sim n^{\lfloor 3d/4 \rfloor}$. With more effort, $C_\zeta(2D, n) \lesssim n^{(\frac{2}{3}+\epsilon)d}$.

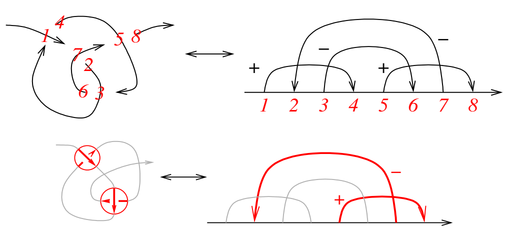
Note that there are some exceptional finite type invariants, e.g. high coefficients of the Alexander polynomial and other poly-time knot polynomials, which can be computed much faster!

Theorem FT3D. If ζ is a finite type invariant of type d then $C_\zeta(3D, V)$ is at most $\sim V^{6d/7+1/7}$. With more effort, $C_\zeta(2D, V) \lesssim V^{(\frac{4}{5}+\epsilon)d}$.

Tentative Conclusion. As
$n^{3d/4} \sim (V^{4/3})^{3d/4} = V \gg V^{6d/7+1/7}$ $\qquad n^{2d/3} \sim (V^{4/3})^{2d/3} = V^{8d/9} \gg V^{4d/5}$
these theorems say "most finite type invariants are probably C3D; the ones in greater doubt are the lucky few that can be computed unusually quickly".

All pre-categorification knot polynomials are power series whose coefficients are finite type invariants. (This is sometimes helpful for the computation of finite type invariants, but rarely helpful for the computation of knot polynomials).

Gauss diagrams and sub-Gauss-diagrams:

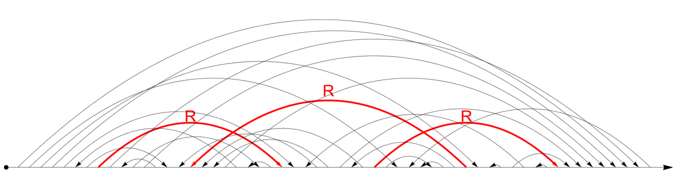


Let φ_d: {knot diagrams} $\to$ ⟨Gauss diagrams⟩ map every knot diagram to the sum of all the sub-diagrams of its Gauss diagram which have at most d arrows.

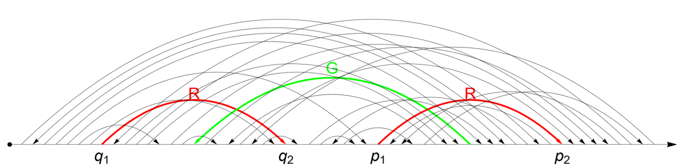
Under-Explained Theorem (Goussarov-Polyak-Viro). A knot invariant ζ is of type d iff there is a linear functional ω on ⟨Gauss diagrams⟩ such that $\zeta = \omega \circ \varphi_d$.

Theorem FT2D. If ζ is a finite type invariant of type d then $C_\zeta(2D, n)$ is at most $\sim n^{\lfloor 3d/4 \rfloor}$. With more effort, $C_\zeta(2D, n) \lesssim n^{(\frac{2}{3}+\epsilon)d}$.

Proof of Theorem FT2D.



We need to count how many times a diagram such as the red appears within a bigger diagram, having n arrows. Clearly this can be done in time $\sim n^3$, and in general, in time $\sim n^d$.

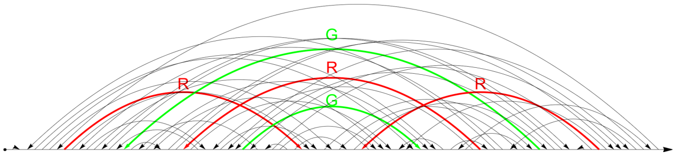


With an appropriate look-up table, it can also be done in time $\sim n^2$ (in general, $\sim n^{d-1}$). That look-up table $(T^{p_1,p_2}_{q_1,q_2})$ is of size (and production cost) $\sim n^4$ if you are naive, and $\sim n^2$ if you are just a bit smarter. Indeed

$$T^{p_1,p_2}_{q_1,q_2} = T^{0,p_2}_{0,q_2} - T^{0,p_1}_{0,q_2} - T^{0,p_2}_{0,q_1} + T^{0,p_1}_{0,q_1},$$

and $(T^{0,p}_{0,q})$ is easy to compute.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/KOS-211021/



With multiple uses of the same lookup table, what naively takes $\sim n^5$ can be reduced to $\sim n^3$.

In general within a big d-arrow diagram we need to find an as-large-as possible collection of arrows to delay. These must be non-adjacent to each other. As the adjacency graph for the arrows is at worst quadrivalent, we can always find $\lceil \frac{d}{4} \rceil$ non-adjacent arrows, and hence solve the counting problem in time $\sim n^{d-\lceil \frac{d}{4} \rceil} = n^{\lfloor 3d/4 \rfloor}$.

Note that this counting argument works equally well if each of the d arrows is pulled from a different set!

It follows that we can compute φ_d in time $\sim n^{\lfloor 3d/4 \rfloor}$.

$\square$

With bigger look-up tables that allow looking up "clusters" of G arrows, we can reduce this to $\sim n^{(\frac{2}{3}+\epsilon)d}$.

$\square$

On to

Theorem FT3D. If ζ is a finite type invariant of type d then $C_\zeta(3D, V)$ is at most $\sim V^{6d/7+1/7}$. With more effort, $C_\zeta(2D, V) \lesssim V^{(\frac{4}{5}+\epsilon)d}$.

An image editing problem:

(Yarn ball and background coutesy of Heather Young)

The line/feather method:

Accurate but takes forever.

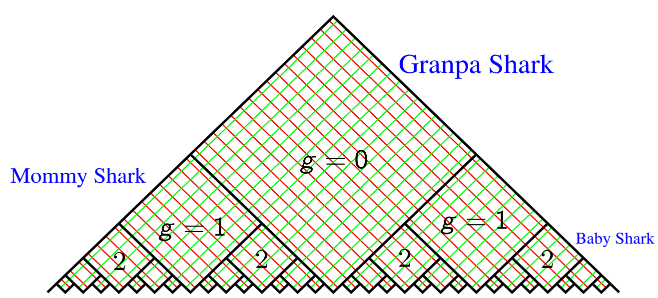
The rectangle/shark method:

Coarse but fast.

In reality, you take a few shark bites and feather the rest ...

... and then there's an optimization problem to solve: when to stop biting and start feathering.

The structure of a crossing field.



There are about $\log_2 L$ "generations". There are 2^g bites in generation g, and the total number of crossings in them is $\sim L^2/2^g$. Let's go hunt!

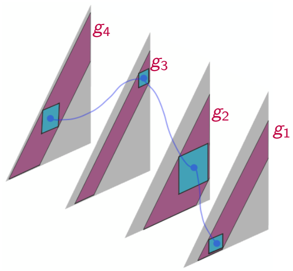
Video and more at http://www.math.toronto.edu/~drorbn/Talks/KOS-211021/

Multi-feathers and multi-sharks.

For a type d invariant we need to count d-tuples of crossings, and each has its own "generation" g_i. So we have the "multi-generation"

$$\bar{g} = (g_1, \dots, g_d).$$

Let $G := \sum g_i$ be the "overall generation". We will choose between a "multi-feather" method and a "multi-shark" method based on the size of G.



The effort to take a single multi-bite is tiny. Indeed,

Lemma Given $2d$ finite sets $B_i = \{t_{i1}, t_{i2}, \dots\} \subset [1..L^3]$ and a permutation $\pi \in S_{2n}$ the quantity

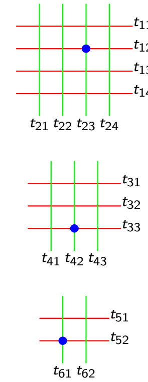
$$N = \left| \left\{ (b_i) \in \prod_{i=1}^{2d} B_i \colon \text{the } b_i\text{'s are ordered as } \pi \right\} \right|$$

can be computed in time $\sim \sum |B_i| \sim \max |B_i|$.

Proof. WLOG $\pi = Id$. For $\iota \in [1..2d]$ and $\beta \in B := \cup B_i$ let

$$N_{\iota,\beta} = \left| \left\{ (b_i) \in \prod_{i=1}^{\iota} B_i \colon b_1 < b_2 < \ldots < b_\iota \leq \beta \right\} \right|.$$

We need to know $N_{2d, \max B}$; compute it inductively using $N_{\iota,\beta} = N_{\iota,\beta'} + N_{\iota-1,\beta'}$, where β' is the predecessor of β in B. $\square$



Conclusion. We wish to compute the contribution to φ_d coming from d-tuples of crossings of multi-generation $\bar{g}$.

- The multi-shark method does it in time

$$\sim (\text{no. of bites}) \cdot (\text{time per bite}) = L^{2d} 2^G \cdot \frac{L}{2^{\min \bar{g}}} < L^{2d+1} 2^G$$

(increases with G).

- The multi-feather method (project and use the 2D algorithm) does it in time

$$\sim (\text{no. of crossings})^{\lfloor \frac{3}{4} d \rfloor} = \left(\prod_{i=1}^{d} L^2 \frac{L^2}{2^{g_i}} \right)^{\lfloor \frac{3}{4} d \rfloor} < \frac{L^{3d}}{(2^G)^{3/4}}$$

(decreases with G).

Of course, for any specific G we are free to choose whichever is better, shark or feather.

The two methods agree (and therefore are at their worst) if $2^G = L^{\frac{4}{7}(d-1)}$, and in that case, they both take time $\sim L^{\frac{18}{7}d + \frac{3}{7}} = V^{\frac{6}{7}d + \frac{1}{7}}$.

The same reasoning, with the $n^{(\frac{2}{3}+\epsilon)d}$ feather, gives $V^{(\frac{4}{5}+\epsilon)d}$.

$\square$

If time — a word about braids.

Thank You!

Video and more at http://www.math.toronto.edu/~drorbn/Talks/KOS-211021/

I Still Don't Understand the Alexander Polynomial

Dror Bar-Natan, http://drorbn.net/mo21

Moscow by Web, April 2021

Abstract. As an algebraic knot theorist, I still don't understand the Alexander polynomial. There are two conventions as for how to present tangle theory in algebra: one may name the strands of a tangle, or one may name their ends. The distinction might seem too minor to matter, yet it leads to a completely different view of the set of tangles as an algebraic structure. There are lovely formulas for the Alexander polynomial as viewed from either perspective, and they even agree where they meet. But the "strands" formulas know about strand doubling while the "ends" ones don't, and the "ends" formulas know about skein relations while the "strands" ones don't. There ought to be a common generalization, but I don't know what it is.

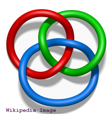
Thanks for inviting me to Moscow! As most of you have never seen it, here's a picture of the lecture room:

If you can, please turn your video on! (And mic, whenever needed).

I use talks to self-motivate; so often I choose a topic and write an abstract when I know I can do it, yet when I haven't done it yet. This time it turns out my abstract was wrong — I'm still uncomfortable with the Alexander polynomial, but in slightly different ways than advertised two slides before.

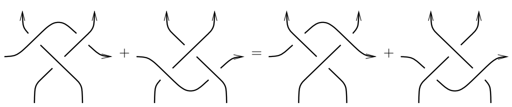
My discomfort.

- I can compute the multivariable Alexander polynomial real fast:



$\longrightarrow (uvw)^{-1/2}(u-1)(v-1)(w-1).$

- But I can only prove "skein relations" real slow:



This talk is to a large extent an elucidation of the Ph.D. theses of my former students Jana Archibald and Iva Halacheva. See [Ar, Ha1, Ha2].

Also thanks to Roland van der Veen for comments.

A technicality. There's supposed to be fire alarm testing in my building today. Don't panic!

1. Virtual Skein Theory Heaven

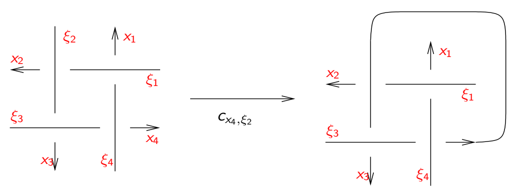
Definition. A "Contraction Algebra" assigns a set $\mathcal{T}(\mathcal{X}, X)$ to any pair of finite sets $\mathcal{X} = \{\xi \dots\}$ and $X = \{x, \dots\}$ provided $|\mathcal{X}| = |X|$, and has operations

- "Disjoint union" $\sqcup\colon \mathcal{T}(\mathcal{X}, X) \times \mathcal{T}(\mathcal{Y}, Y) \to \mathcal{T}(\mathcal{X} \sqcup \mathcal{Y}, X \sqcup Y)$, provided $\mathcal{X} \cap \mathcal{Y} = X \cap Y = \emptyset$.
- "Contractions" $c_{x,\xi}\colon \mathcal{T}(\mathcal{X}, X) \to \mathcal{T}(\mathcal{X} \setminus \xi, X \setminus x)$, provided $x \in X$ and $\xi \in \mathcal{X}$.
- Renaming operations $\sigma^{\xi}_{\eta}\colon \mathcal{T}(\mathcal{X} \sqcup \{\xi\}, X) \to \mathcal{T}(\mathcal{X} \sqcup \{\eta\}, X)$ and $\sigma^{x}_{y}\colon \mathcal{T}(\mathcal{X}, X \sqcup \{x\}) \to \mathcal{T}(\mathcal{X}, X \sqcup \{y\})$.

Subject to axioms that will be specified right after the two examples in the next three slides.

If R is a ring, a contraction algebra is said to be "R-linear" if all the $\mathcal{T}(\mathcal{X}, X)$'s are R-modules, if the disjoint union operations are R-bilinear, and if the contractions $c_{x,\xi}$ and the renamings $\sigma^{\cdot}_{\cdot}$ are R-linear.

(Contraction algebras with some further "unit" properties are called "wheeled props" in [MMS, DHR])

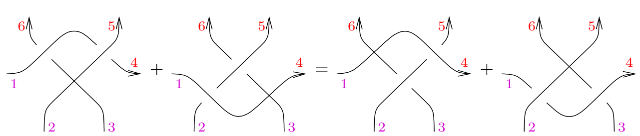


Example 1. Let $\mathcal{T}(\mathcal{X}, X)$ be the set of virtual tangles with incoming ends ("tails") labeled by $\mathcal{X}$ and outgoing ends ("heads") labeled by X, with $\sqcup$ and $\sigma^{\cdot}_{\cdot}$ the obvious disjoint union and end-renaming operations, and with $c_{x,\xi}$ the operation of attaching a head x to a tail ξ while introducing no new crossings.

Note 1. $\mathcal{T}$ can be made linear by allowing formal linear combinations.

Note 2. $\mathcal{T}$ is finitely presented, with generators the positive and negative crossings, and with relations the Reidemeister moves! (If you want, you can take this to be the definition of "virtual tangles").

Note 3. A contraction algebra morphism out of $\mathcal{T}$ is an invariant of virtual tangles (and hence of virtual knots and links) and would be an ideal tool to prove Skein Relations:



Example 2. Let V be a finite dimensional vector space and set $\mathcal{V}(\mathcal{X}, X) := (V^*)^{\otimes\mathcal{X}} \otimes V^{\otimes X}$, with $\sqcup = \otimes$, with $\sigma^{\cdot}_{\cdot}$ the operation of renaming a factor, and with $c_{x,\xi}$ the operation of contraction: the evaluation of tensor factor ξ (which is a V^*) on tensor factor x (which is a V).

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/

Axioms. One axiom is primary and interesting,

- Contractions commute! Namely, $c_{x,\xi}/\!\!/c_{y,\eta} = c_{y,\eta}/\!\!/c_{x,\xi}$ (or in old-speak, $c_{y,\eta} \circ c_{x,\xi} = c_{x,\xi} \circ c_{y,\eta}$).

And the rest are just what you'd expect:

- $\sqcup$ is commutative and associative, and it commutes with $c_{\cdot,\cdot}$ and with $\sigma^{\cdot}_{\cdot}$ whenever that makes sense.
- $c_{\cdot,\cdot}$ is "natural" relative to renaming: $c_{x,\xi} = \sigma^x_y/\!\!/\sigma^\xi_\eta/\!\!/c_{y,\eta}$.
- $\sigma^\xi_\xi = \sigma^x_x = Id$, $\sigma^\xi_\eta/\!\!/\sigma^\eta_\zeta = \sigma^\xi_\zeta$, $\sigma^x_y/\!\!/\sigma^y_z = \sigma^x_z$, and renaming operations commute where it makes sense.

Comments.

- We can relax $|\mathcal{X}| = |X|$ at no cost.
- We can lose the distinction between $\mathcal{X}$ and X and get "circuit algebras".
- There is a "coloured version", where $\mathcal{T}(\mathcal{X}, X)$ is replaced with $\mathcal{T}(\mathcal{X}, X, \lambda, l)$ where $\lambda\colon \mathcal{X} \to C$ and $l\colon X \to C$ are "colour functions" into some set C of "colours", and contractions $c_{x,\xi}$ are allowed only if x and ξ are of the same colour, $l(x) = \lambda(\xi)$. In the world of tangles, this is "coloured tangles".

2. Heaven is a Place on Earth

(A version of the main results of Archibald's thesis, [Ar]).

Let us work over the base ring $\mathcal{R} = \mathbb{Q}[\{T^{\pm 1/2}\colon T \in C\}]$. Set

$$\mathcal{A}(\mathcal{X}, X) := \{w \in \Lambda(\mathcal{X} \sqcup X)\colon \deg_{\mathcal{X}} w = \deg_X w\}$$

(so in particular the elements of $\mathcal{A}(\mathcal{X}, X)$ are all of even degree). The union operation is the wedge product, the renaming operations are changes of variables, and $c_{x,\xi}$ is defined as follows. Write $w \in \mathcal{A}(\mathcal{X}, X)$ as a sum of terms of the form uw' where $u \in \Lambda(\xi, x)$ and $w' \in \mathcal{A}(\mathcal{X} \setminus \xi, X \setminus x)$, and map u to 1 if it is 1 or $x\xi$ and to 0 is if is ξ or x:

$$1w' \mapsto w', \qquad \xi w' \mapsto 0, \qquad xw' \mapsto 0, \qquad x\xi w' \mapsto w'.$$

Proposition. $\mathcal{A}$ is a contraction algebra.

Alternative Formulations.

- $$c_{x,\xi} w = \iota_\xi \iota_x \mathbb{e}^{x\xi} w,$$ where $\iota_\cdot$ denotes interior multiplication.
- Using Fermionic integration, $$c_{x,\xi} w = \int \mathbb{e}^{x\xi} w \, d\xi dx.$$
- $c_{x,\xi}$ represents composition in exterior algebras! With $X^* := \{x^*\colon x \in X\}$, we have that $\operatorname{Hom}(\Lambda X, \Lambda Y) \cong \Lambda(X^* \sqcup Y)$ and the following square commutes:

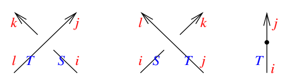
$$\begin{tikzcd}
\operatorname{Hom}(\Lambda X, \Lambda Y) \otimes \operatorname{Hom}(\Lambda Y, \Lambda Z) \arrow[r, "/\!\!/"] \arrow[d, leftrightarrow] & \operatorname{Hom}(\Lambda X, \Lambda Z) \arrow[d, leftrightarrow] \\
\Lambda(X^* \sqcup Y \sqcup Y^* \sqcup Z) \arrow[r, "\prod_{y \in Y} c_{y,y^*}"] & \Lambda(X^*, Z)
\end{tikzcd}$$

- Similarly, $\Lambda(\mathcal{X} \sqcup X) \cong (H^*)^{\otimes \mathcal{X}} \otimes H^{\otimes X}$ where H is a 2-dimensional "state space" and H^* is its dual. Under this identification, $c_{x,\xi}$ becomes the contraction of an H factor with an H^* factor.

We construct a morphism of coloured contraction algebras $\mathcal{A}\colon \mathcal{T} \to \mathcal{A}$ by declaring

$$\begin{aligned}
X_{ijkl}[S, T] &\mapsto T^{-1/2} \exp\left(\begin{pmatrix} \xi_l & \xi_i \end{pmatrix} \begin{pmatrix} 1 & 1 - T \\ 0 & T \end{pmatrix} \begin{pmatrix} x_j \\ x_k \end{pmatrix} \right) \\
\bar{X}_{ijkl}[S, T] &\mapsto T^{1/2} \exp\left(\begin{pmatrix} \xi_i & \xi_j \end{pmatrix} \begin{pmatrix} T^{-1} & 0 \\ 1 - T^{-1} & 1 \end{pmatrix} \begin{pmatrix} x_k \\ x_l \end{pmatrix} \right) \\
P_{ij}[T] &\mapsto \exp(\xi_i x_j)
\end{aligned}$$

with



$X_{ijkl}[S, T]$ $\qquad$ $\bar{X}_{ijkl}[S, T]$ $\qquad$ $P_{ij}[T]$

(Note that the matrices appearing in these formulas are the Burau matrices).

Theorem.

If D is a classical link diagram with k components coloured $T_1, \ldots, T_k$ whose first component is open and the rest are closed, if MVA is the multivariable Alexander polynomial of the closure of D (with these colours), and if ρ_j is the counterclockwise rotation number of the jth component of D, then

$$\mathcal{A}(D) = T_1^{-1/2}(T_1 - 1) \left(\prod_j T_j^{\rho_j/2} \right) \cdot MVA \cdot (1 + \xi_{\text{in}} \wedge x_{\text{out}}).$$

($\mathcal{A}$ vanishes on closed links).

3. An Implementation of $\mathcal{A}$

If I didn't implement I wouldn't believe myself.

Written in Mathematica [Wo], available as the notebook `Alpha.nb` at http://drorbn.net/mo21/ap. Code lines are highlighted in grey, demo lines are plain. We start with an implementation of elements ("`Wedge`") of exterior algebras, and of the wedge product ("`WP`"):

```
WP[Wedge[u___], Wedge[v___]] := Signature[{u, v}] * Wedge @@ Sort[{u, v}];
WP[0, _] = WP[_, 0] = 0;
WP[A_, B_] :=
  Expand[Distribute[A ** B] /.
    (a_. * u_Wedge) ** (b_. * v_Wedge) :> a b WP[u, v]];
```

```
WP[Wedge[] + Wedge[a] - 2 b ∧ a, Wedge[] - 3 Wedge[b] + 7 c ∧ d]
```

```
Wedge[] + Wedge[a] - 3 Wedge[b] - a ∧ b + 7 c ∧ d + 7 a ∧ c ∧ d + 14 a ∧ b ∧ c ∧ d
```

We then define the exponentiation map in exterior algebras ("`WExp`") by summing the series and stopping the sum once the current term ("`t`") vanishes:

```
WExp[A_] := Module[{s = Wedge[], t = Wedge[], k = 0},
  While[t =!= 0, s += (t = Expand[WP[t, A] / (++k)])]; s]
```

```
WExp[a ∧ b + c ∧ d + e ∧ f]
```

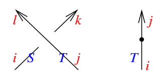
```
Wedge[] + a ∧ b + c ∧ d + e ∧ f + a ∧ b ∧ c ∧ d + a ∧ b ∧ e ∧ f + c ∧ d ∧ e ∧ f + a ∧ b ∧ c ∧ d ∧ e ∧ f
```

Contractions!

```
c_{x_,y_}[w_Wedge] := Module[{i, j},
  {i} = FirstPosition[w, x, {0}]; {j} = FirstPosition[w, y, {0}];
  Piecewise[{{w, (i == 0) ∧ (j == 0)},
    {(-1)^(i+j+If[i>j,0,1]) Delete[w, {{i}, {j}}], (i > 0) ∧ (j > 0)}}]
 ];
c_{x_,y_}[ℰ_] := ℰ /. w_Wedge :> c_{x,y}[w]
WExp[a∧b + 2 c∧d]
c_{d,c}@WExp[a∧b + 2 c∧d]
```

```
Wedge[] + a∧b + 2 c∧d + 2 a∧b∧c∧d
-Wedge[] - a∧b
```

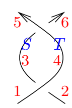
The negative crossing and the "point":



$\bar{X}_{ijkl}[S,T]$ $P_{ij}[T]$

```
𝒜[X̄_{i_,j_,k_,l_}[S_, T_]] := 𝒜[{i, j}, {k, l}, ⟨|ξ_i → S, ξ_j → T, x_k → S, x_l → T|⟩,
   Expand[T^{1/2} WExp[Expand[{ξ_i, ξ_j}.( T^-1  0 ; 1 - T^-1  1 ).{x_k, x_l}] /. ξ_a_ x_b_ :> ξ_a∧x_b]]];
𝒜[X̄_{i_,j_,k_,l_}] := 𝒜[X̄_{i,j,k,l}[τ_i, τ_j]];
𝒜[P_{i_,j_}[T_]] := 𝒜[{i}, {j}, ⟨|ξ_i → T, x_j → T|⟩, WExp[ξ_i∧x_j]];
𝒜[P_{i_,j_}] := 𝒜[P_{i,j}[τ_i]]
```

The union operation on $\mathcal{A}$'s (implemented as "multiplication"):




```
𝒜 /: 𝒜[is1_, os1_, cs1_, w1_] × 𝒜[is2_, os2_, cs2_, w2_] :=
 𝒜[is1 ⋃ is2, os1 ⋃ os2, Join[cs1, cs2], WP[w1, w2]]
Short[𝒜[X_{2,4,3,1}[S, T]] × 𝒜[X̄_{3,4,6,5}], 5]
```

```
𝒜[{1, 2, 3, 4}, {3, 4, 5, 6},
 ⟨|ξ2 → S, x4 → T, x3 → S, ξ1 → T, ξ3 → τ3, ξ4 → τ4, x6 → τ3, x5 → τ4|⟩, √τ4 Wedge[]/√T -
  √τ4 x3∧ξ1/√T + √T √τ4 x3∧ξ1 - √T √τ4 x3∧ξ2 - √τ4 x4∧ξ1/√T - √τ4 x5∧ξ4/√T -
  x6∧ξ3/(√T √τ4) + <<40>> + √T x3∧x5∧x6∧ξ1∧ξ3∧ξ4/√τ4 - √T x3∧x5∧x6∧ξ2∧ξ3∧ξ4/√τ4 -
  x4∧x5∧x6∧ξ1∧ξ3∧ξ4/(√T √τ4) + √T x3∧x4∧x5∧x6∧ξ1∧ξ2∧ξ3∧ξ4/√τ4]
```

Automatic and intelligent multiple contractions:

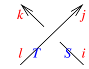
```
c@𝒜[is_, os_, cs_, w_] := Fold[c_{#2,#2}[#1] &, 𝒜[is, os, cs, w], is ⋂ os]
𝒜[{A_𝒜}] := c[A];
𝒜[{A1_𝒜, As__𝒜}] := Module[{A2},
  A2 = First@MaximalBy[{As}, Length[A1[[1]] ⋂ #[[2]]] + Length[A1[[2]] ⋂ #[[1]]] &];
  𝒜[Join[{c[A1 A2]}, DeleteCases[{As}, A2]]]]
𝒜[𝒪s_List] := 𝒜[𝒜 /@ 𝒪s]
c[𝒜[X_{2,4,3,1}[S, T]] × 𝒜[X̄_{3,4,6,5}]]
```

```
𝒜[{1, 2}, {5, 6}, ⟨|ξ2 → S, ξ1 → T, x6 → S, x5 → T|⟩,
 Wedge[] - x5∧ξ1 - x6∧ξ2 - x5∧x6∧ξ1∧ξ2]
```

```
𝒜@{𝒜[X_{2,4,3,1}[S, T]], 𝒜[X̄_{3,4,6,5}]}
```

```
𝒜[{1, 2}, {5, 6}, ⟨|ξ2 → S, ξ1 → T, x6 → S, x5 → T|⟩,
 Wedge[] - x5∧ξ1 - x6∧ξ2 - x5∧x6∧ξ1∧ξ2]
```

`𝒜[is,os,cs,w]` is also a container for the values of the $\mathcal{A}$-invariant of a tangle. In it, `is` are the labels of the input strands, `os` are the labels of the output strands, `cs` is an assignment of colours (namely, variables) to all the ends $\{\xi_i\}_{i\in\mathrm{is}}\sqcup\{x_j\}_{j\in\mathrm{os}}$, and `w` is the "payload": an element of $\Lambda(\{\xi_i\}_{i\in\mathrm{is}}\sqcup\{x_j\}_{j\in\mathrm{os}})$.



$X_{ijkl}[S,T]$

```
𝒜[X_{i_,j_,k_,l_}[S_, T_]] := 𝒜[{l, i}, {j, k}, ⟨|ξ_i → S, x_j → T, x_k → S, ξ_l → T|⟩,
   Expand[T^{-1/2} WExp[Expand[{ξ_l, ξ_i}.( 1  1 - T ; 0  T ).{x_j, x_k}] /. ξ_a_ x_b_ :> ξ_a∧x_b]]];
𝒜[X_{1,2,3,4}[u, v]]
```

```
𝒜[{4, 1}, {2, 3}, ⟨|ξ1 → u, x2 → v, x3 → u, ξ4 → v|⟩,
 Wedge[]/√v - x2∧ξ4/√v - √v x3∧ξ1 - x3∧ξ4/√v + √v x3∧ξ4 + √v x2∧x3∧ξ1∧ξ4]
```

```
𝒜[X_{i_,j_,k_,l_}] := 𝒜[X_{i,j,k,l}[τ_i, τ_l]]
```

The linear structure on $\mathcal{A}$'s:

```
𝒜 /: α_ × 𝒜[is_, os_, cs_, w_] := 𝒜[is, os, cs, Expand[α w]]
𝒜 /: 𝒜[is1_, os1_, cs1_, w1_] + 𝒜[is2_, os2_, cs2_, w2_] /;
  (Sort@is1 == Sort@is2) ∧ (Sort@os1 == Sort@os2) ∧
   (Sort@Normal@cs1 == Sort@Normal@cs2) := 𝒜[is1, os1, cs1, w1 + w2]
```

Deciding if two $\mathcal{A}$'s are equal:

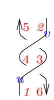
```
𝒜 /: 𝒜[is1_, os1_, _, w1_] ≡ 𝒜[is2_, os2_, _, w2_] :=
 TrueQ[(Sort@is1 === Sort@is2) ∧ (Sort@os1 === Sort@os2) ∧
   PowerExpand[w1 == w2]]
```

Contractions of $\mathcal{A}$-objects:

```
c_{h_,t_}@𝒜[is_, os_, cs_, w_] := 𝒜[
   DeleteCases[is, t], DeleteCases[os, h], KeyDrop[cs, {x_h, ξ_t}], c_{x_h,ξ_t}[w]
  ] /. If[MatchQ[cs[ξ_t], τ_], cs[ξ_t] → cs[x_h], cs[x_h] → cs[ξ_t]];
c_{4,4}[𝒜[X_{2,4,3,1}[S, T]] × 𝒜[X̄_{3,4,6,5}]]
```

```
𝒜[{1, 2, 3}, {3, 5, 6}, ⟨|ξ2 → S, x3 → S, ξ1 → T, ξ3 → τ3, x6 → τ3, x5 → T|⟩,
 Wedge[] - x3∧ξ1 + T x3∧ξ1 - T x3∧ξ2 - x5∧ξ1 - x6∧ξ1 + x6∧ξ1/T - x6∧ξ3/T +
  T x3∧x5∧ξ1∧ξ2 - x3∧x6∧ξ1∧ξ2 + T x3∧x6∧ξ1∧ξ2 + x3∧x6∧ξ1∧ξ3 -
  x3∧x6∧ξ1∧ξ3/T - x3∧x6∧ξ2∧ξ3 - x5∧x6∧ξ1∧ξ3/T - x3∧x5∧x6∧ξ1∧ξ2∧ξ3]
```

4. Skein relations and evaluations for $\mathcal{A}$

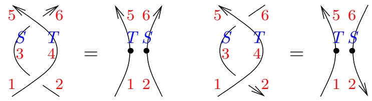



```
𝒜@{X̄_{4,1,6,3}[v, u], X̄_{3,2,5,4}}
```

```
𝒜[{1, 2}, {5, 6}, ⟨|ξ2 → v, x5 → u, ξ1 → u, x6 → v|⟩,
 √u √v Wedge[] - √u x5∧ξ1/√v + √u x5∧ξ2/√v - √u √v x5∧ξ2 + √v x6∧ξ1/√u - √u √v x6∧ξ1 -
  √v x6∧ξ2/√u - √u x5∧x6∧ξ1∧ξ2/√v - √v x5∧x6∧ξ1∧ξ2/√u + √u √v x5∧x6∧ξ1∧ξ2]
```

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/

Reidemeister 2



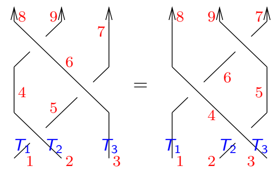
$\mathcal{A}$@$\{X_{2,4,3,1}[S, T], \overline{X}_{3,4,6,5}\} \equiv \mathcal{A}$@$\{P_{1,5}[T], P_{2,6}[S]\}$

True

$\mathcal{A}$@$\{\overline{X}_{3,1,2,4}[S, T], X_{6,5,3,4}\} \equiv \mathcal{A}$@$\{P_{1,5}[T], P_{6,2}[S]\}$

True

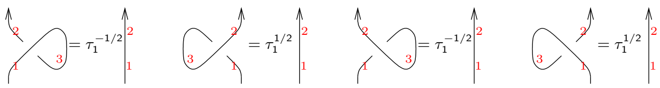
Reidemeister 3



$\mathcal{A}$@$\{X_{2,5,4,1}[T_2, T_1], X_{3,7,6,5}[T_3, T_1], X_{6,9,8,4}\} \equiv$
$\mathcal{A}$@$\{X_{3,5,4,2}[T_3, T_2], X_{4,6,8,1}[T_3, T_1], X_{5,7,9,6}\}$

True

Reidemeister 1

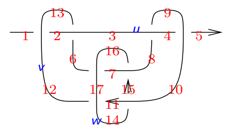


$\{\mathcal{A}$@$\{X_{3,3,2,1}\} \equiv \tau_1^{-1/2}\,\mathcal{A}$@$\{P_{1,2}\},\ \mathcal{A}$@$\{X_{1,2,3,3}\} \equiv \tau_1^{1/2}\,\mathcal{A}$@$\{P_{1,2}\},$
$\mathcal{A}$@$\{\overline{X}_{1,3,3,2}\} \equiv \tau_1^{-1/2}\,\mathcal{A}$@$\{P_{1,2}\},\ \mathcal{A}$@$\{\overline{X}_{3,1,2,3}\} \equiv \tau_1^{1/2}\,\mathcal{A}$@$\{P_{1,2}\}\}$

{True, True, True, True}

(So we have an invariant, up to rotation numbers).

The Relation with the Multivariable Alexander Polynomial



MVA = $u^{-1/2}\,v^{-1/2}\,w^{-1/2}\,(u-1)\,(v-1)\,(w-1)$;

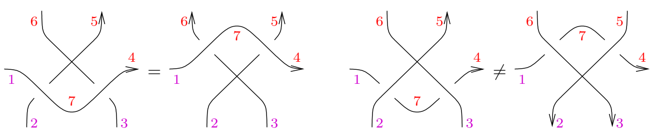
A = $\{\overline{X}_{1,12,2,13}[u, v], \overline{X}_{13,2,6,3}, X_{8,4,9,3}, X_{4,10,5,9}, X_{6,17,7,16}[v, w],$
$X_{15,8,16,7}, \overline{X}_{14,10,15,11}, \overline{X}_{11,17,12,14}\}$ // $\mathcal{A}$ // Last // Factor

$$\frac{(-1+u)^2\,(-1+v)\,(-1+w)\,(\mathrm{Wedge}[\,] - x_5 \wedge \xi_1)}{u\,v}$$

A == $u^{-1/2}\,(u-1)\,u^{0}\,v^{-1/2}\,w^{1/2}$ MVA (Wedge[$\wedge$] $- x_5 \wedge \xi_1$)

True

Overcrossings Commute but Undercrossings don't



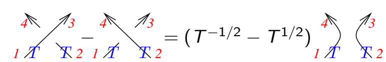
$\mathcal{A}$@$\{X_{2,7,5,1}, X_{3,4,6,7}\} \equiv \mathcal{A}$@$\{X_{3,7,6,1}, X_{2,4,5,7}\}$

True

$\mathcal{A}$@$\{\overline{X}_{1,2,7,5}, \overline{X}_{7,3,4,6}\} \equiv \mathcal{A}$@$\{\overline{X}_{1,3,7,6}, \overline{X}_{7,2,4,5}\}$

False

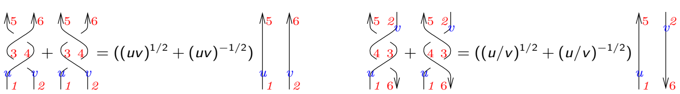
The Conway Relation (see [Co])



$\mathcal{A}$@$\{X_{2,3,4,1}[T, T]\} - \mathcal{A}$@$\{\overline{X}_{1,2,3,4}[T, T]\} \equiv (T^{-1/2} - T^{1/2})\,\mathcal{A}$@$\{P_{1,4}[T], P_{2,3}[T]\}$

True

Conway's Second Set of Identities (see [Co])



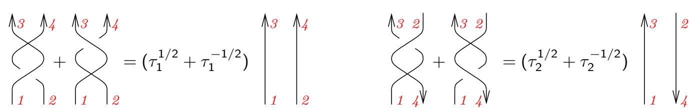
$\mathcal{A}$@$\{X_{2,4,3,1}[v, u], X_{4,6,5,3}\} + \mathcal{A}$@$\{\overline{X}_{1,2,4,3}[u, v], \overline{X}_{3,4,6,5}\} \equiv$
$(u^{1/2}\,v^{1/2} + u^{-1/2}\,v^{-1/2})\,\mathcal{A}$@$\{P_{1,5}[u], P_{2,6}[v]\}$

True

$\mathcal{A}$@$\{\overline{X}_{4,1,6,3}[v, u], \overline{X}_{3,2,5,4}\} + \mathcal{A}$@$\{X_{1,6,3,4}[u, v], X_{2,5,4,3}\} \equiv$
$(u^{1/2}\,v^{-1/2} + u^{-1/2}\,v^{1/2})\,\mathcal{A}$@$\{P_{1,5}[u], P_{2,6}[v]\}$

True

Virtual versions (Archibald, [Ar])



$\mathcal{A}$@$\{X_{2,3,4,1}\} + \mathcal{A}$@$\{\overline{X}_{2,1,4,3}\} \equiv (\tau_1^{1/2} + \tau_1^{-1/2})\,\mathcal{A}$@$\{P_{1,3}, P_{2,4}\}$

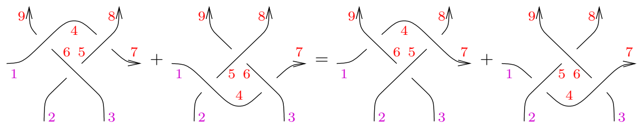
True

$\mathcal{A}$@$\{\overline{X}_{1,2,3,4}\} + \mathcal{A}$@$\{X_{1,4,3,2}\} \equiv (\tau_2^{1/2} + \tau_2^{-1/2})\,\mathcal{A}$@$\{P_{1,3}, P_{2,4}\}$

True

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/

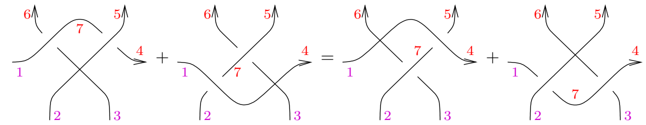
Conway's Third Identity (see [Co])



$\mathcal{A}@\{X_{6,4,9,1}, \overline{X}_{4,5,7,8}, \overline{X}_{2,3,5,6}\} + \mathcal{A}@\{X_{2,4,5,1}, \overline{X}_{4,3,7,6}, X_{6,8,9,5}\} \equiv$
$\mathcal{A}@\{\overline{X}_{1,6,4,9}, X_{5,7,8,4}, X_{3,5,6,2}\} + \mathcal{A}@\{\overline{X}_{1,2,4,5}, X_{3,7,6,4}, \overline{X}_{5,6,8,9}\}$

True

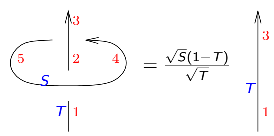
Virtual version (Archibald, [Ar])



$\mathcal{A}@\{X_{3,7,6,1}, \overline{X}_{7,2,4,5}\} + \mathcal{A}@\{X_{2,4,7,1}, X_{3,5,6,7}\} \equiv$
$\mathcal{A}@\{X_{3,7,6,2}, X_{7,4,5,1}\} + \mathcal{A}@\{\overline{X}_{1,2,7,5}, X_{3,4,6,7}\}$

True

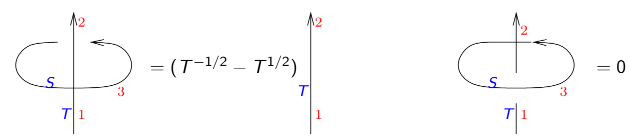
Jun Murakami's Fifth Axiom (see [Mu])



$\mathcal{A}@\{X_{1,4,2,5}[T, S], X_{4,3,5,2}\} \equiv \frac{\sqrt{S}\,(1-T)}{\sqrt{T}}\,\mathcal{A}@\{P_{1,3}[T]\}$

True

Virtual versions (Archibald, [Ar])



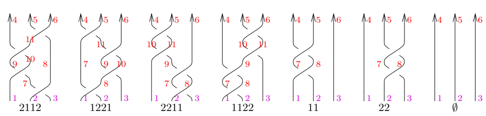
$\mathcal{A}@\{X_{3,2,3,1}[S, T]\} \equiv \left(T^{-1/2} - T^{1/2}\right)\mathcal{A}@\{P_{1,2}[T]\}$

True

$\mathcal{A}@\{X_{1,3,2,3}\}$

$\mathcal{A}[\{1\}, \{2\}, \langle| \xi_1 \to \tau_1, x_2 \to \tau_1 |\rangle, 0]$

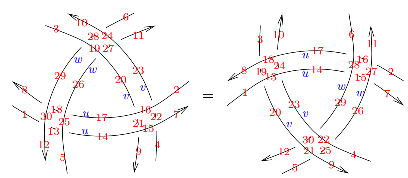
Jun Murakami's Third Axiom (see [Mu])



$\mathcal{A}_{2112} = \mathcal{A}@\{X_{3,8,7,2}, X_{7,10,9,1}, X_{10,11,4,9}, X_{8,6,5,11}\};$
$\mathcal{A}_{1221} = \mathcal{A}@\{X_{2,8,7,1}, X_{3,10,9,8}, X_{10,6,11,9}, X_{11,5,4,7}\};$
$\mathcal{A}_{2211} = \mathcal{A}@\{X_{3,8,7,2}, X_{8,6,9,7}, X_{9,11,10,1}, X_{11,5,4,10}\};$
$\mathcal{A}_{1122} = \mathcal{A}@\{X_{2,8,7,1}, X_{8,9,4,7}, X_{3,11,10,9}, X_{11,6,5,10}\};$
$\mathcal{A}_{11} = \mathcal{A}@\{X_{2,8,7,1}, X_{8,5,4,7}, P_{3,6}\};\quad \mathcal{A}_{22} = \mathcal{A}@\{X_{3,8,7,2}, X_{8,6,5,7}, P_{1,4}\};$
$\mathcal{A}_{\emptyset} = \mathcal{A}@\{P_{1,4}, P_{2,5}, P_{3,6}\};$
$g_+[z\_] := z^{1/2} + z^{-1/2};\quad g_-[z\_] := z^{1/2} - z^{-1/2};$
$g_+[\tau_1]\,g_-[\tau_2]\,\mathcal{A}_{2112} - g_-[\tau_2]\,g_+[\tau_3]\,\mathcal{A}_{1221} - g_-[\tau_3/\tau_1]\,(\mathcal{A}_{2211} + \mathcal{A}_{1122}) +$
$g_-[\tau_2\,\tau_3/\tau_1]\,g_+[\tau_3]\,\mathcal{A}_{11} - g_+[\tau_1]\,g_-[\tau_1\,\tau_2/\tau_3]\,\mathcal{A}_{22} \equiv g_-[\tau_3^2/\tau_1^2]\,\mathcal{A}_{\emptyset}$

True

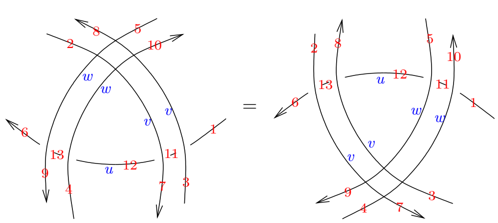
The Naik-Stanford Double Delta Move (see [NS])



Timing$[\mathcal{A}@\{X_{6,10,28,24}[w, v], \overline{X}_{28,3,29,19}[w, v], X_{26,20,27,19}[w, v], \overline{X}_{27,23,11,24}[w, v],$
$X_{1,12,13,30}[u, w], \overline{X}_{13,5,14,25}[u, w], X_{17,26,18,25}[u, w], \overline{X}_{18,29,8,30}[u, w],$
$X_{4,7,22,15}[v, u], \overline{X}_{22,2,23,16}[v, u], X_{20,17,21,16}[v, u], \overline{X}_{21,14,9,15}[v, u]\} \equiv$
$\mathcal{A}@\{X_{5,9,25,21}[w, v], \overline{X}_{25,4,26,22}[w, v], X_{29,23,30,22}[w, v], \overline{X}_{30,20,12,21}[w, v],$
$X_{2,11,16,27}[u, w], \overline{X}_{16,6,17,28}[u, w], X_{14,29,15,28}[u, w], \overline{X}_{15,26,7,27}[u, w],$
$X_{3,8,19,18}[v, u], \overline{X}_{19,1,20,13}[v, u], X_{23,14,24,13}[v, u], \overline{X}_{24,17,10,18}[v, u]\}]$

{190.422, True}

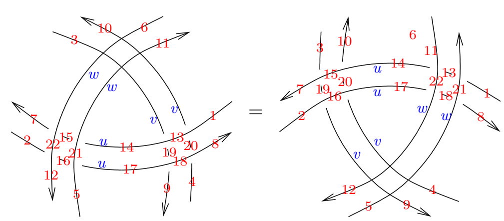
Virtual Version 1 (Archibald, [Ar])



$\mathcal{A}@\{X_{1,8,11,3}[u, v], \overline{X}_{11,2,12,7}[u, v], X_{12,10,13,4}[u, w], \overline{X}_{13,5,6,9}[u, w]\} \equiv$
$\mathcal{A}@\{X_{1,10,11,4}[u, w], \overline{X}_{11,5,12,9}[u, w], X_{12,8,13,3}[u, v], \overline{X}_{13,2,6,7}[u, v]\}$

True

Virtual Version 2 (Archibald, [Ar])



$\mathcal{A}@\{\overline{X}_{20,1,10,13}[v, u], X_{3,14,19,13}[v, u], X_{14,11,15,21}[u, w], \overline{X}_{15,6,7,22}[u, w],$
$X_{2,12,16,22}[u, w], \overline{X}_{16,5,17,21}[u, w], \overline{X}_{19,17,9,18}[v, u], X_{4,8,20,18}[v, u]\} \equiv$
$\mathcal{A}@\{X_{1,11,13,21}[u, w], \overline{X}_{13,6,14,22}[u, w], \overline{X}_{20,14,10,15}[v, u], X_{3,7,19,15}[v, u],$
$\overline{X}_{19,2,9,16}[v, u], X_{4,17,20,16}[v, u], X_{17,12,18,22}[u, w], \overline{X}_{18,5,8,21}[u, w]\}$

True

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/

5. Some Problems in Heaven

Unfortunately, $\dim\mathcal{A}(\mathcal{X},X)=\dim\Lambda(\mathcal{X},X)=4^{|X|}$ is big. Fortunately, we have the following theorem, a version of one of the main results in Halacheva's thesis, [Ha1, Ha2]:

Theorem. Working in $\Lambda(\mathcal{X}\cup X)$, if $w=\omega\mathbb{e}^{\lambda}$ is a balanced Gaussian (namely, a scalar ω times the exponential of a quadratic $\lambda=\sum_{\zeta\in\mathcal{X},z\in X}\alpha_{\zeta,z}\zeta z$), then generically so is $c_{x,\xi}\mathbb{e}^{\lambda}$.

(This is great news! The space of balanced quadratics is only $|\mathcal{X}||X|$-dimensional!)

Proof. Recall that $c_{x,\xi}\colon(1,\xi,x,x\xi)w'\mapsto(1,0,0,1)w'$, write $\lambda=\mu+\eta x+\xi y+\alpha\xi x$, and ponder $\mathbb{e}^{\lambda}=$

$$\ldots+\frac{1}{k!}\underbrace{(\mu+\eta x+\xi y+\alpha\xi x)(\mu+\eta x+\xi y+\alpha\xi x)\cdots(\mu+\eta x+\xi y+\alpha\xi x)}_{k\text{ factors}}+\ldots.$$

Then $c_{x,\xi}\mathbb{e}^{\lambda}$ has three contributions:

- $\mathbb{e}^{\mu}$, from the term proportional to 1 (namely, independent of ξ and x) in $\mathbb{e}^{\lambda}$
- $-\alpha\mathbb{e}^{\mu}$, from the term proportional to $x\xi$, where the x and the ξ come from the same factor above.
- $\eta y\mathbb{e}^{\mu}$, from the term proportional to $x\xi$, where the x and the ξ come from different factors above.

So $c_{x,\xi}\mathbb{e}^{\lambda}=\mathbb{e}^{\mu}(1-\alpha+\eta y)=(1-\alpha)\mathbb{e}^{\mu}(1+\eta y/(1-\alpha))=(1-\alpha)\mathbb{e}^{\mu}\mathbb{e}^{\eta y/(1-\alpha)}=(1-\alpha)\mathbb{e}^{\mu+\eta y/(1-\alpha)}$.

$\square$

Γ-calculus.

Thus we have an almost-always-defined "Γ-calculus": a contraction algebra morphism $\mathcal{T}(\mathcal{X},X)\to R\times(\mathcal{X}\otimes_{R/R}X)$ whose behaviour under contractions is given by

$$c_{x,\xi}(\omega,\lambda=\mu+\eta x+\xi y+\alpha\xi x)=((1-\alpha)\omega,\mu+\eta y/(1-\alpha)).$$

(Γ is fully defined on pure tangles – tangles without closed components – and hence on long knots).

6. An Implementation of Γ.

If I didn't implement I wouldn't believe myself.

Written in Mathematica [Wo], available as the notebook `Gamma.nb` at http://drorbn.net/mo21/ap. Code lines are highlighted in grey, demo lines are plain. We start with canonical forms for quadratics with rational function coefficients:

```
CCF[ℰ_] := Factor[ℰ];
CF[ℰ_] := Module[{vs = Union@Cases[ℰ, (ξ | x)_, ∞]},
   Total[(CCF[#[[2]]] (Times @@ vs^#[[1]])) & /@ CoefficientRules[ℰ, vs]]];
```

Multiplying and comparing Γ objects:

```
Γ /: Γ[is1_, os1_, cs1_, ω1_, λ1_] × Γ[is2_, os2_, cs2_, ω2_, λ2_] :=
  Γ[is1 ⋃ is2, os1 ⋃ os2, Join[cs1, cs2], ω1 ω2, λ1 + λ2]
Γ /: Γ[is1_, os1_, _, ω1_, λ1_] ≡ Γ[is2_, os2_, _, ω2_, λ2_] :=
  TrueQ[(Sort@is1 === Sort@is2) ∧ (Sort@os1 === Sort@os2) ∧
    Simplify[ω1 == ω2] ∧ CF@λ1 == CF@λ2]
```

No rules for linear operations!

Contractions:

```
c_{h_,t_}@Γ[is_, os_, cs_, ω_, λ_] := Module[{α, η, y, μ},
   α = ∂_{ξ_t,x_h} λ;  μ = λ /. ξ_t | x_h → 0;
   η = ∂_{x_h} λ /. ξ_t → 0;  y = ∂_{ξ_t} λ /. x_h → 0;
   Γ[
     DeleteCases[is, t], DeleteCases[os, h], KeyDrop[cs, {x_h, ξ_t}],
     CCF[(1 - α) ω], CF[μ + η y / (1 - α)]
   ] /. If[MatchQ[cs[ξ_t], τ_], cs[ξ_t] → cs[x_h], cs[x_h] → cs[ξ_t]]];
c@Γ[is_, os_, cs_, ω_, λ_] := Fold[c_{#2,#2}[#1] &, Γ[is, os, cs, ω, λ], is ⋂ os]
```

The crossings and the point:

```
Γ[X_{i_,j_,k_,l_}[S_, T_]] := Γ[{l, i}, {j, k}, <|ξ_i → S, x_j → T, x_k → S, ξ_l → T|>,
    T^(-1/2), CF[{ξ_l, ξ_i}.( 1  1 - T
                               0    T  ).{x_j, x_k}]];
Γ[X̄_{i_,j_,k_,l_}[S_, T_]] := Γ[{i, j}, {k, l}, <|ξ_i → S, ξ_j → T, x_k → S, x_l → T|>,
    T^(1/2), CF[{ξ_i, ξ_j}.(   T^-1    0
                             1 - T^-1  1 ).{x_k, x_l}]];
Γ[X_{i_,j_,k_,l_}] := Γ[X_{i,j,k,l}[τ_i, τ_l]];
Γ[X̄_{i_,j_,k_,l_}] := Γ[X̄_{i,j,k,l}[τ_i, τ_j]];
Γ[P_{i_,j_}[T_]] := Γ[{i}, {j}, <|ξ_i → T, x_j → T|>, 1, ξ_i x_j];
Γ[P_{i_,j_}] := Γ[P_{i,j}[τ_i]];
```

Automatic intelligent contractions:

```
Γ[{γ_Γ}] := c[γ];
Γ[{γ1_Γ, γs__Γ}] := Module[{γ2},
  γ2 = First@MaximalBy[{γs}, Length[γ1[[1]] ⋂ #[[2]]] + Length[γ1[[2]] ⋂ #[[1]]] &];
  Γ[Join[{c[γ1 γ2]}, DeleteCases[{γs}, γ2]]] ]
Γ[𝒪s_List] := Γ[Γ /@ 𝒪s]
```

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/

Conversions $\mathcal{A} \leftrightarrow \Gamma$:

```
Γ@𝒜[is_, os_, cs_, w_] := Module[{i, j, ω = Coefficient[w, Wedge[ ]]},
   Γ[is, os, cs, ω, Sum[Cancel[-Coefficient[w, x_j ∧ ξ_i] ξ_i x_j / ω],
     {i, is}, {j, os}]]
  ];
𝒜@Γ[is_, os_, cs_, ω_, λ_] :=
  𝒜[is, os, cs, Expand[ω WExp[Expand[λ] /. ξ_a_ x_b_ :→ ξ_a ∧ x_b]]];
```

The conversions are inverses of each other:

$$\gamma = \Gamma[\{1, 2, 3\}, \{1, 2, 3\}, \{x_1 \to \tau_1, x_2 \to \tau_2, x_3 \to \tau_3, \xi_1 \to \tau_1, \xi_2 \to \tau_2, \xi_3 \to \tau_3\}, \omega, a_{11} x_1 \xi_1 + a_{12} x_2 \xi_1 + a_{13} x_3 \xi_1 + a_{21} x_1 \xi_2 + a_{22} x_2 \xi_2 + a_{23} x_3 \xi_2 + a_{31} x_1 \xi_3 + a_{32} x_2 \xi_3 + a_{33} x_3 \xi_3];$$

$\Gamma @ \mathcal{A} @ \gamma == \gamma$

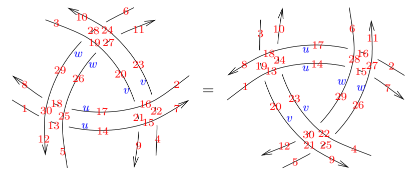
True

The conversions commute with contractions:

$\Gamma @ c_{3,3} @ \mathcal{A} @ \gamma \equiv c_{3,3} @ \gamma$

True

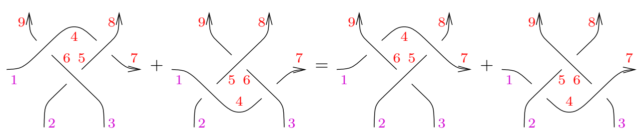
The Naik-Stanford Double Delta Move (again)



$$\text{Timing}\Big[\Gamma @ \big\{X_{6,10,28,24}[w, v], \overline{X}_{28,3,29,19}[w, v], X_{26,20,27,19}[w, v], \overline{X}_{27,23,11,24}[w, v], X_{1,12,13,30}[u, w], \overline{X}_{13,5,14,25}[u, w], X_{17,26,18,25}[u, w], \overline{X}_{18,29,8,30}[u, w], X_{4,7,22,15}[v, u], \overline{X}_{22,2,23,16}[v, u], X_{20,17,21,16}[v, u], \overline{X}_{21,14,9,15}[v, u]\big\} \equiv \Gamma @ \big\{X_{5,9,25,21}[w, v], \overline{X}_{25,4,26,22}[w, v], X_{29,23,30,22}[w, v], \overline{X}_{30,20,12,21}[w, v], X_{2,11,16,27}[u, w], \overline{X}_{16,6,17,28}[u, w], X_{14,29,15,28}[u, w], \overline{X}_{15,26,7,27}[u, w], X_{3,8,19,18}[v, u], \overline{X}_{19,1,20,13}[v, u], X_{23,14,24,13}[v, u], \overline{X}_{24,17,10,18}[v, u]\big\}\Big]$$

{0.703125, True}

Conway's Third Identity



Sorry, Γ has nothing to say about that...'

What I still don't understand.

- What becomes of $c_{x,\xi}e^{\lambda}$ if we have to divide by 0 in order to write it again as an exponentiated quadratic? Does it still live within a very small subset of $\Lambda(\mathcal{X} \sqcup X)$?
- How do cablings and strand reversals fit within $\mathcal{A}$?
- Are there "classicality conditions" satisfied by the invariants of classical tangles (as opposed to virtual ones)?

References

- J. Archibald, *The Multivariable Alexander Polynomial on Tangles*, University of Toronto Ph.D. thesis, 2010, `http://drorbn.net/mo21/AT`.
- J. H. Conway, *An Enumeration of Knots and Links, and some of their Algebraic Properties*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), Pergamon, Oxford, 1970, 329-–358.
- Z. Dancso, I. Halacheva, and M. Robertson, *Circuit Algebras are Wheeled Props*, J. Pure and Appl. Alg., to appear, arXiv:2009.09738.
- I. Halacheva, *Alexander Type Invariants of Tangles, Skew Howe Duality for Crystals and The Cactus Group*, University of Toronto Ph.D. thesis, 2016, `http://drorbn.net/mo21/HT`.
- I. Halacheva, *Alexander Type Invariants of Tangles*, arXiv:1611.09280.
- M. Markl, S. Merkulov, and S. Shadrin, *Wheeled PROPs, Graph Complexes and the Master Equation*, J. Pure and Appl. Alg. **213-4** (2009) 496–535, arXiv:math/0610683.
- J. Murakami, *A State Model for the Multivariable Alexander Polynomial*, Pacific J. Math. **157-1** (1993) 109-–135.
- S. Naik and T. Stanford, *A Move on Diagrams that Generates S-Equivalence of Knots*, J. Knot Theory Ramifications **12-5** (2003) 717-–724, arXiv: math/9911005.
- *Wolfram Language & System Documentation Center*, `https://reference.wolfram.com/language/`.

Thank You!

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/`

The Alexander Polynomial is a Quantum Invariant in a Different Way

▶ On a chat window here I saw a comment "Alexander is the quantum $gl(1|1)$ invariant". I have an opinion about this, and I'd like to share it. First, some stories.

I left the wonderful subject of Categorification nearly 15 years ago. It got crowded, lots of very smart people had things to say, and I feared I will have nothing to add. Also, clearly the next step was to categorify all other "quantum invariants". Except it was not clear what "categorify" means. Worse, I felt that I (perhaps "we all") didn't understand "quantum invariants" well enough to try to categorify them, whatever that might mean.

I still feel that way! I learned a lot since 2006, yet I'm still not comfortable with quantum algebra, quantum groups, and quantum invariants. I still don't feel that I know what God had in mind when She created this topic.

Yet I'm not here to rant about my philosophical quandaries, but only about things that I learned about the Alexander polynomial after 2006.

Yes, the Alexander polynomial fits within the Dogma, "one invariant for every Lie algebra and representation" (it's $gl(1|1)$, I hear). But it's better to think of it as a quantum invariant arising by other means, outside the Dogma.

Alexander comes from (or in) practically any non-Abelian Lie algebra. Foremost from the not-even-semi-simple 2D "$ax+b$" algebra. You get a polynomially-sized extension to tangles using some lovely formulas (can you categorify them?). It generalizes to higher dimensions and it has an organized family of siblings. (There are some questions too, beyond categorification).

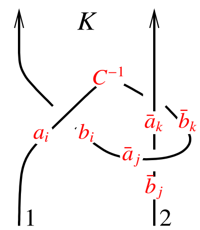
I note the spectacular existing categorification of Alexander by Ozsváth and Szabó. The theorems are proven and a lot they say, the programs run and fast they run. Yet if that's where the story ends, She has abandoned us. Or at least abandoned me: a simpleton will never be able to catch up.

If you care only about categorification, the take-home from my talk will be a challenge: Categorify what I believe is the best Alexander invariant for tangles.

The Yang-Baxter Technique. Given an algebra U (typically some $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and suitable elements R, C,

$$R=\sum a_i\otimes b_i\in U\otimes U \quad\text{with}\quad R^{-1}=\sum \bar a_i\otimes\bar b_i \quad\text{and}\quad C,C^{-1}\in U,$$

form
$$Z(K)=\sum_{i,j,k} a_iC^{-1}\bar b_k\bar a_j b_i\otimes \bar b_j\bar a_k.$$



Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

Example 1. Let $\mathfrak{a}:=L\langle a,x\rangle/([a,x]=x)$, $\mathfrak{b}:=\mathfrak{a}^\star=\langle b,y\rangle$, and $\mathfrak{g}:=\mathfrak{b}\rtimes\mathfrak{a}=\mathfrak{b}\oplus\mathfrak{a}$ with $[a,x]=x$, $[a,y]=-y$, $[b,\cdot]=0$, and $[x,y]=b$ and with $\deg(y,b,a,x)=(1,1,0,0)$. Let $U=\hat{\mathcal{U}}(\mathfrak{g})$ and

$$R:=\mathbb{e}^{b\otimes a+y\otimes x}\in U\otimes U \quad\text{or better}\quad R_{ij}:=\mathbb{e}^{b_ia_j+y_ix_j}\in U_i\otimes U_j,\quad\text{and}\quad C_i=\mathbb{e}^{-b_i/2}.$$

Gentle's Agreement. Everything converges!

Theorem 1. With "scalars":=power series in $\{b_i\}$ which are rational functions in $\{b_i\}$ and $\{B_i:=\mathbb{e}^{b_i}\}$,

$$Z(K)=\mathbb{O}_{ybax}\left(\omega^{-1}\mathbb{e}^{\underbrace{l^{ij}b_ia_j}_{L}+\underbrace{q^{ij}y_ix_j}_{Q}}(1+\epsilon P_1+\epsilon^2P_2+\dots)\right)$$

a tangle w/o closed components

"normal ordering" at $ybax$ order

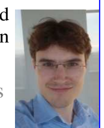
the "i over j" linking numbers (integers)

categorify us! scalars

a scalar; if K is a long knot, the Alexander poly $\Delta(T)$ categorify me!

a docile perturbation for other Lie algebras; semisimple algebras have a hidden parameter ϵ!

With Roland van der Veen



Continues Lev Rozansky

Example 2. Let $\mathfrak{h}:=A\langle p,x\rangle/([p,x]=1)$ be the Heisenberg algebra, with $C_i=\mathbb{e}^{t/2}$ and $R_{ij}=\mathbb{e}^{t/2}\mathbb{e}^{t(p_i-p_j)x_j}$.

I just told you the whole Alexander story! Everything else is details.

Claim. $R_{ij}=\mathbb{O}_{px}\left(\mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j}\right)$.

Theorem 2. $Z(K)=\mathbb{O}_{px}\left(\omega^{-1}\mathbb{e}^{q^{ij}p_ix_j}\right)$ where ω and the q^{ij} are rational functions in $T=\mathbb{e}^t$. In fact ω and ωq^{ij} are Laurent polynomials (categorify us!). When K is a long knot, ω is the Alexander polynomial.

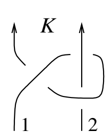
Packaging. Write $\mathbb{O}_{px}\left(\omega^{-1}\mathbb{e}^{q^{ij}p_ix_j}\right)$ as

$$\mathbb{E}_{p_1,\dots,x_1,\dots}[\omega,Q]\leftrightarrow\begin{array}{c|ccc}\omega & x_1 & x_2 & \cdots\\\hline p_1 & q^{11} & q^{12} & \cdots\\ p_2 & q^{21} & q^{22} & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{array}$$

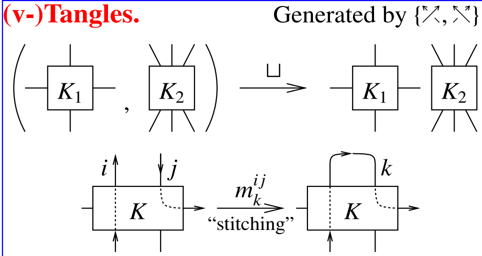
The "First Tangle". $Z(K)=$

$$\mathbb{E}_{12}\left[\frac{2T-1}{T},\frac{(T-1)(p_1-p_2)(Tx_1-x_2)}{2T-1}\right]$$

$$=\begin{array}{c|cc}2-T^{-1} & x_1 & x_2\\\hline p_1 & \frac{T(T-1)}{2T-1} & \frac{1-T}{2T-1}\\ p_2 & \frac{T(1-T)}{2T-1} & \frac{T-1}{2T-1}\end{array}$$



(v-)Tangles. Generated by $\{\nearrow\!\!\!\!\nwarrow,\nwarrow\!\!\!\!\nearrow\}$!



There's also strand doubling and reversal…

Theorem 3. Full evaluation via

$$(\nearrow\!\!\!\!\nwarrow_{ij},\nwarrow\!\!\!\!\nearrow_{ji})\to\begin{array}{c|cc}1 & x_i & x_j\\\hline p_i & 0 & T^{\pm1}-1\\ p_j & 0 & 1-T^{\pm1}\end{array}\qquad(1)\square$$

$$K_1\sqcup K_2\to\begin{array}{c|cc}\omega_1\omega_2 & X_1 & X_2\\\hline P_1 & A_1 & 0\\ P_2 & 0 & A_2\end{array}\qquad(2)\square$$

$$\begin{array}{c|ccc}\omega & x_i & x_j & \cdots\\\hline p_i & \alpha & \beta & \theta\\ p_j & \gamma & \delta & \epsilon\\ \vdots & \phi & \psi & \Xi\end{array}\xrightarrow{hm_k^{ij}}\begin{array}{c|cc}(1+\gamma)\omega & x_k & \cdots\\\hline p_k & 1+\beta-\frac{(1-\alpha)(1-\delta)}{1+\gamma} & \theta+\frac{(1-\alpha)\epsilon}{1+\gamma}\\ \vdots & \psi+\frac{(1-\delta)\phi}{1+\gamma} & \Xi-\frac{\phi\epsilon}{1+\gamma}\end{array}\qquad(3)$$

"Γ-calculus" relates via $A\leftrightarrow I-A^T$ and has slightly simpler formulas: $\omega\to(1-\beta)\omega$,

$$\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}\to\begin{pmatrix}\gamma+\frac{\alpha\delta}{1-\beta}&\epsilon+\frac{\delta\theta}{1-\beta}\\\phi+\frac{\alpha\psi}{1-\beta}&\Xi+\frac{\psi\theta}{1-\beta}\end{pmatrix}$$

Why Should You Categorify This? The simplest and fastest Alexander for tangles, easily generalizes to the multi-variable case, generalizes to v-tangles and w-tangles, generalizes to other Lie algebras. In fact, it's in almost any Lie algebra, and you don't even need to know what is $gl(1|1)$! But you'll have to deal with denominators and/or divisions!

Note. Example 1 ⇝ Example 2 via $\mathfrak{g}\hookrightarrow\mathfrak{h}(t)$ via $(y,b,a,x)\mapsto(-tp,t,px,x)$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LearningSeminarOnCategorification-2006/

The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Convention. For a finite set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$. $(p,x)^* = (\pi,\xi)$

The Generating Series $\mathcal{G}$: $\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \to \mathbb{Q}[\![\zeta_A, z_B]\!]$.

Claim. $L\in\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \xrightarrow[\mathcal{G}]{\sim} \mathbb{Q}[z_B][\![\zeta_A]\!] \ni \mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}L(z_A^n) = L\left(\mathbb{e}^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{L} = {}_{\text{greek}}\mathcal{L}_{\text{latin}},$$

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{L}\right)_{\zeta_a=0} \quad \text{for } p\in\mathbb{Q}[z_A].$$

Claim. If $L\in\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $M\in\mathrm{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0}$.

Examples. • $\mathcal{G}(id\colon \mathbb{Q}[p,x]\to\mathbb{Q}[p,x]) = \mathbb{e}^{\pi p+\xi x}$.

• Consider $R_{ij}\in(\mathfrak{h}_i\otimes\mathfrak{h}_j)[\![t]\!]\cong\mathrm{Hom}\left(\mathbb{Q}[]\to\mathbb{Q}[p_i,x_i,p_j,x_j]\right)[\![t]\!]$. Then $\mathcal{G}(R_{ij}) = \mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j} = \mathbb{e}^{(T-1)(p_i-p_j)x_j}$.

Heisenberg Algebras. Let $\mathfrak{h} = A\langle p,x\rangle/([p,x]=1)$, let $\mathbb{O}_i\colon\mathbb{Q}[p_i,x_i]\to\mathfrak{h}_i$ is the "p before x" PBW normal ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[p_i,x_i,p_j,x_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathfrak{h}_i\otimes\mathfrak{h}_j \xrightarrow{m_k^{ij}} \mathfrak{h}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[p_k,x_k].$$

Then $\mathcal{G}(hm_k^{ij}) = \mathbb{e}^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}$.

Proof. Recall the "Weyl CCR" $\mathbb{e}^{\xi x}\mathbb{e}^{\pi p} = \mathbb{e}^{-\xi\pi}\mathbb{e}^{\pi p}\mathbb{e}^{\xi x}$, and find

$$\begin{aligned}\mathcal{G}(hm_k^{ij}) &= \mathbb{e}^{\pi_ip_i+\xi_ix_i+\pi_jp_j+\xi_jx_j}/\!\!/\mathbb{O}_i\otimes\mathbb{O}_j/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1}\\ &= \mathbb{e}^{\pi_ip_i}\mathbb{e}^{\xi_ix_i}\mathbb{e}^{\pi_jp_j}\mathbb{e}^{\xi_jx_j}/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1} = \mathbb{e}^{\pi_ip_k}\mathbb{e}^{\xi_ix_k}\mathbb{e}^{\pi_jp_k}\mathbb{e}^{\xi_jx_k}/\!\!/\mathbb{O}_k^{-1}\\ &= \mathbb{e}^{-\xi_i\pi_j}\mathbb{e}^{(\pi_i+\pi_j)p_k}\mathbb{e}^{(\xi_i+\xi_j)x_k}/\!\!/\mathbb{O}_k^{-1} = \mathbb{e}^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}.\end{aligned}$$

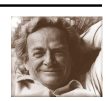
GDO := The category with objects finite sets and

$$\mathrm{mor}(A\to B) = \left\{\mathcal{L} = \omega\mathbb{e}^{Q}\right\}\subset\mathbb{Q}[\![\zeta_A,z_B]\!],$$

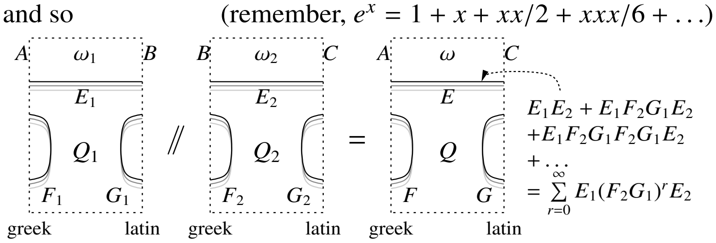
where: • ω is a scalar. • Q is a "small" quadratic in $\zeta_A\cup z_B$.
• Compositions: $\mathcal{L}/\!\!/\mathcal{M} := \left(\mathcal{L}|_{z_i\to\partial_{\zeta_i}}\mathcal{M}\right)_{\zeta_i=0}$.

Compositions. In $\mathrm{mor}(A\to B)$,

$$Q = \sum_{i\in A,j\in B}E_{ij}\zeta_iz_j + \frac{1}{2}\sum_{i,j\in A}F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B}G_{ij}z_iz_j,$$


R. Feynman

and so (remember, $e^x = 1+x+xx/2+xxx/6+\dots$)



where • $E = E_1(I-F_2G_1)^{-1}E_2$ • $F = F_1 + E_1F_2(I-G_1F_2)^{-1}E_1^T$
• $G = G_2 + E_2^TG_1(I-F_2G_1)^{-1}E_2$ • $\omega = \omega_1\omega_2\det(I-F_2G_1)^{-1/2}$

Proof of Claim in Example 2. Let $\Phi_1 := \mathbb{e}^{t(p_i-p_j)x_j}$ and $\Phi_2 := \mathbb{O}_{p_jx_j}\left(\mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j}\right) =: \mathbb{O}(\Psi)$. We show that $\Phi_1=\Phi_2$ in $(\mathfrak{h}_i\otimes\mathfrak{h}_j)[\![t]\!]$ by showing that both solve the ODE $\partial_t\Phi = (p_i-p_j)x_j\Phi$ with $\Phi|_{t=0}=1$. For Φ_1 this is trivial. $\Phi_2|_{t=0}=1$ is trivial, and

$$\partial_t\Phi_2 = \mathbb{O}(\partial_t\Psi) = \mathbb{O}(\mathbb{e}^t(p_i-p_j)x_j\Psi)$$

$$(p_i-p_j)x_j\Phi_2 = (p_i-p_j)x_j\mathbb{O}(\Psi) = (p_i-p_j)\mathbb{O}(x_j\Psi - \partial_{p_j}\Psi)$$
$$= \mathbb{O}\left((p_i-p_j)(x_j\Psi + (\mathbb{e}^t-1)x_j\Psi)\right) = \mathbb{O}(\mathbb{e}^t(p_i-p_j)x_j\Psi) \quad \square$$

Implementation. Without, don't trust!

```
CF = ExpandNumerator@*ExpandDenominator@*PowerExpand@*Factor;
𝔼A1_→B1_[ω1_, Q1_] 𝔼A2_→B2_[ω2_, Q2_] ^:= 𝔼A1⋃A2→B1⋃B2[ω1 ω2, Q1 + Q2]
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I, M = Table},
  I = IdentityMatrix@Length@B1;
  E1 = M[∂i,jQ1, {i, A1}, {j, B1}]; E2 = M[∂i,jQ2, {i, A2}, {j, B2}];
  F1 = M[∂i,jQ1, {i, A1}, {j, A1}]; F2 = M[∂i,jQ2, {i, A2}, {j, A2}];
  G1 = M[∂i,jQ1, {i, B1}, {j, B1}]; G2 = M[∂i,jQ2, {i, B2}, {j, B2}];
  𝔼A1→B2[CF[ω1 ω2 Det[I - F2.G1]^(1/2)], CF@Plus[
    If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
    If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1ᵀ).A1],
    If[B2 === {}, 0, 1/2 B2.(G2 + E2ᵀ.G1.Inverse[I - F2.G1].E2).B2]]]]
```

```
A_ \ B_ := Complement[A, B];
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* =!= A2) :=
 𝔼A1⋃(A2\B1*)→B1⋃A2*[ω1, Q1 + Sum[ξ* ξ, {ξ, A2 \ B1*}]] //
  𝔼B1*⋃A2→B2⋃(B1\A2*)[ω2, Q2 + Sum[z* z, {z, B1 \ A2*}]]
```

```
{p*, x*, π*, ξ*} = {π, ξ, p, x}; (u_i_)* := (u*)i;
l_List* := #* & /@ l;
```

```
Ri_,j_ := 𝔼{}→{pi,xi,pj,xj}[T^(-1/2), (1 - T) pj xj + (T - 1) pi xj];
R̄i_,j_ := 𝔼{}→{pi,xi,pj,xj}[T^(1/2), (1 - T^-1) pj xj + (T^-1 - 1) pi xj];
Ci_ := 𝔼{}→{pi,xi}[T^(-1/2), 0];
C̄i_ := 𝔼{}→{pi,xi}[T^(1/2), 0];
```

```
hmi_,j_→k_ := 𝔼{πi,ξi,πj,ξj}→{pk,xk}[1, -ξi πj + (πi + πj) pk + (ξi + ξj) xk]
```

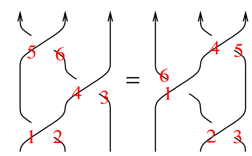
```
𝔼{}→vs_[ωi_, Q_]h := Module[{ps, xs, M},
  ps = Cases[vs, p_]; xs = Cases[vs, x_];
  M = Table[ωi, 1 + Length@ps, 1 + Length@xs];
  M⟦2 ;;, 2 ;;⟧ = Table[CF[∂i,jQ], {i, ps}, {j, xs}];
  M⟦2 ;;, 1⟧ = ps; M⟦1, 2 ;;⟧ = xs;
  MatrixForm[M]h]
```

Proof of Reidemeister 3.

```
(R1,2 R4,3 R5,6 // hm1,4→1 hm2,5→2 hm3,6→3) ==
 (R2,3 R1,6 R4,5 // hm1,4→1 hm2,5→2 hm3,6→3)

True
```

$\square$



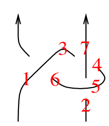
The "First Tangle".

```
Factor /@
 (z = R1,6 C̄3 R̄7,4 R̄5,2 // hm1,3→1 // hm1,4→1 // hm1,5→1 // hm1,6→1 // hm2,7→2)
```

$$\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}\left[\frac{-1+2T}{T},\ \frac{(-1+T)(p_1-p_2)(Tx_1-x_2)}{-1+2T}\right]$$

```
zh
```

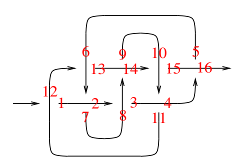
$$\begin{pmatrix}\frac{-1+2T}{T} & x_1 & x_2\\ p_1 & \frac{-T+T^2}{-1+2T} & \frac{1-T}{-1+2T}\\ p_2 & \frac{T-T^2}{-1+2T} & \frac{-1+T}{-1+2T}\end{pmatrix}_h$$



The knot 8_{17}.

```
z = R̄12,1 R̄27 R̄83 R̄4,11 R16,5 R6,13 R14,9 R10,15;
Table[z = z // hm1k→1, {k, 2, 16}] // Last
```

$$\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T^3},\ 0\right]$$



Proof of Theorem 3, (3).

$$\left\{\left(\gamma1 = \mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2,p_3,x_3\}}\left[\omega,\ \{p_1,p_2,p_3\}.\begin{pmatrix}\alpha&\beta&\theta\\ \gamma&\delta&\epsilon\\ \phi&\psi&\Xi\end{pmatrix}.\{x_1,x_2,x_3\}\right]\right)_h,\ (\gamma1 /\!/ hm_{1,2\to0})_h\right\}$$

$$\left\{\begin{pmatrix}\omega & x_1 & x_2 & x_3\\ p_1 & \alpha & \beta & \theta\\ p_2 & \gamma & \delta & \epsilon\\ p_3 & \phi & \psi & \Xi\end{pmatrix}_h,\ \begin{pmatrix}\omega+\gamma\omega & x_0 & x_3\\ p_0 & \frac{\alpha+\beta+\gamma+\beta\gamma+\delta-\alpha\delta}{1+\gamma} & \frac{\epsilon-\alpha\epsilon+\theta+\gamma\theta}{1+\gamma}\\ p_3 & \frac{\phi-\delta\phi+\psi+\gamma\psi}{1+\gamma} & \frac{\Xi+\gamma\Xi-\epsilon\phi}{1+\gamma}\end{pmatrix}_h\right\}$$

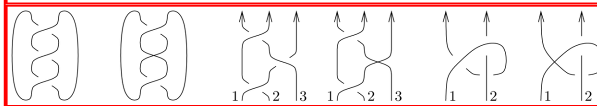
$\square$

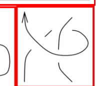
References. On ωεβ=http://drorbn.net/cat20

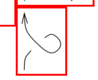
Video and more at http://www.math.toronto.edu/~drorbn/Talks/LearningSeminarOnCategorification-2006/

n-strand (pure) virtual braids with ($\leqslant m$)-xing:

$m\backslash n$	2	3	4	5	6	General n
0	1	1	1	1	1	1
1	5	13	25	41	61	$2n^2-2n+1$
2	17	145	529	1361	2881	$2n^4+4n^3-18n^2+12n+1$
3	53	1561	10873	43121	127021	$\frac{1}{3}\left(4n^6+36n^5-2n^4-546n^3+1066n^2-558n+3\right)$
4	161	16717	222289	1351481	5484721	
5	485	178873	4540201			
6	1457	1913737				
m	$2\cdot 3^m-1$					



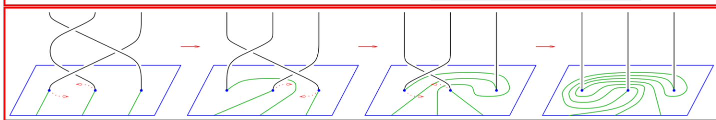


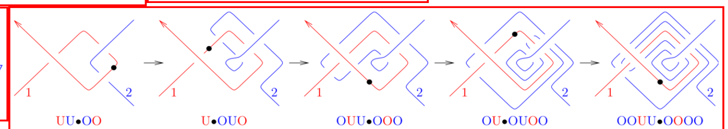


URL: `http://drorbn.net/tldt20`

Zoom Etiquette: If you can see me I should be able to see you!

Credit to Manturov and Chterental!





n-strand classical braids with ($\leqslant m$)-xing:

$m\backslash n$	2	3	4	5	6	General n
0	1	1	1	1	1	1
1	3	5	7	9	11	$2n-1$
2	5	17	33	53	77	$2n^2+2n-7$
3	7	47	131	259	439	$\frac{1}{3}\left(4n^3+18n^2-22n-63\right)$ $(n>2)$
4	9	115	469	1143	2233	
5	11	263	1579	4743	10603	
6	13	577	5121	18941	48209	
7	15	1233	16219	73817	213119	
8	17	2589	50581	283165	924825	
9	19	5371				
m	$2m+1$	$12\cdot 2^m-2F_{m+5}-2m-1$?				

Over then Under Tangles

Trends in Low-Dimensional Topology, online, May 5 2020, noon.

Abstract. Brilliant wrong ideas should not be buried and forgotten. Instead, they should be mined for the gold that lies underneath the layer of wrong. In this paper we explain how "over then under tangles" lead to an easy classification of knots, and under the surface, also to some valid mathematics: an easy classification of braids and virtual braids, an understanding of the Drinfel'd double procedure in quantum algebra, and more.

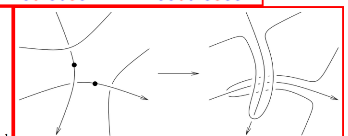
Based on a paper in preparation with Zsuzsanna Dancso and Roland van der Veen.

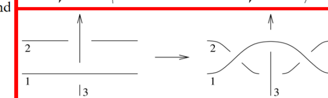
Handout: EGOU.html, EGOU.png.

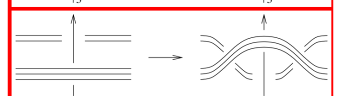
Talk Video.

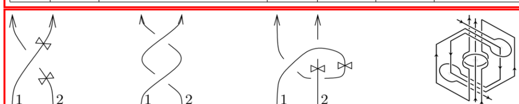
Pensieve.

URL: http://drorbn.net/tldt20.





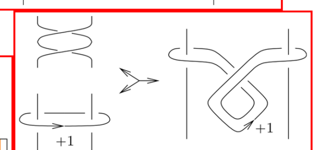


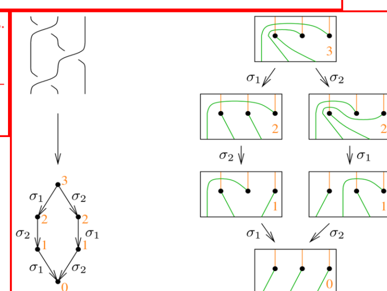


Not what I do, but the tangent I was on for the last few weeks. But first, the tangent to the tangent I was playing with over the last few *days*. Possibly an embarrassment!

Every braid (classical or virtual) has a directed finite graph associated with it. These are cool!

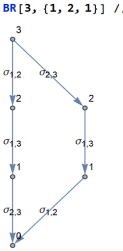
- Enriquez' universal quantization of Lie bi-algebra.
- All else about quantization of Lie bi-algebra.
- PBW / normal ordering.
- Audoux-Meilhan "Characterization of the Reduced Peripheral System of Links".
- B-N's "Balloons and Hoops" paper.



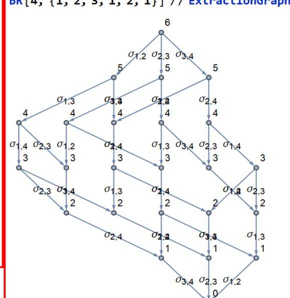


This is Demo.nb at
http://drorbn.net/ap/Talks/TrendsInLDT-2005/.

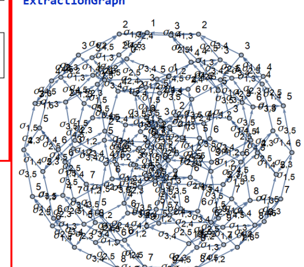
`BR[3, {1, 2, 1}] // ExtractionGraph`



`BR[4, {1, 2, 3, 1, 2, 1}] // ExtractionGraph`



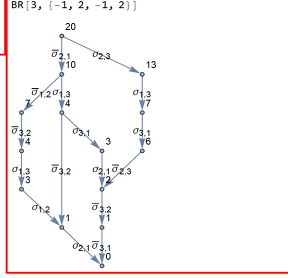
`BR[5, {1, 2, 3, 4, 1, 2, 3, 1, 2, 1}] // ExtractionGraph`



`Knot[4, 1] // BR // Echo // ExtractionGraph`

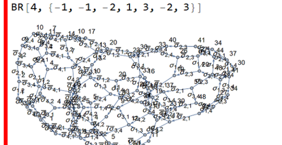
KnotTheory: The minimum braids representing the knots with up to 10 crossings were provided by Thomas Gittings. See arXiv:math.GT/0401051.

`BR[3, {-1, 2, -1, 2}]`



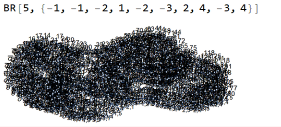
`Knot[6, 1] // BR // Echo // ExtractionGraph`

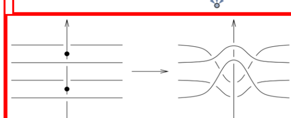
`BR[4, {-1, -1, -2, 1, 3, -2, 3}]`

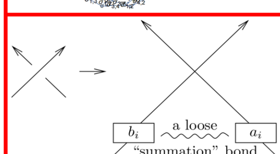


`Knot[8, 1] // BR // Echo // ExtractionGraph`

`BR[5, {-1, -1, -2, 1, -2, -3, 2, 4, -3, 4}]`

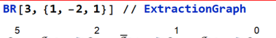




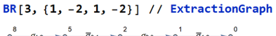


- Subsets.
- Supersets.
- Subsets of supersets.
- Completions.
- Quotients.
- Images.
- Completions of subsets of supersets...

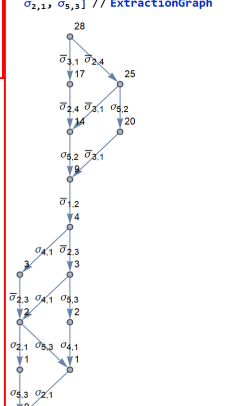
`BR[3, {1, -2, 1}] // ExtractionGraph`



`BR[3, {1, -2, 1, -2}] // ExtractionGraph`

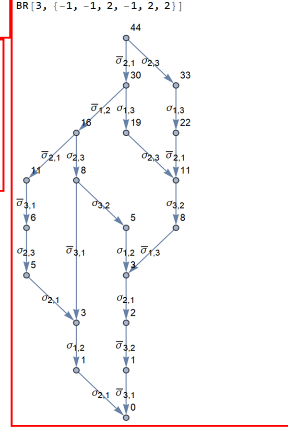


`VPB[5, σ̄3,1, σ̄2,4, σ5,2, σ̄1,2, σ4,1, σ̄2,3, σ2,1, σ5,3] // ExtractionGraph`

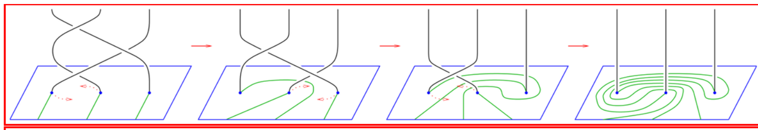


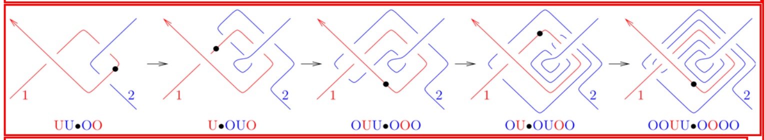
`Knot[6, 3] // BR // Echo // ExtractionGraph`

`BR[3, {-1, -1, 2, -1, 2, 2}]`



Video and more at `http://www.math.toronto.edu/~drorbn/Talks/TrendsInLDT-2005//`





n-strand (pure) virtual braids with ($\leqslant m$)-xing:

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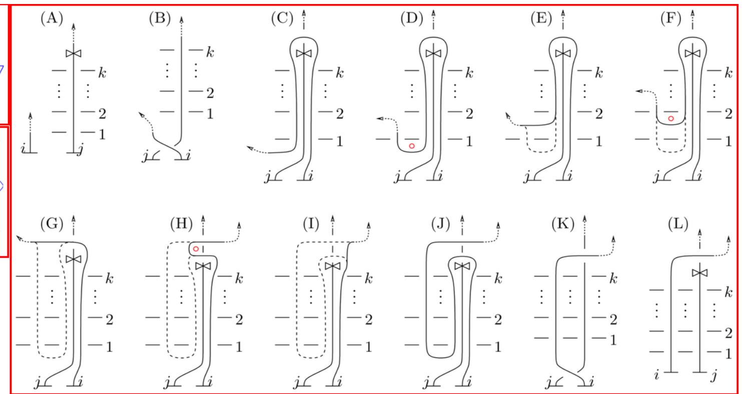
4	161	16717	222289	1351481	5484721	
5	485	178873	4540201			
6	1457	1913737				




```
Γ̄[vd_VD] := FixedPoint[R12Reduce@*γ, vd, 2^8]

Γ̄[T_] /; Head[T] =!= VD := Γ̄[VD[T]]

VPB[n_, {σs___}] := VPB[n, σs];

VD /: vd1_VD ** vd2_VD := Module[{es1, es2, m2},
  es1 = Cases[vd1, EOS[i_] :→ i];
  m2 = Max[es2 = Cases[vd2, EOS[i_] :→ i]];
  Tidy[
   vd1 ⋃ Replace[DeleteCases[vd2, _EOS],
     i_ :→ i / m2 - 1 + es1⟦1 + Count[es2, e_ /; i > e]⟧, {2}]]
 ]

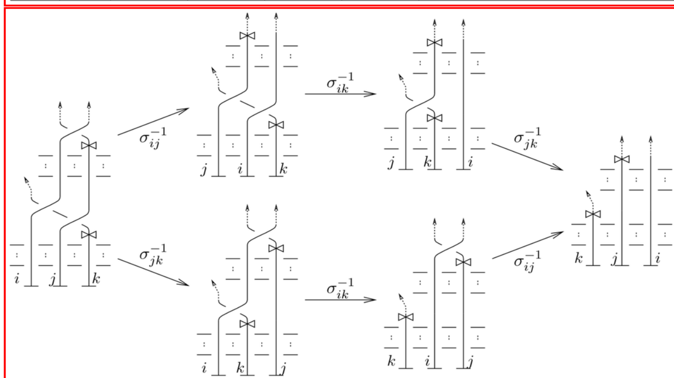
VD[VPB[n_]] := VD @@ (EOS /@ Range[n]);
VD[VPB[n_, σ_{i_,j_}]] := Tidy@Append[VD @@ (EOS /@ Range[n]), X_{+1}[i - 0.5, j - 0.5]];
VD[VPB[n_, σ̄_{i_,j_}]] := Tidy@Append[VD @@ (EOS /@ Range[n]), X_{-1}[i - 0.5, j - 0.5]];
VD[VPB[n_, σ_, σs__]] := VD[VPB[n, σ]] ** VD[VPB[n, σs]]

VPBGenerators[n_] :=
  VPBGenerators[n] =
   Flatten@Table[{σ_{i,j}, σ̄_{i,j}}, {i, n}, {j, DeleteCases[Range@n, i]}];

ProudFollowers[n_, σ_{i_,j_}] := ProudFollowers[n, σ_{i,j}] = Module[{p, q, s},
    Flatten@{σ_{i,j}, σ_{j,i}, σ̄_{j,i},
      Table[{σ_{p,q}, σ_{q,p}, σ̄_{p,q}, σ̄_{q,p}}, {p, {i, j}}, {q, Complement[Range[n], {i, j}]}],
      Table[{σ_{p,q}, σ̄_{p,q}}, {p, Complement[Range[i + 1, n], {j}]},
       {q, Complement[Range[n], {i, j, p}]}]
     }];
ProudFollowers[n_, σ̄_{i_,j_}] :=
 ProudFollowers[n, σ̄_{i,j}] = ProudFollowers[n, σ_{i,j}] /. σ_{i,j} → σ̄_{i,j}

ProudVPBs[n_, 0] := {VPB[n]};
ProudVPBs[n_, 1] := VPB[n, #] & /@ VPBGenerators[n];
ProudVPBs[n_, m_] /; m > 1 :=
 Flatten[ProudVPBs[n, m - 1] /.
   VPB[n, σs___, σ_] :→ (VPB[n, σs, σ, #] & /@ ProudFollowers[n, σ])]

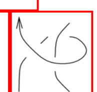
CountOUForms[n_, m_] := Module[{k},
  Length@Union@Flatten@Table[Γ̄@vpb, {k, 0, m}, {vpb, ProudVPBs[n, k]}]]
```



OVER THEN UNDER TANGLES

DROR BAR-NATAN, ZSUZSANNA DANCSO, AND ROLAND VAN DER VEEN

Abstract. Brilliant wrong ideas should not be buried and forgotten. Instead, they should be mined for the gold that lies underneath the layer of wrong. In this paper we explain how "over then under tangles" lead to an easy classification of knots, and under the surface, also to some valid mathematics: a separation theorem for braids and virtual braids, a topological understanding of the Drinfel'd double construction of quantum group theory, and more.



```
SetAttributes[VD, Orderless]

Tidy[vd_VD] := Module[{ps = Union @@ (List @@@ vd)},
  Replace[vd, Thread[ps → Range@Length@ps], {2}] ]

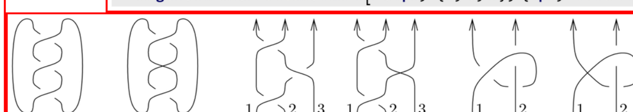
R12Reduce1[vd_VD] := Tidy@Module[{R2s, R2}, Which[
    Length[R2s = Cases[vd, X_s_[i_, j_] :→ X_{-s}[i + 1, j + 1]] ⋂ (List @@ vd)] > 0,
    Complement[vd, VD[R2 = First@R2s, R2 /. X_s_[i_, j_] :→ X_{-s}[i - 1, j - 1]]],
    Length[R2s = Cases[vd, X_s_[i_, j_] :→ X_{-s}[i + 1, j - 1]] ⋂ (List @@ vd)] > 0,
    Complement[vd, VD[R2 = First@R2s, R2 /. X_s_[i_, j_] :→ X_{-s}[i - 1, j + 1]]],
    True, DeleteCases[vd, X_[i_, j_] /; Abs[i - j] == 1] ]]

R12Reduce[vd_VD] := FixedPoint[R12Reduce1, vd]

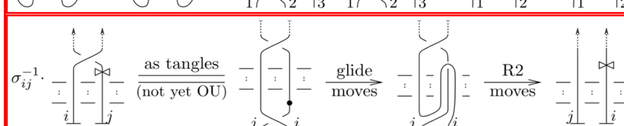
γ[vd_VD] := Module[{js, s1, i1, j1, s2, i2, j2},
  js = Cases[vd, X_[_, j_] :→ j] ⋂ Cases[vd, X_[i_, _] :→ i - 1];
  If[Length[js] == 0, vd,
   j1 = RandomChoice[js]; i2 = j1 + 1;
   Cases[vd, X_s_[i_, j1] :→ (s1 = s; i1 = i)];
   Cases[vd, X_s_[i2, j_] :→ (s2 = s; j2 = j)];
   Tidy@Join[Complement[vd, VD[X_{s1}[i1, j1], X_{s2}[i2, j2]]],
     VD[X_{s2}[j1, j2], X_{s1}[i1, i2], X_{s1 s2}[i1 - s1 / 3, j2 + s2 / 3],
      X_{-s1 s2}[i1 + s1 / 3, j2 - s2 / 3]]
    ] ]]

Γ[vd_VD] := FixedPoint[γ, vd, 2^8]

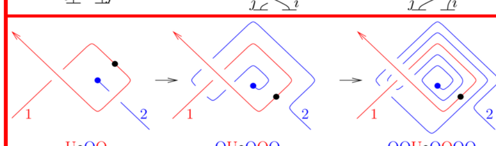
Γ[T_] /; Head[T] =!= VD := Γ[VD[T]]
```

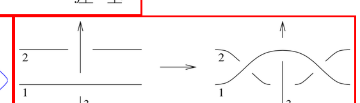


- Subsets.
- Supersets.
- Subsets of supersets.
- Completions.
- Images.
- Completions of subsets of supersets...

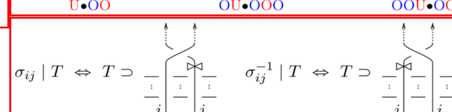


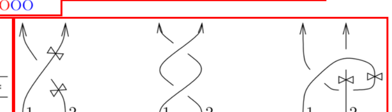
Credit to Manturov and Chterental!





See http://drorbn.net/mo20.





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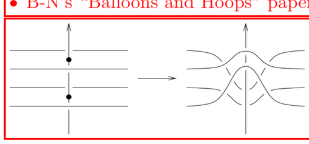
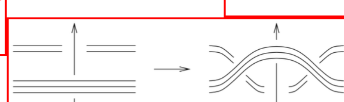
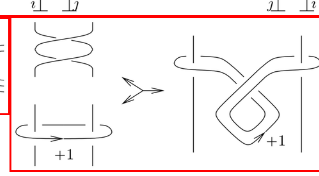
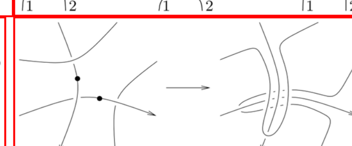
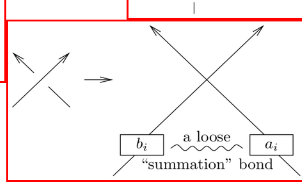
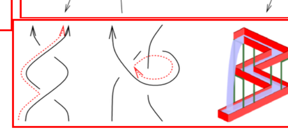












Audio and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2004//

Dror Bar-Natan: Talks: Toronto-1912:

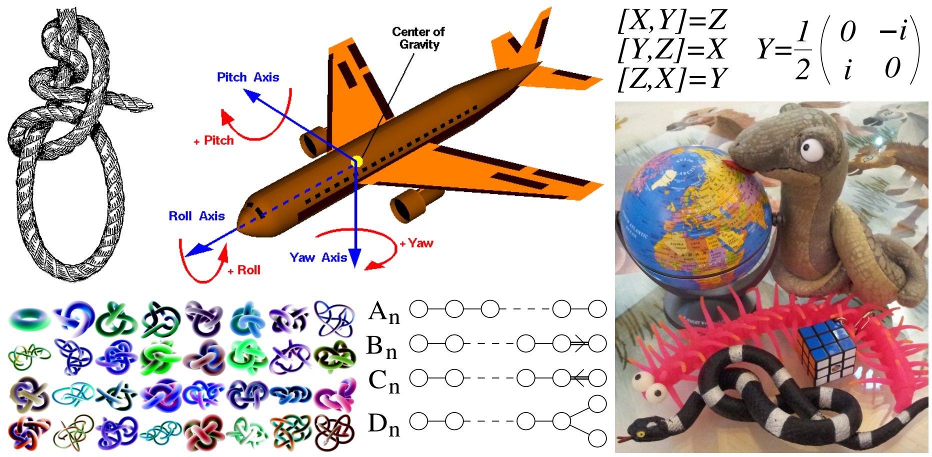
Chord Diagrams, Knots, and Lie Algebras

Thanks for inviting me to the *Chord Diagrams Everywhere* session / Winter 2019 CMS meeting! ωεβ:=http://drorbn.net/to19

Abstract. This will be a service talk on ancient material — I will briefly describe how the exact same type of chord diagrams (and relations between them) occur in a natural way in both knot theory and in the theory of Lie algebras.

While preparing for this talk I realized that I've done it before, much better, within a book review. So here's that review! It has been modified from its original version: it had been formatted to fit this page, parts were highlighted, and commentary had been added in green italics.

[Book] *Introduction to Vassiliev Knot Invariants,* by S. Chmutov, S. Duzhin, and J. Mostovoy, Cambridge University Press, Cambridge UK, 2012, xvi+504 pp., hardback, \$70.00, ISBN 978-1-10702-083-2.



A knot and a Lie algebra, a list of knots and a list of Lie algebras, and an unusual conference of the symmetric and the knotted.

Merely ~~30~~ *36* years ago, if you had asked even the best informed mathematician about the relationship between knots and Lie algebras, she would have laughed, for there isn't and there can't be. Knots are flexible; Lie algebras are rigid. Knots are irregular; Lie algebras are symmetric. The list of knots is a lengthy mess; the collection of Lie algebras is well-organized. Knots are useful for sailors, scouts, and hangmen; Lie algebras for navigators, engineers, and high energy physicists. Knots are blue collar; Lie algebras are white. They are as similar as worms and crystals: both well-studied, but hardly ever together.

Then in the 1980s came Jones, and Witten, and Reshetikhin and Turaev [Jo, Wi, RT] and showed that if you really are the best informed, and you know your quantum field theory and conformal field theory and quantum groups, then you know that the two disjoint fields are in fact intricately related. This "quantum" approach remains the most powerful way to get computable knot invariants out of (certain) Lie algebras (and representations thereof). Yet shortly later, in the late 80s and early 90s, an alternative perspective arose, that of "finite-type" or "Vassiliev-Goussarov" invariants [Va1, Va2, Go1, Go2, BL, Ko1, Ko2, BN1], which made the surprising relationship between knots and Lie algebras appear simple and almost inevitable.

The reviewed [Book] is about that alternative perspective, the one reasonable sounding but not entirely trivial theorem that is crucially needed within it (the "Fundamental Theorem" or the "Kontsevich integral"), and the many threads that begin with that perspective. Let me start with a brief summary of the mathematics, and even before, an even briefer summary.

In briefest, a certain space $\mathcal{A}$ of chord diagrams is the dual to the dual of the space of knots, and at the same time, it is dual to Lie algebras.

The briefer summary is that in some combinatorial sense it is possible to "differentiate" knot invariants, and hence it makes sense to talk about "polynomials" on the space of knots — these are functions on the set of knots (namely, these are knot invariants) whose sufficiently high derivatives vanish. Such polynomials can be fairly conjectured to separate knots — elsewhere in math in lucky cases polynomials separate points, and in our case, specific computations are encouraging. Also, such polynomials are determined by their "coefficients", and each of these, by the one-side-easy "Fundamental Theorem", is a linear functional on some finite space of

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2010 *Mathematics Subject Classification.* Primary 57M25.

Published Bull. Amer. Math. Soc. **50** (2013) 685–690. TEX at http://drorbn.net/AcademicPensieve/2013-01/CDMReview/, copyleft at http://www.math.toronto.edu/~drorbn/Copyleft/. This review was written while I was a guest at the Newton Institute, in Cambridge, UK. I wish to thank N. Bar-Natan, I. Halacheva, and P. Lee for comments and suggestions.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/

graphs modulo relations. These same graphs turn out to parameterize formulas that make sense in a wide class of Lie algebras, and the said relations match exactly with the relations in the definition of a Lie algebra — anti-symmetry and the Jacobi identity. Hence what is more or less dual to knots (invariants), is also, after passing to the coefficients, dual to certain graphs which are more or less dual to Lie algebras. QED, and on to the less brief summary[1].

Let V be an arbitrary invariant of oriented knots in oriented space with values in (say) $\mathbb{Q}$. Extend V to be an invariant of 1-singular knots, knots that have a single singularity that locally looks like a double point $\times$, using the formula

(1) $$V(\times) = V(\nearrow\!\!\!\nwarrow) - V(\nwarrow\!\!\!\nearrow).$$

Further extend V to the set $\mathcal{K}^m$ of m-singular knots (knots with m such double points) by repeatedly using (1).

Definition 1. We say that V is of type m (or "Vassiliev of type m") if its extension $V|_{\mathcal{K}^{m+1}}$ to $(m+1)$-singular knots vanishes identically. We say that V is of finite type (or "Vassiliev") if it is of type m for some m.

Repeated differences are similar to repeated derivatives and hence it is fair to think of the definition of $V|_{\mathcal{K}^m}$ as repeated differentiation. With this in mind, the above definition imitates the definition of polynomials of degree m. Hence finite type invariants can be thought of as "polynomials" on the space of knots[2]. It is known (see e.g. [Book]) that the class of finite type invariants is large and powerful. Yet the first question on finite type invariants remains unanswered:

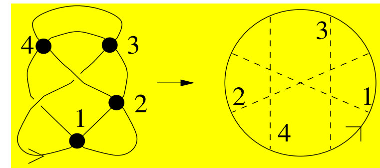
Problem 2. *Honest polynomials are dense in the space of functions. Are finite type invariants dense within the space of all knot invariants? Do they separate knots?*

The top derivatives of a multi-variable polynomial form a system of constants that determine that polynomial up to polynomials of lower degree. Likewise the mth derivative[3] $V^{(m)} = V|_{\mathcal{K}^m} = V\left(\times\overset{m}{\cdots}\times\right)$ of a type m invariant V is a constant in the sense that it does not see the difference between overcrossings and undercrossings and so it is blind to 3D topology. Indeed

$$V\left(\times\overset{m}{\cdots}\times\nearrow\!\!\!\nwarrow\right) - V\left(\times\overset{m}{\cdots}\times\nwarrow\!\!\!\nearrow\right) = V\left(\times\overset{m+1}{\cdots}\times\right) = 0.$$

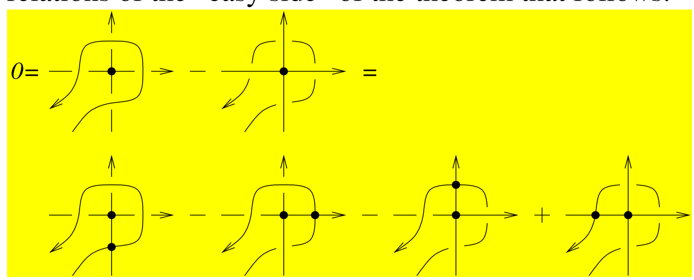
Also, clearly $V^{(m)}$ determines V up to invariants of lower type. Hence a primary tool in the study of finite type invariants is the study of the "top derivative" $V^{(m)}$, also known as "the weight system of V".

Blind to 3D topology, $V^{(m)}$ only sees the combinatorics of the circle that parameterizes an m-singular knot.



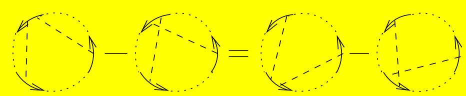
On this circle there are m pairs of points that are pairwise identified in the image; standardly one indicates those by drawing a circle with m chords marked (an "m-chord diagram") as above. Let $\mathcal{D}_m$ denote the space of all formal linear combinations with rational coefficients of m-chord diagrams. Thus $V^{(m)}$ is a linear functional on $\mathcal{D}_m$.

I leave it for the reader to figure out or read in [Book, pp. 88] how the following figure easily implies the "$4T$" relations of the "easy side" of the theorem that follows:



Theorem 3. *(The Fundamental Theorem, details in [Book]).*

- *(Easy side) If V is a rational valued type m invariant then $V^{(m)}$ satisfies the "$4T$" relations shown above, and hence it descends to a linear functional on $\mathcal{A}_m := \mathcal{D}_m/4T$. If in addition $V^{(m)} \equiv 0$, then V is of type $m-1$.*



- *(Hard side, slightly misstated by avoiding "framings") For any linear functional W on $\mathcal{A}_m$ there is a rational valued type m invariant V so that $V^{(m)} = W$.*

Thus to a large extent the study of finite type invariants is reduced to the finite (though super-exponential in m) algebraic study of $\mathcal{A}_m$.

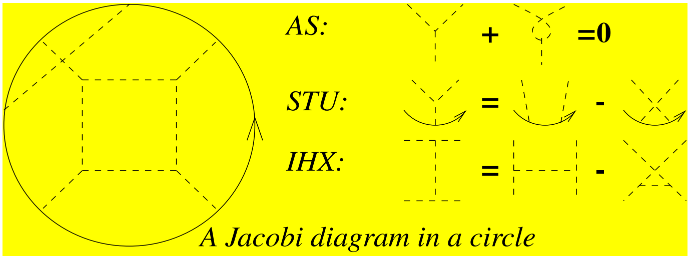
Much of the richness of finite type invariants stems from their relationship with Lie algebras. Theorem 4 below suggests this relationship on an abstract level and Theorem 5 makes that relationship concrete.

[1] Partially self-plagiarized from [BN2].

[2] Keep this apart from invariants of knots whose values are polynomials, such as the Alexander or the Jones polynomial. A posteriori related, these are a priori entirely different.

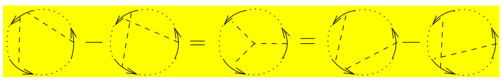
[3] As common in the knot theory literature, in the formulas that follow a picture such as $\times\overset{m}{\cdots}\times\nearrow\!\!\!\nwarrow$ indicates "some knot having m double points and a further (right-handed) crossing". Furthermore, when two such pictures appear within the same formula, it is to be understood that the parts of the knots (or diagrams) involved *outside* of the displayed pictures are to be taken as the same.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/



A Jacobi diagram in a circle

Theorem 4. [BN1] *The space $\mathcal{A}_m$ is isomorphic to the space $\mathcal{A}_m^t$ generated by "Jacobi diagrams in a circle" (chord diagrams that are also allowed to have oriented internal trivalent vertices) that have exactly $2m$ vertices, modulo the AS, STU and IHX relations. See the figure above.*

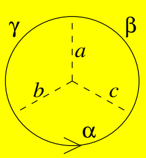


The key to the proof of Theorem 4 is the figure above, which shows that the $4T$ relation is a consequence of two STU relations. The rest is more or less an exercise in induction.

Thinking of internal trivalent vertices as graphical analogs of the Lie bracket, the AS relation becomes the anti-commutativity of the bracket, STU becomes the equation $[x, y] = xy - yx$ and IHX becomes the Jacobi identity. This analogy is made concrete within the following construction, originally due to Penrose [Pe] and to Cvitanović [Cv]. Given a finite dimensional metrized Lie algebra $\mathfrak{g}$ (e.g., any semi-simple Lie algebra) and a finite-dimensional representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of $\mathfrak{g}$, choose an orthonormal basis[4] $\{X_a\}_{a=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}$ and some basis $\{v_\alpha\}_{\alpha=1}^{\dim V}$ of V, let f_{abc} and $r_{\alpha\beta}^{\gamma}$ be the "structure constants" defined by

$$f_{abc} := \langle [X_a, X_b], X_c \rangle \qquad \text{and} \qquad \rho(X_a)(v_\beta) = \sum_\gamma r_{a\beta}^{\gamma} v_\gamma.$$

Now given a Jacobi diagram D label its circle-arcs with Greek letters $\alpha, \beta, \ldots$, and its chords with Latin letters a, $b, \ldots$, and map it to a sum as suggested by the following example:



$$\longrightarrow \sum_{a,b,c,\alpha,\beta,\gamma} f_{abc} r_{a\gamma}^{\beta} r_{b\alpha}^{\gamma} r_{c\beta}^{\alpha}$$

$\left(\begin{array}{l}\text{internal vertices go to } f\text{'s,} \\ \text{circle-vertices to } r\text{'s}\end{array}\right)$

Theorem 5. *This construction is well defined, and the basic properties of Lie algebras imply that it respects the AS, STU, and IHX relations. Therefore it defines a linear functional $W_{\mathfrak{g},\rho} : \mathcal{A}_m \to \mathbb{Q}$, for any m.*

The last assertion along with Theorem 3 show that associated with any $\mathfrak{g}$, ρ and m there is a weight system and hence a knot invariant. Thus knots are indeed linked with Lie algebras.

The above is of course merely a sketch of the beginning of a long story. You can read the details, and some of the rest, in [Book].

What I like about [Book]. Detailed, well thought out, and carefully written. Lots of pictures! Many excellent exercises! A complete discussion of "the algebra of chord diagrams". A nice discussion of the pairing of diagrams with Lie algebras, including examples aplenty. The discussion of the Kontsevich integral (meaning, the proof of the hard side of Theorem 3) is terrific — detailed and complete and full of pictures and examples, adding a great deal to the original sources. The subject of "associators" is huge and worthy of its own book(s); yet in as much as they are related to Vassiliev invariants, the discussion in [Book] is excellent. A great many further topics are touched — multiple ζ-values, the relationship of the Hopf link with the Duflo isomorphism, intersection graphs and other combinatorial aspects of chord diagrams, Rozansky's rationality conjecture, the Melvin-Morton conjecture, braids, n-equivalence, etc.

For all these, I'd certainly recommend [Book] to any newcomer to the subject of knot theory, starting with my own students.

However, some proofs other than that of Theorem 3 are repeated as they appear in original articles with only a superficial touch-up, or are omitted altogether, thus missing an opportunity to clarify some mysterious points. This includes Vogel's construction of a non-Lie-algebra weight system and the Goussarov-Polyak-Viro proof of the existence of "Gauss diagram formulas".

What I wish there was in the book, but there isn't. The relationship with Chern-Simons theory, Feynman diagrams, and configuration space integrals, culminating in an alternative (and more "3D") proof of the Fundamental Theorem. This is a major omission.

Why I hope there will be a continuation book, one day. There's much more to the story! There are finite type invariants of 3-manifolds, and of certain classes of 2-dimensional knots in $\mathbb{R}^4$, and of "virtual knots", and they each have their lovely yet non-obvious theories, and these theories link with each other and with other branches of Lie theory, algebra, topology, and quantum field theory. Volume 2 is sorely needed.

References

[BN1] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[4] This requirement can easily be relaxed.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/

[BN2] D. Bar-Natan, *Finite Type Invariants,* in *Encyclopedia of Mathematical Physics,* (J.-P. Francoise, G. L. Naber and Tsou S. T., eds.) Elsevier, Oxford, 2006 (vol. 2 p. 340).

[Book] The reviewed book.

[BL] J. S. Birman and X-S. Lin, *Knot polynomials and Vassiliev's invariants,* Invent. Math. **111** (1993) 225–270.

[Cv] P. Cvitanović, *Group Theory, Birdtracks, Lie's, and Exceptional Groups,* Princeton University Press, Princeton 2008 and http://www.birdtracks.eu.

[Go1] M. Goussarov, *A new form of the Conway-Jones polynomial of oriented links,* Zapiski nauch. sem. POMI **193** (1991) 4–9 (English translation in *Topology of manifolds and varieties* (O. Viro, editor), Amer. Math. Soc., Providence 1994, 167–172).

[Go2] M. Goussarov, *On n-equivalence of knots and invariants of finite degree,* Zapiski nauch. sem. POMI **208** (1993) 152–173 (English translation in *Topology of manifolds and varieties* (O. Viro, editor), Amer. Math. Soc., Providence 1994, 173–192).

[Jo] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras,* Bull. Amer. Math. Soc. **12** (1985) 103–111.

[Ko1] M. Kontsevich, *Vassiliev's knot invariants,* Adv. in Sov. Math., **16(2)** (1993) 137–150.

[Ko2] M. Kontsevich, *Feynman diagrams and low-dimensional topology,* First European Congress of Mathematics **II** 97–121, Birkhäuser Basel 1994.

[Pe] R. Penrose, *Applications of negative dimensional tensors,* Combinatorial mathematics and its applications (D. J. A. Welsh, ed.), Academic Press, San-Diego 1971, 221–244.

[RT] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups,* Commun. Math. Phys. **127** (1990) 1–26.

[Va1] V. A. Vassiliev, *Cohomology of knot spaces,* in *Theory of Singularities and its Applications (Providence)* (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

[Va2] V. A. Vassiliev, *Complements of discriminants of smooth maps: topology and applications,* Trans. of Math. Mono. **98**, Amer. Math. Soc., Providence, 1992.

[Wi] E. Witten, *Quantum field theory and the Jones polynomial,* Commun. Math. Phys. **121** (1989) 351–399.

Dror Bar-Natan
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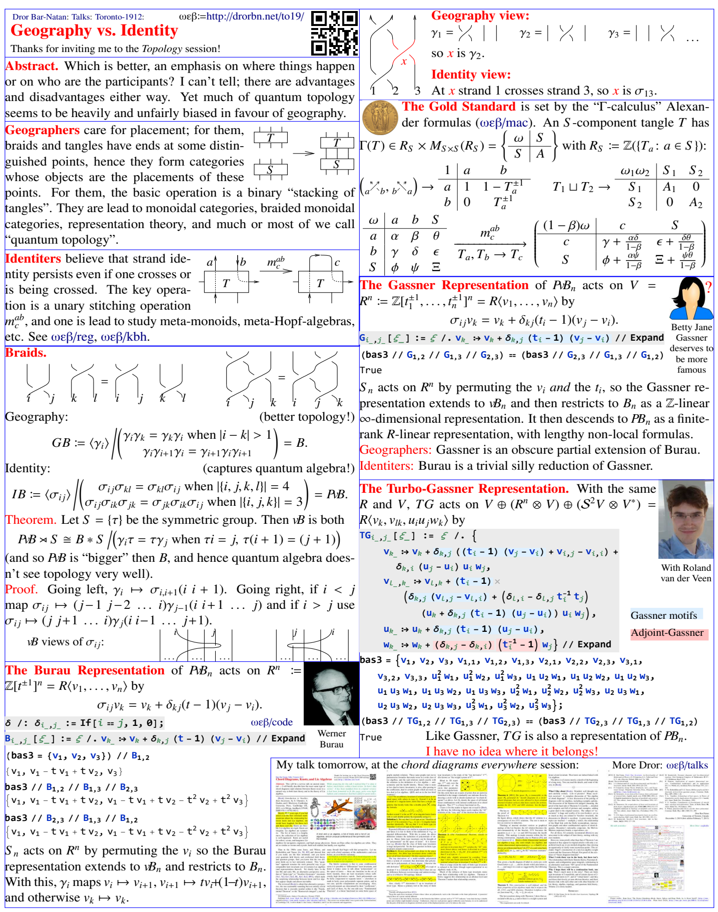
December 6, 2019 (first edition February 7, 2013)

My talk yesterday:

More Dror: ωεβ/talks


Dror Bar-Natan: Talks: Toronto-1912: ωεβ:=http://drorbn.net/to19/

Geography vs. Identity

Thanks for inviting me to the *Topology* session!

Abstract. Which is better, an emphasis on where things happen or on who are the participants? I can't tell; there are advantages and disadvantages either way. Yet much of quantum topology seems to be heavily and unfairly biased in favour of geography.

Geographers care for placement; for them, braids and tangles have ends at some distinguished points, hence they form categories whose objects are the placements of these points. For them, the basic operation is a binary "stacking of tangles". They are lead to monoidal categories, braided monoidal categories, representation theory, and much or most of we call "quantum topology".

Identiters believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation m^{ab}_c, and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See ωεβ/reg, ωεβ/kbh.

Braids.

Geography: (better topology!)

$$GB := \langle\gamma_i\rangle \Big/ \left(\begin{array}{c}\gamma_i\gamma_k = \gamma_k\gamma_i \text{ when } |i-k|>1\\ \gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1}\end{array}\right) = B.$$

Identity: (captures quantum algebra!)

$$IB := \langle\sigma_{ij}\rangle \Big/ \left(\begin{array}{c}\sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \text{ when } |\{i,j,k,l\}|=4\\ \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \text{ when } |\{i,j,k\}|=3\end{array}\right) = PvB.$$

Theorem. Let $S=\{\tau\}$ be the symmetric group. Then vB is both

$$PvB \rtimes S \cong B * S \Big/ \left(\gamma_i\tau = \tau\gamma_j \text{ when } \tau i = j,\ \tau(i+1)=(j+1)\right)$$

(and so PvB is "bigger" then B, and hence quantum algebra doesn't see topology very well).

Proof. Going left, $\gamma_i \mapsto \sigma_{i,i+1}(i\ i+1)$. Going right, if $i<j$ map $\sigma_{ij} \mapsto (j{-}1\ j{-}2\ \dots\ i)\gamma_{j-1}(i\ i{+}1\ \dots\ j)$ and if $i>j$ use $\sigma_{ij} \mapsto (j\ j{+}1\ \dots\ i)\gamma_j(i\ i{-}1\ \dots\ j{+}1)$.

vB views of σ_{ij}:

The Burau Representation of PvB_n acts on $R^n := \mathbb{Z}[t^{\pm 1}]^n = R\langle v_1,\dots,v_n\rangle$ by

$$\sigma_{ij}v_k = v_k + \delta_{kj}(t-1)(v_j-v_i).$$

```
δ /: δi_,j_ := If[i == j, 1, 0];
Bi_,j_[ℰ_] := ℰ /. vk_ :> vk + δk,j (t - 1) (vj - vi) // Expand
(bas3 = {v1, v2, v3}) // B1,2
{v1, v1 - t v1 + t v2, v3}
bas3 // B1,2 // B1,3 // B2,3
{v1, v1 - t v1 + t v2, v1 - t v1 + t v2 - t^2 v2 + t^2 v3}
bas3 // B2,3 // B1,3 // B1,2
{v1, v1 - t v1 + t v2, v1 - t v1 + t v2 - t^2 v2 + t^2 v3}
```

ωεβ/code

Werner Burau

S_n acts on R^n by permuting the v_i so the Burau representation extends to vB_n and restricts to B_n. With this, γ_i maps $v_i \mapsto v_{i+1}$, $v_{i+1} \mapsto tv_i{+}(1{-}t)v_{i+1}$, and otherwise $v_k \mapsto v_k$.

Geography view: $\gamma_1 =$ … $\gamma_2 =$ … $\gamma_3 =$ … …

so x is γ_2.

Identity view: At x strand 1 crosses strand 3, so x is σ_{13}.

The Gold Standard is set by the "Γ-calculus" Alexander formulas (ωεβ/mac). An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{T_a : a \in S\})$:

$$\left({}_a\!\nearrow_b,\ {}_b\!\nwarrow_a\right) \to \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[T_a,T_b\to T_c]{m^{ab}_c} \left(\begin{array}{c|cc}(1-\beta)\omega & c & S\\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta}\\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

The Gassner Representation of PvB_n acts on $V = R^n := \mathbb{Z}[t_1^{\pm1},\dots,t_n^{\pm1}]^n = R\langle v_1,\dots,v_n\rangle$ by

$$\sigma_{ij}v_k = v_k + \delta_{kj}(t_i-1)(v_j-v_i).$$

```
Gi_,j_[ℰ_] := ℰ /. vk_ :> vk + δk,j (ti - 1) (vj - vi) // Expand
(bas3 // G1,2 // G1,3 // G2,3) == (bas3 // G2,3 // G1,3 // G1,2)
True
```

? Betty Jane Gassner deserves to be more famous

S_n acts on R^n by permuting the v_i *and* the t_i, so the Gassner representation extends to vB_n and then restricts to B_n as a $\mathbb{Z}$-linear ∞-dimensional representation. It then descends to PB_n as a finite-rank R-linear representation, with lengthy non-local formulas.

Geographers: Gassner is an obscure partial extension of Burau.

Identiters: Burau is a trivial silly reduction of Gassner.

The Turbo-Gassner Representation. With the same R and V, TG acts on $V \oplus (R^n \otimes V) \oplus (\mathcal{S}^2 V \otimes V^*) = R\langle v_k, v_{lk}, u_i u_j w_k\rangle$ by

```
TGi_,j_[ℰ_] := ℰ /. {
   vk_ :> vk + δk,j ((ti - 1) (vj - vi) + vi,j - vi,i) +
      δk,i (uj - ui) ui wj,
   vl_,k_ :> vl,k + (ti - 1) ×
      (δk,j (vl,j - vl,i) + (δl,i - δl,j ti^-1 tj)
        (uk + δk,j (ti - 1) (uj - ui)) ui wj),
   uk_ :> uk + δk,j (ti - 1) (uj - ui),
   wk_ :> wk + (δk,j - δk,i) (ti^-1 - 1) wj} // Expand
bas3 = {v1, v2, v3, v1,1, v1,2, v1,3, v2,1, v2,2, v2,3, v3,1,
   v3,2, v3,3, u1^2 w1, u1^2 w2, u1^2 w3, u1 u2 w1, u1 u2 w2, u1 u2 w3,
   u1 u3 w1, u1 u3 w2, u1 u3 w3, u2^2 w1, u2^2 w2, u2^2 w3, u2 u3 w1,
   u2 u3 w2, u2 u3 w3, u3^2 w1, u3^2 w2, u3^2 w3};
(bas3 // TG1,2 // TG1,3 // TG2,3) == (bas3 // TG2,3 // TG1,3 // TG1,2)
True
```

With Roland van der Veen

Gassner motifs

Adjoint-Gassner

Like Gassner, TG is also a representation of PB_n.

I have no idea where it belongs!

My talk tomorrow, at the *chord diagrams everywhere* session:

More Dror: ωεβ/talks

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Video and more at http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/

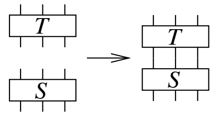
Dror Bar-Natan: Talks: Toronto-1912: ωεβ:=http://drorbn.net/to19/

Geography vs. Identity

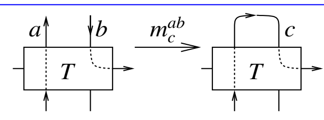
Thanks for inviting me to the *Topology* session!

Abstract. Which is better, an emphasis on where things happen or on who are the participants? I can't tell; there are advantages and disadvantages either way. Yet much of quantum topology seems to be heavily and unfairly biased in favour of geography.

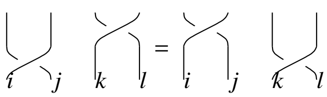
Geographers care for placement; for them, braids and tangles have ends at some distinguished points, hence they form categories whose objects are the placements of these points. For them, the basic operation is a binary "stacking of tangles". They are lead to monoidal categories, braided monoidal categories, representation theory, and much or most of we call "quantum topology".

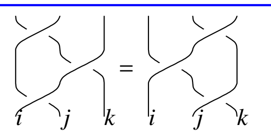


Identiters believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation m^{ab}_c, and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See ωεβ/reg, ωεβ/kbh.



Braids.





Geography: (better topology!)

$$GB := \langle \gamma_i \rangle \Big/ \left(\begin{matrix} \gamma_i\gamma_k = \gamma_k\gamma_i \text{ when } |i-k|>1 \\ \gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1} \end{matrix} \right) = B.$$

Identity: (captures quantum algebra!)

$$IB := \langle \sigma_{ij} \rangle \Big/ \left(\begin{matrix} \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \text{ when } |\{i,j,k,l\}| = 4 \\ \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \text{ when } |\{i,j,k\}| = 3 \end{matrix} \right) = PvB.$$

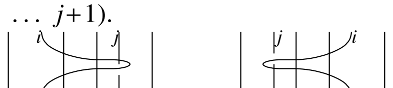
Theorem. Let $S = \{\tau\}$ be the symmetric group. Then vB is both

$$PvB \rtimes S \cong B * S \Big/ \left(\gamma_i\tau = \tau\gamma_j \text{ when } \tau i = j,\ \tau(i+1) = (j+1) \right)$$

(and so PvB is "bigger" then B, and hence quantum algebra doesn't see topology very well).

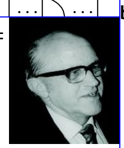
Proof. Going left, $\gamma_i \mapsto \sigma_{i,i+1}(i\ i+1)$. Going right, if $i<j$ map $\sigma_{ij} \mapsto (j{-}1\ j{-}2\ \dots\ i)\gamma_{j-1}(i\ i{+}1\ \dots\ j)$ and if $i>j$ use $\sigma_{ij} \mapsto (j\ j{+}1\ \dots\ i)\gamma_j(i\ i{-}1\ \dots\ j{+}1)$.

vB views of σ_{ij}:



The Burau Representation of PvB_n acts on $R^n := \mathbb{Z}[t^{\pm 1}]^n = R\langle v_1,\dots,v_n\rangle$ by

$$\sigma_{ij}v_k = v_k + \delta_{kj}(t-1)(v_j - v_i).$$

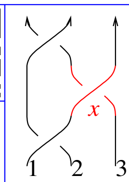


Werner Burau

ωεβ/code

```
δ /: δi_,j_ := If[i == j, 1, 0];
Bi_,j_[ξ_] := ξ /. vk_ :> vk + δk,j (t - 1) (vj - vi) // Expand
(bas3 = {v1, v2, v3}) // B1,2
{v1, v1 - t v1 + t v2, v3}
bas3 // B1,2 // B1,3 // B2,3
{v1, v1 - t v1 + t v2, v1 - t v1 + t v2 - t^2 v2 + t^2 v3}
bas3 // B2,3 // B1,3 // B1,2
{v1, v1 - t v1 + t v2, v1 - t v1 + t v2 - t^2 v2 + t^2 v3}
```

S_n acts on R^n by permuting the v_i so the Burau representation extends to vB_n and restricts to B_n. With this, γ_i maps $v_i \mapsto v_{i+1}$, $v_{i+1} \mapsto tv_i + (1-t)v_{i+1}$, and otherwise $v_k \mapsto v_k$.



Geography view:

$\gamma_1 =$  $\quad \gamma_2 =$  $\quad \gamma_3 =$  $\dots$

so x is γ_2.

Identity view:

At x strand 1 crosses strand 3, so x is σ_{13}.

The Gold Standard is set by the "Γ-calculus" Alexander formulas (ωεβ/mac). An S-component tangle T has

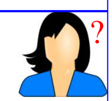
$$\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\} \text{ with } R_S := \mathbb{Z}(\{T_a : a \in S\}):$$

$$\left({}_{a}{\nearrow}_{b},\ {}_{b}{\nwarrow}_{a} \right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-T_a^{\pm 1} \\ b & 0 & T_a^{\pm 1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[T_a, T_b \to T_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array} \right)$$

The Gassner Representation of PvB_n acts on $V = R^n := \mathbb{Z}[t_1^{\pm 1},\dots,t_n^{\pm 1}]^n = R\langle v_1,\dots,v_n\rangle$ by

$$\sigma_{ij}v_k = v_k + \delta_{kj}(t_i - 1)(v_j - v_i).$$



Betty Jane Gassner deserves to be more famous

```
Gi_,j_[ξ_] := ξ /. vk_ :> vk + δk,j (ti - 1) (vj - vi) // Expand
(bas3 // G1,2 // G1,3 // G2,3) == (bas3 // G2,3 // G1,3 // G1,2)
True
```

S_n acts on R^n by permuting the v_i *and* the t_i, so the Gassner representation extends to vB_n and then restricts to B_n as a $\mathbb{Z}$-linear ∞-dimensional representation. It then descends to PB_n as a finite-rank R-linear representation, with lengthy non-local formulas.

Geographers: Gassner is an obscure partial extension of Burau.

Identiters: Burau is a trivial silly reduction of Gassner.

The Turbo-Gassner Representation. With the same R and V, TG acts on $V \oplus (R^n \otimes V) \oplus (\mathcal{S}^2V \otimes V^*) = R\langle v_k, v_{lk}, u_iu_jw_k\rangle$ by

With Roland van der Veen

```
TGi_,j_[ξ_] := ξ /. {
    vk_ :> vk + δk,j ((ti - 1) (vj - vi) + vi,j - vi,i) +
       δk,i (uj - ui) ui wj,
    vl_,k_ :> vl,k + (ti - 1) ×
         (δk,j (vl,j - vl,i) + (δl,i - δl,j ti^-1 tj)
            (uk + δk,j (ti - 1) (uj - ui)) ui wj),
    uk_ :> uk + δk,j (ti - 1) (uj - ui),
    wk_ :> wk + (δk,j - δk,i) (ti^-1 - 1) wj} // Expand
bas3 = {v1, v2, v3, v1,1, v1,2, v1,3, v2,1, v2,2, v2,3, v3,1,
   v3,2, v3,3, u1^2 w1, u1^2 w2, u1^2 w3, u1 u2 w1, u1 u2 w2, u1 u2 w3,
   u1 u3 w1, u1 u3 w2, u1 u3 w3, u2^2 w1, u2^2 w2, u2^2 w3, u2 u3 w1,
   u2 u3 w2, u2 u3 w3, u3^2 w1, u3^2 w2, u3^2 w3};
(bas3 // TG1,2 // TG1,3 // TG2,3) == (bas3 // TG2,3 // TG1,3 // TG1,2)
True
```

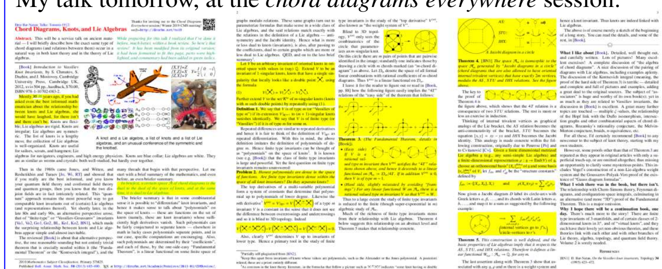
Gassner motifs

Adjoint-Gassner

Like Gassner, TG is also a representation of PB_n.

I have no idea where it belongs!

My talk tomorrow, at the *chord diagrams everywhere* session:



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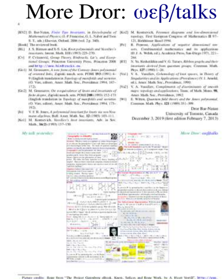


Video and more at http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/

Some Feynman Diagrams in Pure Algebra

With Roland van der Veen

Thanks for allowing me in Columbia U! ωεβ:=http://drorbn.net/co19/ Slides w/ no handout/URL should be banned!

Abstract. I will explain how the computation of compositions of maps of a certain natural class, from one polynomial ring into another, naturally leads to a certain composition operation of quadratics and to Feynman diagrams. I will also explain, with very little detail, how this is used in the construction of some very well-behaved poly-time computable knot polynomials.

The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Gentle Agreement. Everything converges!

Convention. For a finite set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$. $(y,b,a,x)^* = (\eta,\beta,\alpha,\xi)$

The Generating Series $\mathcal{G}$: $\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \to \mathbb{Q}[\![\zeta_A, z_B]\!]$.

Claim. $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \xrightarrow[\mathcal{G}]{\sim} \mathbb{Q}[z_B][\![\zeta_A]\!] \ni \mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n\in\mathbb{N}^A} \frac{\zeta_A^n}{n!} L(z_A^n) = L\left(e^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{L} = {}_{\text{greek}}\mathcal{L}_{\text{latin}},$$

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{L}\right)_{\zeta_a=0} \quad \text{for } p\in\mathbb{Q}[z_A].$$

Claim. If $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $M\in\operatorname{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0}$.

Basic Examples. 1. $\mathcal{G}(id\colon \mathbb{Q}[y,a,x]\to\mathbb{Q}[y,a,x]) = e^{\eta y+\alpha a+\xi x}$.

2. The standard commutative product m_k^{ij} of polynomials is given by $z_i, z_j \to z_k$. Hence $\mathcal{G}(m_k^{ij}) = m_k^{ij}(e^{\zeta_i z_i+\zeta_j z_j}) = e^{(\zeta_i+\zeta_j)z_k}$.

$$\begin{tikzcd} \mathbb{Q}[z]_i\otimes\mathbb{Q}[z]_j \arrow[r, "m_k^{ij}"] \arrow[d, equal] & \mathbb{Q}[z]_k \arrow[d, equal] \\ \mathbb{Q}[z_i,z_j] \arrow[r, "m_k^{ij}"] & \mathbb{Q}[z_k] \end{tikzcd}$$

3. The standard co-commutative coproduct Δ^i_{jk} of polynomials is given by $z_i\to z_j+z_k$. Hence $\mathcal{G}(\Delta^i_{jk}) = \Delta^i_{jk}(e^{\zeta_i z_i}) = e^{\zeta_i(z_j+z_k)}$.

$$\begin{tikzcd} \mathbb{Q}[z]_i \arrow[r, "\Delta^i_{jk}"] \arrow[d, equal] & \mathbb{Q}[z]_j\otimes\mathbb{Q}[z]_k \arrow[d, equal] \\ \mathbb{Q}[z_i] \arrow[r, "\Delta^i_{jk}"] & \mathbb{Q}[z_j,z_k] \end{tikzcd}$$

Heisenberg Algebras. Let $\mathbb{H} = \langle x,y\rangle/[x,y] = \hbar$ (with $\hbar$ a scalar), let $\mathbb{O}_i\colon \mathbb{Q}[x_i,y_i]\to\mathbb{H}_i$ is the "x before y" PBW ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[x_i,y_i,x_j,y_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathbb{H}_i\otimes\mathbb{H}_j \xrightarrow{m_k^{ij}} \mathbb{H}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[x_k,y_k].$$

Then $\mathcal{G}(hm_k^{ij}) = e^{\Lambda_\hbar}$, where $\Lambda_\hbar = -\hbar\eta_i\xi_j + (\xi_i+\xi_j)x_k + (\eta_i+\eta_j)y_k$.

Proof 1. Recall the "Weyl form of the CCR" $e^{\eta y}e^{\xi x} = e^{-\hbar\eta\xi}e^{\xi x}e^{\eta y}$, and compute

$$\begin{aligned}\mathcal{G}(hm_k^{ij}) &= e^{\xi_i x_i+\eta_i y_i+\xi_j x_j+\eta_j y_j}/\!\!/\mathbb{O}_i\otimes\mathbb{O}_j/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1} \\ &= e^{\xi_i x_i}e^{\eta_i y_i}e^{\xi_j x_j}e^{\eta_j y_j}/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1} = e^{\xi_i x_k}e^{\eta_i y_k}e^{\xi_j x_k}e^{\eta_j y_k}/\!\!/\mathbb{O}_k^{-1} \\ &= e^{-\hbar\eta_i\xi_j}e^{(\xi_i+\xi_j)x_k}e^{(\eta_i+\eta_j)y_k}/\!\!/\mathbb{O}_k^{-1} = e^{\Lambda_\hbar}.\end{aligned}$$

Proof 2. We compute in a faithful 3D representation ρ of $\mathbb{H}$: (ωεβ/hm)

```
{x̂ = (0 1 0; 0 0 0; 0 0 0), ŷ = (0 0 0; 0 0 ℏ; 0 0 0), ĉ = (0 0 1; 0 0 0; 0 0 0)};
{x̂.ŷ - ŷ.x̂ == ℏ ĉ, x̂.ĉ == ĉ.x̂, ŷ.ĉ == ĉ.ŷ}
{True, True, True}
Λ = -ℏ ηi ξj ck + (ξi + ξj) xk + (ηi + ηj) yk;
Simplify@With[{𝔼 = MatrixExp},
  𝔼[x̂ ξi].𝔼[ŷ ηi].𝔼[x̂ ξj].𝔼[ŷ ηj] ==
   𝔼[x̂ ∂xk Λ].𝔼[ŷ ∂yk Λ].𝔼[ĉ ∂ck Λ]]
True
```

A Real DoPeGDO Example (DoPeGDO:=Docile Perturbed Gaussian Differential Operators). Let $sl_{2+}^\epsilon := L\langle y,b,a,x\rangle$ subject to $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $[x,y]=\epsilon a+b$. So $t:=\epsilon a-b$ is central and if $\exists\epsilon^{-1}$, $sl_{2+}^\epsilon\cong sl_2\oplus\langle t\rangle$. Let $CU:=\mathcal{U}(sl_{2+}^\epsilon)$, and let cm_k^{ij} be the composition below, where $\mathbb{O}_i\colon\mathbb{Q}[y_i,b_i,a_i,x_i]\to CU_i$ be the PBW ordering map in the order $ybax$:

$$\begin{tikzcd} CU_i\otimes CU_j \arrow[r, "m_k^{ij}"] & CU_k \\ \mathbb{Q}[y_i,b_i,a_i,x_i,y_j,b_j,a_j,x_j] \arrow[u, "\mathbb{O}_{i,j}"] \arrow[r, "cm_k^{ij}"] & \mathbb{Q}[y_k,b_k,a_k,x_k] \arrow[u, "\mathbb{O}_k"] \end{tikzcd}$$

Claim. Let (all brawn and no brains)

$$\Lambda = \left(\eta_i + \frac{e^{-\alpha_i-\epsilon\beta_i}\eta_j}{1+\epsilon\eta_j\xi_i}\right)y_k + \left(\beta_i+\beta_j+\frac{\log\left(1+\epsilon\eta_j\xi_i\right)}{\epsilon}\right)b_k +$$
$$\left(\alpha_i+\alpha_j+\log\left(1+\epsilon\eta_j\xi_i\right)\right)a_k + \left(\frac{e^{-\alpha_j-\epsilon\beta_j}\xi_i}{1+\epsilon\eta_j\xi_i}+\xi_j\right)x_k$$

Then $e^{\eta_i y_i+\beta_i b_i+\alpha_i a_i+\xi_i x_i+\eta_j y_j+\beta_j b_j+\alpha_j a_j+\xi_j x_j}/\!\!/\mathbb{O}_{i,j}/\!\!/cm_k^{ij} = e^{\Lambda}/\!\!/\mathbb{O}_k$, and hence $\mathcal{G}(cm_k^{ij}) = e^\Lambda$.

Proof. We compute in a faithful 2D representation ρ of CU: (ωεβ/sl2)

```
{ŷ = (0 0; ∈ 0), b̂ = (0 0; 0 -∈), â = (1 0; 0 0), x̂ = (0 1; 0 0)};
{â.x̂ - x̂.â == x̂, â.ŷ - ŷ.â == -ŷ, b̂.ŷ - ŷ.b̂ == -∈ ŷ,
 b̂.x̂ - x̂.b̂ == ∈ x̂, x̂.ŷ - ŷ.x̂ == b̂ + ∈ â}
{True, True, True, True, True}
Simplify@With[{𝔼 = MatrixExp},
  𝔼[ηi ŷ].𝔼[βi b̂].𝔼[αi â].𝔼[ξi x̂].𝔼[ηj ŷ].𝔼[βj b̂].
   𝔼[αj â].𝔼[ξj x̂] == 𝔼[ŷ ∂yk Λ].𝔼[b̂ ∂bk Λ].𝔼[â ∂ak Λ].
   𝔼[x̂ ∂xk Λ]]
True
Series[Λ, {∈, 0, 2}]
(ak (αi + αj) + yk (ηi + e^-αi ηj) +
   bk (βi + βj + ηj ξi) + xk (e^-αj ξi + ξj)) +
 (ak ηj ξi - 1/2 bk ηj^2 ξi^2 - e^-αi yk ηj (βi + ηj ξi) -
    e^-αj xk ξi (βj + ηj ξi)) ∈ +
 (-1/2 ak ηj^2 ξi^2 + 1/3 bk ηj^3 ξi^3 + 1/2 e^-αi yk ηj (βi^2 + 2 βi ηj ξi + 2 ηj^2 ξi^2) +
    1/2 e^-αj xk ξi (βj^2 + 2 βj ηj ξi + 2 ηj^2 ξi^2)) ∈^2 + O[∈]^3
```

Note 1. If the lower half of the alphabet (a,b,α,β) is regarded as constants, then $\Lambda = C+Q+\sum_{k\geq1}\epsilon^k P^{(k)}$ is a docile perturbed Gaussian relative to the upper half of the alphabet (x,y,ξ,η): C is a scalar, Q is a quadratic, and $\deg P^{(k)}\leq 2k+2$.

Note 2. $\operatorname{wt}(x,y,\xi,\eta;a,b,\alpha,\beta;\epsilon) = (1,1,1,1;2,0,0,2;-2)$.

Quadratic Casimirs. If $t\in\mathfrak{g}\otimes\mathfrak{g}$ is the quadratic Casimir of a semi-simple Lie algebra $\mathfrak{g}$, then e^t, regarded by PBW as an element of $\mathcal{S}^{\otimes 2} = \operatorname{Hom}\left(\mathcal{S}(\mathfrak{g})^{\otimes 0}\to\mathcal{S}(\mathfrak{g})^{\otimes 2}\right)$, has a latin-latin dominant Gaussian factor. Likewise for R-matrices.

(Baby) **DoPeGDO** := The category with objects finite sets[†1] and

$$\operatorname{mor}(A\to B) = \{\mathcal{L} = \omega\exp(Q+P)\}\subset\mathbb{Q}[\![\zeta_A, z_B, \epsilon]\!],$$

where: • ω is a scalar.[†2] • Q is a "small" ϵ-free quadratic in $\zeta_A\cup z_B$.[†3] • P is a "docile perturbation": $P=\sum_{k\geq1}\epsilon^k P^{(k)}$, where $\deg P^{(k)}\leq 2k+2$.[†4] • Compositions:[†6] $\mathcal{L}/\!\!/\mathcal{M} := \left(\mathcal{L}|_{z_i\to\partial_{\zeta_i}}\mathcal{M}\right)_{\zeta_i=0}$.

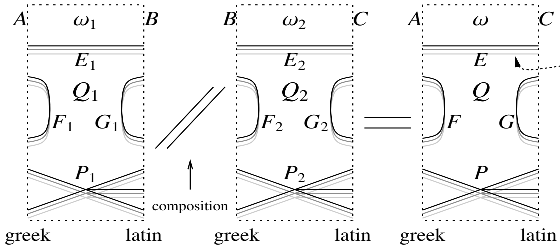
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Columbia-191125/

So What? If V is a representation, then $V^{\otimes n}$ explodes as a function of n, while in **DoPeGDO** up to a fixed power of ϵ, the ranks of $\mathrm{mor}(A \to B)$ grow polynomially as a function of $|A|$ and $|B|$.

Compositions. In $\mathrm{mor}(A \to B)$,

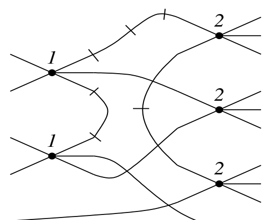
$$Q = \sum_{i\in A, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j,$$

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \dots$)



$E_1E_2 + E_1F_2G_1E_2$
$+E_1F_2G_1F_2G_1E_2$
$+\dots$
$= \sum_{r=0}^{\infty} E_1(F_2G_1)^r E_2$

where • $E = E_1(I - F_2G_1)^{-1}E_2$.
• $F = F_1 + E_1F_2(I - G_1F_2)^{-1}E_1^T$.
• $G = G_2 + E_2^T G_1(I - F_2G_1)^{-1}E_2$.
• $\omega = \omega_1\omega_2 \det(I - F_2G_1)^{-1}$.
• P is computed as the solution of a messy PDE or using "connected Feynman diagrams" (yet we're still in pure algebra!). Docility is preserved.



DoPeGDO Footnotes. Each variable has a "weight"$\in \{0, 1, 2\}$, and always $\mathrm{wt}\, z_i + \mathrm{wt}\,\zeta_i = 2$.

†1. Really, "weight-graded finite sets" $A = A_0 \sqcup A_1 \sqcup A_2$.

†2. Really, a power series in the weight-0 variables[†5].

†3. The weight of Q must be 2, so it decomposes as $Q = Q_{20}+Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series[†5].

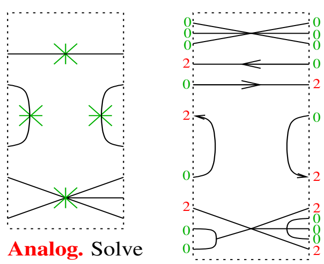
†4. Setting $\mathrm{wt}\,\epsilon = -2$, the weight of P is ≤ 2 (so the powers of the weight-0 variables are not constrained)[†5].

†5. In the knot-theoretic case, all weight-0 power series are rational functions of bounded degree in the exponentials of the weight-0 variables.

†6. There's also an obvious product

$$\mathrm{mor}(A_1 \to B_1)\times\mathrm{mor}(A_2 \to B_2) \to \mathrm{mor}(A_1\sqcup A_2 \to B_1\sqcup B_2).$$

Full DoPeGDO. Compute compositions in two phases:
• A 1-1 phase over the ring of power series in the weight-0 variables, in which the weight-2 variables are spectators.
• A (slightly modified) 2-0 phase over $\mathbb{Q}$, in which the weight-1 variables are spectators.

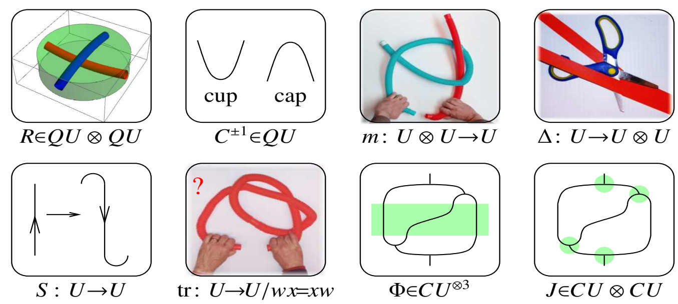


Analog. Solve $Ax = a,\ B(x)y = b$

Questions. • Are there QFT precedents for "two-step Gaussian integration"?
• In QFT, one saves even more by considering "one-particle-irreducible" diagrams and "effective actions". Does this mean anything here?
• Understanding $\mathrm{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$ seems like a good cause. Can you find other applications for the technology here?

$QU = \mathcal{U}_\hbar(sl^\epsilon_{2+}) = A\langle y, b, a, x\rangle[\![\hbar]\!]$ with $[a, x] = x$, $[b, y] = -\epsilon y$, $[a, b] = 0$, $[a, y] = -y$, $[b, x] = \epsilon x$, and $xy-qyx = (1-AB)/\hbar$, where $q = e^{\hbar\epsilon}$, $A = e^{-\hbar\epsilon a}$, and $B = e^{-\hbar b}$. Also $\Delta(y, b, a, x) = (y_1 + B_1y_2, b_1 + b_2, a_1 + a_2, x_1 + A_1x_2)$, $S(y, b, a, x) = (-B^{-1}y, -b, -a, -A^{-1}x)$, and $R = \sum \hbar^{j+k}y^kb^j \otimes a^jx^k/j![k]_q!$.

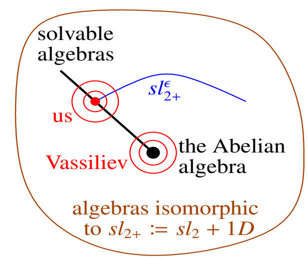
Theorem. Everything of value regrading $U = CU$ and/or its quantization $U = QU$ is **DoPeGDO**:

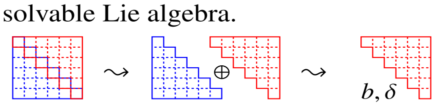


also Cartan's θ, the $\mathbb{D}$equantizator, and more, and all of their compositions.

Solvable Approximation. In sl_n, half is enough! Indeed $sl_n \oplus \mathfrak{a}_{n-1} = \mathcal{D}(◹, b, \delta)$. Now define $sl^\epsilon_{n+} := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon◺$, and $[◹, ◺] = ◺ + \epsilon◹$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1} = 0$ always yields a solvable Lie algebra.

4D Metrized Lie Algebras





$b(◹) = b\colon ◹ \otimes ◹ \to ◹$
$b(◺) \rightsquigarrow \delta\colon ◹ \to ◹ \otimes ◹$

Conclusion. There are lots of poly-time-computable well-behaved near-Alexander knot invariants: • They extend to tangles with appropriate multiplicative behaviour. • They have cabling and strand reversal formulas. ωεβ/akt
The invariant for $sl^\epsilon_{2+}/(\epsilon^2 = 0)$ (prior art: ωεβ/Ov) attains 2,883 distinct values on the 2,978 prime knots with $\leq$ 12 crossings. HOMFLY-PT and Khovanov homology together attain only 2,786 distinct values.

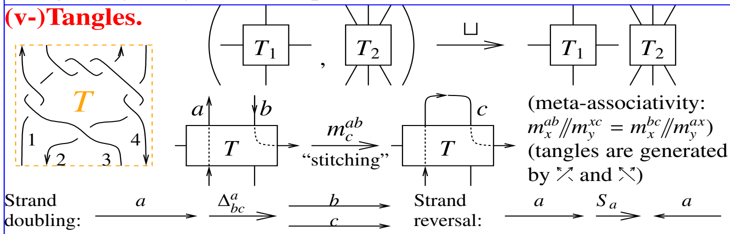
knot diag	n^t_k / $(\rho'_1)^+$	Alexander's ω^+	genus / ribbon; unknotting # / amphi?	$(\rho'_2)^+$
	0^a_1 / 0	1	0 / ✔; 0 / ✔	0
	3^a_1 / T	$T-1$	1 / ✘; 1 / ✘	$3T^3-12T^2+26T-38$
	4^a_1 / 0	$3-T$	1 / ✘; 1 / ✔	$T^4-3T^3-15T^2+74T-110$
	5^a_1 / $2T^3+3T$	T^2-T+1	2 / ✘; 2 / ✘	$5T^7-20T^6+55T^5-120T^4+217T^3-338T^2+450T-510$
	5^a_2 / $5T-4$	$2T-3$	1 / ✘; 1 / ✘	$-10T^4+120T^3-487T^2+1054T-1362$
	6^a_1 / $T-4$	$5-2T$	1 / ✔; 1 / ✘	$14T^4-16T^3-293T^2+1098T-1598$
	6^a_2 / T^3-4T^2+4T-4	$-T^2+3T-3$	2 / ✘; 1 / ✘	$3T^8-21T^7+49T^6+15T^5-433T^4+1543T^3-3431T^2+5482T-6410$
	6^a_3 / 0	T^2-3T+5	2 / ✘; 1 / ✔	$4T^8-33T^7+121T^6-203T^5-111T^4+1499T^3-4210T^2+7186T-8510$
	7^a_1 / $3T^5+5T^3+6T$	T^3-T^2+T-1	3 / ✘; 3 / ✘	$7T^{11}-28T^{10}+77T^9-168T^8+322T^7-560T^6+891T^5-1310T^4+1777T^3-2238T^2+2604T-2772$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Columbia-191125/

Algebraic Knot Theory

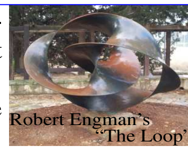
Abstract. This will be a very "light" talk: I will explain why about 13 years ago, in order to have a say on some problems in knot theory, I've set out to find tangle invariants with some nice compositional properties. In other talks in Sydney (ωεβ/talks) I have explained / will explain how such invariants were found - though they are yet to be explored and utilized.

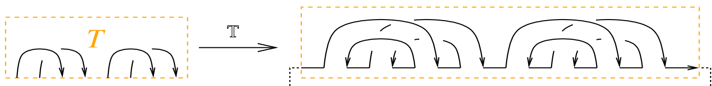
(v-)Tangles.

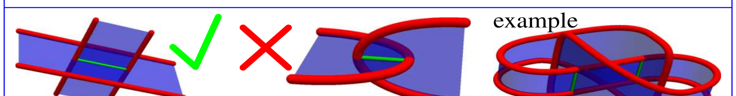


Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.

Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via


Robert Engman's "The Loop"





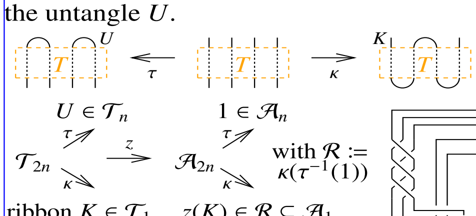
a ribbon singularity a clasp singularity

A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

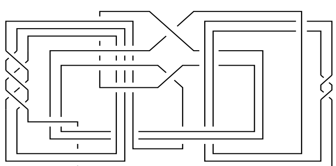
Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

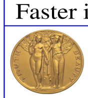
Theorem. K is ribbon iff it is κT for a tangle T for which τT is the untangle U.




Gompf, Scharlemann, Thompson [GST]



Faster is better, leaner is meaner!



The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{T_a : a \in S\})$:

$$\left({}_a{\nwarrow\!\!\nearrow}_b, {}_b{\nwarrow\!\!\nearrow}_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-T_a^{\pm 1} \\ b & 0 & T_a^{\pm1}\end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[T_a, T_b \to T_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta}\end{array}\right)$$

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

Strand Doubling and Reversal.

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow[\substack{\mu := T_a - 1\\ \nu := \alpha - \sigma_a \\ T_a \mapsto T_bT_c}]{q\Delta^a_{bc}} \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu \\ c & (T_c-1)\nu/\mu & (\alpha - \sigma_aT_a - \nu T_c)/\mu & (T_c-1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array}\right)$$

$$dS^a \Big\downarrow T_a \to T_a^{-1}$$

$$\left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha \end{array}\right)$$

Where σ assigns to every $a \in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left({}_a{\nwarrow\!\!\nearrow}_b, {}_b{\nwarrow\!\!\nearrow}_a\right) = (a \to 1, b \to t_a^{\pm 1})$, $\sigma(T_1 \sqcup T_2) = \sigma(T_1) \sqcup \sigma(T_2)$, and $\sigma /\!/ m^{ab}_c = (\sigma \setminus \{a,b\}) \cup (c \to \sigma_a\sigma_b)|_{t_a,t_b \to t_c}$.

Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

Implementation key idea: ωεβ/AlexDemo

$$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \textstyle\sum \alpha_{ab} t_a h_b)$$

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_,b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, φ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; φ ψ Ξ) = (∂_{t_a,h_a}λ ∂_{t_a,h_b}λ ∂_{t_a}λ; ∂_{t_b,h_a}λ ∂_{t_b,h_b}λ ∂_{t_b}λ; ∂_{h_a}λ ∂_{h_b}λ λ) /. (t | h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + α δ / μ  ϵ + δ θ / μ; φ + α ψ / μ  Ξ + ψ θ / μ).{h_c, 1}]
    /. {T_a → T_c, T_b → T_c} // ΓCollect];
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_{a_} :→ a, ∞];
   M = Outer[Factor[∂_{h_{#1} t_{#2}} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.(1  1 - T_a; 0  T_a).{h_a, h_b}];
Rm_{a_,b_} := Rp_{ab} /. T_a → 1 / T_a;
```

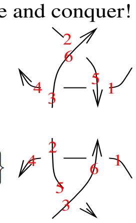
Meta-Associativity **Runs.**

```
ζ = Γ[ω, {t1, t2, t3, ts}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; φ1 φ2 φ3 Ξ).{h1, h2, h3, hs}];
(ζ // m_{12→1} // m_{13→1}) == (ζ // m_{23→2} // m_{12→1})
True
```

R3 . . . divide and conquer!

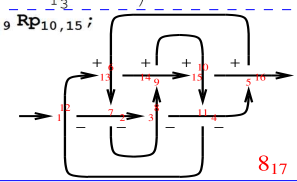
```
{Rm51 Rm62 Rp34 // m_{14→1} // m_{25→2} // m_{36→3},
  Rp61 Rm24 Rm35 // m_{14→1} // m_{25→2} // m_{36→3}}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}\right\}$$



```
z = Rm_{12,1} Rm_{27} Rm_{83} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15};
Do[z = z // m_{1k→1}, {k, 2, 16}];
z
```

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1 \\ t_1 & 1 \end{pmatrix}$$


8_{17}

Fact. Γ is better viewed as an invariant of a certain class of 2D knotted objects in $\mathbb{R}^4$ [BND, BN].

Fact. Γ is the "0-loop" part of an invariant that generalizes to "n-loops" (1D tangles only, see further talks and future publications with van der Veen).

Speculation. Stepping stones to categorification?


M. Polyak & T. Ohtsuki @ Heian Shrine, Kyoto

Ask me about geography vs. identity!

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: w-Knots and the Alexander Polynomial,* Alg. and Geom. Top. **16-2** (2016) 1063–1133, arXiv:1405.1956, ωεβ/WKO1.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[Vo] H. Vo, *Alexander Invariants of Tangles via Expansions,* University of Toronto Ph.D. thesis, ωεβ/Vo.

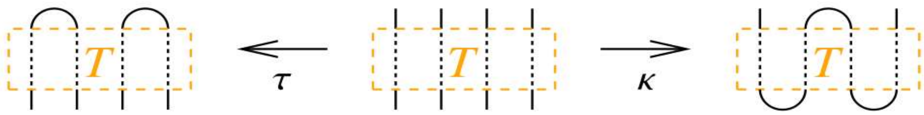
"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Macquarie-191016/

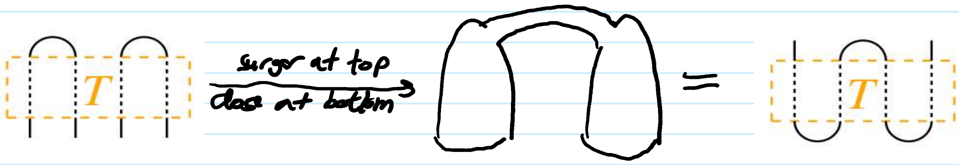
Proof of the Tangle Characterization of Ribbon Knots



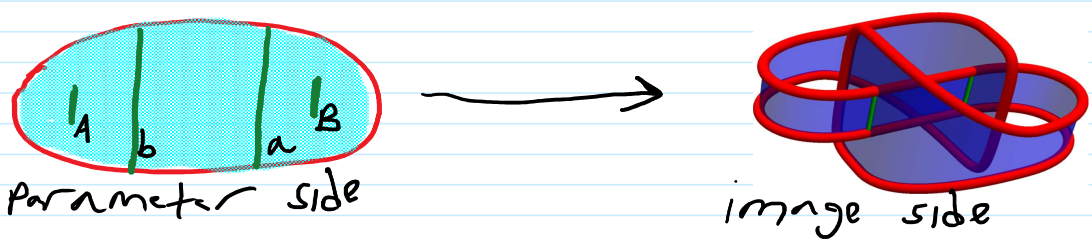
(tangle has 2n strands, here n=2)

Theorem. A knot K is ribbon iff there exists a tangle T whose τ closure is the untangle and whose κ closure is K.

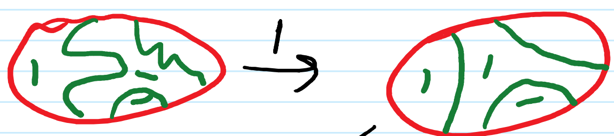
Proof. The backward ⟸ implication is easy:



For the forward implication, follow the following 5 steps:

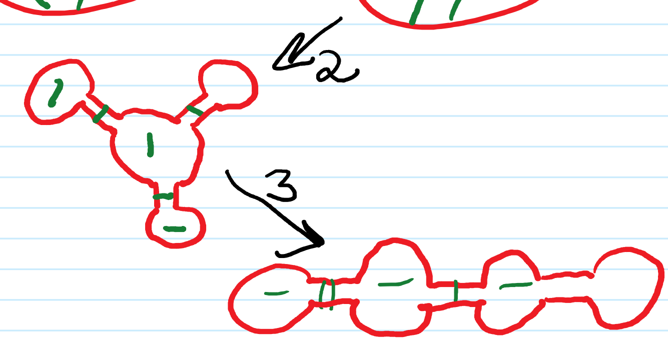


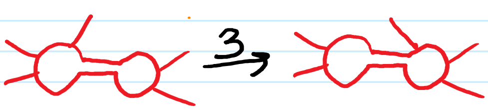
Step I: In-situ cosmetics.
At end: D is a tree of chord-and-arc polygons.



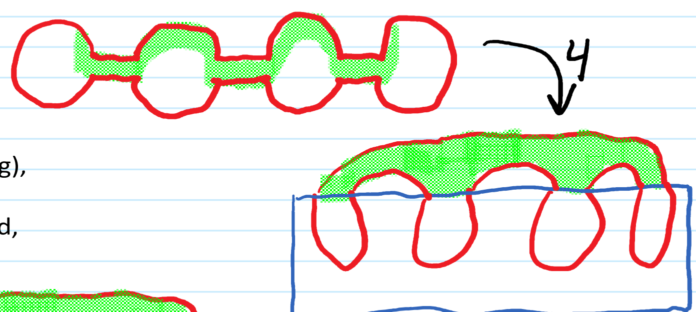
Step 2: Near-situ cosmetics.
At end: D is tree-band-sum of n unknotted disks.

Step 3: Slides.
At end: D is a linear-band-sum of n unknotted disks.

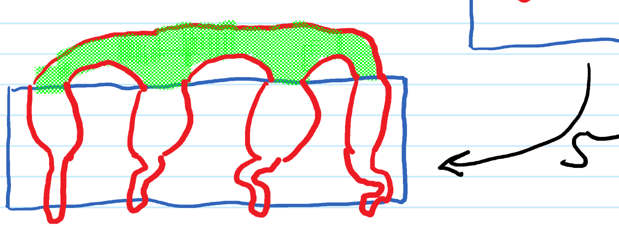




Step 4: Exposure!
The green domain is contractible - so it can be shrank, moved at will (with the blue membrane following along), and expanded back again.
At end: D has (n-1) exposed bridges which when turned, make D a union of n unknotted disks.



Step 5: Pulling bottom handles avoiding the obstacles.
At end: Theorem is proven.



Everything around sl^{ϵ}_{2+} is **DoPeGDO**. So what?

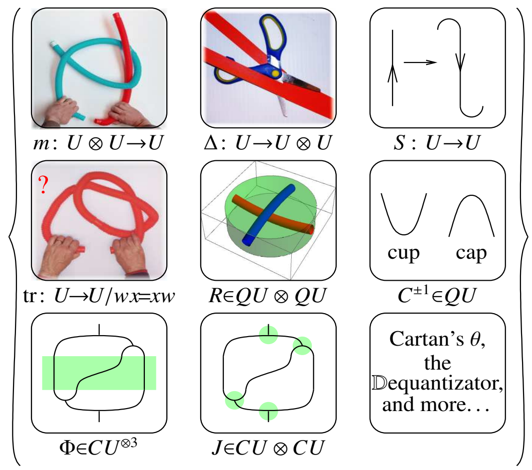
Thanks for inviting me to UCLA!
ωεβ:=http://drorbn.net/la19/
More at ωεβ/talks

Continues Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen [BV].

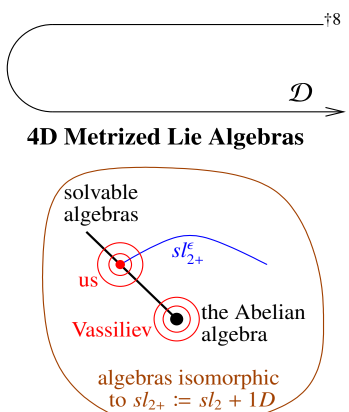
Abstract. I'll explain what "everything around" means: classical and quantum m, Δ, S, tr, R, C, and θ, as well as P, Φ, J, $\mathbb{D}$, and more, and all of their compositions. What **DoPeGDO** means: the category of Docile Perturbed Gaussian Differential Operators. And what sl^{ϵ}_{2+} means: a solvable approximation of the semi-simple Lie algebra sl_2.

Knot theorists should rejoice because all this leads to very powerful and well-behaved poly-time-computable knot invariants. Quantum algebraists should rejoice because it's a realistic playground for testing complicated equations and theories.

Conventions. 1. For a set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$.†1 2. Everything converges!



Less Abstract



DoPeGDO := The category with objects finite sets†2 and mor$(A\to B)$:

$$\{\mathcal{F} = \omega\exp(Q+P)\} \subset \mathbb{Q}[\![\zeta_A, z_B, \epsilon]\!]$$

Where: • ω is a scalar.†3 • Q is a "small" ϵ-free quadratic in $\zeta_A\cup z_B$.†4 • P is a "docile perturbation": $P=\sum_{k\geq 1}\epsilon^k P^{(k)}$, where $\deg P^{(k)}\leq 2k+2$.†5
• Compositions:†6

$$\mathcal{F}/\!/\mathcal{G} = \mathcal{G}\circ\mathcal{F} := \left(\mathcal{G}|_{\zeta_i\to\partial_{z_i}}\mathcal{F}\right)_{z_i=0} = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

Cool! $(V^*)^{\otimes\Sigma}\otimes V^{\otimes S}$ explodes; the ranks of quadratics and bounded-degree polynomials grow slowly!†7 Representation theory is over-rated!
Cool! How often do you see a computational toolbox so successful?

Our Algebras. Let $sl^{\epsilon}_{2+} := L\langle y,b,a,x\rangle$ subject to $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $[x,y]=\epsilon a+b$. So $t:=\epsilon a-b$ is central and if $\exists\epsilon^{-1}$, $sl^{\epsilon}_{2+}/\langle t\rangle\cong sl_2$. ωεβ/oa
U is either $CU = \mathcal{U}(sl^{\epsilon}_{2+})[\![\hbar]\!]$ or $QU = \mathcal{U}_\hbar(sl^{\epsilon}_{2+}) = A\langle y,b,a,x\rangle[\![\hbar]\!]$ with $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $xy-qyx=(1-AB)/\hbar$, where $q=e^{\hbar\epsilon}$, $A=e^{-\hbar\epsilon a}$, and $B=e^{-\hbar b}$. Set also $T=A^{-1}B=e^{\hbar t}$.
The Quantum Leap. Also decree that in QU,

$$\Delta(y,b,a,x) = (y_1+B_1y_2, b_1+b_2, a_1+a_2, x_1+A_1x_2),$$
$$S(y,b,a,x) = (-B^{-1}y, -b, -a, -A^{-1}x),$$

and $R=\sum\hbar^{j+k}y^kb^j\otimes a^jx^k/j![k]_q!$.

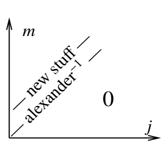
Mid-Talk Debts. • What is this good for in quantum algebra?
• In knot theory?
• How does the "inclusion" $\mathcal{D}\colon \mathrm{Hom}(U^{\otimes\Sigma}\to U^{\otimes S})\rightsquigarrow$ **DoPeGDO** work?
• Proofs that everything around sl^{ϵ}_{2+} really is **DoPeGDO**.
• Relations with prior art.
• The rest of the "compositions" story.

Theorem ([BG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing


Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

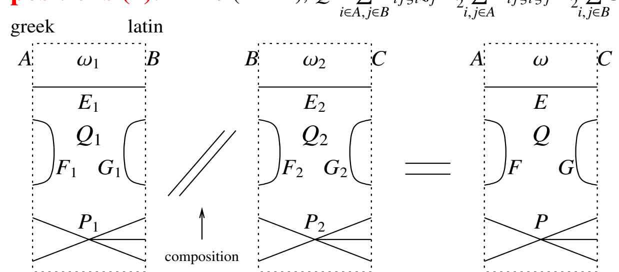
"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

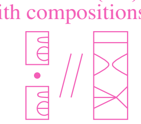
$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$

Compositions (1). In mor$(A\to B)$, $Q=\sum_{i\in A,j\in B}E_{ij}\zeta_iz_j+\frac12\sum_{i,j\in A}F_{ij}\zeta_i\zeta_j+\frac12\sum_{i,j\in B}G_{ij}z_iz_j$



Where • $E=E_1(I-F_2G_1)^{-1}E_2$.
• $F=F_1+E_1F_2(I-G_1F_2)^{-1}E_1^T$.
• $G=G_2+E_2^TG_1(I-F_2G_1)^{-1}E_2$.
• $\omega=\omega_1\omega_2\det(I-F_2G_1)^{-1}$.
Alexander lives here
• P is computed using "connected Feynman diagrams" or as the solution of a messy PDE (yet we're still in algebra!).

One abstraction level up from tangles!
{tangles} → {…}
with compositions:



DoPeGDO Footnotes. †1. Each variable has a "weight"$\in\{0,1,2\}$, and always $\mathrm{wt}\,z_i+\mathrm{wt}\,\zeta_i=2$.
†2. Really, "weight-graded finite sets" $A=A_0\sqcup A_1\sqcup A_2$.
†3. Really, a power series in the weight-0 variables†9.
†4. The weight of Q must be 2, so it decomposes as $Q=Q_{20}+Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series†9.
†5. Setting $\mathrm{wt}\,\epsilon=-2$, the weight of P is ≤ 2 (so the powers of the weight-0 variables are not constrained†9).
†6. There's also an obvious product
$$\mathrm{mor}(A_1\to B_1)\times\mathrm{mor}(A_2\to B_2)\to\mathrm{mor}(A_1\sqcup A_2\to B_1\sqcup B_2).$$
†7. That is, if the weight-0 variables are ignored. Otherwise more care is needed yet the conclusion remains.
†8. $\mathrm{Hom}(U^{\otimes\Sigma}\to U^{\otimes S})\rightsquigarrow\mathrm{mor}(\{\eta_i,\beta_i,\tau_i,\alpha_i,\xi_i\}_{i\in\Sigma}\to\{y_i,b_i,t_i,a_i,x_i\}_{i\in S})$, where $\mathrm{wt}(\eta_i,\xi_i,y_i,x_i)=1$ and $\mathrm{wt}(\beta_i,\tau_i,\alpha_i;b_i,t_i,a_i)=(2,2,0;0,0,2)$.
†9. For tangle invariants the wt-0 power series are always rational functions in the exponentials of the wt-0 variables (for knots: just one variable), with degrees bounded linearly by the crossing number.

Video and more: http://www.math.toronto.edu/~drorbn/Talks/CRM-1907, http://www.math.toronto.edu/~drorbn/Talks/UCLA-191101.

$\mathcal{D}\colon \mathrm{Hom}(U^{\otimes\Sigma}\to U^{\otimes S})\to\mathbb{Q}[\![\eta_\Sigma,\beta_\Sigma,\alpha_\Sigma,\xi_\Sigma,y_S,b_S,a_S,x_S]\!]$. The PBW theorem for CU (always in the *ybax* order), or its quantum analog for QU, say that if $U=CU$ or QU then $U^{\otimes S}$ is isomorphic as a vector space to $\mathbb{Q}[y_i,b_i,a_i,x_i]_{i\in S}[\![\hbar]\!]$; so it is enough to understand $\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$ for finite sets A and B.

Claim. $F\in\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])\xrightarrow[\mathcal{D}]{\sim}\mathbb{Q}[z_B][\![\zeta_A]\!]\ni\mathcal{F}$ via

$$\mathcal{D}(F):=\sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}F(z_A^n)=F\left(\mathbb{e}^{\sum_{a\in A}\zeta_a z_a}\right)=\mathcal{F},$$

$$\mathcal{D}^{-1}(\mathcal{F})(p)=\left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{F}\right)_{\zeta_a=0}\quad\text{for } p\in\mathbb{Q}[z_A].$$

Claim. Assuming convergence, if $F\in\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $G\in\mathrm{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, $\mathcal{F}=\mathcal{D}(F)$, and $\mathcal{G}=\mathcal{D}(G)$, then

$$\mathcal{D}(F/\!\!/G)=\left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

And so the title of the talk finally makes sense!

Example. $\mathcal{D}(id\colon U\to U)=\mathbb{e}^{\eta y+\beta b+\alpha a+\xi x}$.

Example. Let $c\Delta^i_{jk}\colon CU^{\otimes\{i\}}\to CU^{\otimes\{j,k\}}$ be the standard coproduct, given by $c\Delta^i_{jk}(y_i,b_i,a_i,x_i)=(y_j+y_k,b_j+b_k,a_j+a_k,x_j+x_k)$. Then

$$\mathcal{D}(c\Delta^i_{jk})=c\Delta^i_{jk}(\mathbb{e}^{\eta_i y_i+\beta_i b_i+\alpha_i a_i+\xi_i x_i})$$
$$=\mathbb{e}^{\eta_i(y_j+y_k)+\beta_i(b_j+b_k)+\alpha_i(a_j+a_k)+\xi_i(x_j+x_k)}.$$

Example. The standard commutative product m^{ij}_k of polynomials is given by $z_i,z_j\to z_k$. Hence $\mathcal{D}(m^{ij}_k)=m^{ij}_k(\mathbb{e}^{\zeta_i z_i+\zeta_j z_j})=\mathbb{e}^{(\zeta_i+\zeta_j)z_k}$.

$$\begin{array}{ccc}\mathbb{Q}[z]_i\otimes\mathbb{Q}[z]_j & \xrightarrow{m^{ij}_k} & \mathbb{Q}[z]_k\\ \| & & \|\\ \mathbb{Q}[z_i,z_j] & \xrightarrow{m^{ij}_k} & \mathbb{Q}[z_k]\end{array}$$

A real DoPeGDO Example. Let $cm^{ij}_k\colon CU_i\otimes CU_j\to CU_k$ be "classical multiplication" for sl^ϵ_{2+}, and let $\mathbb{O}_i\colon\mathbb{Q}[y_i,b_i,a_i,x_i]\to CU_i$ be the PBW ordering map.

$$\begin{array}{ccc}CU_i\otimes CU_j & \xrightarrow{cm^{ij}_k} & CU_k\\ \uparrow{\scriptstyle\mathbb{O}_{i,j}} & & \uparrow{\scriptstyle\mathbb{O}_k}\\ \mathbb{Q}[y_i,b_i,a_i,x_i,y_j,b_j,a_j,x_j] & & \mathbb{Q}[y_k,b_k,a_k,x_k]\end{array}$$

Claim. Let (all brawn and no brains)

$$\Lambda=\left(\eta_i+\frac{e^{-\alpha_i-\epsilon\beta_i}\eta_j}{1+\epsilon\eta_j\xi_i}\right)y_k+\left(\beta_i+\beta_j+\frac{\log\left(1+\epsilon\eta_j\xi_i\right)}{\epsilon}\right)b_k+$$
$$\left(\alpha_i+\alpha_j+\log\left(1+\epsilon\eta_j\xi_i\right)\right)a_k+\left(\frac{e^{-\alpha_j-\epsilon\beta_j}\xi_i}{1+\epsilon\eta_j\xi_i}+\xi_j\right)x_k$$

Then $\mathbb{e}^{\eta_i y_i+\beta_i b_i+\alpha_i a_i+\xi_i x_i+\eta_j y_j+\beta_j b_j+\alpha_j a_j+\xi_j x_j}/\!\!/\mathbb{O}_{i,j}/\!\!/cm^{ij}_k=\mathbb{e}^\Lambda/\!\!/\mathbb{O}_k$, and hence $\mathcal{D}(cm^{ij}_k)=\mathbb{e}^\Lambda$ and cm^{ij}_k is DoPeGDO.

Proof. We compute in a faithful 2D representation $z\mapsto\hat{z}$ of CU: (ωεβ/cm)

```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
{ŷ = (0 0; ϵ 0), b̂ = (0 0; 0 -ϵ), â = (1 0; 0 0), x̂ = (0 1; 0 0)};
HL /@ {â.x̂ - x̂.â == x̂, â.ŷ - ŷ.â == -ŷ, b̂.ŷ - ŷ.b̂ == -ϵ ŷ,
  b̂.x̂ - x̂.b̂ == ϵ x̂, x̂.ŷ - ŷ.x̂ == b̂ + ϵ â}
{True, True, True, True, True}
HL@Simplify@With[{𝔼 = MatrixExp},
  𝔼[ηi ŷ].𝔼[βi b̂].𝔼[αi â].𝔼[ξi x̂].𝔼[ηj ŷ].𝔼[βj b̂].
   𝔼[αj â].𝔼[ξj x̂] == 𝔼[ŷ ∂yk Λ].𝔼[b̂ ∂bk Λ].𝔼[â ∂ak Λ].
   𝔼[x̂ ∂xk Λ]]
True
Series[Λ, {ϵ, 0, 1}]
```

$$(a_k(\alpha_i+\alpha_j)+y_k(\eta_i+\mathbb{e}^{-\alpha_i}\eta_j)+b_k(\beta_i+\beta_j+\eta_j\xi_i)+x_k(\mathbb{e}^{-\alpha_j}\xi_i+\xi_j))+$$
$$\left(a_k\eta_j\xi_i-\frac{1}{2}b_k\eta_j^2\xi_i^2-\mathbb{e}^{-\alpha_i}y_k\eta_j(\beta_i+\eta_j\xi_i)-\mathbb{e}^{-\alpha_j}x_k\xi_i(\beta_j+\eta_j\xi_i)\right)\epsilon+O[\epsilon]^2$$

(Shame, but this technique fails for QU).

Claim. In QU, R is DoPeGDO.

Proof. Recall that with $q=\mathbb{e}^{\hbar\epsilon}$,

$$R=\sum\hbar^{j+k}y^kb^j\otimes a^jx^k/j![k]_q!=\mathbb{O}\left(\mathbb{e}^{\hbar b_1a_2}\mathbb{e}_q^{\hbar y_1x_2}\right).$$

Now expand $\mathbb{e}_q^{\hbar y_1x_2}$ in powers of ϵ using:

Faddeev's Formula (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q:=\frac{q^n-1}{q-1}$, with $[n]_q!:=[1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}_q^x:=\sum_{n\geq0}\frac{x^n}{[n]_q!}$, we have

$$\log\mathbb{e}_q^x=\sum_{k\geq1}\frac{(1-q)^kx^k}{k(1-q^k)}=x+\frac{(1-q)^2x^2}{2(1-q^2)}+\ldots.$$

Proof. We have that $\mathbb{e}_q^x=\frac{\mathbb{e}_q^{qx}-\mathbb{e}_q^x}{qx-x}$ ("the q-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx}=(1+(1-q)x)\mathbb{e}_q^x$, and

$$\log\mathbb{e}_q^{qx}=\log(1+(1-q)x)+\log\mathbb{e}_q^x.$$

Writing $\log\mathbb{e}_q^x=\sum_{k\geq1}a_kx^k$ and comparing powers of x, we get $q^ka_k=-(1-q)^k/k+a_k$, or $a_k=\frac{(1-q)^k}{k(1-q^k)}$. □

Compositions (2). Recall that with all indices i running in some set B,

$$\mathcal{F}/\!\!/\mathcal{G}=\left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}\overset{(1)}{=}\mathbb{e}^{\sum\partial_{z_i}\partial_{\zeta_i}}(\mathcal{F}\mathcal{G})\big|_{z_i=\zeta_i=0},$$

(1) Strictly speaking, true only when $B\cap(A\cup C)=\emptyset$.

so in general we wish to understand

$$[F\colon\mathcal{E}]_B:=\mathbb{e}^{\frac12\sum_{i,j\in B}F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E}\quad\text{and}\quad\langle F\colon\mathcal{E}\rangle_B:=[F\colon\mathcal{E}]_B|_{z_B\to0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F\colon\mathcal{E}\mathbb{e}^{\frac12\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B=\det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}\colon\mathcal{E}\right\rangle_B.$$

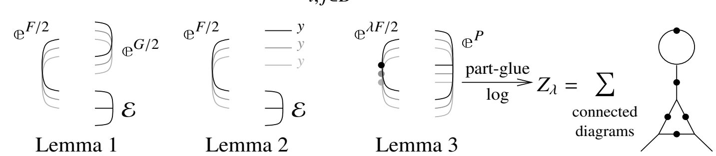
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F\colon\mathcal{E}\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B=\mathbb{e}^{\frac12\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F\colon\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda:=\log[\lambda F\colon\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0=P\quad\text{and}\quad\partial_\lambda Z_\lambda=\frac12\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda+(\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Complexity to ϵ^k, for an n-xing width w knot (by [LT], $w\in O(\sqrt{n})$), is $O(n^2w^{2k+2}\log n)=O(n^{k+3}\log n)$ integer operations.

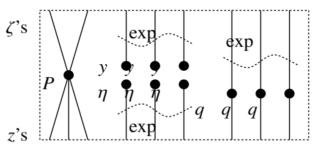
Video and more: http://www.math.toronto.edu/~drorbn/Talks/CRM-1907, http://www.math.toronto.edu/~drorbn/Talks/UCLA-191101.

A Partial To Do List.

- Understand tr and links.
- Implement Φ, J. Determine the appropriate wt-0 ground ring.
- Implement the "dequantizators".
- Understand denominators and get rid of them.
- Implement zipping at the log-level.
- Clean the program and make it efficient.
- Run it for all small knots and links, at $k = 3, 4$.
- Understand the centre and figure out how to read the output.
- Is the "+" really necessary in sl_{2+}^{ϵ}? Why?
- Extend to sl_3 and beyond.
- Describe a genus bound and a Seifert formula.
- Obtain "Gauss-Gassner formulas" (ωεβ/NCSU).
- Relate with the representation theory dogma, with Melvin-Morton-Rozansky and with Rozansky-Overbay.
- Understand the braid group representations that arise.
- Relate with finite-type (Vassiliev) invariants.
- Find a topological interpretation/foundation. The Garoufalidis - Rozansky "loop expansion" [GR]?
- Figure out the action of the Cartan automorphism.
- Understand "the subspace of classical knots / tangles".
- Disprove the ribbon-slice conjecture!
- Figure out the action of the Weyl group.
- Use to study "Ševera quantization".
- Do everything at the "arrow diagram" level of finite-type invariants of (rotational) virtual tangles.
- Find "internal" proofs of consistency.
- What else can you do with the "solvable approximations"?
- And with the "Gaussian compositions" technology?

Warning. Some implementation details match earlier versions of the theory.

The Zipping Theorem. If P has a finite ζ-degree and $\tilde{q}$ is the inverse matrix of $1-q$: $(\delta^i_j - q^i_j)\tilde{q}^j_k = \delta^i_k$, then



$$\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i + y_j\zeta^j + q^i_j z_i \zeta^j}\right\rangle$$
$$= |\tilde{q}|e^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k), \zeta^j + \eta^i\tilde{q}^j_i\right)\right\rangle.$$

The "Speedy" Engine

ωεβ/engine

Internal Utilities

Canonical Form:

```
CCF[ℰ_] :=
  PP_CCF@ExpandDenominator@
    ExpandNumerator@PP_Together@Together[PP_Exp[
        Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^CCF[x]]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := PP_CF@Module[
    {vs = Cases[ℰ, (y | b | t | a | x | η | β | τ | α | ξ)_, ∞] ⋃
        {y, b, t, a, x, η, β, τ, α, ξ}},
    Total[CoefficientRules[Expand[ℰ], vs] /.
      (ps_ → c_) :→ CCF[c] (Times @@ vs^ps)]
  ];
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

The Kronecker δ:

```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q}P$:

```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] × 𝔼[L2_, Q2_, P2_] :=
  𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

Variables and their duals:

```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, 𝓏};
{τ*, β*, η*, α*, ξ*, 𝓏*} = {t, b, y, a, x, z};
(u_i_)* := (u*)_i;
```

Upper to lower and lower to Upper:

```
U2l = {B_i_^p_. :→ e^(-p ℏ γ b_i), B^p_. :→ e^(-p ℏ γ b), T_i_^p_. :→ e^(p ℏ t_i),
    T^p_. :→ e^(p ℏ t), 𝒜_i_^p_. :→ e^(p γ α_i), 𝒜^p_. :→ e^(p γ α)};
l2U = {e^(c_. b_i_ + d_.) :→ B_i^(-c/(ℏ γ)) e^d, e^(c_. b + d_.) :→ B^(-c/(ℏ γ)) e^d,
    e^(c_. t_i_ + d_.) :→ T_i^(c/ℏ) e^d, e^(c_. t + d_.) :→ T^(c/ℏ) e^d,
    e^(c_. α_i_ + d_.) :→ 𝒜_i^(c/γ) e^d, e^(c_. α + d_.) :→ 𝒜^(c/γ) e^d,
    e^ℰ_ :→ e^(Expand@ℰ)};
```

Derivatives in the presence of exponentiated variables:

```
D_b[f_] := ∂_b f - ℏ γ B ∂_B f; D_b_i_[f_] := ∂_b_i f - ℏ γ B_i ∂_B_i f;
D_t[f_] := ∂_t f + ℏ T ∂_T f; D_t_i_[f_] := ∂_t_i f + ℏ T_i ∂_T_i f;
D_α[f_] := ∂_α f + γ 𝒜 ∂_𝒜 f; D_α_i_[f_] := ∂_α_i f + γ 𝒜_i ∂_𝒜_i f;
D_v_[f_] := ∂_v f; D_{v_,0}[f_] := f; D_{}[f_] := f;
D_{v_,n_Integer}[f_] := D_v[D_{v,n-1}[f]];
D_{l_List,ls___}[f_] := D_{ls}[D_l[f]];
```

Finite Zips:

```
collect[sd_SeriesData, 𝓏_] :=
  MapAt[collect[#, 𝓏] &, sd, 3];
collect[ℰ_, 𝓏_] := PP_Collect@Collect[ℰ, 𝓏];
Zip_{}[P_] := P;
Zip_𝓏s_[Ps_List] := Zip_𝓏s /@ Ps;
Zip_{𝓏_,𝓏s___}[P_] := PP_Zip[
  (collect[P // Zip_{𝓏s}, 𝓏] /. f_. 𝓏^d_. :→ (D_{𝓏*,d}[f])) /.
    𝓏* → 0 /. ((𝓏* /. {b → B, t → T, α → 𝒜}) → 1)]
```

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = e^{L+Q}P(\epsilon)$. Such zips regard the L variables as scalars.

```
QZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  PP_{QZip}@Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule, out},
    zs = Table[ζ*, {ζ, ζs}];
    c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
    ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0),
        {ζ, ζs}];
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = CF@Inverse@Table[Kδ_{z,ζ*} − ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ζrule = Thread[ζs → ζs + ηs.qt];
    CF /@ 𝔼[L, c + ηs.qt.ys,
      Det[qt] Zip_{ζs}[P /. (zrule ⋃ ζrule)]] ];
```

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = \mathrm{P}e^{L+Q}$. Such zips regard all of $\mathrm{P}e^{Q}$ as a single”P”. Here the z’s are b and α and the ζ’s are β and a.

```
LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  PP_{LZip}@Module[{ζ, z, zs, Zs, c, ys, ηs, lt, zrule,
      Zrule, ζrule, Q1, EEQ, EQ},
    zs = Table[ζ*, {ζ, ζs}];
    Zs = zs /. {b → B, t → T, α → 𝒜};
    c = L /. Alternatives @@ (ζs ⋃ zs) → 0 /.
      Alternatives @@ Zs → 1;
    ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z,ζ*} − ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    Zrule = Join[zrule,
      zrule /.
       r_Rule :> ((U = r[[1]] /. {b → B, t → T, α → 𝒜}) →
          (U /. U2l /. r //. l2U))];
    ζrule = Thread[ζs → ζs + ηs.lt];
    Q1 = Q /. (Zrule ⋃ ζrule);
    EEQ[ps___] :=
     EEQ[ps] =
      PP_{"EEQ"}@(CF[e^{-Q1} D_{Thread[{zs,{ps}}]}[e^{Q1}]] /.
         {Alternatives @@ zs → 0, Alternatives @@ Zs → 1});
    CF@𝔼[c + ηs.lt.ys,
      Q1 /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1},
      Det[lt]
       (Zip_{ζs}[(EQ @@ zs) (P /. (Zrule ⋃ ζrule))] /.
          Derivative[ps___][EQ][___] :> EEQ[ps] /.
         _EQ → 1) ] ];
```

```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := PP_B@Module[{n},
    Times[
      L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i},
        {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
     ] // LZip_{Join@@Table[{β_{n@i},τ_{n@i},a_{n@i}},{i,{is}}]} //
   QZip_{Join@@Table[{ξ_{n@i},y_{n@i}},{i,{is}}]} ];
B_{is__}[L_, R_] := B_{{is}}[L, R];
```

𝔼 morphisms with domain and range.

```
B_{is_List}[𝔼_{d1_→r1_}[L1_, Q1_, P1_], 𝔼_{d2_→r2_}[L2_, Q2_, P2_]] :=
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@
   B_{is}[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] // 𝔼_{d2_→r2_}[L2_, Q2_, P2_] :=
  B_{r1⋂d2}[𝔼_{d1→r1}[L1, Q1, P1], 𝔼_{d2→r2}[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] ≡ 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_{d1_→r1_}[L1_, Q1_, P1_] 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[L1, Q1, P1] × 𝔼[L2, Q2, P2]);
𝔼_{dr_}[L_, Q_, P_]_{$k_} := 𝔼_{dr} @@ 𝔼[L, Q, P]_{$k};
𝔼_[ℰ___][i_] := {ℰ}[[i]];
```

𝔼[Λ]

```
𝔼_{dr_}[Λ_] :=
 CF@Module[{L, Λ0 = Limit[Λ, ϵ → 0]},
   𝔼_{dr}[L = Λ0 /. (η | y | ξ | x)_ → 0, Λ0 − L, e^{Λ−Λ0}]_{$k} /. l2U]
```

Exponentials as needed.

Task. Define $\mathrm{Exp}_{m,i,k}[P]$ to compute $e^{\mathbb{O}(P)}$ to ϵ^{k} in the using the $m_{i,j\to i}$ multiplication, where P is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\lambda\,\mathbb{O}(P)} = \mathbb{O}\big(e^{\lambda P_0} F(\lambda)\big)$, then $F(\lambda = 0) = 1$ and we have:

$$\mathbb{O}\big(e^{\lambda P_0}(P_0\,F(\lambda) + \partial_\lambda F)\big) = \mathbb{O}\big(\partial_\lambda e^{\lambda P_0}\,F(\lambda)\big) =$$

$$\partial_\lambda \mathbb{O}\big(e^{\lambda P_0}\,F(\lambda)\big) = \partial_\lambda e^{\lambda\,\mathbb{O}(P)} = e^{\lambda\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\big(e^{\lambda P_0}\,F(\lambda)\big)\,\mathbb{O}(P)$$

This is a linear ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^{k}\,\varphi$ we find that $F_0 = 1$ and solve for φ.

```
(* Bug: The first line is valid only if 𝕆(e^{P0}) == e^{𝕆(P0)}. *)
Exp_{m_,i_,0}[P_] := Module[{LQ = Normal@P /. ϵ → 0},
   𝔼[LQ /. (x | y)_i → 0, LQ /. (b | a | t)_i → 0, 1] ];
```

```
Exp_{m_,i_,k_}[P_] := Block[{$k = k},
  Module[{P0, λ, φ, φs, F, j, rhs, eqn, pows, at0, atλ},
   P0 = Normal@P /. ϵ → 0;
   F = Normal@Last@Exp_{m,i,k-1}[λ P];
   While[
    rhs =
     m_{i,j→i}[
        𝔼_{{}→{i}}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0,
          F]_k sσ_{i→j}@𝔼_{{}→{i}}[0, 0, P]_k] // Last // Normal;
    eqn = CF[(∂_λ F) + P0 F − rhs];
    eqn =!= 0, (*do*)
    pows = First /@ CoefficientRules[eqn, {y_i, b_i, a_i, x_i}];
    F += Sum[ϵ^k φ_{js}[λ] Times @@ {y_i, b_i, a_i, x_i}^{js},
      {js, pows}];
    rhs =
     m_{i,j→i}[
        𝔼_{{}→{i}}[λ P0 /. (x | y)_i → 0, λ P0 /. (b | a | t)_i → 0,
          F]_k sσ_{i→j}@𝔼_{{}→{i}}[0, 0, P]_k] // Last // Normal;
    eqn = CF[(∂_λ F) + P0 F − rhs];
    φs = Table[φ_{js}[λ], {js, pows}];
    at0 = Table[φ_{js}[0] == 0, {js, pows}];
    atλ = (# == 0) & /@
      (pows /. CoefficientRules[eqn, {y_i, b_i, a_i, x_i}]);
    F = F /. DSolve[And @@ (at0 ⋃ atλ), φs, λ][[1]]
   ];
   𝔼_{{}→{i}}[P0 /. (x | y)_i → 0, P0 /. (b | a | t)_i → 0,
    F + O[ϵ]^{k+1} /. λ → 1]] ]
```

“Define” Code

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] :=
 Module[{SD, ii, jj, kk, isp, nis, nisp, sis},
  Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_{nisp,$k_Integer}, PP_Boot@Block[{i, j, k}, op_{isp,$k} = ℰ;
         op_{nis,$k}]]; 
      SD[op_isp, op_{{is},$k}]; SD[op_sis__, op_{sis}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

The Objects

ωεβ/objects

Symmetric Algebra Objects

```
sm_{i_,j_→k_} :=
  𝔼_{{i,j}→{k}}[b_k (β_i + β_j) + t_k (τ_i + τ_j) + a_k (α_i + α_j) +
    y_k (η_i + η_j) + x_k (ξ_i + ξ_j)];
sΔ_{i_→j_,k_} :=
  𝔼_{{i}→{j,k}}[β_i (b_j + b_k) + τ_i (t_j + t_k) + α_i (a_j + a_k) +
    η_i (y_j + y_k) + ξ_i (x_j + x_k)];
sS_{i_} := 𝔼_{{i}→{i}}[-β_i b_i - τ_i t_i - α_i a_i - η_i y_i - ξ_i x_i];
sϵ_{i_} := 𝔼_{{}→{i}}[0];
sη_{i_} := 𝔼_{{i}→{}}[0];
sσ_{i_→j_} := 𝔼_{{i}→{j}}[β_i b_j + τ_i t_j + α_i a_j + η_i y_j + ξ_i x_j];
sϒ_{i_→j_,k_,l_,m_} := 𝔼_{{i}→{j,k,l,m}}[β_i b_k + τ_i t_k + α_i a_l + η_i y_j + ξ_i x_m];
```

The CU Definitions

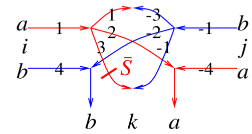
```
cΛ = (η_i + e^(-γ α_i - ϵ β_i) η_j / (1 + γ ϵ η_j ξ_i)) y_k + (β_i + β_j + Log[1 + γ ϵ η_j ξ_i]/ϵ) b_k +
   (α_i + α_j + Log[1 + γ ϵ η_j ξ_i]/γ) a_k + (e^(-γ α_j - ϵ β_j) ξ_i / (1 + γ ϵ η_j ξ_i) + ξ_j) x_k;
Define[cm_{i,j→k} = 𝔼_{{i,j}→{k}}[cΛ]]
Define[cσ_{i→j} = sσ_{i,j} /. τ_i → 0, cϵ_i = sϵ_i, cη_i = sη_i,
  cΔ_{i→j,k} = sΔ_{i→j,k},
  cS_i = sS_i // sϒ_{i→1,2,3,4} // cm_{4,3→i} // cm_{i,2→i} // cm_{i,1→i}];
```

Booting Up QU

```
Define[aσ_{i→j} = 𝔼_{{i}→{j}}[a_j α_i + x_j ξ_i],
 bσ_{i→j} = 𝔼_{{i}→{j}}[b_j β_i + y_j η_i]]
Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k + (𝒜_j^(-1) ξ_i + ξ_j) x_k],
 bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k + (η_i + e^(-ϵ β_i) η_j) y_k]]
Define[R_{i,j} = 𝔼_{{}→{i,j}}[ℏ a_j b_i + Sum_{k=1}^{$k+1} (1 - e^(γ ϵ ℏ))^k (ℏ y_i x_j)^k / (k (1 - e^(k γ ϵ ℏ)))],
 R̄_{i,j} = CF@𝔼_{{}→{i,j}}[-ℏ a_j b_i, -ℏ x_j y_i / B_i,
    1 + If[$k == 0, 0, (R̄_{{i,j},$k-1})_$k[3] -
       (((R̄_{{i,j},0})_$k R_{1,2} (R̄_{{3,4},$k-1})_$k) // (bm_{i,1→i} am_{j,2→j}) //
          (bm_{i,3→i} am_{j,4→j}))[3]]],
 P_{i,j} = 𝔼_{{i,j}→{}}[β_i α_j / ℏ, η_i ξ_j / ℏ,
   1 + If[$k == 0, 0, (P_{{i,j},$k-1})_$k[3] -
      (R_{1,2} // ((P_{{1,j},0})_$k (P_{{i,2},$k-1})_$k))[3]]]]
```

```
Define[aS_i = (aσ_{i→2} R̄_{1,i}) // P_{1,2},
 aS̄_i = 𝔼_{{i}→{i}}[-a_i α_i, -x_i 𝒜_i ξ_i,
   1 + If[$k == 0, 0, (aS̄_{{i},$k-1})_$k[3] -
      ((aS̄_{{i},0})_$k // aS_i // (aS̄_{{i},$k-1})_$k)[3]]]]
```

```
Define[bS_i = bσ_{i→1} R_{i,2} // aS_2 // P_{1,2},
 bS̄_i = bσ_{i→1} R_{i,2} // aS̄_2 // P_{1,2},
 aΔ_{i→j,k} = (R_{1,j} R_{2,k}) // bm_{1,2→3} // P_{3,i},
 bΔ_{i→j,k} = (R_{j,1} R_{k,2}) // am_{1,2→3} // P_{i,3}]
```

The Drinfel'd double: 


```
Define[
 dm_{i,j→k} =
  ((sϒ_{i→4,4,1,1} // aΔ_{1→1,2} // aΔ_{2→2,3} // aS̄_3)
     (sϒ_{j→-1,-1,-4,-4} // bΔ_{-1→-1,-2} // bΔ_{-2→-2,-3})) //
   (P_{-1,3} P_{-3,1} am_{2,-4→k} bm_{4,-2→k})]
```

```
Define[dσ_{i→j} = aσ_{i→j} bσ_{i→j},
 dϵ_i = sϵ_i, dη_i = sη_i,
 dS_i = sϒ_{i→1,1,2,2} // (bS̄_1 aS_2) // dm_{2,1→i},
 dS̄_i = sϒ_{i→1,1,2,2} // (bS_1 aS̄_2) // dm_{2,1→i},
 dΔ_{i→j,k} = (bΔ_{i→3,1} aΔ_{i→2,4}) // (dm_{3,4→k} dm_{1,2→j})]
```

```
Define[C_i = 𝔼_{{}→{i}}[0, 0, B_i^(1/2) e^(-ℏ ϵ a_i/2)]_$k,
 C̄_i = 𝔼_{{}→{i}}[0, 0, B_i^(-1/2) e^(ℏ ϵ a_i/2)]_$k,
 Kink_i = (R_{1,3} C̄_2) // dm_{1,2→1} // dm_{1,3→i},
 Kink̄_i = (R̄_{1,3} C_2) // dm_{1,2→1} // dm_{1,3→i}]
```

Note. $t == \epsilon a - \gamma b$ and $b == -t/\gamma + \epsilon a/\gamma$.

```
Define[b2t_i = 𝔼_{{i}→{i}}[α_i a_i + β_i (ϵ a_i - t_i) / γ + ξ_i x_i + η_i y_i],
 t2b_i = 𝔼_{{i}→{i}}[α_i a_i + τ_i (ϵ a_i - γ b_i) + ξ_i x_i + η_i y_i]]
```

The Knot Tensors

```
Define[kR_{i,j} = R_{i,j} // (b2t_i b2t_j) /. t_{i|j} → t,
 kR̄_{i,j} = R̄_{i,j} // (b2t_i b2t_j) /. {t_{i|j} → t, T_{i|j} → T},
 km_{i,j→k} = (t2b_i t2b_j) // dm_{i,j→k} //
   b2t_k /. {t_k → t, T_k → T, τ_{i|j} → 0},
 kC_i = C_i // b2t_i /. T_i → T,
 kC̄_i = C̄_i // b2t_i /. T_i → T,
 kKink_i = Kink_i // b2t_i /. {t_i → t, T_i → T},
 kKink̄_i = Kink̄_i // b2t_i /. {t_i → t, T_i → T}]
```

Some of the Atoms.

ωεβ/atoms

With $\mathcal{A}_i := e^{\alpha_i}$ and $B_i = e^{-b_i}$,

```
PP_ := Identity; $k = 1; ℏ = γ = 1;
Column[
 (# → (ℰ = ToExpression[#];
       Normal@Simplify[ℰ⟦1⟧ + ℰ⟦2⟧ + Log@ℰ⟦3⟧])) & /@
  {"dm_{i,j→k}", "dΔ_{i→j,k}", "dS_i", "R_{i,j}", "P_{i,j}"}]
```

Video and more: http://www.math.toronto.edu/~drorbn/Talks/CRM-1907, http://www.math.toronto.edu/~drorbn/Talks/UCLA-191101.

$$
\begin{aligned}
dm_{i,j\to k} &\to a_k\,(\alpha_i+\alpha_j)+b_k\,(\beta_i+\beta_j)+y_k\,\eta_i+\frac{y_k\,\eta_j}{\mathcal{A}_i}+\frac{x_k\,\xi_i}{\mathcal{A}_j}+\eta_j\,\xi_i-\\
&\quad B_k\,\eta_j\,\xi_i+\frac{1}{4\,\mathcal{A}_i\,\mathcal{A}_j}\in\Big(2\,y_k\,\eta_j\,\big(2\,x_k\,\xi_i+\mathcal{A}_j\,(-2\,\beta_i+(1-3\,B_k)\,\eta_j\,\xi_i)\big)+\\
&\qquad \mathcal{A}_i\,\xi_i\,\Big(x_k\,(-4\,\beta_j+2\,(1-3\,B_k)\,\eta_j\,\xi_i)+\\
&\qquad\quad \mathcal{A}_j\,\eta_j\,\big(4\,a_k\,B_k+(1-4\,B_k+3\,B_k^2)\,\eta_j\,\xi_i\big)\Big)\Big)+x_k\,\xi_j\\
d\triangle_{i\to j,k} &\to a_j\,\alpha_i+a_k\,\alpha_i+b_j\,\beta_i+b_k\,\beta_i+y_j\,\eta_i+B_j\,y_k\,\eta_i+\\
&\quad x_j\,\xi_i+x_k\,\xi_i+\frac{1}{2}\in\big(B_j\,y_j\,y_k\,\eta_i^2+x_k\,\xi_i\,(-2\,a_j+x_j\,\xi_i)\big)\\
dS_i &\to -a_i\,\alpha_i-b_i\,\beta_i-\frac{\mathcal{A}_i\,(y_i\,\eta_i+(-\eta_i+B_i\,(x_i+\eta_i))\,\xi_i)}{B_i}-\\
&\quad \frac{1}{4\,B_i^2}\in\mathcal{A}_i\,\Big(\mathcal{A}_i\,\eta_i^2\,(2\,y_i^2-6\,y_i\,\xi_i+3\,\xi_i^2)+B_i^2\,\xi_i\,\big(4\,a_i\,x_i+2\,x_i^2\,\mathcal{A}_i\,\xi_i+\\
&\qquad 2\,x_i\,(2\,\beta_i+\mathcal{A}_i\,\eta_i\,\xi_i)+\eta_i\,(-4+4\,\beta_i+\mathcal{A}_i\,\eta_i\,\xi_i)\big)+\\
&\qquad 2\,B_i\,\eta_i\,\big(y_i\,(-2+2\,\beta_i+2\,x_i\,\mathcal{A}_i\,\xi_i+\mathcal{A}_i\,\eta_i\,\xi_i)-\\
&\qquad\quad \xi_i\,(-2+2\,a_i+2\,\beta_i+3\,x_i\,\mathcal{A}_i\,\xi_i+2\,\mathcal{A}_i\,\eta_i\,\xi_i)\big)\Big)\\
R_{i,j} &\to a_j\,b_i+x_j\,y_i-\frac{1}{4}\in x_j^2\,y_i^2\\
P_{i,j} &\to \alpha_j\,\beta_i+\eta_i\,\xi_j+\frac{1}{4}\in\eta_i^2\,\xi_j^2
\end{aligned}
$$

A Quantum Algebra Example. ωεβ/qa

Proto-Proposition†0 (with Jesse Frohlich and Roland van der Veen, near [Ma, Proposition 1.7.3]). Let H be a finite dimensional Hopf algebra and let $U=H^{*cop}\otimes H$ be its Drinfel'd double, with R-matrix $R\in H^*\otimes H\subset U\otimes U$. Write $R^{\dagger 1}=\sum\rho_a\otimes r_a$, and let $\langle\cdot\mid\cdot\rangle\colon H^*\otimes H\to\mathbb{F}$ be the duality pairing. Then the functional $\int\in U^*$ defined by

$$\int\phi\otimes x:=\sum\langle\phi\rho_a^{\dagger 2}\mid x r_a^{\dagger 3}\rangle$$

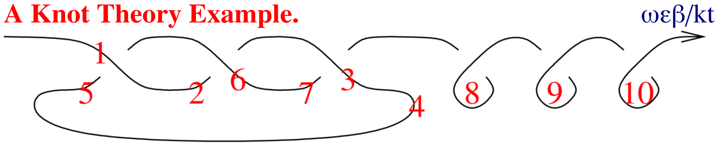
is a right†4 integral in U^*. (Meaning $\Delta^i_{jk}/\!\!/\int_j=\int_i/\!\!/\epsilon_k$ in $\mathrm{Hom}(U^{\otimes\{i\}}\to U^{\otimes\{k\}})$).

†0 A "proto-proposition" is something that will become a proposition once you figure out the correct statement. †1 Or did we want it to be $R/\!\!/S_1^2$? Or $R/\!\!/S_2^2$? †2 Or is it $\rho_a\phi$? †3 Or is it $r_a x$? †4 Or maybe "left"?

```
inp = E_{}→{1}[3 a_1 b_1, 5 x_1 y_1, 1] // dm_{i,1→i};
Table[
  HL@TrueQ[
    (inp // (sϒ_{i→1,1,2,2} RR) // BM // AM // P_{1,2} ) dϵ_j ≡
     (inp // ΔΔ // (sϒ_{i→1,1,2,2} RR) // BM // AM // P_{1,2})],
  {ΔΔ, {dΔ_{i→i,j}, dΔ_{i→j,i}}}, {AM, {dm_{2,4→2}, dm_{4,2→2}}},
  {BM, {dm_{1,3→1}, dm_{3,1→1}}},
  {RR, {R_{3,4}, R_{3,4} // dS_3 // dS_3, R_{3,4} // dS_4 // dS_4}}
 ] // MatrixForm
```

$$
\begin{pmatrix}
\begin{pmatrix}\text{False}&\text{False}&\text{False}\\ \text{False}&\text{False}&\text{False}\end{pmatrix} & \begin{pmatrix}\text{False}&\text{False}&\text{True}\\ \text{False}&\text{False}&\text{False}\end{pmatrix}\\
\begin{pmatrix}\text{False}&\text{False}&\text{False}\\ \text{False}&\text{False}&\text{True}\end{pmatrix} & \begin{pmatrix}\text{False}&\text{False}&\text{False}\\ \text{False}&\text{False}&\text{False}\end{pmatrix}
\end{pmatrix}
$$

A Knot Theory Example. ωεβ/kt




```
$k = 2;
Simplify[
  R_{1,5} R_{6,2} R_{3,7} C̄_4 Kink̄_8 Kink̄_9 Kink̄_10 // dm_{1,2→1} // dm_{1,3→1} //
      dm_{1,4→1} // dm_{1,5→1} // dm_{1,6→1} // dm_{1,7→1} // dm_{1,8→1} //
   dm_{1,9→1} // dm_{1,10→1}] /. v_1 :→ v
```

$$
\begin{aligned}
&\mathbb{E}_{\{\}\to\{1\}}\Big[0,\,0,\,\frac{B}{1-B+B^2}+\\
&\quad\frac{B\,\big(-B+2\,B^2+2\,B^4+a\,(-1+B-B^3+B^4)-2\,x\,y-B^3\,(3+2\,x\,y)\big)\in}{(1-B+B^2)^3}+\\
&\quad\frac{1}{2\,(1-B+B^2)^5}\\
&\quad B\,\Big(4\,B^8+a^2\,(1-B+B^2)^2\,(1+B-6\,B^2+B^3+B^4)+6\,B^5\,x^2\,y^2+\\
&\qquad 2\,x\,y\,(-2+3\,x\,y)-B^7\,(11+4\,x\,y)-2\,B^2\,(1+6\,x^2\,y^2)-\\
&\qquad 2\,B^4\,(1-2\,x\,y+6\,x^2\,y^2)+B\,(1+8\,x\,y+6\,x^2\,y^2)+\\
&\qquad B^6\,(6+8\,x\,y+6\,x^2\,y^2)+B^3\,(4+4\,x\,y+30\,x^2\,y^2)+\\
&\qquad 2\,a\,(1-B+B^2)\,\big(2\,B^6+2\,x\,y+8\,B^3\,(1+x\,y)-5\,B^2\,(1+2\,x\,y)-\\
&\qquad\quad 2\,B^5\,(1+2\,x\,y)-B^4\,(7+2\,x\,y)+B\,(2+4\,x\,y)\big)\Big)\in^2+O[\in]^3\Big]
\end{aligned}
$$

References.

[BG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BV] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* arXiv:1708.04853.

[Fa] L. Faddeev, *Modular Double of a Quantum Group,* arXiv:math/9912078.

[GR] S. Garoufalidis and L. Rozansky, *The Loop Expansion of the Kontsevich Integral, the Null-Move, and S-Equivalence,* arXiv:math.GT/0003187.

[LT] R. J. Lipton and R. E. Tarjan, *A Separator Theorem for Planar Graphs,* SIAM J. Appl. Math. **36-2** (1979) 177–189.

[Ma] S. Majid, *Foundations of Quantum Group Theory,* Cambridge University Press, 1995.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Qu] C. Quesne, *Jackson's q-Exponential as the Exponential of a Series,* arXiv:math-ph/0305003.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

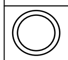
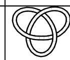
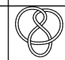
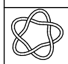
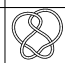
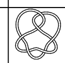
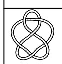
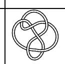
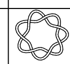
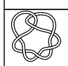
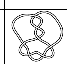
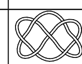
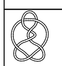
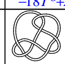
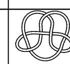
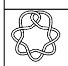
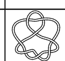
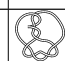
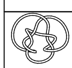
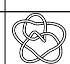
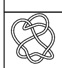
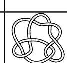
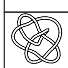
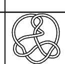
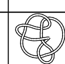
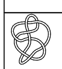
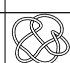
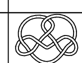
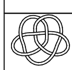
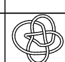
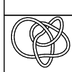
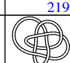
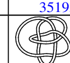
[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

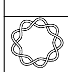
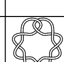
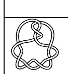
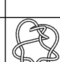
[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Za] D. Zagier, *The Dilogarithm Function,* in Cartier, Moussa, Julia, and Vanhove (eds) *Frontiers in Number Theory, Physics, and Geometry II.* Springer, Berlin, Heidelberg, and ωεβ/Za.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

Observations. • Separates the Rolfsen table; does better than Khovanov plus HOMFLY-PT on knots with up to 12 crossings (not tested beyond). • The degrees are bounded by the genus! • ρ_1 vanishes for amphichiral knots. • Has a chance of detecting non-ribbonness (ωεβ/akt)!

knot diag	n^t_k	Alexander's ω^+ $(\rho'_1)^+$	genus / ribbon unknotting # / amphi?	$(\rho'_2)^+$
	0^a_1	1 0	0 / ✔ 0 / ✔	0
	3^a_1	$T-1$ T	1 / ✘ 1 / ✘	$3T^3-12T^2+26T-38$
	4^a_1	$3-T$ 0	1 / ✘ 1 / ✔	$T^4-3T^3-15T^2+74T-110$
	5^a_1	T^2-T+1 $2T^3+3T$	2 / ✘ 2 / ✘	$5T^7-20T^6+55T^5-120T^4+217T^3-338T^2+450T-510$
	5^a_2	$2T-3$ $5T-4$	1 / ✘ 1 / ✘	$-10T^4+120T^3-487T^2+1054T-1362$
	6^a_1	$5-2T$ $T-4$	1 / ✔ 1 / ✘	$14T^4-16T^3-293T^2+1098T-1598$
	6^a_2	$-T^2+3T-3$ T^3-4T^2+4T-4	2 / ✘ 1 / ✘	$3T^8-21T^7+49T^6+15T^5-433T^4+1543T^3-3431T^2+5482T-6410$
	6^a_3	T^2-3T+5 0	2 / ✘ 1 / ✔	$4T^8-33T^7+121T^6-203T^5-111T^4+1499T^3-4210T^2+7186T-8510$
	7^a_1	T^3-T^2+T-1 $3T^5+5T^3+6T$	3 / ✘ 3 / ✘	$7T^{11}-28T^{10}+77T^9-168T^8+322T^7-560T^6+891T^5-1310T^4+1777T^3-2238T^2+2604T-2772$
	7^a_2	$3T-5$ $14T-16$	1 / ✘ 1 / ✘	$-129T^4+1177T^3-4421T^2+9226T-11718$
	7^a_3	$2T^2-3T+3$ $-9T^3+8T^2-16T+12$	2 / ✘ 2 / ✘	$-18T^8+208T^7-917T^6+2666T^5-6049T^4+11283T^3-17671T^2+23356T-25736$
	7^a_4	$4T-7$ $32-24T$	1 / ✘ 2 / ✘	$-352T^4+3616T^3-14378T^2+30700T-39188$
	7^a_5	$2T^2-4T+5$ $9T^3-16T^2+29T-28$	2 / ✘ 2 / ✘	$-18T^8+264T^7-1548T^6+5680T^5-15107T^4+31152T^3-51476T^2+69252T-76414$
	7^a_6	$-T^2+5T-7$ $T^3-8T^2+19T-20$	2 / ✘ 1 / ✘	$3T^8-35T^7+128T^6+105T^5-2610T^4+11225T^3-28031T^2+47186T-55946$
	7^a_7	T^2-5T+9 $8-3T$	2 / ✘ 1 / ✘	$4T^8-55T^7+310T^6-805T^5+86T^4+6349T^3-22686T^2+43610T-53622$
	8^a_1	$7-3T$ $5T-16$	1 / ✘ 1 / ✘	$42T^4+215T^3-2542T^2+7562T-10542$
	8^a_2	$-T^3+3T^2-3T+3$ $2T^5-8T^4+10T^3-12T^2+13T-12$	3 / ✘ 2 / ✘	$5T^{12}-39T^{11}+119T^{10}-139T^9-249T^8+1660T^7-4959T^6+11131T^5-20813T^4+33595T^3-47521T^2+58988T-63556$
	8^a_3	$9-4T$ 0	1 / ✘ 2 / ✔	$224T^4-224T^3-3910T^2+14100T-20364$
	8^a_4	$-2T^2+5T-5$ $3T^3-8T^2+6T-4$	2 / ✘ 2 / ✘	$54T^8-344T^7+865T^6-650T^5-2723T^4+12243T^3-28461T^2+45792T-53540$
	8^a_5	$-T^3+3T^2-4T+5$ $-2T^5+8T^4-13T^3+20T^2-22T+24$	3 / ✘ 2 / ✘	$5T^{12}-39T^{11}+128T^{10}-182T^9-274T^8+2476T^7-8642T^6+21517T^5-42924T^4+71719T^3-102448T^2+126480T-135628$
	8^a_6	$-2T^2+6T-7$ $5T^3-20T^2+28T-32$	2 / ✘ 2 / ✘	$38T^8-216T^7+112T^6+2880T^5-14787T^4+42444T^3-85415T^2+128406T-146916$
	8^a_7	T^3-3T^2+5T-5 $-T^5+4T^4-10T^3+12T^2-13T+12$	3 / ✘ 1 / ✘	$8T^{12}-75T^{11}+343T^{10}-979T^9+1821T^8-1782T^7-1623T^6+12083T^5-33001T^4+64599T^3-101194T^2+131404T-143216$
	8^a_8	$2T^2-6T+9$ $-T^3+4T^2-12T+16$	2 / ✔ 2 / ✘	$62T^8-504T^7+1736T^6-2408T^5-3717T^4+26492T^3-68493T^2+113418T-133180$
	8^a_9	$-T^3+3T^2-5T+7$ 0	3 / ✔ 1 / ✔	$9T^{12}-87T^{11}+417T^{10}-1305T^9+2858T^8-4134T^7+2114T^6+8285T^5-31925T^4+69235T^3-112773T^2+148508T-162396$
	8^a_{10}	T^3-3T^2+6T-7 $-T^5+4T^4-11T^3+16T^2-21T+20$	3 / ✘ 2 / ✘	$8T^{12}-75T^{11}+362T^{10}-1122T^9+2306T^8-2540T^7-2198T^6+18817T^5-54380T^4+110103T^3-175694T^2+230080T-251346$
	8^a_{11}	$-2T^2+7T-9$ $5T^3-24T^2+39T-44$	2 / ✘ 1 / ✘	$38T^8-264T^7+301T^6+3514T^5-21716T^4+68785T^3-146898T^2+227828T-263172$
	8^a_{12}	$T^2-7T+13$ 0	2 / ✘ 2 / ✔	$4T^8-77T^7+583T^6-1991T^5+987T^4+17311T^3-71802T^2+147914T-185846$
	8^a_{13}	$2T^2-7T+11$ $-T^3+4T^2-14T+20$	2 / ✘ 1 / ✘	$62T^8-592T^7+2351T^6-3918T^5-4235T^4+40079T^3-111533T^2+191500T-227432$
	8^a_{14}	$-2T^2+8T-11$ $5T^3-28T^2+57T-68$	2 / ✘ 1 / ✘	$38T^8-312T^7+444T^6+5096T^5-34777T^4+116368T^3-255750T^2+401632T-465478$
	8^a_{15}	$3T^2-8T+11$ $21T^3-64T^2+120T-140$	2 / ✘ 2 / ✘	$-123T^8+2128T^7-15241T^6+66120T^5-199999T^4+451912T^3-792414T^2+1101720T-1228222$
	8^a_{16}	T^3-4T^2+8T-9 $T^5-6T^4+17T^3-28T^2+35T-36$	3 / ✘ 2 / ✘	$8T^{12}-100T^{11}+598T^{10}-2205T^9+5292T^8-7164T^7-2380T^6+43100T^5-137314T^4+291750T^3-478742T^2+636488T-698666$
	8^a_{17}	$-T^3+4T^2-8T+11$ 0	3 / ✘ 1 / ✔	$9T^{12}-116T^{11}+722T^{10}-2843T^9+7656T^8-13668T^7+11117T^6+21968T^5-113086T^4+273778T^3-475622T^2+649064T-717954$
	8^a_{18}	$-T^3+5T^2-10T+13$ 0	3 / ✘ 2 / ✔	$9T^{12}-145T^{11}+1075T^{10}-4842T^9+14504T^8-28560T^7+27957T^6+35195T^5-225204T^4+573797T^3-1021641T^2+1411484T-1567262$
	8^n_{19}	T^3-T^2+1 $-3T^5-4T^2-3T$	3 / ✘ 3 / ✘	$7T^{11}-19T^{10}+6T^9+48T^8-52T^7-91T^6+211T^5+16T^4-431T^3+289T^2+536T-1060$
	8^n_{20}	T^2-2T+3 $4T-4$	2 / ✔ 1 / ✘	$4T^8-22T^7+66T^6-124T^5+52T^4+478T^3-1652T^2+3014T-3640$
	8^n_{21}	$-T^2+4T-5$ $T^3-8T^2+16T-20$	2 / ✘ 1 / ✘	$3T^8-28T^7+49T^6+352T^5-2489T^4+8164T^3-17530T^2+27092T-31226$

knot diag	n^t_k	Alexander's ω^+ $(\rho'_1)^+$	genus / ribbon unknotting # / amphi?	$(\rho'_2)^+$
	9^a_1	$T^4-T^3+T^2-T+1$ $4T^7+7T^5+9T^3+10T$	4 / ✘ 4 / ✘	$9T^{15}-36T^{14}+99T^{13}-216T^{12}+414T^{11}-720T^{10}+1170T^9-1800T^8+2630T^7-3662T^6+4853T^5-6142T^4+7423T^3-8572T^2+9420T-9780$
	9^a_2	$4T-7$ $30T-40$	1 / ✘ 1 / ✘	$-728T^4+6088T^3-21946T^2+44788T-56420$
	9^a_3	$2T^3-3T^2+3T-3$ $-13T^5+12T^4-25T^3+20T^2-32T+24$	3 / ✘ 3 / ✘	$-26T^{12}+296T^{11}-1311T^{10}+3838T^9-8867T^8+17613T^7-31407T^6+51061T^5-76085T^4+104297T^3-131779T^2+152840T-160976$
	9^a_4	$3T^2-5T+5$ $23T^3-28T^2+46T-44$	2 / ✘ 2 / ✘	$-219T^8+1999T^7-8389T^6+23799T^5-52835T^4+96723T^3-149121T^2+194698T-213338$

Video and more: http://www.math.toronto.edu/~drorbn/Talks/CRM-1907, http://www.math.toronto.edu/~drorbn/Talks/UCLA-191101.

Computation without Representation

Thanks for inviting me to Ohio!
ωεβ:=http://drorbn.net/o19/

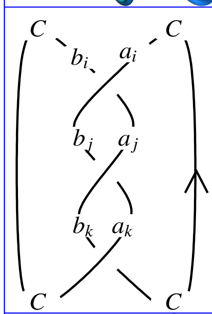
Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at [BV] and at ωεβ/talks.

Abstract. A major part of "quantum topology" is the definition and computation of various knot invariants by carrying out computations in quantum groups. Traditionally these computations are carried out "in a representation", but this is very slow: one has to use tensor powers of these representations, and the dimensions of powers grow exponentially fast.

In my talk, I will describe a direct method for carrying out such computations without having to choose a representation and explain why in many ways the results are better and faster. The two key points we use are a technique for composing infinite-order "perturbed Gaussian" differential operators, and the little-known fact that every semi-simple Lie algebra can be approximated by solvable Lie algebras, where computations are easier.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know. (experimental analysis @ωεβ/kiw)

Knotted Candies ωεβ/kc



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R=\sum a_i\otimes b_i\in U\otimes U \quad\text{and}\quad C\in U,$$

form $$Z=\sum_{i,j,k} Ca_ib_ja_kC^2b_ia_jb_kC.$$

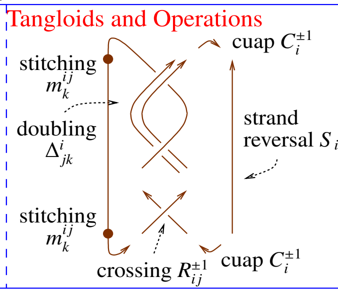
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

A Knot Theory Portfolio.

- Has operations $\sqcup$, m^{ij}_k, Δ^i_{jk}, S_i.
- All tangloids are generated by $R^{\pm1}$ and $C^{\pm1}$ (so "easy" to produce invariants).
- Makes some knot properties ("genus", "ribbon") become "definable".

Tangloids and Operations

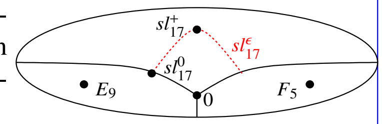


A "Quantum Group" Portfolio consists of a vector space U along with maps (and some axioms...)

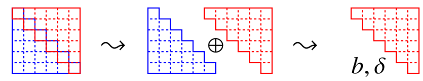
$$\begin{array}{ccccccc}
\sqcup\to\otimes & & S_i & & & & \\
\mathbb{Q}=\hat{U}^{\otimes\emptyset} & \xrightarrow{C_i} & \hat{U}^{\otimes\{i\}} & \underset{m^{jk}_i}{\overset{\Delta^i_{jk}}{\rightleftarrows}} & \hat{U}^{\otimes\{j,k\}} & \xleftarrow{R_{jk}} & \mathbb{Q}=\hat{U}^{\otimes\emptyset} \\
\uparrow\mathbb{O}_0 & & \uparrow\mathbb{O}_{y_ix_i\cdots} & & \uparrow\mathbb{O}_{y_jx_j\cdots\otimes y_kx_k\cdots} & & \uparrow\mathbb{O}_0 \\
\hat{\mathcal{S}}(\emptyset) & \xrightarrow{C_i} & \hat{\mathcal{S}}(B_i) & \underset{m^{jk}_i}{\overset{\Delta^i_{jk}}{\rightleftarrows}} & \hat{\mathcal{S}}(B_j,B_k) & \xleftarrow{R_{jk}} & \hat{\mathcal{S}}(\emptyset) \\
\sqcup\to\cdot & & S_i & & & &
\end{array}$$

Our Way. For certain algebras, work in a homomorphic polydimensional "space of formulas".

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes2}\otimes V$ is on the right. We care about $sl^k_{17}:=sl^\epsilon_{17}/(\epsilon^{k+1}=0)$.



Solvable Approximation. In gl_n, half is enough! Indeed $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(◹,b,\delta)$:



$b(◹)=b\colon ◹\otimes◹\to◹$
$b(◺)\rightsquigarrow\delta\colon ◹\to◹\otimes◹$

Now define $gl^\epsilon_n:=\mathcal{D}(◹,b,\epsilon\delta)$. Schematically, this is $[◹,◹]=◹$, $[◺,◺]=\epsilon◺$, and $[◹,◺]=◺+\epsilon◹$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1}=0$ always yields a solvable Lie algebra.

CU and *QU*. Starting from sl_2, get $CU_\epsilon=\langle y,a,x,t\rangle/([t,-]=0,\ [a,y]=-y,\ [a,x]=x,\ [x,y]=2\epsilon a-t)$. Quantize using standard tools (I'm sorry) and get $QU_\epsilon=\langle y,a,x,t\rangle/([t,-]=0,\ [a,y]=-y,\ [a,x]=x,\ xy-\mathbb{e}^{\hbar\epsilon}yx=(1-T\mathbb{e}^{-2\hbar\epsilon a})/\hbar)$.

PBW Bases. The U's we care about always have "Poincaré-Birkhoff-Witt" bases; there is some finite set $B=\{y,x,\ldots\}$ of "generators" and isomorphisms $\mathbb{O}_{y,x,\ldots}\colon\hat{\mathcal{S}}(B)\to U$ defined by "ordering monomials" to some fixed $y,x,\ldots$ order. The quantum group portfolio now becomes a "symmetric algebra" portfolio, or a "power series" portfolio.

Operations are Objects.

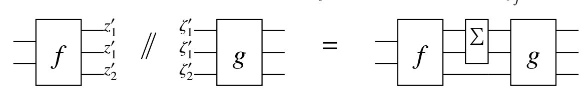
★ $$B^*:=\{z_i^*=\zeta_i\colon z_i\in B\},$$
$$\langle z_i^m,\zeta_i^n\rangle=\delta_{mn}n!,$$
$$\left\langle\prod z_i^{m_i},\prod\zeta_i^{n_i}\right\rangle=\prod\delta_{m_in_i}n_i!,$$
in general, for $f\in\mathcal{S}(z_i)$ and $g\in\mathcal{S}(\zeta_i)$,
$$\langle f,g\rangle=f(\partial_{\zeta_i})g\big|_{\zeta_i=0}=g(\partial_{z_i})f\big|_{z_i=0}.$$

$$\begin{array}{c}
f\in\mathrm{Hom}_{\mathbb{Q}}(S(B)\to S(B'))\\
\parallel\\
S(B)^*\otimes S(B')\\
\parallel\star\\
S(B^*)\otimes S(B')\\
\parallel\\
S(B^*\sqcup B')\\
\parallel\\
\tilde{f}\in\mathbb{Q}[\zeta_i,z_i']
\end{array}$$

The Composition Law. If

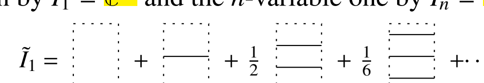
$$\mathcal{S}(B)\xrightarrow[\tilde{f}\in\mathbb{Q}[\![\zeta_i,z'_j]\!]]{f}\mathcal{S}(B')\xrightarrow[\tilde{g}\in\mathbb{Q}[\![\zeta'_j,z''_k]\!]]{g}\mathcal{S}(B'')$$

then $\widetilde{(f/\!\!/g)}=(\widetilde{g\circ f})=\left(\tilde{g}\big|_{\zeta'_j\to\partial_{z'_j}}\tilde{f}\right)_{z'_j=0}=\left(\tilde{f}\big|_{z'_j\to\partial_{\zeta'_j}}\tilde{g}\right)_{\zeta'_j=0}$:



Examples

1. The 1-variable identity map $I\colon\mathcal{S}(z)\to\mathcal{S}(z)$ is given by $\tilde{I}_1=\mathbb{e}^{z\zeta}$ and the n-variable one by $\tilde{I}_n=\mathbb{e}^{z_1\zeta_1+\cdots+z_n\zeta_n}$:



2. The "archetypal multiplication map $m^{ij}_k\colon\mathcal{S}(z_i,z_j)\to\mathcal{S}(z_k)$" has $\tilde{m}=\mathbb{e}^{z_k(\zeta_i+\zeta_j)}$.
3. The "archetypal coproduct $\Delta^i_{jk}\colon\mathcal{S}(z_i)\to\mathcal{S}(z_j,z_k)$", given by $z_i\to z_j+z_k$ or $\Delta z=z\otimes1+1\otimes z$, has $\tilde{\Delta}=\mathbb{e}^{(z_j+z_k)\zeta_i}$.
4. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1x_2}\in\mathcal{U}_q\otimes\mathcal{U}_q$. The "baby R-matrix" is $\tilde{R}=\mathbb{e}^{\hbar yx}\in\mathcal{S}(y,x)$.
5. The "Weyl form of the canonical commutation relations" states that if $[y,x]=tI$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y}=\mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. So with $SW_{xy}\circlearrowright\mathcal{S}(y,x)\underset{\mathbb{O}_{yx}}{\overset{\mathbb{O}_{xy}}{\rightrightarrows}}\mathcal{U}(y,x)$ we have $\widetilde{SW}_{xy}=\mathbb{e}^{\eta y+\xi x-\eta\xi t}$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ohio-1901

The Real Thing. In the algebra QU_ϵ, over $\mathbb{Q}[\![\hbar]\!]$ using the *yaxt* order, $T=\mathbb{e}^{\hbar t}$, $\bar{T}=T^{-1}$, $\mathcal{A}=\mathbb{e}^{\alpha}$, and $\bar{\mathcal{A}}=\mathcal{A}^{-1}$, we have

$$\tilde{R}_{ij}=\mathbb{e}^{\hbar(y_ix_j-t_ia_j)}\left(1+\epsilon\hbar\left(a_ia_j-\hbar^2y_i^2x_j^2/4\right)+O(\epsilon^2)\right)$$

in $\mathcal{S}(B_i,B_j)$, and in $\mathcal{S}(B_1^*,B_2^*,B)$ we have

$$\tilde{m}=\mathbb{e}^{(\alpha_1+\alpha_2)a+\eta_2\xi_1(1-T)/\hbar+(\xi_1\bar{\mathcal{A}}_2+\xi_2)x+(\eta_1+\eta_2\bar{\mathcal{A}}_1)y}\left(1+\epsilon\lambda+O(\epsilon^2)\right),$$

where $\lambda=2a\eta_2\xi_1T+\eta_2^2\xi_1^2\left(3T^2-4T+1\right)/4\hbar-\eta_2\xi_1^2(3T-1)x\bar{\mathcal{A}}_2/2$
$-\eta_2^2\xi_1(3T-1)y\bar{\mathcal{A}}_1/2+\eta_2\xi_1xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$.

Finally,

$$\tilde{\Delta}=\mathbb{e}^{\tau(t_1+t_1)+\eta(y_1+T_1y_2)+\alpha(a_1+a_2)+\xi(x_1+x_2)}\left(1+O(\epsilon)\right)\in\mathcal{S}(B^*,B_1,B_2),$$

and $\tilde{S}=\mathbb{e}^{-\tau t-\alpha a-\eta\xi(1-\bar{T})\mathcal{A}/\hbar-\bar{T}\eta y\mathcal{A}-\xi x\mathcal{A}}\left(1+O(\epsilon)\right)\in\mathcal{S}(B^*,B)$.

The Zipping Issue. (between unbound and bound lies half-zipped).

Zipping. If $P(\zeta^j,z_i)$ is a polynomial, or whenever otherwise convergent, set $\left\langle P(\zeta^j,z_i)\right\rangle_{(\zeta^j)}=P\left(\partial_{z_j},z_i\right)\Big|_{z_i=0}$. (E.g., if $P=\sum a_{nm}\zeta^nz^m$ then $\langle P\rangle_\zeta=\sum a_{nm}\,\partial_z^nz^m\Big|_{z=0}=\sum n!a_{nn}$).

The Zipping / Contraction Theorem. If $P=P(\zeta^j,z_i)$ has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P\mathbb{e}^{c+\eta^iz_i+y_j\zeta^j+q^i_jz_i\zeta^j}\right\rangle_{(\zeta^j)}=\det(\tilde{q})\mathbb{e}^{c+\eta^i\tilde{q}^k_iy_k}\left\langle P\,\middle|\,\begin{smallmatrix}\zeta^j\to\zeta^j+\eta^i\tilde{q}^j_i\\ z_i\to\tilde{q}^k_i(z_k+y_k)\end{smallmatrix}\right\rangle_{(\zeta^j)}$$

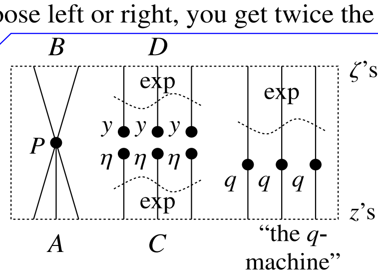
where $\tilde{q}$ is the inverse matrix of $1-q$: $(\delta^i_j-q^i_j)\tilde{q}^j_k=\delta^i_k$.

Exponential Reservoirs. The true Hilbert hotel is exp! Remove one x from an "exponential reservoir" of x's and you are left with the same exponential reservoir:

$$\mathbb{e}^x=\left[\ldots+\frac{xxxxx}{120}+\ldots\right]\xrightarrow{\partial_x}\left[\ldots+\frac{x\cancel{x}xxx}{120}+\ldots\right]=(\mathbb{e}^x)'=\mathbb{e}^x,$$

and if you let each element choose left or right, you get twice the same reservoir:

$$\mathbb{e}^x\xrightarrow{x\to x_l+x_r}\mathbb{e}^{x_l+x_r}=\mathbb{e}^{x_l}\mathbb{e}^{x_r}.$$



A Graphical Proof. Glue top to bottom on the right, in all possible ways. Several scenarios occur:

1. Start at A, go through the q-machine $k\geq 0$ times, stop at B. Get $\left\langle P\left(\zeta,\sum_{k\geq 0}q^kz\right)\right\rangle=\langle P(\zeta,\tilde{q}z)\rangle$.
2. Loop through the q-machine and swallow your own tail. Get $\exp\left(\sum q^k/k\right)=\exp(-\log(1-q))=\tilde{q}$. 3. …

By the reservoir splitting principle, these scenarios contribute multiplicatively. □

Implementation. ($\mathbb{E}[Q,P]$ means $\mathbb{e}^QP$) ωεβ/Zip

```
Zip_{ζs_List}@𝔼[Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   ζrule = Thread[ζs → ζs + ηs.qt];
   Simplify /@
    𝔼[c + ηs.qt.ys, Det[qt] Zip_{ζs}[P /. (zrule ⋃ ζrule)]] ];
```

Real Zipping is a minor mess, and is done in two phases:

	τa-phase		ξy-phase	
ζ-like variables	τ	a	ξ	y
z-like variables	t	α	x	η

Already at $\epsilon=0$ we get the best known formulas for the Alexander polynomial!

Generic Docility. A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form

$$\mathbb{e}^{q^{ij}z_iz_j}P=\mathbb{e}^{q^{ij}z_iz_j}\left(\sum_{k\geq 0}\epsilon^kP_k\right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k\leq 4k$.

Our Docility. In the case of QU_ϵ, all invariants and operations are of the form $\mathbb{e}^{L+Q}P$, where

- L is a quadratic of the form $\sum l_{z\zeta}z\zeta$, where z runs over $\{t_i,\alpha_i\}_{i\in S}$ and ζ over $\{\tau_i,a_i\}_{i\in S}$, with integer coefficients $l_{z\zeta}$.
- Q is a quadratic of the form $\sum q_{z\zeta}z\zeta$, where z runs over $\{x_i,\eta_i\}_{i\in S}$ and ζ over $\{\xi_i,y_i\}_{i\in S}$, with coefficients $q_{z\zeta}$ in the ring R_S of rational functions in $\{T_i,\mathcal{A}_i\}_{i\in S}$.
- P is a docile power series in $\{y_i,a_i,x_i,\eta_i,\xi_i\}_{i\in S}$ with coefficients in R_S, and where $\deg(y_i,a_i,x_i,\eta_i,\xi_i)=(1,2,1,1,1)$.

Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$. !!!!!

- At $\epsilon^2=0$ we get the Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] invariant, which is stronger than HOMFLY-PT polynomial and Khovanov homology taken together!
- In general, get "higher diagonals in the Melvin-Morton-Rozansky expansion of the coloured Jones polynomial" [MM, BNG], but why spoil something good?

References.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BV] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* arXiv:1708.04853.

[Fa] L. Faddeev, *Modular Double of a Quantum Group,* arXiv:math/9912078.

[GR] S. Garoufalidis and L. Rozansky, *The Loop Exapnsion of the Kontsevich Integral, the Null-Move, and S-Equivalence,* arXiv:math.GT/0003187.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

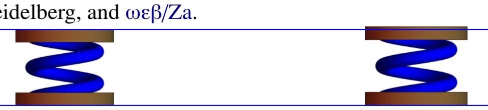
[Qu] C. Quesne, *Jackson's q-Exponential as the Exponential of a Series,* arXiv: math-ph/0305003.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Za] D. Zagier, *The Dilogarithm Function,* in Cartier, Moussa, Julia, and Vanhove (eds) *Frontiers in Number Theory, Physics, and Geometry II.* Springer, Berlin, Heidelberg, and ωεβ/Za.



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ohio-1901

The Algebras H and H^*. Let $q = \mathbb{e}^{\hbar\epsilon\gamma}$ and set $H = \langle a, x\rangle/([a, x] = \gamma x)$ with

$$A = \mathbb{e}^{-\hbar\epsilon a},\quad xA = qAx,\quad S_H(a, A, x) = (-a, A^{-1}, -A^{-1}x),$$
$$\Delta_H(a, A, x) = (a_1 + a_2, A_1A_2, x_1 + A_1x_2)$$

and dual $H^* = \langle b, y\rangle/([b, y] = -\epsilon y)$ with

$$B = \mathbb{e}^{-\hbar\gamma b},\quad By = qyB,\quad S_{H^*}(b, B, y) = (-b, B^{-1}, -yB^{-1}),$$
$$\Delta_{H^*}(b, B, y) = (b_1 + b_2, B_1B_2, y_1B_2 + y_2).$$

Pairing by $(a, x)^* = (b, y)$ ($\Rightarrow \langle B, A\rangle = q$) making $\langle y^lb^i, a^jx^k\rangle = \delta_{ij}\delta_{kl}j![k]_q!$ so $R = \sum \frac{y^kb^j\otimes a^jx^k}{j![k]_q!}$.

The Algebra QU. Using the Drinfel'd double procedure, $QU_{\gamma,\epsilon} := H^{*cop}\otimes H$ with $(\phi f)(\psi g) = \langle\psi_1S^{-1}f_3\rangle\langle\psi_3, f_1\rangle(\phi\psi_2)(f_2g)$ and $S(y, b, a, x) = (-B^{-1}y, -b, -a, -A^{-1}x)$,

$$\Delta(y, b, a, x) = (y_1 + y_2B_1, b_1 + b_2, a_1 + a_2, x_1 + A_1x_2).$$

Note also that $t := \epsilon a - \gamma b$ is central and can replace b, and set $QU = QU_\epsilon = QU_{1,\epsilon}$.

The 2D Lie Algebra. One may show* that if $[a, x] = \gamma x$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$. Ergo with

$$\begin{tikzcd} \mathcal{S}(a, x) \arrow[loop left, "SW_{ax}"] \arrow[r, shift left, "\mathbb{O}_{ax}"] \arrow[r, shift right, "\mathbb{O}_{xa}"'] & \mathcal{U}(a, x) \end{tikzcd}$$

we have $\widetilde{SW}_{ax} = \mathbb{e}^{\alpha a + \mathbb{e}^{-\gamma\alpha}\xi x}$.

$*$ Indeed $xa = (a - \gamma)x$ thus $xa^n = (a - \gamma)^nx$ thus $x\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha(a-\gamma)}x = \mathbb{e}^{-\gamma\alpha}\mathbb{e}^{\alpha a}x$ thus $x^n\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}(\mathbb{e}^{-\gamma\alpha})^nx^n$ thus $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$.

Faddeev's Formula (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n-1}{q-1}$, with $[n]_q! := [1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}_q^x := \sum_{n\geq 0}\frac{x^n}{[n]_q!}$, we have

$$\log\mathbb{e}_q^x = \sum_{k\geq 1}\frac{(1-q)^kx^k}{k(1-q^k)} = x + \frac{(1-q)^2x^2}{2(1-q^2)} + \dots.$$

Proof. We have that $\mathbb{e}_q^x = \frac{\mathbb{e}_q^{qx}-\mathbb{e}_q^x}{qx-x}$ ("the q-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx} = (1 + (1 - q)x)\mathbb{e}_q^x$, and

$$\log\mathbb{e}_q^{qx} = \log(1 + (1 - q)x) + \log\mathbb{e}_q^x.$$

Writing $\log\mathbb{e}_q^x = \sum_{k\geq 1}a_kx^k$ and comparing powers of x, we get $q^ka_k = -(1 - q)^k/k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. □

A Full Implementation. ωεβ/Full

Utilities

```
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
    Expand[ℰ] //. 𝕖^x_ 𝕖^y_ :> 𝕖^(x+y) /. 𝕖^x_ :> 𝕖^CF[x]];
```

```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] :=
  𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]_{$k_} := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z};
(u_{i_})* := (u*)_i;
```

```
collect[sd_SeriesData, ζ_] :=
  MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip_{}[P_] := P; Zip_{ζ_,ζs___}[P_] :=
 (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :> ∂_{ζ*,d} f) /. ζ* → 0
```

```
QZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule},
   zs = Table[ζ*, {ζ, ζs}];
   c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
   ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → CF[qt.(zs + ys)]];
   ζrule = Thread[ζs → ζs + ηs.qt];
   CF /@ 𝔼[L, c + ηs.qt.ys,
     Det[qt] Zip_{ζs}[P /. (zrule ⋃ ζrule)]] ];
```

```
U2l = {B_{i_}^p_. → 𝕖^(-p ℏ γ b_i), B^p_. → 𝕖^(-p ℏ γ b), T_{i_}^p_. → 𝕖^(p ℏ t_i),
   T^p_. → 𝕖^(p ℏ t), 𝒜_{i_}^p_. → 𝕖^(p γ α_i), 𝒜^p_. → 𝕖^(p γ α)};
l2U = {𝕖^(c_. b_{i_} + d_.) :> B_i^(-c/(ℏ γ)) 𝕖^d, 𝕖^(c_. b + d_.) :> B^(-c/(ℏ γ)) 𝕖^d,
   𝕖^(c_. t_{i_} + d_.) :> T_i^(c/ℏ) 𝕖^d, 𝕖^(c_. t + d_.) :> T^(c/ℏ) 𝕖^d,
   𝕖^(c_. α_{i_} + d_.) :> 𝒜_i^(c/γ) 𝕖^d, 𝕖^(c_. α + d_.) :> 𝒜^(c/γ) 𝕖^d,
   𝕖^ℰ_ :> 𝕖^(Expand@ℰ)};
```

```
LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Q2 = (Q1 = Q /. U2l /. zrule) /. Alternatives @@ zs → 0;
   CF /@ 𝔼[L2, Q2, Det[lt] 𝕖^(-L2-Q2)
      Zip_{ζs}[𝕖^(L1+Q1) (P /. U2l /. zrule)]] //. l2U ];
```

```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := Module[{n}, Times[
     L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
     R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
    ] // LZip_{Join@@Table[{β_{n@i}, τ_{n@i}, a_{n@i}}, {i, {is}}]} //
   QZip_{Join@@Table[{ξ_{n@i}, y_{n@i}}, {i, {is}}]} ];
B_{is___}[L_, R_] := B_{{is}}[L, R];
```

𝔼 morphisms with domain and range.

```
B_{is_List}[𝔼_{d1_→r1_}[L1_, Q1_, P1_], 𝔼_{d2_→r2_}[L2_, Q2_, P2_]] :=
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@
   B_{is}[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] // 𝔼_{d2_→r2_}[L2_, Q2_, P2_] :=
  B_{r1⋂d2}[𝔼_{d1→r1}[L1, Q1, P1], 𝔼_{d2→r2}[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] ≡ 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_{d1_→r1_}[L1_, Q1_, P1_] 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼_{d_→r_}[L_, Q_, P_]_{$k_} := 𝔼_{d→r} @@ 𝔼[L, Q, P]_{$k};
𝔼_[ℰ___][i_] := {ℰ}[[i]];
```

"Define" code

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
```

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ohio-1901

```
Define[op_is__ = ℰ_] :=
 Module[{SD, ii, jj, kk, isp, nis, nisp, sis},
  Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_{nisp,$k_Integer}, Block[{i, j, k}, op_{isp,$k} = ℰ;
        op_{nis,$k}]];
      SD[op_isp, op_{{is},$k}]; SD[op_{sis__}, op_{{sis}}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
    }] ]]
```

The Fundamental Tensors

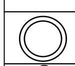
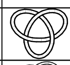
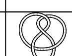
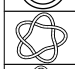
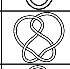
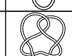
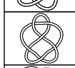
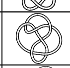
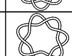
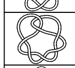
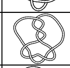
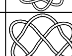
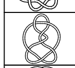
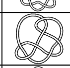
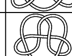
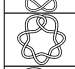
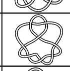
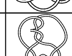
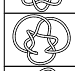
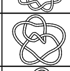
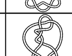
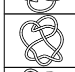
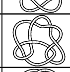
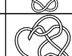
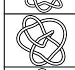
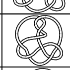
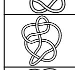
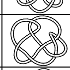
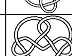
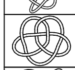
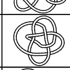
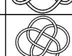
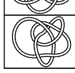
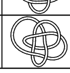
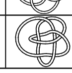
```
Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k, (e^{-γ α_j} ξ_i + ξ_j) x_k, 1]_{$k},
 bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k, (η_i + η_j) y_k, e^{(e^{-ε β_i} - 1) η_j y_k}]_{$k}]
Define[R_{i,j} =
 𝔼_{{}→{i,j}}[ℏ a_j b_i, ℏ x_j y_i, e^(Sum_{k=2}^{$k+1} (1 - e^{γ ε ℏ})^k (ℏ y_i x_j)^k / (k (1 - e^{k γ ε ℏ})))]_{$k}]
Define[R̄_{i,j} = 𝔼_{{}→{i,j}}[-ℏ a_j b_i, -ℏ x_j y_i / B_i,
   1 + If[$k == 0, 0, (R̄_{{i,j},$k-1})_{$k}[3] -
     (((R̄_{{i,j},0})_{$k} R_{1,2} (R̄_{{3,4},$k-1})_{$k}) // (bm_{i,1→i} am_{j,2→j}) //
       (bm_{i,3→i} am_{j,4→j}))[3]]],
 P_{i,j} = 𝔼_{{i,j}→{}}[β_i α_j / ℏ, η_i ξ_j / ℏ,
   1 + If[$k == 0, 0, (P_{{i,j},$k-1})_{$k}[3] -
     (R_{1,2} // ((P_{{1,j},0})_{$k} (P_{{i,2},$k-1})_{$k}))[3]]]]
Define[aS_j = R̄_{i,j} ~ B_i ~ P_{i,j},
 aS̄_i = 𝔼_{{i}→{i}}[-a_i α_i, -x_i 𝒜_i ξ_i,
   1 + If[$k == 0, 0, (aS̄_{{i},$k-1})_{$k}[3] -
     ((aS̄_{{i},0})_{$k} ~ B_i ~ aS_i ~ B_i ~ (aS̄_{{i},$k-1})_{$k})[3]]]]
Define[bS_i = R_{i,1} ~ B_1 ~ aS_1 ~ B_1 ~ P_{i,1},
 bS̄_i = R_{i,1} ~ B_1 ~ aS̄_1 ~ B_1 ~ P_{i,1},
 aΔ_{i→j,k} = (R_{1,j} R_{2,k}) // bm_{1,2→3} // P_{3,i},
 bΔ_{i→j,k} = (R_{j,1} R_{k,2}) // am_{1,2→3} // P_{i,3}]
```

```
Define[
 dm_{i,j→k} =
  (𝔼_{{i,j}→{i,j}}[β_i b_i + α_j a_j, η_i y_i + ξ_j x_j, 1]
     (aΔ_{i→1,2} // aΔ_{2→2,3} // aS̄_3) (bΔ_{j→-1,-2} // bΔ_{-2→-2,-3})) //
   (P_{-1,3} P_{-3,1} am_{2,j→k} bm_{i,-2→k}),
 dS_i = 𝔼_{{i}→{1,2}}[β_i b_1 + α_i a_2, η_i y_1 + ξ_i x_2, 1] // (bS̄_1 aS_2) //
   dm_{2,1→i},
 dΔ_{i→j,k} = (bΔ_{i→3,1} aΔ_{i→2,4}) // (dm_{3,4→k} dm_{1,2→j})]
Define[C_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/2} e^{-ℏ ε a_i/2}]_{$k},
 C̄_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/2} e^{ℏ ε a_i/2}]_{$k},
 Kink_i = (R_{1,3} C̄_2) // dm_{1,2→1} // dm_{1,3→i},
 Kink̄_i = (R̄_{1,3} C_2) // dm_{1,2→1} // dm_{1,3→i}]
Define[
 b2t_i = 𝔼_{{i}→{i}}[α_i a_i - β_i t_i / γ, ξ_i x_i + η_i y_i, e^{ε β_i a_i/γ}]_{$k},
 t2b_i = 𝔼_{{i}→{i}}[α_i a_i - τ_i γ b_i, ξ_i x_i + η_i y_i, e^{ε τ_i a_i}]_{$k}]
Define[kR_{i,j} = R_{i,j} // (b2t_i b2t_j) /. t_{i|j} → t,
 kR̄_{i,j} = R̄_{i,j} // (b2t_i b2t_j) /. {t_{i|j} → t, T_{i|j} → T},
 km_{i,j→k} = (t2b_i t2b_j) // dm_{i,j→k} //
   b2t_k /. {t_k → t, T_k → T, τ_{i|j} → 0},
 kC_i = C_i // b2t_i /. T_i → T, kC̄_i = C̄_i // b2t_i /. T_i → T,
 kKink_i = Kink_i // b2t_i /. {t_i → t, T_i → T},
 kKink̄_i = Kink̄_i // b2t_i /. {t_i → t, T_i → T}]
```

The Trefoil

```
$k = 2; Z = kR_{1,5} kR_{6,2} kR_{3,7} kC̄_4 kKink̄_8 kKink̄_9 kKink̄_10;
Do[Z = Z ~ B_{1,r} ~ km_{1,r→1}, {r, 2, 10}];
Simplify /@ Z /. v_1 :> v
```

$$\mathbb{E}_{\{\}\to\{1\}}\Big[0,\,0,\,\frac{T}{1-T+T^2}+\frac{1}{(1-T+T^2)^3}T\hbar\,\big(2a\,(-1+T-T^3+T^4)+T\,(-1+2T-3T^2+2T^3)\,\gamma-2\,(1+T^3)\,x\,y\,\gamma\,\hbar\big)\,\epsilon+\frac{1}{2\,(1-T+T^2)^5}T\hbar^2\,\big(4a^2\,(1-T+T^2)^2\,(1+T-6T^2+T^3+T^4)+4a\,(1-T+T^2)\,\gamma\,\big(T\,(2-5T+8T^2-7T^3-2T^4+2T^5)-2\,(-1-2T+5T^2-4T^3+T^4+2T^5)\,x\,y\,\hbar\big)+\gamma^2\,\big(T\,(1-2T+4T^2-2T^3+6T^5-11T^6+4T^7)+4\,(-1+2T+T^3+T^4+2T^6-T^7)\,x\,y\,\hbar+6\,(1-T+T^2)^2\,(1+3T+T^2)\,x^2\,y^2\,\hbar^2\big)\big)\,\epsilon^2+O[\epsilon]^3\Big]$$

diagram	n^t_k Today's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting # / amphi?	diagram	n^t_k Today's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting # / amphi?	diagram	n^t_k Today's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting # / amphi?
	0^a_1 0	1	0 / ✔ 0 / ✔		3^a_1 t	$t-1$	1 / ✘ 1 / ✘		4^a_1 0	$3-t$	1 / ✘ 1 / ✔
	5^a_1 $2t^3+3t$	t^2-t+1	2 / ✘ 2 / ✘		5^a_2 $5t-4$	$2t-3$	1 / ✘ 1 / ✘		6^a_1 $t-4$	$5-2t$	1 / ✔ 1 / ✘
	6^a_2 t^3-4t^2+4t-4	$-t^2+3t-3$	2 / ✘ 1 / ✘		6^a_3 0	t^2-3t+5	2 / ✘ 1 / ✔		7^a_1 $3t^5+5t^3+6t$	t^3-t^2+t-1	3 / ✘ 3 / ✘
	7^a_2 $14t-16$	$3t-5$	1 / ✘ 1 / ✘		7^a_3 $-9t^3+8t^2-16t+12$	$2t^2-3t+3$	2 / ✘ 2 / ✘		7^a_4 $32-24t$	$4t-7$	1 / ✘ 2 / ✘
	7^a_5 $9t^3-16t^2+29t-28$	$2t^2-4t+5$	2 / ✘ 2 / ✘		7^a_6 $t^3-8t^2+19t-20$	$-t^2+5t-7$	2 / ✘ 1 / ✘		7^a_7 $8-3t$	t^2-5t+9	2 / ✘ 1 / ✘
	8^a_1 $5t-16$	$7-3t$	1 / ✘ 1 / ✘		8^a_2 $2t^5-8t^4+10t^3-12t^2+13t-12$	$-t^3+3t^2-3t+3$	3 / ✘ 2 / ✘		8^a_3 0	$9-4t$	1 / ✘ 2 / ✔
	8^a_4 $3t^3-8t^2+6t-4$	$-2t^2+5t-5$	2 / ✘ 2 / ✘		8^a_5 $-2t^5+8t^4-13t^3+20t^2-22t+24$	$-t^3+3t^2-4t+5$	3 / ✘ 2 / ✘		8^a_6 $5t^3-20t^2+28t-32$	$-2t^2+6t-7$	2 / ✘ 2 / ✘
	8^a_7 $-t^5+4t^4-10t^3+12t^2-13t+12$	t^3-3t^2+5t-5	3 / ✘ 1 / ✘		8^a_8 $-t^3+4t^2-12t+16$	$2t^2-6t+9$	2 / ✔ 2 / ✘		8^a_9 0	$-t^3+3t^2-5t+7$	3 / ✔ 1 / ✔
	8^a_{10} $-t^5+4t^4-11t^3+16t^2-21t+20$	t^3-3t^2+6t-7	3 / ✘ 2 / ✘		8^a_{11} $5t^3-24t^2+39t-44$	$-2t^2+7t-9$	2 / ✘ 1 / ✘		8^a_{12} 0	$t^2-7t+13$	2 / ✘ 2 / ✔
	8^a_{13} $-t^3+4t^2-14t+20$	$2t^2-7t+11$	2 / ✘ 1 / ✘		8^a_{14} $5t^3-28t^2+57t-68$	$-2t^2+8t-11$	2 / ✘ 1 / ✘		8^a_{15} $21t^3-64t^2+120t-140$	$3t^2-8t+11$	2 / ✘ 2 / ✘
	8^a_{16} $t^5-6t^4+17t^3-28t^2+35t-36$	t^3-4t^2+8t-9	3 / ✘ 2 / ✘		8^a_{17} 0	$-t^3+4t^2-8t+11$	3 / ✘ 1 / ✔		8^a_{18} 0	$-t^3+5t^2-10t+13$	3 / ✘ 2 / ✔
	8^n_{19} $-3t^5-4t^2-3t$	t^3-t^2+1	3 / ✘ 3 / ✘		8^n_{20} $4t-4$	t^2-2t+3	2 / ✔ 1 / ✘		8^n_{21} $t^3-8t^2+16t-20$	$-t^2+4t-5$	2 / ✘ 1 / ✘

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ohio-1901

Do Not Turn Over Until Instructed

Dror Bar-Natan: Talks: MAASeaway-1810:

Thanks for inviting me to the fall 2018 MAA Seaway Section meeting!

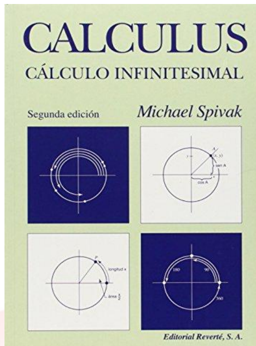
My Favourite First-Year Analysis Theorem

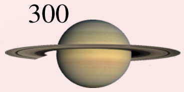
Handout, video, links at ωεβ:=http://drorbn.net/maa18/

Abstract. Whatever it may be, it should say something useful and exciting and it should not be *about* rigour, yet it should *demand* rigour. You can't guess. You probably think it the dreariest. You are wrong.

Contents Ⓢ

Several excerpts here are from Spivak's "Calculus" Ⓢ. I believe they fall under "fair use".





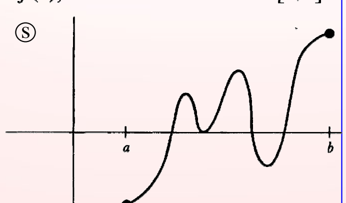
for every $\varepsilon > 0$ there is $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Ⓢ

If f and g are continuous at a, then

(1) $f + g$ is continuous at a,
(2) $f \cdot g$ is continuous at a.

6 Continuous Functions.

If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some x in $[a, b]$ such that $f(x) = 0$. Ⓢ



7 Three Hard Theorems.

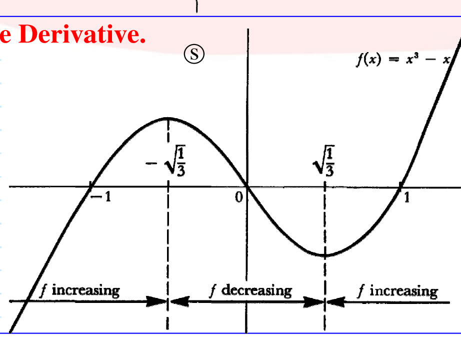
11 Significance of the Derivative. Ⓢ

$y = x^3 - x$

$y' = 3x^2 - 1$

$= (\sqrt{3}x+1)(\sqrt{3}x-1)$

$= \begin{cases} >0 & x > \sqrt{1/3} \\ <0 & -\sqrt{1/3} < x < \sqrt{1/3} \\ >0 & x < -\sqrt{1/3} \end{cases}$



14 The Fundamental Theorem of Calculus.

If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

Ⓢ
$$\int_a^b f = g(b) - g(a).$$

Tweets **Tweets & replies** ***16 π is Irrational.**

Dror Bar-Natan @drorbarnatan · 2 Apr 2013
π=a/b, f(x)=x^n(a-bx)^n/n!, n large => 0<V=∫(0,π)f(x)sin(x)dx<1. Repeated integration by parts & f(x)=f(π-x) => V∈Z. So π is irrational.

1

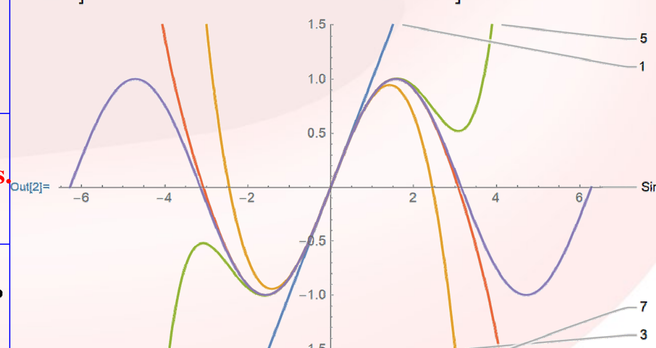
20 Approximation by Polynomial Functions.

Suppose that f is a function for which
$$f'(a), \ldots, f^{(n)}(a)$$
all exist. Let
$$a_k = \frac{f^{(k)}(a)}{k!}, \quad 0 \le k \le n,$$
and define
$$P_{n,a}(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n.$$
Then
$$\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = 0.$$ Ⓢ

For example for $f(x) = \sin(x)$ at $a = 0$, $f^{(k)} = \sin, \cos, -\sin, -\cos, \sin, \ldots$, so
$$a_k = \begin{cases} \frac{(-1)^{(k-1)/2}}{k!} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

```
In[1]:= a_k_ := { (-1)^((k-1)/2) / k!   OddQ[k] ;
                   0                     EvenQ[k]
Plot[Evaluate@Append[
   Table[Labeled[Sum[a_k x^k, {k, 0, n}], n], {n, {1, 3, 5, 7}}],
   Labeled[Sin[x], Sin]
  ], {x, -2π, 2π}, PlotRange -> {-1.5, 1.5}]
```

Out[2]=



```
In[3]:= Column@Table[k -> N[a_k 157^k], {k, {0, 3, 9, 13, 29, 35, 157, 223, 457}}]

Out[3]= 0 -> 0.
        3 -> -644982.
        9 -> 1.59711×10^14
        13 -> 5.65477×10^18
        29 -> 5.42689×10^32
        35 -> -6.95433×10^36
        157 -> 4.86366×10^66
        223 -> -1.94045×10^61
        457 -> 4.87404×10^-16

In[4]:= {N[Sum[a_k 157^k, {k, 0, 457}]], Sum[N[a_k 157^k], {k, 0, 457}]}

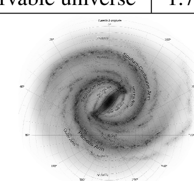
Out[4]= {-0.0795485, 5.10624×10^50}

In[5]:= N@Sin[157]

Out[5]= -0.0795485
```

Some sizes (in multiples of the diameter of a Hydrogen atom:

A red blood cell	1.56×10^5
The CN Tower	1.11×10^{13}
The rings of Saturn	5.6×10^{18}
The Milky Way galaxy	1.89×10^{31}
The observable universe	1.76×10^{37}



Do Not Turn Over Until Instructed

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MAASeaway-1810/

The Taylor Remainder Formulas. Let f be a smooth function, let $P_{n,a}(x)$ be the nth order Taylor polynomial of f around a and evaluated at x, so with $a_k = f^{(k)}(a)/k!$,

$$P_{n,a}(x) := \sum_{k=0}^{n} a_k(x-a)^k,$$

and let $R_{n,a}(x) := f(x) - P_{n,a}(x)$ be the "mistake" or "remainder term". Then

Brook Taylor

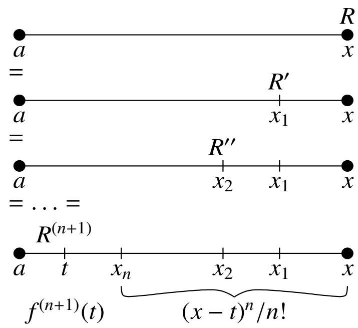
$$R_{n,a}(x) = \int_a^x dt\, \frac{f^{(n+1)}(t)}{n!}(x-t)^n, \tag{1}$$

or alternatively, for some t between a and x,

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}. \tag{2}$$

(In particular, the Taylor expansions of sin, cos, exp, and of several other lovely functions converges to these functions *everywhere*, no matter the odds.)

Proof of (1) (for adults; I learned it from my son Itai). The fundamental theorem of calculus says that if $g(a) = 0$ then $g(x) = \int_a^x dx_1 g(x_1)$. By design, $R^{(k)}_{n,a}(a) = 0$ for $0 \le k \le n$. Therefore



$$R_{n,a}(x) = \int_a^x dx_1 R'_{n,a}(x_1)$$

$$= \int_a^x dx_1 \int_a^{x_1} dx_2 R''_{n,a}(x_2)$$

$$= \ldots = \int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_n} dx_n \int_a^t dt\, R^{(n+1)}_{n,a}(t)$$

$$= \int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_n} dx_n \int_a^t dt\, f^{(n+1)}(t),$$

when $x > a$, and with similar logic when $x < a$,

$$= \int_{a\le t\le x_n\le\ldots\le x_1\le x} f^{(n+1)}(t) = \int_a^x dt\, f^{(n+1)}(t) \int_{t\le x_n\le\ldots\le x_1\le x} 1$$

$$= \int_a^x dt \frac{f^{(n+1)}(t)}{n!} \int_{(x_1,\ldots,x_n)\in[t,x]^n} 1 = \int_a^x dt \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

□

de-Fubini (obfuscation in the name of simplicity). Prematurely aborting the above chain of equalities, we find that for any $1 \le k \le n+1$,

$$R(x) = \int_a^x dt\, R^{(k)}(t)\frac{(x-t)^{k-1}}{(k-1)!}.$$

But these are easy to prove by induction using integration by parts, and there's no need to invoke Fubini.

Guido Fubini

space left blank for creative doodling

Partial Derivatives Commute. Make Fubini Smile Again!

If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is C^2 near $a \in \mathbb{R}^2$, then $f_{12}(a) = f_{21}(a)$.

Proof. Let $x \in \mathbb{R}^2$ be small, and let $R := [a_1, a_1+x_1]\times[a_2, a_2+x_2]$.

$$f_{12}(a) \sim \boxed{\int f_{12}} = \sum f = \boxed{\int f_{21}} \sim f_{21}(a)$$

$$f_{12}(a) \sim \frac{1}{|R|}\int_R f_{12} = \frac{1}{|R|}\int_{a_1}^{a_1+x_1} dt_1\,(f_1(t_1, a_2+x_2) - f_1(t_1, a_2))$$

$$= \frac{1}{|R|}\left(\begin{matrix} f(a_1+x_1, a_2+x_2) - f(a_1+x_1, a_2) \\ -f(a_1, a_2+x_2) + f(a_1, a_2)\end{matrix}\right).$$

But the answer here is the same as in

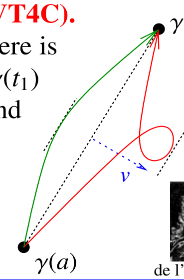
$$f_{21}(a) \sim \frac{1}{|R|}\int_R f_{21} = \frac{1}{|R|}\int_{a_2}^{a_2+x_2} dt_2\,(f_2(a_1+x_1, t_2) - f_2(a_1, t_2))$$

$$= \frac{1}{|R|}\left(\begin{matrix} f(a_1+x_1, a_2+x_2) - f(a_1, a_2+x_2) \\ -f(a_1+x_1, a_2) + f(a_1, a_2)\end{matrix}\right),$$

and both of these approximations get better and better as $x \to 0$. □

The Mean Value Theorem for Curves (MVT4C). If $\gamma\colon [a,b] \to \mathbb{R}^2$ is a smooth curve, then there is some $t_1 \in (a,b)$ for which $\gamma(b) - \gamma(a)$ and $\dot\gamma(t_1)$ are linearly dependent. If also $\gamma(a) = 0$, and $\gamma = \begin{pmatrix}\xi\\ \eta\end{pmatrix}$ and $\eta \ne 0 \ne \dot\eta$ on (a,b), then

$$\frac{\xi(b)}{\eta(b)} = \frac{\dot\xi(t_1)}{\dot\eta(t_1)} \quad \left(\text{when lucky, } = \frac{\ddot\xi(t_2)}{\ddot\eta(t_2)} \ldots\right).$$



de l'Hôpital

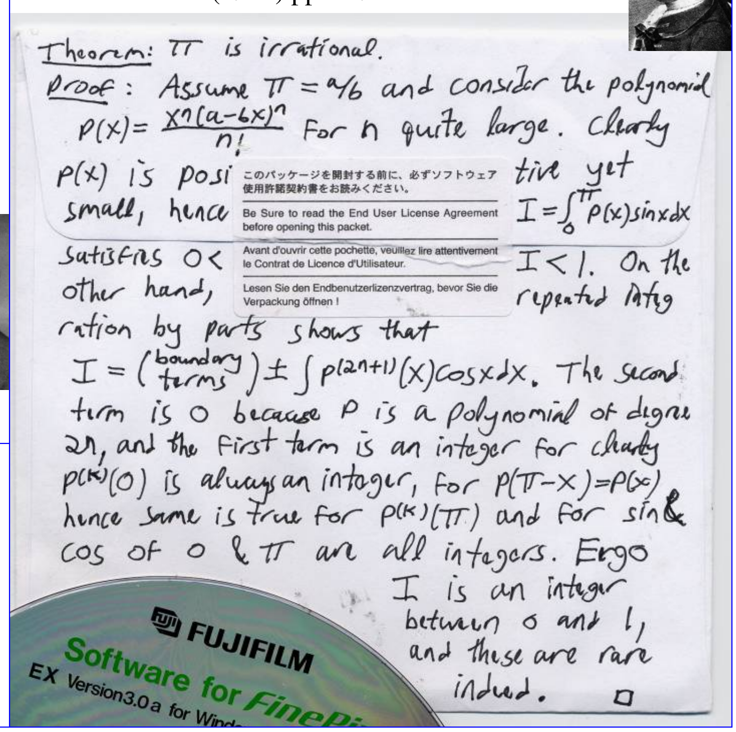
Proof of (2). Iterate the lucky MVT4C as follows:

$$\frac{R_{n,a}(x)}{(x-a)^{n+1}} = \frac{R'_{n,a}(t_1)}{(n+1)(t_1-a)^n} = \ldots = \frac{R^{(n+1)}_{n,a}(t_{n+1})}{(n+1)!} = \frac{f^{(n+1)}(t)}{(n+1)!}.$$

□

J.H. Lambert

***π* is Irrational** following Ivan Niven, Bull. Amer. Math. Soc. (1947) pp. 509:



Theorem: π is irrational.

Proof: Assume $\pi = a/b$ and consider the polynomial $P(x) = \frac{x^n(a-bx)^n}{n!}$ for n quite large. Clearly $P(x)$ is positive yet small, hence $I = \int_0^\pi P(x)\sin x\,dx$ satisfies $0 < I < 1$. On the other hand, repeated integration by parts shows that $I = \binom{\text{boundary}}{\text{terms}} \pm \int P^{(2n+1)}(x)\cos x\,dx$. The second term is 0 because P is a polynomial of degree $2n$, and the first term is an integer for clearly $P^{(k)}(0)$ is always an integer, for $P(\pi - x) = P(x)$ & hence same is true for $P^{(k)}(\pi)$ and for sin & cos of 0 & π are all integers. Ergo I is an integer between 0 and 1, and these are rare indeed. □

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MAASeaway-1810/

Solvable Approximations of the Quantum sl_2 Portfolio

Our Main Theorem (loosely stated). Everything that matters in the quantum sl_2 portfolio can be continuously expressed in terms of docile perturbed Gaussians using solvable approximations. ◯

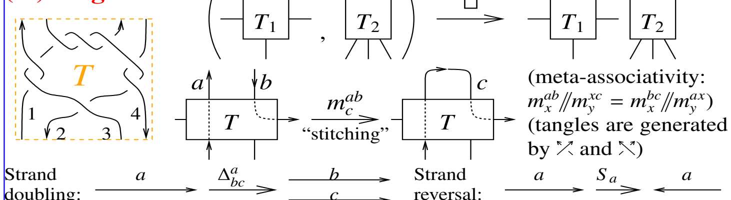
Our Main Points.

- What's the "quantum sl_2 portfolio"?
- What in it "matters" and why? (the most important question)
- What's "solvable approximation"? What's "continuously"?
- What are "docile perturbed Gaussians"?
- Why do they matter? (2^{nd} most important)
- How proven? (docile)
- How implemented? (sacred; the work of unsung heroes)
- Some context and background.
- What's next?

The quantum sl_2 Portfolio includes a classical universal enveloping algebra CU, its quantization QU, their tensor powers $CU^{\otimes S}$ and $QU^{\otimes S}$ with the "tensor operations" $\otimes$, their products m_k^{ij}, coproducts Δ^i_{jk} and antipodes S_i, their Cartan automophisms $C\theta\colon CU\to CU$ and $Q\theta\colon QU\to QU$, the "dequantizators" $A\mathbb{D}\colon QU\to CU$ and $S\mathbb{D}\colon QU\to CU$, and most importantly, the R-matrix R and the Drinfel'd element s. All this in any PBW basis, and change of basis maps are included.

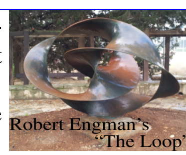
$$R, s\in\{QU^{\otimes S}\}\circlearrowleft_{\otimes, m^{ij}_k,\Delta^i_{jk},S_i,\theta} \xrightarrow{A\mathbb{D},S\mathbb{D}} \{CU^{\otimes S}\}\circlearrowleft_{\otimes, m^{ij}_k,\Delta^i_{jk},S_i,\theta}$$

(v-)Tangles.

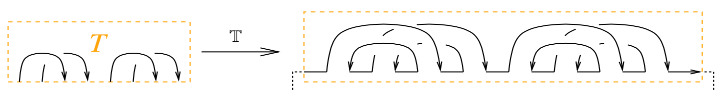


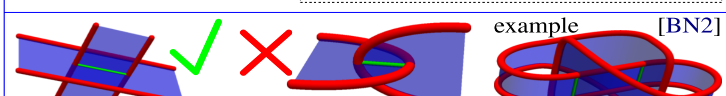
(meta-associativity: $m^{ab}_x /\!\!/ m^{xc}_y = m^{bc}_x /\!\!/ m^{ax}_y$) (tangles are generated by ⤯ and ⤮)

Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.


Robert Engman's "The Loop"

Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via



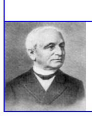


a ribbon singularity a clasp singularity

A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3=\partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

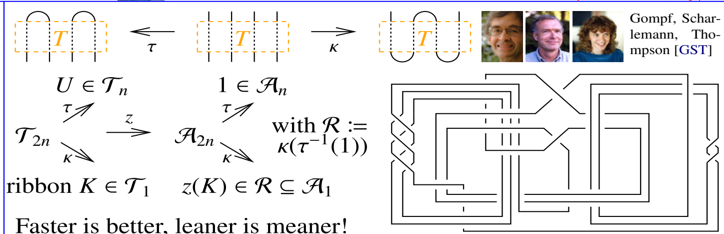
Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t)=f(t)f(1/t)$. (also for slice)



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org The Knot Atlas



Gompf, Scharlemann, Thompson [GST]

$$U\in\mathcal{T}_n \qquad 1\in\mathcal{A}_n$$
$$\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n} \qquad \text{with } \mathcal{R} := \kappa(\tau^{-1}(1))$$

ribbon $K\in\mathcal{T}_1$ $\quad z(K)\in\mathcal{R}\subseteq\mathcal{A}_1$

Faster is better, leaner is meaner!

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T)\in R_S\times M_{S\times S}(R_S)=\left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S:=\mathbb{Z}(\{t_a\colon a\in S\})$:

$$({}_a\!\nearrow_b, {}_b\!\nwarrow_a)\to\begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-t_a^{\pm1}\\ b & 0 & t_a^{\pm1}\end{array} \qquad T_1\sqcup T_2\to\begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[t_a,t_b\to t_c]{m^{ab}_c} \left(\begin{array}{c|cc}(1-\beta)\omega & c & S\\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta}\\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

(Roland: "add to A the product of column b and row a, divide by $(1-A_{ab})$, delete column b and row a".)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

$$\begin{array}{c|cc}\omega & a & S\\ \hline a & \alpha & \theta\\ S & \phi & \Xi\end{array} \xrightarrow[\substack{\mu=T_a-1\\ \nu=\alpha-\sigma_a\\ T_a\mapsto T_bT_c}]{q\Delta^a_{bc}} \left(\begin{array}{c|ccc}\omega & b & c & S\\ \hline b & (\sigma_a-\alpha T_a-\nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu\\ c & (T_c-1)\nu/\mu & (\alpha-\sigma_aT_a-\nu T_c)/\mu & (T_c-1)\theta/\mu\\ S & \phi & \phi & \Xi\end{array}\right)$$

$$\Big\downarrow dS^a\,,\ T_a\to T_a^{-1}$$

$$\left(\begin{array}{c|cc}\alpha\omega/\sigma_a & a & S\\ \hline a & 1/\alpha & \theta/\alpha\\ S & -\phi/\alpha & (\alpha\Xi-\phi\theta)/\alpha\end{array}\right)$$

Where σ assigns to every $a\in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma({}_a\!\nearrow_b, {}_b\!\nwarrow_a)=(a\to 1, b\to t_a^{\pm1})$, $\sigma(T_1\sqcup T_2)=\sigma(T_1)\sqcup\sigma(T_2)$, and $\sigma/\!\!/m^{ab}_c=(\sigma\setminus\{a,b\})\cup(c\to\sigma_a\sigma_b)|_{t_a,t_b\to t_c}$.

Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

Implementation key idea: ωεβ/AlexDemo

$(\omega, A=(\alpha_{ab}))\leftrightarrow$
$(\omega, \lambda=\sum\alpha_{ab}t_ah_b)$

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_ b_ → c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂_{t_a,h_a}λ ∂_{t_a,h_b}λ ∂_{t_a}λ; ∂_{t_b,h_a}λ ∂_{t_b,h_b}λ ∂_{t_b}λ; ∂_{h_a}λ ∂_{h_b}λ λ) /. (t|h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + α δ / μ  ϵ + δ θ / μ; ϕ + α ψ / μ  Ξ + ψ θ / μ).{h_c, 1}]
   /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_ b_} := Γ[1, {t_a, t_b}.(1  1 - T_a; 0  T_a).{h_a, h_b}];
Rm_{a_ b_} := Rp_{ab} /. T_a → 1 / T_a;

ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)_{a_} ⧴ a, ∞];
  M = Outer[Factor[∂_{h_#1 t_#2} λ] &, S, S];
  M = Prepend[M, t_# & /@ S] // Transpose;
  M = Prepend[M, Prepend[h_# & /@ S, ω]];
  M // MatrixForm];
```

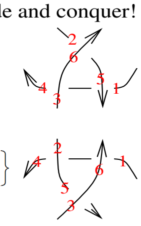
Meta-Associativity **Runs.**

```
ξ = Γ[ω, {t1, t2, t3, ts}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hs}];
(ξ // m12→1 // m13→1) == (ξ // m23→2 // m12→1)
True
```

R3 . . . divide and conquer!

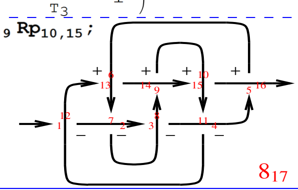
```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$



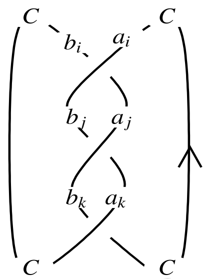
```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

$$\begin{pmatrix}11-\frac{1}{T_1^3}+\frac{4}{T_1^2}-\frac{8}{T_1}-8T_1+4T_1^2-T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$



8_{17}

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/



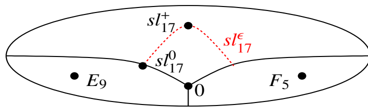
The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i \otimes b_i \in U \otimes U \quad \text{and} \quad C \in U,$$

form
$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

Problem. Extract information from Z.
The Dogma. Use representation theory. In principle finite, but *slow*.

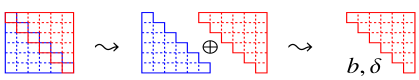
The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl^k_{17} := sl^\epsilon_{17}/(\epsilon^{k+1} = 0)$.



Solvable Approximation. A quantized universal enveloping algebra (aka "quantum group") is an ∞-dimensional inverse limit.

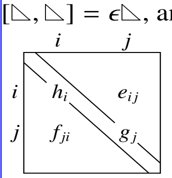
untrimmed	$\longrightarrow$	halfway-trimmed	$\longrightarrow$	almost fully-trimmed
$\updownarrow$		$\updownarrow$		$\updownarrow$
too hard		"solvable approximation" (a parameter is hidden)		"finite-type"

Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b\colon ◹ \otimes ◹ \to ◹$
$b(◺) \rightsquigarrow \delta\colon ◹ \to ◹ \otimes ◹$

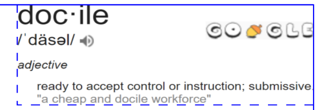
Now define $gl^\epsilon_n := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon ◺$, and $[◹, ◺] = ◺ + \epsilon ◹$. In detail, it is



$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} \qquad [f_{ij}, f_{kl}] = \epsilon \delta_{jk} f_{il} - \epsilon \delta_{li} f_{kj}$$
$$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon \delta_{j<k} e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l} f_{il})$$
$$-\delta_{li}(\epsilon \delta_{k<j} e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j} f_{kj})$$
$$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik}) e_{jk} \qquad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) e_{jk}$$
$$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik}) f_{jk} \qquad [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) f_{jk}$$

Solvable Approximation (2). At $\epsilon = 1$ and modulo $h = g$, the above is just gl_n. By rescaling at $\epsilon \neq 0$, gl^ϵ_n is independent of ϵ. We let gl^k_n be gl^ϵ_n regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1} = 0$. It is the "$k$-smidgen solvable approximation" of gl_n!
Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_3 := [\mathfrak{g}_2, \mathfrak{g}_2]$, $\ldots$, $\mathfrak{g}_d = 0$. Equivalently, if it is a subalgebra of some large-size ◹ algebra.
Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

Definition. A "docile perturbed Gaussian" in the variables $(z_i)_{i \in S}$ over the ring R is an expression of the form



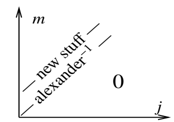
$$\mathbb{e}^{q^{ij} z_i z_j} P = \mathbb{e}^{q^{ij} z_i z_j} \left(\sum_{k \geq 0} \epsilon^k P_k \right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k \leq 4k$.
Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$.

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

$$\left. \frac{(q^{1/2} - q^{-1/2}) J_d(K)}{q^{d/2} - q^{-d/2}} \right|_{q=e^\hbar} = \sum_{j,m \geq 0} a_{jm}(K) d^j \hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot \omega(K)(e^\hbar) = 1$.

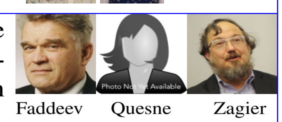


"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1})\omega(K)(q^d)} \left(1 + \sum_{k=1}^\infty \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)} \right).$$

Prior art. Some amazing computations by Rozansky and Overbay in [Ro2, Ro3] and in [Ov].

Faddeev's Formula (In as much as we can tell, first appeared w/o proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n - 1}{q - 1}$, with $[n]_q! := [1]_q [2]_q \cdots [n]_q$ and with $\mathbb{e}^x_q := \sum_{n \geq 0} \frac{x^n}{[n]_q!}$, we have



$$\log \mathbb{e}^x_q = \sum_{k \geq 1} \frac{(1-q)^k x^k}{k(1 - q^k)} = x + \frac{(1-q)^2 x^2}{2(1-q^2)} + \ldots .$$

Proof. We have that $\mathbb{e}^x_q = \frac{\mathbb{e}^{qx}_q - \mathbb{e}^x_q}{qx - x}$ ("the q-derivative of $\mathbb{e}^x_q$ is itself"), and hence $\mathbb{e}^{qx}_q = (1 + (1-q)x)\mathbb{e}^x_q$, and

$$\log \mathbb{e}^{qx}_q = \log(1 + (1-q)x) + \log \mathbb{e}^x_q.$$

Writing $\log \mathbb{e}^x_q = \sum_{k \geq 1} a_k x^k$ and comparing powers of x, we get $q^k a_k = -(1-q)^k/k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. □

***GDO*-Categories.** Given $\mathfrak{g}$ with basis $B = \{x, y, \ldots\}$, consider the following diagram:

$$\begin{tikzcd}
\mathbb{Q} = \hat{\mathcal{U}}_{(q)}\left(\bigoplus_0 \mathfrak{g}\right) \arrow[r, "Z"] & \hat{\mathcal{U}}_{(q)}(\mathfrak{g}) \arrow[r, shift left, "\Delta"] & \hat{\mathcal{U}}_{(q)}\left(\bigoplus_2 \mathfrak{g}\right) \arrow[l, shift left, "m"] \\
\hat{\mathcal{S}}(\emptyset) \arrow[u] \arrow[r, "Z"] & \hat{\mathcal{S}}(B) \arrow[u, shift right, "\mathbb{O}(xy\ldots:\,\cdot)"'] \arrow[u, shift left, "\mathbb{O}(yx\ldots:\,\cdot)"] \arrow[loop below, "SW_{xy}"] \arrow[r, shift left, "\Delta"] & \hat{\mathcal{S}}(B_1, B_2) \arrow[l, shift left, "m"] \arrow[u, "\mathbb{O}(y_1x_1\ldots\otimes y_2x_2\ldots:\,\cdot)"']
\end{tikzcd}$$

Hence Z, SW_{xy}, m, Δ, (and likewise S and θ) are morphisms in the *completion* of the monoidal category $\mathcal{F}$ whose objects are finite sets B and whose morphisms are $\mathrm{mor}_{\mathcal{F}}(B, B') := \mathrm{Hom}_{\mathbb{Q}}(\mathcal{S}(B) \to \mathcal{S}(B')) = \mathcal{S}(B^*, B')$ (by convention, $x^* = \xi$, $y^* = \eta$, etc.). Ergo we need to *consolidate* (at least parts of) said completion.

Aside. "Consolidate" means "give a finite name to an infinite object, and figure out how to sufficiently manipulate such finite names". E.g., solving $f'' = -f$ we encounter and set $\sum \frac{(-1)^k x^{2k}}{(2k)!} \rightsquigarrow \cos x$, $\sum \frac{(-1)^k x^{2k+1}}{(2k+1)!} \rightsquigarrow \sin x$, and then $\cos^2 x + \sin^2 x = 1$ and $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

The Composition Law. If

$$\mathcal{S}(B_0) \xrightarrow[{}^t f \in \mathbb{Q}[\![\zeta_{0i}, z_{1j}]\!]]{f} \mathcal{S}(B_1) \xrightarrow[{}^t g \in \mathbb{Q}[\![\zeta_{1j}, z_{2k}]\!]]{g} \mathcal{S}(B_2)$$

then ${}^t(f /\!\!/ g) = {}^t(g \circ f) = \left(g|_{\zeta_{1j} \to \partial_{z_{1j}}} f \right)_{z_{1j} = 0}$.

Examples.

1. The 1-variable identity map $I\colon \mathcal{S}(z) \to \mathcal{S}(z)$ is given by ${}^t I_1 = \mathbb{e}^{z\zeta}$ and the n-variable one by ${}^t I_n = \mathbb{e}^{z_1\zeta_1 + \cdots + z_n\zeta_n}$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/

2. The "$z_i \to z_j$ variable rename map $\sigma^i_j\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j)$ becomes ${}^t\sigma^i_j = \mathbb{e}^{z_j\zeta_i}$, and it's easy to rename several variables simultaneously.
3. The "archetypal multiplication map $m^{ij}_k\colon \mathcal{S}(z_i, z_j) \to \mathcal{S}(z_k)$" has ${}^tm = \mathbb{e}^{z_k(\zeta_i+\zeta_j)}$.
4. The "archetypal coproduct $\Delta^i_{jk}\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j, z_k)$", given by $z_i \to z_j + z_k$ or $\Delta z = z \otimes 1 + 1 \otimes z$, has ${}^t\Delta = \mathbb{e}^{(z_j+z_k)\zeta_i}$.
5. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1 x_2} \in \mathcal{U}_q \otimes \mathcal{U}_q$. The "baby R-matrix" is ${}^tR = \mathbb{e}^{\hbar yx} \in \mathcal{S}(y, x)$.

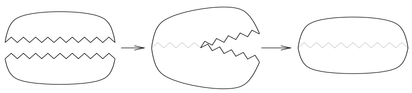
Proposition. If $F\colon \mathcal{S}(B) \to \mathcal{S}(B')$ is linear and "continuous", then ${}^tF = \exp\left(\sum_{z_i\in B}\zeta_i z_i\right) /\!\!/ F$.

The Heisenberg Example. The "Weyl form of the canonical commutation relations" states that if $[y, x] = t$ and t is central, then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. Thus with

$$\begin{tikzcd} \mathcal{S}(t,y,x) \arrow[loop left, "SW_{xy}"] \arrow[r, shift left, "\mathbb{O}_{xy}"] \arrow[r, shift right, "\mathbb{O}_{yx}"'] & \mathcal{U}(t,y,x) \end{tikzcd}$$

we have ${}^tSW_{xy} = \mathbb{e}^{\tau t+\eta y+\xi x-\eta\xi t}$.

The Zipping Issue (between unbound and bound lies half-zipped).



Zipping. If $P(\zeta^j, z_i)$ is a polynomial, or whenever otherwise convergent, set

$$\left\langle P(\zeta^j, z_i)\right\rangle_{(\zeta^j)} = P\left(\partial_{z_j}, z_i\right)\Big|_{z_i=0}.$$

(E.g., if $P = \sum a_{nm}\zeta^n z^m$ then $\langle P\rangle_\zeta = \sum n! a_{nn}$).

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i, \zeta^j)\mathbb{e}^{\eta^i z_i + y_j\zeta^j}\right\rangle_{(\zeta^j)} = \left\langle P(z_i + y_i, \zeta^j)\mathbb{e}^{\eta^i(z_i+y_i)}\right\rangle_{(\zeta^j)},$$

(proof: replace $y_j \to \hbar y_j$ and test at $\hbar = 0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i, \zeta^j)\mathbb{e}^{c+\eta^i z_i + y_j\zeta^j + q^i_j z_i\zeta^j}\right\rangle_{(\zeta^j)} = \det(\tilde{q})\left\langle P(\tilde{q}^k_i(z_k + y_k), \zeta^j)\mathbb{e}^{c+\eta^i\tilde{q}^k_i(z_k+y_k)}\right\rangle_{(\zeta^j)}$$

where $\tilde{q}$ is the inverse matrix of $1 - q$: $(\delta^i_j - q^i_j)\tilde{q}^j_k = \delta^i_k$ (proof: replace $q^i_j \to \hbar q^i_j$ and test at $\hbar = 0$ and at $\partial_\hbar$).

Implementation. ωεβ/ZipBindDemo

```
Kδ /: Kδi_,j_ := If[i === j, 1, 0];
{z*, x*, y*} = {ζ, ξ, η}; {ζ*, ξ*, η*} = {z, x, y};
(u_i_)* := (u*)i;
```

```
Zip{}[P_] := P;
Zip{ζ_,ζs___}[P_] :=
  (Expand[P // Zip{ζs}] /. f_. ζ^d_. :> ∂{ζ*,d}f) /. ζ* → 0
```

```
Zip{ζ}[(a ζ^6 + ζ + 3) (z^5 e^z + 7 z) + 99 b]
```

```
7 + 720 a + 99 b
```

```
Zip{ξ,η}[ξ^3 η^3 e^(a x + b y + c x y)]
```

```
a^3 b^3 + 9 a^2 b^2 c + 18 a b c^2 + 6 c^3
```

```
(* 𝔼[Q,P] means e^Q P *)
𝔼 /: Zipζs_List@𝔼[Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδz,ζ* - ∂z,ζQ, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q1 = c + ηs.zs /. zrule;
   Q2 = Q1 /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[Q2, Det[qt] e^-Q2 Zipζs[e^Q1 (P /. zrule)]] ];
```

```
Eh = 𝔼[h Sum[Sum[a10 i+j xi ξj, {j,1,3}], {i,1,3}], Sum[fi[x1, x2, x3] ξi, {i,1,3}]];
E1 = Eh /. h → 1
```

```
𝔼[a11 x1 ξ1 + a21 x2 ξ1 + a31 x3 ξ1 + a12 x1 ξ2 +
  a22 x2 ξ2 + a32 x3 ξ2 + a13 x1 ξ3 + a23 x2 ξ3 + a33 x3 ξ3,
 ξ1 f1[x1, x2, x3] + ξ2 f2[x1, x2, x3] + ξ3 f3[x1, x2, x3]]
```

```
Short[lhs = Zip{ξ1,ξ2}@E1, 5]
```

$$\mathbb{E}\Big[\big((a_{13}\,((-1+a_{22})\,a_{31}-a_{21}\,a_{32})+a_{12}\,(-a_{23}\,a_{31}+a_{21}\,a_{33})+(-1+a_{11})\,(a_{23}\,a_{32}-(-1+a_{22})\,a_{33}))\,x_3\,\xi_3\big)\big/(-1+a_{12}\,a_{21}-a_{11}\,(-1+a_{22})+a_{22}),\ \frac{\ll 17\gg+a_{21}\ll 1\gg}{(-1+a_{12}\,a_{21}-a_{11}\,(-1+a_{22})+a_{22})^2}\Big]$$

```
lhs == Zip{ξ1}@Zip{ξ2}@E1 == Zip{ξ2}@Zip{ξ1}@E1
```

```
True
```

```
Short[
 lhs = Normal[Eh /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 3}]] //
    Zip{ξ1,ξ2}, 5]
```

$$h\,a_{13}\,\xi_3\,f_1[0,0,x_3] + 2h^2\,a_{11}\,a_{13}\,\xi_3\,f_1[0,0,x_3] + 3h^3\,a_{11}^2\,a_{13}\,\xi_3\,f_1[0,0,x_3] + 2h^3\,a_{12}\,a_{13}\,a_{21}\,\xi_3\,f_1[0,0,x_3] + h^2\,a_{13}\,a_{22}\,\xi_3\,f_1[0,0,x_3] + \ll 337\gg + \frac{1}{6}h^3\,a_{31}^3\,x_3^3\,\xi_3\,f_3^{(3,0,0)}[0,0,x_3] + \frac{1}{2}h^3\,a_{31}^2\,a_{32}\,x_3^3\,f_1^{(3,1,0)}[0,0,x_3] + \frac{1}{6}h^3\,a_{31}^3\,x_3^3\,f_2^{(3,1,0)}[0,0,x_3] + \frac{1}{6}h^3\,a_{31}^3\,x_3^3\,f_1^{(4,0,0)}[0,0,x_3]$$

```
rhs =
  Normal[Zip{ξ1,ξ2}@Eh /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 3}]];
Simplify[lhs == rhs]
```

```
True
```

```
𝔼 /: 𝔼[Q1_, P1_] 𝔼[Q2_, P2_] := 𝔼[Q1 + Q2, P1 * P2];
Bindζs_List[L_𝔼, R_𝔼] := Module[{n, hideζs, hidezs},
   hideζs = Table[ζs[[i]] → ζn@i, {i, Length@ζs}];
   hidezs = Table[ζs[[i]]* → zn@i, {i, Length@ζs}];
   Zipζs/.hideζs[(L /. hidezs) (R /. hideζs)] ];
```

```
Bind{ξ2}[𝔼[ξ (x1 + x2), 1], 𝔼[ξ2 (x2 + x3), 1]]
```

```
𝔼[ξ (x1 + x2 + x3), 1]
```

```
Bind{ξ2}[𝔼[(ξ2 + ξ3) x2, 1], 𝔼[(ξ1 + ξ2) x, 1]]
```

```
𝔼[x (ξ1 + ξ2 + ξ3), 1]
```

The 2D Lie Algebra. Clever people know* that if $[a, x] = \gamma x$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$. Ergo with

$$\begin{tikzcd} \mathcal{S}(a,x) \arrow[loop left, "SW_{ax}"] \arrow[r, shift left, "\mathbb{O}_{ax}"] \arrow[r, shift right, "\mathbb{O}_{xa}"'] & \mathcal{U}(a,x) \end{tikzcd}$$

we have ${}^tSW_{ax} = \mathbb{e}^{\alpha a+\mathbb{e}^{-\gamma\alpha}\xi x}$.

$*$ Indeed $xa = (a - \gamma)x$ thus $xa^n = (a-\gamma)^n x$ thus $x\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha(a-\gamma)}x = \mathbb{e}^{-\gamma\alpha}\mathbb{e}^{\alpha a}x$ thus $x^n\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}(\mathbb{e}^{-\gamma\alpha})^n x^n$ thus $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$.

The Real Thing. In $QU/(\epsilon^2 = 0)$ over $\mathbb{Q}[\![\hbar]\!]$ using the yax order, $T = \mathbb{e}^{\hbar t}$, $\bar{T} = T^{-1}$, $\mathcal{A} = \mathbb{e}^{\gamma\alpha}$, and $\bar{\mathcal{A}} = \mathcal{A}^{-1}$, we have

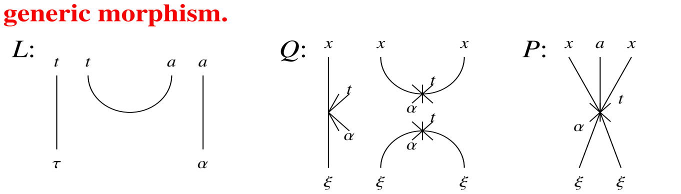
$${}^tR_{ij} = \mathbb{e}^{\hbar(y_ix_j - t_ia_j/\gamma)}\left(1 + \epsilon\hbar\left(a_ia_j/\gamma - \gamma\hbar^2y_i^2x_j^2/4\right)\right)$$

in $\mathcal{S}(B_i, B_j)$, and in $\mathcal{S}(B_1^*, B_2^*, B)$ we have

$${}^tm = \mathbb{e}^{(\alpha_1+\alpha_2)a + \eta_2\xi_1(1-T)/\hbar + (\xi_1\bar{\mathcal{A}}_2+\xi_2)x + (\eta_1+\eta_2\bar{\mathcal{A}}_1)y}\,(1+\epsilon\lambda_m),$$

where $\lambda_m = 2a\eta_2\xi_1T + \frac{1}{4}\gamma\eta_2^2\xi_1^2\left(3T^2-4T+1\right)/\hbar - \frac{1}{2}\gamma\eta_2\xi_1^2(3T-1)x\bar{\mathcal{A}}_2 - \frac{1}{2}\gamma\eta_2^2\xi_1(3T-1)y\bar{\mathcal{A}}_1 + \gamma\eta_2\xi_1xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$. Similar formulas delight us for ${}^t\Delta$ and ${}^tS$.

A generic morphism.



Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/

Implementation.

ωεβ/SL2Portfolio

```
QZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   simp /@ 𝔼[L, Q2, Det[qt] e^(-Q2) Zip_ζs[e^Q1 (P /. zrule)]]];
QZip_{ζs_List} := QZip_{ζs,CF};
```

```
LZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Q2 = (Q1 = Q /. T2t /. zrule) /. Alternatives @@ zs → 0;
   simp /@
    𝔼[L2, Q2, Det[lt] e^(-L2-Q2)
      Zip_ζs[e^(L1+Q1) (P /. T2t /. zrule)]] //. t2T];
LZip_{ζs_List} := LZip_{ζs,CF};
```

```
Bind_{}[L_, R_] := L R;
Bind_{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : T | t | a | x | y)_i → v_{n@i}, {i, {is}}],
     R /. Table[(v : τ | α | ξ | η)_i → v_{n@i}, {i, {is}}]
    ] // LZip_{Flatten@Table[{τ_{n@i},a_{n@i}},{i,{is}}]} //
   QZip_{Flatten@Table[{ξ_{n@i},y_{n@i}},{i,{is}}]} ];
B_{L_List} := Bind_L; B_{is___} := Bind_{is};
Bind[ℰ_𝔼] := ℰ;
Bind[Ls__, ζs_List, R_] := Bind_ζs[Bind[Ls], R];
```

A Partial To Do List.

- Complete all "docility" arguments by identifying a "contained" docile substructure.
- Understand denominators and get rid of them.
- See if much can be gained by including P in the exponential: $e^{L+Q}P \rightsquigarrow e^{L+Q+P}$?
- Clean the program and make it efficient.
- Run it for all small knots and links, at $k = 2, 3$.
- Understand the centre and figure out how to read the output.
- Execute the Drinfel'd double procedure at $\mathbb{E}$-level (and thus get rid of `DeclareAlgebra` and all that is around it!).
- Extend to sl_3 and beyond.
- Do everything with `Zip` and `Bind` as the fundamentals, without ever referring back to (quantized) Lie algebras.
- Prove a genus bound and a Seifert formula.
- Obtain "Gauss-Gassner formulas" (ωεβ/NCSU).
- Relate with Melvin-Morton-Rozansky and with Rozansky-Overbay.
- Understand the braid group representations that arise.
- Find a topological interpretation. The Garoufalidis-Rozansky "loop expansion" [GR]?
- Figure out the action of the Cartan automorphism.
- Disprove the ribbon-slice conjecture!
- Figure out the action of the Weyl group.
- Do everything at the "arrow diagram" level of finite-type invariants of (rotational) virtual tangles.
- What else can you do with the "solvable approximations"?
- And with the "Gaussian zip and bind" technology?

References.

[BN1] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* ωεβ/KBH, arXiv:1308.1721.

[BN2] D. Bar-Natan, *Polynomial Time Knot Polynomial,* research proposal for the 2017 Killam Fellowship, ωεβ/K17.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[BV] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* Proc. Amer. Math. Soc., to appear, arXiv:1708.04853.

[Fa] L. Faddeev, *Modular Double of a Quantum Group,* arXiv:math/9912078.

[GR] S. Garoufalidis and L. Rozansky, *The Loop Exapnsion of the Kontsevich Integral, the Null-Move, and S-Equivalence,* arXiv:math.GT/0003187.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Qu] C. Quesne, *Jackson's q-Exponential as the Exponential of a Series,* arXiv:math-ph/0305003.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Vo] H. Vo, *Alexander Invariants of Tangles via Expansions,* University of Toronto Ph.D. thesis, in preparation.

[Za] D. Zagier, *The Dilogarithm Function,* in Cartier, Moussa, Julia, and Vanhove (eds) *Frontiers in Number Theory, Physics, and Geometry II.* Springer, Berlin, Heidelberg, and ωεβ/Za.

The Complete Implementation.

ωεβ/SL2Portfolio

An even fuller implementation is at ωεβ/FullImp.

Initialization / Utilities

```
$p = 2; $k = 1; $U = QU; $E := {$k, $p};
$trim := {ℏ^(p_.) /; p > $p → 0, ϵ^(k_.) /; k > $k → 0};
q_ℏ = e^(γ ϵ ℏ);
T2t = {T_{i_}^(p_.) → e^(p ℏ t_i), T^(p_.) → e^(p ℏ t)};
t2T = {e^(c_. t_{i_} + b_.) :→ T_i^(c/ℏ) e^b, e^(c_. t + b_.) :→ T^(c/ℏ) e^b, e^(ℰ_) :→ e^(Expand@ℰ)};
SetAttributes[SS, HoldAll];
SS[ℰ_, op_] := Collect[
   Normal@Series[If[$p > 0, ℰ, ℰ /. T2t], {ℏ, 0, $p}],
   ℏ, op];
SS[ℰ_] := SS[ℰ, Together];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
c_Integer_{k_Integer} := c + O[ϵ]^(k+1);
```

```
CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@
    Together[Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^(CF[x])];
```

```
Unprotect[SeriesData];
SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
SeriesData /: Expand[sd_SeriesData] :=
  MapAt[Expand, sd, 3];
SeriesData /: Simplify[sd_SeriesData] :=
  MapAt[Simplify, sd, 3];
SeriesData /: Together[sd_SeriesData] :=
  MapAt[Together, sd, 3];
SeriesData /: Collect[sd_SeriesData, specs__] :=
  MapAt[Collect[#, specs] &, sd, 3];
Protect[SeriesData];
```

DeclareAlgebra

```
Unprotect[NonCommutativeMultiply];
Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x;
x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

```
DeclareAlgebra[U_Symbol, opts__Rule] :=
 Module[{gp, sr, g, cp, M, CE, k = 0,
    gs = Generators /. {opts},
    cs = Centrals /. {opts} /. Centrals → {} },
   (#_U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
  sr = Flatten@Table[{g → ++k, g_{i_} → {i, k}}, {g, gs}];
  (* sorting → *)
  cp = Alternatives @@ cs; (* cents *)
  SetAttributes[M, HoldRest]; M[0, _] = 0;
  M[a_, x_] := a x;
  CE[ℰ_] := Collect[ℰ, _U, Expand] /. $trim;
  U_{i_}[ℰ_] := ℰ /. {t : cp :→ t_i, u_U :→ (#_i &) /@ u};
  U_{i_}[NCM[]] = U@{} = 1_U = U[];
  B[U@(x_)_{i_}, U@(y_)_{i_}] := U_i@B[U@x, U@y];
  B[U@(x_)_{i_}, U@(y_)_{j_}] /; i =!= j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** (c_. 1_U) := CE[c x]; (c_. 1_U) ** x_ := CE[c x];
  (a_. U[xx___, x_]) ** (b_. U[y_, yy___]) :=
   If[OrderedQ[{x, y} /. sr],
    CE@M[a b /. $trim, U[xx, x, y, yy]],
    U@xx **
     CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] **
     U@yy ];
  U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] :=
   CE[c U@Table[l, {n}] ** U@{r}];
  U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
  U@{l_, r___} := U@{Expand[l], r};
  U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
  𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, l_List :→ l_null, {1}];
    vs = Join @@ (First /@ sp);
    us = Join @@ (sp /. l_{s_} :→ (l /. x_{i_} :→ x_s));
    CE[Total[
       CoefficientRules[poly, vs] /. (p_ → c_) :→ c U@(us^p)
      ]] /. x_null :→ x];
  𝕆_U[specs___, 𝔼[L_, Q_, P_]] :=
   𝕆_U[specs, SS@Normal[P e^(L+Q)]];
  σ_{rs___}[c_. * u_U] :=
   (c /. (t : cp)_{j_} :→ t_{j/.{rs}}) U[List @@ (u /. v_{j_} :→ v_{j/.{rs}})];
  m_{j_→k_}[c_. * u_U] :=
   CE[((c /. (t : cp)_j → t_k) DeleteCases[u, _{j|k}]) **
     U @@ Cases[u, w_j :→ w_k] ** U @@ Cases[u, _k]];
  U /: c_. * u_U * v_U := CE[c u ** v];
  S_{i_}[c_. * u_U] :=
   CE[((c /. S_i[U, Centrals]) DeleteCases[u, _i]) **
     U_i[NCM @@ Reverse@Cases[u, x_i :→ S@U@x]]];
  Δ_{i_→j_,k_}[c_. * u_U] :=
   CE[((c /. Δ_{i→j,k}[U, Centrals]) DeleteCases[u, _i]) **
     (NCM @@ Cases[u, x_i :→ σ_{1→j,2→k}@Δ@U@x] /.
       NCM[] → U[])]; ]
```

DeclareMorphism

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs, {(g_ → img_) :→ (m[U[g]] = img),
     (g_ :→ img_) :→ (m[U[g]] := img /. $trim)}, {1}];
  m[1_U] = 1_V;
  m[U[g_{i_}]] := V_i[m[U@g]];
  m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /. u_U :→ m[u]] /. $trim; )
```

Meta-Operations

```
σ_{rs___}[ℰ_Plus] := σ_rs /@ ℰ;
m_{j_→j_} = Identity; m_{j_→k_}[0] = 0;
m_{j_→k_}[ℰ_Plus] := Simp[m_{j→k} /@ ℰ];
m_{is___,i_,j_→k_}[ℰ_] := m_{j→k}@m_{is,i→j}@ℰ;
S_{i_}[ℰ_Plus] := Simp[S_i /@ ℰ];
Δ_{is___}[ℰ_Plus] := Simp[Δ_is /@ ℰ];
```

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
(S@y_CU = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_{i_}[CU, Centrals] = {t_i → -t_i};
Δ@y_CU = CU@y_1 + CU@y_2; Δ@a_CU = CU@a_1 + CU@a_2;
Δ@x_CU = CU@x_1 + CU@x_2;
Δ_{i_→j_,k_}[CU, Centrals] = {t_i → t_j + t_k};
```

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

```
DeclareAlgebra[QU, Generators → {y, a, x},
  Centrals → {t, T}];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] := SS[q_ħ - 1] QU@{y, x} +
   𝕆_QU[{a}, SS[(1 - T e^(-2 ϵ a ħ)) / ħ]];
(S@y_QU := 𝕆_QU[{a, y}, SS[-T^-1 e^(ħ ϵ a) y]]; S@a_QU = -a_QU;
 S@x_QU := 𝕆_QU[{a, x}, SS[-e^(ħ ϵ a) x]];)
S_{i_}[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
Δ@y_QU := 𝕆_QU[{y_1, a_1}_1, {y_2}_2, SS[y_1 + T_1 e^(-ħ ϵ a_1) y_2]];
Δ@a_QU = QU@a_1 + QU@a_2;
Δ@x_QU := 𝕆_QU[{a_1, x_1}_1, {x_2}_2, SS[x_1 + e^(-ħ ϵ a_1) x_2]];
Δ_{i_→j_,k_}[QU, Centrals] = {t_i → t_j + t_k, T_i → T_j T_k};
```

The representation ρ

```
ρ@y_CU = ρ@y_QU = ( 0  0
                    ϵ  0 ); ρ@a_CU = ρ@a_QU = ( γ  0
                                                0  0 );
ρ@x_CU = ( 0  γ
           0  0 ); ρ@x_QU = ( 0  (1 - e^(-γ ϵ ħ)) / (ϵ ħ)
                              0  0 );
ρ[e^ℰ_] := MatrixExp[ρ[ℰ]];
ρ[ℰ_] :=
 (ℰ /. T2t /. t → γ ϵ /.
   (U : CU | QU)[u___] :→ Fold[Dot, ( 1  0
                                      0  1 ), ρ /@ U /@ {u}])
```

tSW

Goal. In either U, compute $F = e^{-\eta y}\, e^{\xi x}\, e^{\eta y}\, e^{-\xi x}$. First compute $G = e^{\xi x}\, y\, e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta\left(e^{-\eta y}\, e^{\eta G}\right) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
SW_xy[U_, kk_] :=
  SW_xy[U, kk] = Block[{$U = U, $k = kk, $p = kk},
    Module[{G, F, fs, f, bs, e, b, es},
      G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].
          NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
      fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, l},
          {j, 0, l}, {k, 0, l}];
      F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k});
      es = Flatten[Table[Coefficient[e, b] == 0,
          {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}},
          {b, bs}]];
      F = F /. DSolve[es, fs, η][[1]];
      𝔼[0,
        ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
        F + O_$k /. {e- → 1, U → Times}
       ] /. (v : η | ξ | t | T | y | a | x) → v_1
    ]];
tSW_{xy_, i_, j_→k_} :=
  SW_xy[$U, $k] /. {ξ_1 → ξ_i, η_1 → η_j, (v : t | T | y | a | x)_1 → v_k};
tSW_{xa, i_, j_→k_} := 𝔼[α_j a_k, e^(-γ α_j) ξ_i x_k, 1];
tSW_{ay, i_, j_→k_} := 𝔼[α_i a_k, e^(-γ α_i) η_j y_k, 1];
```

Exponentials as needed.

Task. Define $\mathrm{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi\,\mathbb{O}(P)}$ to ϵ^k in the algebra U_i, where ξ is a scalar, X is x_i or y_i, and P is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form. Should satisfy $U@\mathrm{Exp}_{U_i,k}[\xi, P] == \mathbb{S}_U\left[e^{\xi X},\ x \to \mathbb{O}(P)\right]$.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\xi\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)$, then $F(\xi = 0) = 1$ and we have:

$$\mathbb{O}\left(e^{\xi P_0}(P_0 F(\xi) + \partial_\xi F\right) = \mathbb{O}\left(\partial_\xi e^{\xi P_0} F(\xi)\right) = \\ \partial_\xi \mathbb{O}\left(e^{\xi P_0} F(\xi)\right) = \partial_\xi e^{\xi\,\mathbb{O}(P)} = e^{\xi\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)\mathbb{O}(P)$$

This is an ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ.

```
(* Bug: The first line is valid only if O(e^P0) == e^O(P0). *)
(* Bug: ξ must be a symbol. *)
Exp_{U_i_, 0}[ξ_, P_] := Module[{LQ = Normal@P /. ϵ → 0},
   𝔼[ξ LQ /. (x | y)_i → 0, ξ LQ /. (t | a)_i → 0, 1] ];
Exp_{U_i_, k_}[ξ_, P_] := Block[{$U = U, $k = k},
  Module[{P0, φ, φs, F, j, rhs, at0, atξ},
    P0 = Normal@P /. ϵ → 0;
    φs = Flatten@Table[φ_{j1,j2,j3}[ξ], {j2, 0, k},
       {j1, 0, 2 k + 1 - j2}, {j3, 0, 2 k + 1 - j2 - j1}];
    F = Normal@Last@Exp_{U_i, k-1}[ξ, P] +
      ϵ^k φs.(φs /. φ_{js__}[ξ] :> Times @@ {y_i, a_i, x_i}^{js});
    rhs =
     Normal@
      Last@
       m_{i, j→i}[𝔼[ξ P0 /. (x | y)_i → 0, ξ P0 /. (t | a)_i → 0, F + O_k]
         m_{i→j}@𝔼[0, 0, P + O_k]];
    at0 = (# == 0) & /@
      Flatten@CoefficientList[F - 1 /. ξ → 0, {y_i, a_i, x_i}];
    atξ = (# == 0) & /@
      Flatten@CoefficientList[(∂_ξ F) + P0 F - rhs,
       {y_i, a_i, x_i}];
    𝔼[ξ P0 /. (x | y)_i → 0, ξ P0 /. (t | a)_i → 0, F + O_k] /.
     DSolve[And @@ (at0 ⋃ atξ), φs, ξ][[1]] ]]
```

Zip and Bind

```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] :=
  𝔼[L1 + L2, Q1 + Q2, P1 * P2];
```

```
{t*, y*, a*, x*, z*} = {τ, η, α, ξ, ζ};
{τ*, η*, α*, ξ*, ζ*} = {t, y, a, x, z};
(u_i_)* := (u*)_i;
```

```
Zip_{}[P_] := P;
Zip_{ζ_, ζs___}[P_] :=
 (Expand[P // Zip_{ζs}] /. f_. ζ^d_. :> ∂_{ζ*, d} f) /. ζ* → 0
```

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

```
QZip_{ζs_List, simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[K δ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   simp /@ 𝔼[L, Q2, Det[qt] e^(-Q2) Zip_{ζs}[e^Q1 (P /. zrule)]] ];
QZip_{ζs_List} := QZip_{ζs, CF};
```

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^Q as a single"P". Here the z's are t and α and the ζ's are τ and a.

```
LZip_{ζs_List, simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[K δ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Q2 = (Q1 = Q /. T2t /. zrule) /. Alternatives @@ zs → 0;
   simp /@
    𝔼[L2, Q2, Det[lt] e^(-L2-Q2)
      Zip_{ζs}[e^(L1+Q1) (P /. T2t /. zrule)]] //. t2T ];
LZip_{ζs_List} := LZip_{ζs, CF};
```

```
Bind_{}[L_, R_] := L R;
Bind_{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : T | t | a | x | y)_i → v_{n@i}, {i, {is}}],
     R /. Table[(v : τ | α | ξ | η)_i → v_{n@i}, {i, {is}}]
    ] // LZip_{Flatten@Table[{τ_{n@i}, a_{n@i}}, {i, {is}}]} //
   QZip_{Flatten@Table[{ξ_{n@i}, y_{n@i}}, {i, {is}}]} ];
B_{l_List} := Bind_l;  B_{is___} := Bind_{{is}};
Bind[ℰ_𝔼] := ℰ;
Bind[Ls__, ζs_List, R_] := Bind_{ζs}[Bind[Ls], R];
```

Tensorial Representations

```
tη = t𝟙 = 𝔼[0, 0, 1 + O_$k];
```

```
tm_{i_, j_→k_} := Module[{tk},
   𝔼[(τ_i + τ_j) t_k + α_i a_k + α_j a_k, η_i y_k + ξ_j x_k, 1]
    (tSW_{xy, i, j→tk} /. {t_tk → t_k, T_tk → T_k, y_tk → e^(-γ α_i) y_k,
       a_tk → a_k, x_tk → e^(-γ α_j) x_k})];
m_{j_→k_}[ℰ_𝔼] := ℰ ~ B_{j,k} ~ tm_{j,k→k};
```

```
tm_{1,2→3}
```

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/

$$\mathbb{E}\Big[a_3\alpha_1 + a_3\alpha_2 + t_3(\tau_1+\tau_2),\ y_3\eta_1 + e^{-\gamma\alpha_1}y_3\eta_2 + e^{-\gamma\alpha_2}x_3\xi_1 + \frac{(1-T_3)\eta_2\xi_1}{\hbar} + x_3\xi_2,\ 1+\frac{1}{4\hbar}\eta_2\xi_1\big(8\hbar a_3T_3 + 4e^{-\gamma\alpha_1-\gamma\alpha_2}\gamma\hbar^2x_3y_3 + 2e^{-\gamma\alpha_1}\gamma\hbar y_3\eta_2 - 6e^{-\gamma\alpha_1}\gamma\hbar T_3y_3\eta_2 + 2e^{-\gamma\alpha_2}\gamma\hbar x_3\xi_1 - 6e^{-\gamma\alpha_2}\gamma\hbar T_3x_3\xi_1 + \gamma\eta_2\xi_1 - 4\gamma T_3\eta_2\xi_1 + 3\gamma T_3^2\eta_2\xi_1\big)\epsilon + O[\epsilon]^2\Big]$$

```
S[U_, kk_] := S[U, kk] = Module[{OE},
   OE = m3,2,1→1[ExpQU1,$k[η, S1[QU[y1]] /. QU → Times]
       ExpQU2,$k[α, S2[QU[a2]] /. QU → Times]
       ExpQU3,$k[ξ, S3[QU[x3]] /. QU → Times]];
   𝔼[-t1 τ1 + OE⟦1⟧, OE⟦2⟧, OE⟦3⟧] /.
    {η → η1, α → α1, ξ → ξ1}];
tSi_ := S[$U, $k] /. {(v : τ | η | α | ξ)1 → vi,
    (v : t | T | y | a | x)1 → vi};
```

```
tS1
```

$$\mathbb{E}\Big[-a_1\alpha_1 - t_1\tau_1,\ \frac{-e^{\gamma\alpha_1}\hbar y_1\eta_1 - e^{\gamma\alpha_1}\hbar T_1x_1\xi_1 + e^{\gamma\alpha_1}\eta_1\xi_1 - e^{\gamma\alpha_1}T_1\eta_1\xi_1}{\hbar T_1},\ 1 + \frac{1}{4\hbar T_1^2}\big(4e^{\gamma\alpha_1}\gamma\hbar^2T_1y_1\eta_1 - 4e^{\gamma\alpha_1}\hbar^2a_1T_1y_1\eta_1 - 2e^{2\gamma\alpha_1}\gamma\hbar^2y_1^2\eta_1^2 - 4e^{\gamma\alpha_1}\hbar^2a_1T_1^2x_1\xi_1 - 4e^{\gamma\alpha_1}\gamma\hbar T_1\eta_1\xi_1 + 8e^{\gamma\alpha_1}\hbar a_1T_1\eta_1\xi_1 + 4e^{\gamma\alpha_1}\gamma\hbar T_1^2\eta_1\xi_1 - 4e^{2\gamma\alpha_1}\gamma\hbar^2T_1x_1y_1\eta_1\xi_1 + 6e^{2\gamma\alpha_1}\gamma\hbar y_1\eta_1^2\xi_1 - 2e^{2\gamma\alpha_1}\gamma\hbar T_1y_1\eta_1^2\xi_1 - 2e^{2\gamma\alpha_1}\gamma\hbar^2T_1^2x_1^2\xi_1^2 + 6e^{2\gamma\alpha_1}\gamma\hbar T_1x_1\eta_1\xi_1^2 - 2e^{2\gamma\alpha_1}\gamma\hbar T_1^2x_1\eta_1\xi_1^2 - 3e^{2\gamma\alpha_1}\gamma\eta_1^2\xi_1^2 + 4e^{2\gamma\alpha_1}\gamma T_1\eta_1^2\xi_1^2 - e^{2\gamma\alpha_1}\gamma T_1^2\eta_1^2\xi_1^2\big)\epsilon + O[\epsilon]^2\Big]$$

```
Δ[U_, kk_] := Δ[U, kk] = Module[{OE},
   OE = Block[{$k = kk, $p = kk + 1},
     m1,3,5→1@
      m2,4,6→2@Times[ (* Warning:
         wrong unless $p≥$k+1! *)
        ReplacePart[1 → 0]@
         ExpQU1,$k[η, Δ1→1,2[QU[y1]] /. QU → Times],
        ReplacePart[2 → 0]@
         ExpQU3,$k[α, Δ3→3,4[QU[a3]] /. QU → Times],
        ReplacePart[1 → 0]@
         ExpQU5,$k[ξ, Δ5→5,6[QU[x5]] /. QU → Times]
       ] /. {η → η1, α → α1, ξ → ξ1}];
   𝔼[τ1 (t1 + t2) + α1 (a1 + a2), OE⟦2⟧, OE⟦3⟧]];
tΔi_→j_,k_ :=
  Δ[$U, $k] /. {(v : τ | η | α | ξ)1 → vi,
    (v : t | T | y | a | x)1 → vj, (v : t | T | y | a | x)2 → vk};
```

```
tΔ1→1,2
```

$$\mathbb{E}\Big[(a_1+a_2)\alpha_1 + (t_1+t_2)\tau_1,\ y_1\eta_1 + T_1y_2\eta_1 + x_1\xi_1 + x_2\xi_1,\ 1 + \frac{1}{2}\big(-2\hbar a_1T_1y_2\eta_1 + \gamma\hbar T_1y_1y_2\eta_1^2 - 2\hbar a_1x_2\xi_1 + \gamma\hbar x_1x_2\xi_1^2\big)\epsilon + O[\epsilon]^2\Big]$$

The Faddeev-Quesne formula:

```
𝕖q_,k_[x_] := 𝕖^(Sum[(1 - q)^j x^j / (j (1 - q^j)), {j, 1, k+1}]); 𝕖q_[x_] := 𝕖q,$k[x]
```

```
R[QU, kk_] :=
  R[QU, kk] = 𝔼[-ℏ a2 t1/γ, ℏ x2 y1,
    Series[𝕖^(ℏ γ^-1 t1 a2 - ℏ y1 x2)
      (𝕖^(ℏ b1 a2) 𝕖qℏ,kk[ℏ y1 x2] /. b1 → γ^-1 (ϵ a1 - t1)),
     {ϵ, 0, kk}]];
tRi_,j_ :=
  R[$U, $k] /. {(v : t | T | y | a | x)1 → vi,
    (v : t | T | y | a | x)2 → vj};
t̄Ri_,j_ := t̄Ri,j = tRi,j ~Bj~ tSj;
```

```
{tR1,2, t̄R1,2}
```

$$\Big\{\mathbb{E}\Big[-\frac{\hbar a_2t_1}{\gamma},\ \hbar x_2y_1,\ 1 + \Big(\frac{\hbar a_1a_2}{\gamma} - \frac{1}{4}\gamma\hbar^3x_2^2y_1^2\Big)\epsilon + O[\epsilon]^2\Big],$$
$$\mathbb{E}\Big[\frac{\hbar a_2t_1}{\gamma},\ -\frac{\hbar x_2y_1}{T_1},\ 1 + \frac{1}{4\gamma T_1^2}\big(-4\hbar a_1a_2T_1^2 - 4\gamma\hbar^2a_1T_1x_2y_1 - 4\gamma\hbar^2a_2T_1x_2y_1 - 3\gamma^2\hbar^3x_2^2y_1^2\big)\epsilon + O[\epsilon]^2\Big]\Big\}$$

tC is the counterclockwise spinner; $\overline{\text{tC}}$ is its inverse.

```
tCi_ := 𝔼[0, 0, Ti^(1/2) 𝕖^(-ϵ ai ℏ) + O$k];
t̄Ci_ := 𝔼[0, 0, Ti^(-1/2) 𝕖^(ϵ ai ℏ) + O$k];
```

```
Block[{$k = 3}, {tC1, t̄C2}]
```

$$\Big\{\mathbb{E}\Big[0,0,\ \sqrt{T_1} - \hbar a_1\sqrt{T_1}\,\epsilon + \frac{1}{2}\hbar^2a_1^2\sqrt{T_1}\,\epsilon^2 - \frac{1}{6}\big(\hbar^3a_1^3\sqrt{T_1}\big)\epsilon^3 + O[\epsilon]^4\Big],$$
$$\mathbb{E}\Big[0,0,\ \frac{1}{\sqrt{T_2}} + \frac{\hbar a_2\epsilon}{\sqrt{T_2}} + \frac{\hbar^2a_2^2\epsilon^2}{2\sqrt{T_2}} + \frac{\hbar^3a_2^3\epsilon^3}{6\sqrt{T_2}} + O[\epsilon]^4\Big]\Big\}$$

```
Kink[QU, kk_] :=
  Kink[QU, kk] =
   Block[{$k = kk}, (tR1,3 t̄C2) ~B1,2~ tm1,2→1 ~B1,3~ tm1,3→1];
tKinki_ := Kink[$U, $k] /. {(v : t | T | y | a | x)1 → vi};
K̄ink[QU, kk_] :=
  K̄ink[QU, kk] =
   Block[{$k = kk}, (t̄R1,3 tC2) ~B1,2~ tm1,2→1 ~B1,3~ tm1,3→1];
t̄Kinki_ := K̄ink[$U, $k] /. {(v : t | T | y | a | x)1 → vi}
```

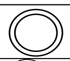
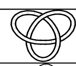
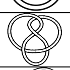
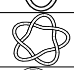
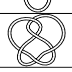
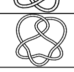
Alternative Algorithms

```
λalt,k_[CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
    eq = ρ@𝕖^(ξ xCU).ρ@𝕖^(η yCU) == ρ@𝕖^(d yCU).ρ@𝕖^(c (t 1CU - 2 ϵ aCU)).ρ@𝕖^(b xCU);
    {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /.
      C@1 → 0;
    Series[𝕖^(-η y - ξ x + η ξ t + c t + d y - 2 ϵ c a + b x) /. so, {ϵ, 0, k}]]];
```

The Trefoil

```
Block[{$k = 1},
 Z = tR1,5 tR6,2 tR3,7 t̄C4 t̄Kink8 t̄Kink9 t̄Kink10;
 Do[Z = Z ~B1,k~ tm1,k→1, {k, 2, 10}]; Z]
```

$$\mathbb{E}\Big[0,0,\ \frac{T_1}{1-T_1+T_1^2} + \big(\big(-2\hbar a_1T_1 - \gamma\hbar T_1^2 + 2\hbar a_1T_1^2 + 2\gamma\hbar T_1^3 - 3\gamma\hbar T_1^4 - 2\hbar a_1T_1^4 + 2\gamma\hbar T_1^5 + 2\hbar a_1T_1^5 - 2\gamma\hbar^2T_1x_1y_1 - 2\gamma\hbar^2T_1^4x_1y_1\big)\epsilon\big)\Big/\big(1 - 3T_1 + 6T_1^2 - 7T_1^3 + 6T_1^4 - 3T_1^5 + T_1^6\big) + O[\epsilon]^2\Big]$$

diagram	n_k^t Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^t Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral
	0_1^a 0	1	0 / ✔ 0 / ✔		3_1^a t	$t-1$	1 / ✘ 1 / ✘
	4_1^a 0	$3-t$	1 / ✘ 1 / ✔		5_1^a $2t^3+3t$	t^2-t+1	2 / ✘ 2 / ✘
	5_2^a $5t-4$	$2t-3$	1 / ✘ 1 / ✘		6_1^a $t-4$	$5-2t$	1 / ✔ 1 / ✘

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/

The Dogma is Wrong

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. Preliminary writeup [BV1], fuller writeup [BV2]. More at ωεβ/talks.

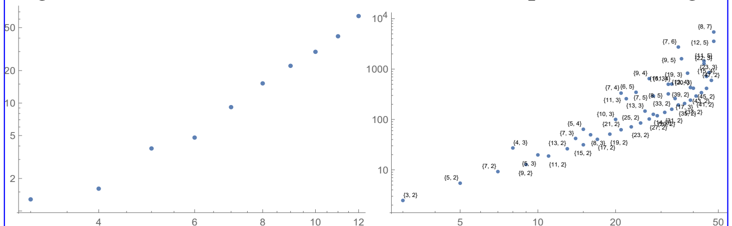
Happy Birthday Anton!

ωεβ:=http://drorbn.net/ld17/

Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: "quantize and use representation theory". We present an alternative and better procedure: "centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra". While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

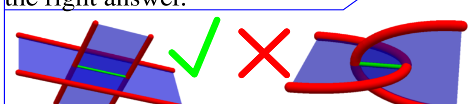
Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Power. On the 250 knots with at most 10 crossings, the pair (ω,ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

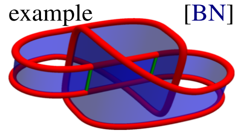
Genus. Up to 12 xings, always ρ_1 is symmetric under $t\leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg\rho_1^+\leq 2g-1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

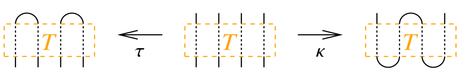
Ribbon Knots.

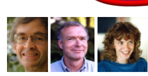


a ribbon singularity a clasp singularity

example [BN]





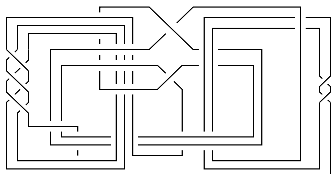


Gompf, Scharlemann, Thompson [GST]

$$\begin{array}{ccc} U\in\mathcal{T}_n & & 1\in\mathcal{A}_n \\ {\scriptstyle\tau}\nearrow & & {\scriptstyle\tau}\nearrow \\ \mathcal{T}_{2n} & \xrightarrow{z} & \mathcal{A}_{2n} \\ {\scriptstyle\kappa}\searrow & & {\scriptstyle\kappa}\searrow \\ \text{ribbon } K\in\mathcal{T}_1 & & z(K)\in\mathcal{R}\subseteq\mathcal{A}_1 \end{array}$$

with $\mathcal{R}:=\kappa(\tau^{-1}(1))$

[Vo]: Works for Alexander!



$A^+=-t^8+2t^7-t^6-2t^4+5t^3-2t^2-7t+13$

$\rho_1^+=5t^{15}-18t^{14}+33t^{13}-32t^{12}+2t^{11}+42t^{10}-62t^9-8t^8+166t^7-242t^6+108t^5+132t^4-226t^3+148t^2-11t-36$

Faster is better, leaner is meaner!

Ordering Symbols. $\mathbb{O}(poly\mid specs)$ plants the variables of *poly* in $S(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,

$$\mathbb{O}\left(a_1^3y_1a_2e^{y_3}x_3^9\mid x_3a_1\otimes y_1y_3a_2\right)=x^9a^3\otimes ye^ya\in\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$$

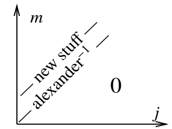
This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

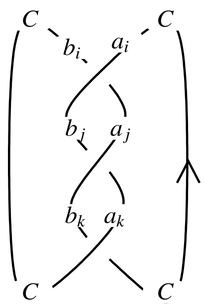
$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar}=\sum_{j,m\geq 0}a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q)=\frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R=\sum a_i\otimes b_i\in U\otimes U\quad\text{and}\quad C\in U,$$

form

$$Z=\sum_{i,j,k}Ca_ib_ja_kC^2b_ia_jb_kC.$$

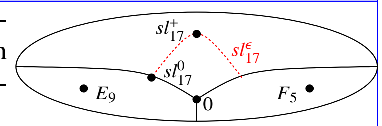
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

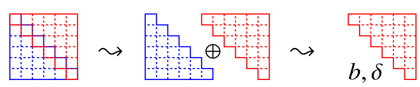
The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

$$m_k^{ij}\circlearrowright\{\mathcal{F}_S\}\xrightarrow{\mathbb{E}}\{U^{\otimes S}\}\circlearrowleft m_k^{ij}$$

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2}\otimes V$ is on the right. We care about $sl_{17}^k:=sl_{17}^\epsilon/(\epsilon^{k+1}=0)$.



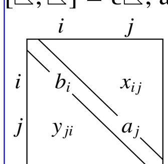
Recomposing gl_n. Half is enough! $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(◹,b,\delta)$:



$b(◹)=b$: $◹\otimes◹\to◹$

$b(◺)\rightsquigarrow\delta$: $◹\to◹\otimes◹$

Now define $gl_n^\epsilon:=\mathcal{D}(◹,b,\epsilon\delta)$. Schematically, this is $[◹,◹]=◹$, $[◺,◺]=\epsilon◺$, and $[◹,◺]=◺+\epsilon◹$. In detail, it is



$$[x_{ij},x_{kl}]=\delta_{jk}x_{il}-\delta_{li}x_{kj}\qquad[y_{ij},y_{kl}]=\epsilon\delta_{jk}y_{il}-\epsilon\delta_{li}y_{kj}$$

$$[x_{ij},y_{kl}]=\delta_{jk}(\epsilon\delta_{j<k}x_{il}+\delta_{il}(b_i+\epsilon a_i)/2+\delta_{i>l}y_{il})$$
$$-\delta_{li}(\epsilon\delta_{k<j}x_{kj}+\delta_{kj}(b_j+\epsilon a_j)/2+\delta_{k>j}y_{kj})$$

$$[a_i,x_{jk}]=(\delta_{ij}-\delta_{ik})x_{jk}\qquad[b_i,x_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})x_{jk}$$

$$[a_i,y_{jk}]=(\delta_{ij}-\delta_{ik})y_{jk}\qquad[b_i,y_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})y_{jk}$$

The Main sl_2 Theorem. Let $\mathfrak{g}^\epsilon=\langle t,y,a,x\rangle/([t,\cdot]=0,\ [a,x]=x,\ [a,y]=-y,\ [x,y]=t-2\epsilon a)$ and let $\mathfrak{g}_k=\mathfrak{g}^\epsilon/(\epsilon^{k+1}=0)$. The $\mathfrak{g}_k$-invariant of any S-component tangle K can be written in the form $Z(K)=\mathbb{O}\left(\omega\mathbb{e}^{L+Q+P}\colon\bigotimes_{i\in S}y_ia_ix_i\right)$, where ω is a scalar (a rational function in the variables t_i and their exponentials $T_i:=\mathbb{e}^{t_i}$), where $L=\sum l_{ij}t_ia_j$ is a quadratic in t_i and a_j with integer coefficients l_{ij}, where $Q=\sum q_{ij}y_ix_j$ is a quadratic in the variables y_i and x_j with scalar coefficients q_{ij}, and where P is a polynomial in $\{\epsilon,y_i,a_i,x_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\{y_i,\sqrt{a_i},x_i\}$. Furthermore, after setting $t_i=t$ and $T_i=T$ for all i, the invariant $Z(K)$ is poly-time computable.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1708/

The PBW Problem. In $\mathcal{U}(\mathfrak{g}^\epsilon)$, bring $Z = y^3a^2x^2 \cdot y^2a^2x$ to yax-order. In other words, find $g \in \mathbb{Z}[\epsilon, t, y, a, x]$ such that $Z = \mathbb{O}(f = y_1^3y_2^2a_1^2a_2^2x_1^2x_2 \colon y_1a_1x_1y_2a_2x_2) = \mathbb{O}(g \colon yax)$.

Solution, Part 1. In $\hat{\mathcal{U}}(\mathfrak{g}^\epsilon)$ we have

$$X_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2} := \mathbb{e}^{\tau_1 t}\mathbb{e}^{\eta_1 y}\mathbb{e}^{\alpha_1 a}\mathbb{e}^{\xi_1 x}\mathbb{e}^{\tau_2 t}\mathbb{e}^{\eta_2 y}\mathbb{e}^{\alpha_2 a}\mathbb{e}^{\xi_2 x} = \mathbb{e}^{\tau t}\mathbb{e}^{\eta y}\mathbb{e}^{\alpha a}\mathbb{e}^{\xi x} =: Y_{\tau,\eta,\alpha,\xi},$$

where τ, η, α, ξ are ugly functions of $\tau_1, \eta_i, \alpha_i, \xi_i$:

$$\begin{aligned}
\tau &= \tau_1 + \tau_2 - \frac{\log(1-\epsilon\eta_2\xi_1)}{\epsilon} = \tau_1 + \tau_2 + \eta_2\xi_1 + \frac{\epsilon}{2}\eta_2^2\xi_1^2 + \dots, \\
\eta &= \eta_1 + \frac{\mathbb{e}^{-\alpha_1}\eta_2}{(1-\epsilon\eta_2\xi_1)} = \eta_1 + \mathbb{e}^{-\alpha_1}\eta_2 + \epsilon\mathbb{e}^{-\alpha_1}\eta_2^2\xi_1 + \dots, \\
\alpha &= \alpha_1 + \alpha_2 + 2\log(1-\epsilon\eta_2\xi_1) = \alpha_1 + \alpha_2 - 2\epsilon\eta_2\xi_1 + \dots, \\
\xi &= \frac{\mathbb{e}^{-\alpha_2}\xi_1}{(1-\epsilon\eta_2\xi_1)} + \xi_2 = \mathbb{e}^{-\alpha_2}\xi_1 + \xi_2 + \epsilon\mathbb{e}^{-\alpha_2}\eta_2\xi_1^2 + \dots.
\end{aligned}$$

Note 1. This defines a mapping $\Phi \colon \mathbb{R}^8_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2} \to \mathbb{R}^4_{\tau,\eta,\alpha,\xi}$.

Proof. $\mathfrak{g}^\epsilon$ has a 2D representation ρ:

```
ρt = (1 0
      0 1); ρy = ( 0  0
                  -ϵ  0);
ρa = ((1 + 1/ϵ)/2        0
          0        -(1 - 1/ϵ)/2); ρx = (0 1
                                         0 0);
Simplify@{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy,
  ρx.ρy - ρy.ρx == ρt - 2 ϵ ρa}

{True, True, True}
```

It is enough to verify the desired identity in ρ:

```
ME = MatrixExp;
Simplify[
 ME[τ1 ρt].ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[τ2 ρt].
    ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] ==
   ME[τ0 ρt].ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx] /.
  {τ0 → -Log[1-ϵ η2 ξ1]/ϵ + τ1 + τ2, η0 → η1 + e^-α1 η2/(1-ϵ η2 ξ1),
   α0 → 2 Log[1 - ϵ η2 ξ1] + α1 + α2, ξ0 → e^-α2 ξ1/(1-ϵ η2 ξ1) + ξ2}]

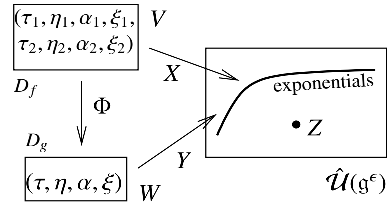
True
```

Solution, Part 2. But now, with $D_f = f(z \mapsto \partial_\zeta) = \partial^3_{\eta_1}\partial^2_{\alpha_1}\partial^2_{\xi_1}\partial^2_{\eta_2}\partial^2_{\alpha_2}\partial_{\xi_2}$,

$$\begin{aligned}
Z &= D_f X_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2}\big|_{vs=0} = D_f Y_{\tau,\eta,\alpha,\xi}\big|_{vs=0} \\
&= \mathbb{O}\left(D_f \mathbb{e}^{\tau t}\mathbb{e}^{\eta y}\mathbb{e}^{\alpha a}\mathbb{e}^{\xi x}\big|_{vs=0} \colon yax\right) = \mathbb{O}(g \colon yax) :
\end{aligned}$$

```
Expand[∂{η1,3} ∂{α1,2} ∂{ξ1,2} ∂{η2,2} ∂{α2,2} ∂{ξ2,1} Exp[
    (-Log[1-ϵ η2 ξ1]/ϵ + τ1 + τ2) t + (η1 + e^-α1 η2/(1-ϵ η2 ξ1)) y +
     (2 Log[1 - ϵ η2 ξ1] + α1 + α2) a + (e^-α2 ξ1/(1-ϵ η2 ξ1) + ξ2) x
   ] /. (τ | η | α | ξ)1|2 → 0]

2 a^4 t^2 x y^3 + 4 t x^2 y^4 - 16 a t x^2 y^4 + 24 a^2 t x^2 y^4 - 16 a^3 t x^2 y^4 +
 4 a^4 t x^2 y^4 + 16 x^3 y^5 - 32 a x^3 y^5 + 24 a^2 x^3 y^5 - 8 a^3 x^3 y^5 + a^4 x^3 y^5 +
 2 a^4 t x y^3 ϵ - 8 a^5 t x y^3 ϵ + 8 x^2 y^4 ϵ - 40 a x^2 y^4 ϵ + 80 a^2 x^2 y^4 ϵ -
 80 a^3 x^2 y^4 ϵ + 40 a^4 x^2 y^4 ϵ - 8 a^5 x^2 y^4 ϵ - 4 a^5 x y^3 ϵ^2 + 8 a^6 x y^3 ϵ^2
```



Note 2. Replacing $f \to D_f$ (and likewise $g \to D_g$), we find that $D_g = \Phi_* D_f$.

Note 3. The two great evils of mathematics are non-commutativity and non-linearity. We traded one for the other.

Note 4. We could have done similarly with $\mathbb{e}^{\tau_1 t}\mathbb{e}^{\eta_1 y}\mathbb{e}^{\alpha_1 a}\mathbb{e}^{\xi_1 x} = \mathbb{e}^{\tau t + \eta y + \alpha a + \xi x}$, and with $S(\mathbb{e}^{\tau_1 t}\mathbb{e}^{\eta_1 y}\mathbb{e}^{\alpha_1 a}\mathbb{e}^{\xi_1 x})$, $\Delta(\mathbb{e}^{\tau_1 t}\mathbb{e}^{\eta_1 y}\mathbb{e}^{\alpha_1 a}\mathbb{e}^{\xi_1 x})$, $\prod_{i=1}^5 \mathbb{e}^{\tau_i t}\mathbb{e}^{\eta_i y}\mathbb{e}^{\alpha_i a}\mathbb{e}^{\xi_i x}$.

Fact. $R_{12} \to \exp\left(\partial_{\tau_1}\partial_{\alpha_2} + \partial_{y_1}\partial_{x_2}\right)(1 + \sum_{d\geq 1}\epsilon^d p_d)$, where the p_d are computable polynomials of a-priori bounded degrees.

Moral. We need to understand the pushforwards via maps like Φ of (formally ∞-order) "differential operators at 0", that in themselves are perturbed Gaussians. This turns out to be the same problem as "0-dimensional QFT" (except no integration is ever needed), and if $\epsilon^{k+1} = 0$, it is explicitly soluble.

References.

[BN] D. Bar-Natan, *Polynomial Time Knot Polynomial,* research proposal for the 2017 Killam Fellowship, ωεβ/K17.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BV1] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* arXiv:1708.04853.

[BV2] D. Bar-Natan and R. van der Veen, *Poly-Time Knot Polynomials Via Solvable Approximations,* in preparation.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

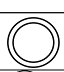
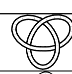
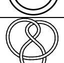
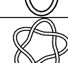
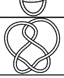
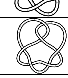
[Vo] H. Vo, University of Toronto Ph.D. thesis, in preparation.

dog·ma (dôg′mə, dŏg′-) — The Free Dictionary, ωεβ/TFD

n. pl. **dog·mas** or **dog·ma·ta** (-mə-tə)

1. A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.

2. A principle or statement of ideas, or a group of such principles or statements, especially when considered to be authoritative or accepted uncritically: *"Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry" (Bertrand Russell).*

diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral
	0_1^a 0	1	0 / ✔ 0 / ✔		3_1^a t	$t-1$	1 / ✘ 1 / ✘
	4_1^a 0	$3-t$	1 / ✘ 1 / ✔		5_1^a $2t^3+3t$	t^2-t+1	2 / ✘ 2 / ✘
	5_2^a $5t-4$	$2t-3$	1 / ✘ 1 / ✘		6_1^a $t-4$	$5-2t$	1 / ✔ 1 / ✘

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1708/`

The Dogma is Wrong

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at ωεβ/talks.

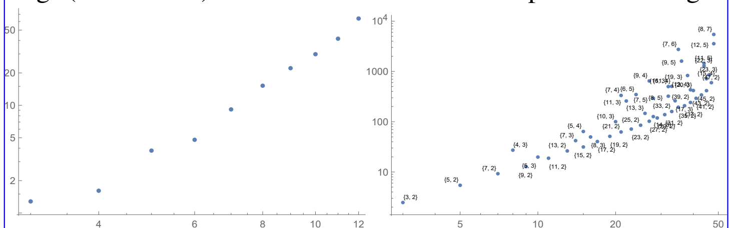
Thanks for the invitation!

ωεβ:=http://drorbn.net/Toulouse-1705/

Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: "quantize and use representation theory". We present an alternative and better procedure: "centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra". While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

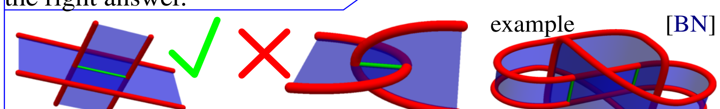
Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



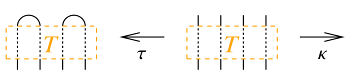
Power. On the 250 knots with at most 10 crossings, the pair (ω,ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg\rho_1^+ \leq 2g-1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

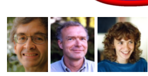
Ribbon Knots.



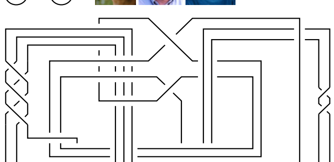
a ribbon singularity — a clasp singularity — example [BN]



$$\begin{array}{ccc} & U\in\mathcal{T}_n & & 1\in\mathcal{A}_n \\ \mathcal{T}_{2n} & \overset{\tau}{\nearrow}\ \underset{\kappa}{\searrow} & \xrightarrow{z} \mathcal{A}_{2n} & \overset{\tau}{\nearrow}\ \underset{\kappa}{\searrow} \\ & \text{ribbon } K\in\mathcal{T}_1 & & z(K)\in\mathcal{R}\subseteq\mathcal{A}_1\end{array}$$



Gompf, Scharlemann, Thompson [GST]



[Vo]: Works for Alexander!  with $\mathcal{R}:=\kappa(\tau^{-1}(1))$

$A^+ = -t^8+2t^7-t^6-2t^4+5t^3-2t^2-7t+13$

$\rho_1^+ = 5t^{15}-18t^{14}+33t^{13}-32t^{12}+2t^{11}+42t^{10}-62t^9-8t^8+166t^7-242t^6+108t^5+132t^4-226t^3+148t^2-11t-36$

Faster is better, leaner is meaner!

dog·ma (dôg′mə, dŏg′-) — The Free Dictionary, ωεβ/TFD

n. pl. **dog·mas** or **dog·ma·ta** (-mə-tə)

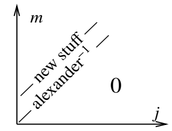
1. A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.
2. A principle or statement of ideas, or a group of such principles or statements, especially when considered to be authoritative or accepted uncritically: *"Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry" (Bertrand Russell).*

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

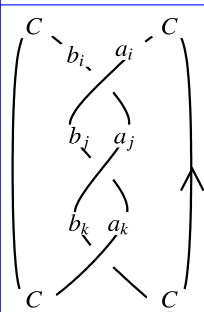
$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.



"Above diagonal" we have **Rozansky's Theorem** [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty \frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$



The Yang-Baxter Technique. Given an algebra A (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R=\sum a_i\otimes b_i\in A\otimes A \quad\text{and}\quad C\in A,$$

form

$$Z=\sum_{i,j,k} Ca_ib_ja_kC^2b_ia_jb_kC.$$

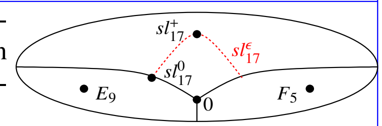
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

$$m_k^{ij} \circlearrowright \{\mathcal{F}_S\} \xrightarrow{\mathbb{E}} \{A^{\otimes S}\} \circlearrowleft m_k^{ij}$$

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2}\otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^\epsilon/(\epsilon^{k+1}=0)$.



Why are "solvable algebras" any good? Contrary to common beliefs, computations in semi-simple Lie algebras are just awful:

```
In[1]:= MatrixExp[( a b
                    c d )] // FullSimplify // MatrixForm        Enter
```

Yet in solvable algebras, exponentiation is fine and even BCH, $z=\log(e^xe^y)$, is bearable:

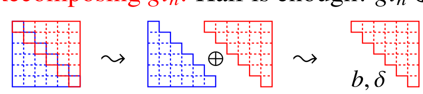
```
In[2]:= MatrixExp[( a b
                    0 c )] // MatrixForm
```

Out[2]//MatrixForm=

$$\begin{pmatrix} e^a & \frac{b(e^a-e^c)}{a-c} \\ 0 & e^c\end{pmatrix}$$

```
In[3]:= MatrixExp[( a1 b1
                    0  c1 )].MatrixExp[( a2 b2
                                         0  c2 )] //
          MatrixLog // PowerExpand // Simplify //
        MatrixForm                                       Enter
```

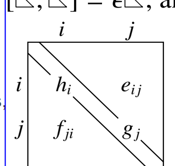
Recomposing gl_n. Half is enough! $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(\lnot,b,\delta)$:



$b(\lnot)=b: \lnot\otimes\lnot\to\lnot$

$b(\llcorner)\rightsquigarrow\delta: \lnot\to\lnot\otimes\lnot$

Now define $gl_n^\epsilon := \mathcal{D}(\lnot,b,\epsilon\delta)$. Schematically, this is $[\lnot,\lnot]=\lnot$, $[\llcorner,\llcorner]=\epsilon\llcorner$, and $[\lnot,\llcorner]=\llcorner+\epsilon\lnot$. In detail, it is



$[e_{ij},e_{kl}]=\delta_{jk}e_{il}-\delta_{li}e_{kj}$ $\quad$ $[f_{ij},f_{kl}]=\epsilon\delta_{jk}f_{il}-\epsilon\delta_{li}f_{kj}$

$[e_{ij},f_{kl}]=\delta_{jk}(\epsilon\delta_{j<k}e_{il}+\delta_{il}(h_i+\epsilon g_i)/2+\delta_{i>l}f_{il})$
$\quad -\delta_{li}(\epsilon\delta_{k<j}e_{kj}+\delta_{kj}(h_j+\epsilon g_j)/2+\delta_{k>j}f_{kj})$

$[g_i,e_{jk}]=(\delta_{ij}-\delta_{ik})e_{jk}$ $\quad$ $[h_i,e_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})e_{jk}$

$[g_i,f_{jk}]=(\delta_{ij}-\delta_{ik})f_{jk}$ $\quad$ $[h_i,f_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})f_{jk}$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Toulouse-1705/

The sl_2 Example. Let $\mathfrak{g}^\epsilon = \langle h,e,l,f\rangle/([h,\cdot]=0,\ [e,l]=-e,\ [f,l]=f,\ [e,f]=h-2\epsilon l)$ and let $\mathfrak{g}_k=\mathfrak{g}^\epsilon/(\epsilon^{k+1}=0)$.

The Main $\mathfrak{g}_k$ Theorem. The $\mathfrak{g}_k$-invariant of any S-component tangle T can be written in the form

$$Z(T)=\mathbb{O}\left(\omega\mathbb{e}^{L+Q+P}\colon \bigotimes_{i\in S} e_il_if_i\right),$$

where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i:=\mathbb{e}^{h_i}$), where $L=\sum a_{ij}h_il_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients, where $Q=\sum b_{ij}e_if_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}, and where P is a polynomial in $\{\epsilon,e_i,l_i,f_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\{e_i,\sqrt{l_i},f_i\}$. Furthermore, after setting $h_i=h$ and $t_i=t$ for all i, the invariant $Z(T)$ is poly-time computable.

The Main $\mathfrak{g}_k$ Lemma. The following "re-ordering relations" hold:

$$\mathbb{O}\left(\mathbb{e}^{\gamma l+\beta e}\colon le\right)=\mathbb{O}\left(\mathbb{e}^{\gamma l+\mathbb{e}^\gamma\beta e}\colon el\right)\quad\text{(and similarly for } fl\to lf\text{)},$$

$$\mathbb{O}\left(\mathbb{e}^{\beta e+\alpha f+\delta ef}\colon fe\right)=\mathbb{O}\left(\nu\mathbb{e}^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta ef)+\lambda_k(\epsilon,e,l,f,\alpha,\beta,\delta)}\colon elf\right),$$

with $\nu=(1+h\delta)^{-1}$ and where $\lambda_k(\epsilon,e,l,f,\alpha,\beta,\delta)$ is some fixed polynomial of degree at most $2k+2$ in $\epsilon,e,\sqrt{l},f,\alpha,\beta,\delta$, with scalar coefficients.

Demo Programs.

ωεβ/Demo

Formatting

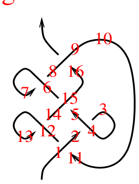
```
CF[ℰ_] := Module[{vars = Union@Cases[ℰ, e_ | l_ | f_, ∞]},
  If[vars === {}, Factor[ℰ],
   Total[CoefficientRules[ℰ, vars] /.
     (p_ → c_) ⧴ Factor[c] Times @@ (vars^p)] ]];
CF[ℰ_𝔼] := CF /@ ℰ;
```

Preparation

```
𝔼[i_, j_, s_] := 𝔼[1, (-1)^s l_j, (-t)^s e_i f_j,
   t^s e_i l_(1+s) i-s j f_j + (-1)^s l_i l_j + (-t^2)^s e_i^2 f_j^2/4];
𝔼[i_, s_] := 𝔼[1, 0, 0, s l_i];
𝔼 /: 𝔼[1, L1_, Q1_, P1_] 𝔼[1, L2_, Q2_, P2_] :=
  𝔼[1, L1 + L2, Q1 + Q2, P1 + P2];
```

Preparing the Trefoil

```
z1 = (𝔼[1, 11, 0] 𝔼[4, 2, -1] 𝔼[15, 5, 0] ×
   𝔼[6, 8, -1] 𝔼[9, 16, 0] 𝔼[12, 14, -1] ×
   𝔼[3, -1] 𝔼[7, +1] 𝔼[10, -1] 𝔼[13, +1])
```



$$\mathbb{E}\Big[1,\ -l_2+l_5-l_8+l_{11}-l_{14}+l_{16},\ -\frac{e_4f_2}{t}+e_{15}f_5-\frac{e_6f_8}{t}+e_1f_{11}-\frac{e_{12}f_{14}}{t}+e_9f_{16},\ -\frac{e_4^2f_2^2}{4t^2}+\frac14 e_{15}^2f_5^2-\frac{e_6^2f_8^2}{4t^2}+\frac14 e_1^2f_{11}^2-\frac{e_{12}^2f_{14}^2}{4t^2}+\frac14 e_9^2f_{16}^2+e_1f_{11}l_1+\frac{e_4f_2l_2}{t}-l_3-l_2l_4+l_7+\frac{e_6f_8l_8}{t}-l_6l_8+e_9f_{16}l_9-l_{10}+l_1l_{11}+l_{13}+\frac{e_{12}f_{14}l_{14}}{t}-l_{12}l_{14}+e_{15}f_5l_{15}+l_5l_{15}+l_9l_{16}\Big]$$

Differential Polynomials

(Implementing $P(\partial_\alpha,\partial_\beta)(f)$)

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] :=
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

le and *fl* Sorts

```
S_{1_j_ (x:e|f)_i_ → k_}[𝔼[ω_, L_, Q_, P_]] :=
 With[{λ = ∂_{1_j} L, α = ∂_{x_i} Q, q = 𝕖^γ β x_k + γ l_k}, CF[
   𝔼[ω, L /. l_j → l_k, t^λ α x_k + (Q /. x_i → 0),
    𝕖^-q DP_{l_j→D_γ, x_i→D_β}[P][𝕖^q] /. {β → α/ω, γ → λ Log[t]}] ]];
```

The Λόγος

```
Λ[k_] := ((t - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 e_k l_k f_k δ^2 μ^2 -
    δ (1 + μ) (f_k^2 α^2 + e_k^2 β^2) - e_k^2 f_k^2 δ^3 (1 + 3 μ) -
    2 (α β + 2 δ μ + e_k f_k δ^2 (1 + 2 μ) + 2 l_k δ μ^2) (f_k α + e_k β) -
    4 (l_k μ^2 + e_k f_k δ (1 + μ)) (α β + δ μ)) (1 + t) / 4;
```

fe Sorts

```
S_{f_i_ e_j_ → k_}[𝔼[ω_, L_, Q_, P_]] :=
 With[{q = ((1 - t) α β + β e_k + α f_k + δ e_k f_k) / μ}, CF[
   𝔼[μ ω, L, μ ω q + μ (Q /. f_i | e_j → 0),
     μ^4 𝕖^-q DP_{f_i→D_α, e_j→D_β}[P][𝕖^q] + ω^4 Λ[k]] /. μ → 1 + (t - 1) δ /.
    {α → ω^-1 (∂_{f_i} Q /. e_j → 0), β → ω^-1 (∂_{e_j} Q /. f_i → 0),
     δ → ω^-1 ∂_{f_i, e_j} Q}]];
```

Elf Merges

```
m_{i_, j_ → k_}[Z_𝔼] := Module[{x, z},
  CF[(Z // S_{f_i e_j → x} // S_{1_i e_x → x} // S_{f_x 1_j → x}) /. z_{-i|j|x} → z_k]]
```

Rewriting the Trefoil

(by merging 16 elves)

```
(Do[z1 = z1 // m_{1,k→1}, {k, 2, 16}]; z1)
```

$$\mathbb{E}\Big[\frac{1-t+t^2}{t},\ 0,\ 0,\ \frac{(-1+t)\left(1-t+t^2\right)^2\left(1-t+2t^2\right)}{t^3}-\frac{2(1+t)\left(1-t+t^2\right)^3e_1f_1}{t^4}-\frac{2(-1+t)(1+t)\left(1-t+t^2\right)^3l_1}{t^4}\Big]$$

Readout

```
ρ1[𝔼[ω_, _, _, P_]] := CF[ t ((P /. e_ | l_ | f_ → 0) - t ω^3 (∂_t ω)) / ((t - 1)^2 ω^2) ]
```

$\rho_1(3_1)$

```
ρ1[z1] // Expand
```

$\frac{1}{t}+t$

References.

[BN] D. Bar-Natan, *Polynomial Time Knot Polynomial,* research proposal for the 2017 Killam Fellowship, ωεβ/K17.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

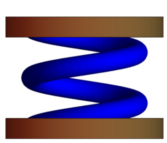
[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Vo] H. Vo, University of Toronto Ph.D. thesis, in preparation.



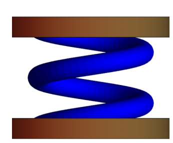
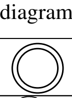
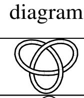
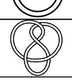
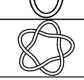


diagram	n_k^t Alexander's ω^+ / Today's / Rozansky's ρ_1^+	genus / ribbon / unknotting number / amphicheiral	diagram	n_k^t Alexander's ω^+ / Today's / Rozansky's ρ_1^+	genus / ribbon / unknotting number / amphicheiral
	0_1^a 1 0	0 / ✔ 0 / ✔		3_1^a $t-1$ t	1 / ✘ 1 / ✘
	4_1^a $3-t$ 0	1 / ✘ 1 / ✔		5_1^a t^2-t+1 $2t^3+3t$	2 / ✘ 2 / ✘

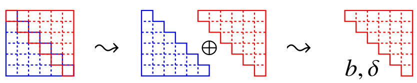
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Toulouse-1705/

What else can you do with solvable approximations?

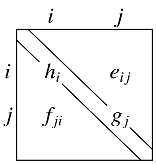
Abstract. Recently, Roland van der Veen and myself found that there are sequences of solvable Lie algebras "converging" to any given semi-simple Lie algebra (such as sl_2 or sl_3 or $E8$). Certain computations are much easier in solvable Lie algebras; in particular, using solvable approximations we can compute in polynomial time certain projections (originally discussed by Rozansky) of the knot invariants arising from the Chern-Simons-Witten topological quantum field theory. This provides us with the first strong knot invariants that are computable for truly large knots.

But sl_2 and sl_3 and similar algebras occur in physics (and in mathematics) in many other places, beyond the Chern-Simons-Witten theory. Do solvable approximations have further applications?

Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon ◺$, and $[◹, ◺] = ◺ + \epsilon ◹$. In detail, it is



$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj} \qquad [f_{ij}, f_{kl}] = \epsilon\delta_{jk}f_{il} - \epsilon\delta_{li}f_{kj}$$
$$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon\delta_{j<k}e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l}f_{il}) - \delta_{li}(\epsilon\delta_{k<j}e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j}f_{kj})$$
$$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik})e_{jk} \qquad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})e_{jk}$$
$$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik})f_{jk} \qquad [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})f_{jk}$$

Solvable Approximation. At $\epsilon = 1$ and modulo $h = g$, the above is just gl_n. By rescaling at $\epsilon \neq 0$, gl_n^ϵ is independent of ϵ. We let gl_n^k be gl_n^ϵ regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1} = 0$. It is the "$k$-smidgen solvable approximation" of gl_n!

Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_3 := [\mathfrak{g}_2, \mathfrak{g}_2]$, …, $\mathfrak{g}_d = 0$. Equivalently, if it is a subalgebra of some large-size ◹ algebra.

Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

Why are "solvable algebras" any good? Contrary to common beliefs, computations in semi-simple Lie algebras are just awful:

```
In[1]:= MatrixExp[(a b
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```

Yet in solvable algebras, exponentiation is fine and even BCH, $z = \log(e^x e^y)$, is bearable:

```
In[2]:= MatrixExp[(a b
                   0 c)] // MatrixForm
```

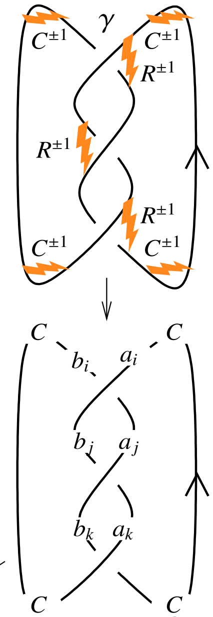
Out[2]//MatrixForm=
$$\begin{pmatrix} e^a & \frac{b(e^a - e^c)}{a-c} \\ 0 & e^c \end{pmatrix}$$

```
In[3]:= MatrixExp[(a1 b1
                   0  c1)].MatrixExp[(a2 b2
                                      0  c2)] //
        MatrixLog // PowerExpand // Simplify //
      MatrixForm                                           Enter
```

Question. What else can you do with solvable approximation? Chern-Simons-Witten theory is often "solved" using ideas from conformal field theory and using quantization of various moduli spaces. Does it make sense to use solvable approximation there too? Elsewhere in physics? Elsewhere in mathematics?

See Also. Talks at George Washington University [ωεβ/gwu], Indiana [ωεβ/ind], and Les Diablerets [ωεβ/ld], and a University of Toronto "Algebraic Knot Theory" class [ωεβ/akt].

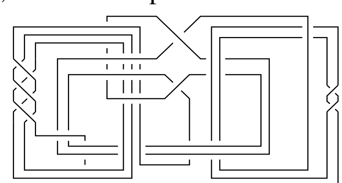
Chern-Simons-Witten. Given a knot $\gamma(t)$ in $\mathbb{R}^3$ and a metrized Lie algebra $\mathfrak{g}$, set $Z(\gamma) :=$
$$\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\, e^{ik\,cs(A)} PExp_\gamma(A),$$
where $cs(A) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(AdA + \frac{2}{3}A^3\right)$ and
$$PExp_\gamma(A) := \prod_0^1 \exp(\gamma^* A) \in \mathcal{U} = \hat{\mathcal{U}}(\mathfrak{g}),$$
and $\mathcal{U}(\mathfrak{g}) := \langle\text{words in } \mathfrak{g}\rangle/(xy - yx = [x, y])$. In a favourable gauge, one may hope that this computation will localize near the crossings and the bends, and all will depend on just two quantities,
$$R = \sum a_i \otimes b_i \in \mathcal{U} \otimes \mathcal{U} \quad \text{and} \quad C \in \mathcal{U}.$$
This was never done formally, yet R and C can be "guessed" and all "quantum knot invariants" arise in this way. So for the trefoil,
$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$



But Z lives in $\mathcal{U}$, a complicated space. How do you extract information out of it?

Solution 1, Representation Theory. Choose a finite dimensional representation ρ of $\mathfrak{g}$ in some vector space V. By luck and the wisdom of Drinfel'd and Jimbo, $\rho(R) \in V^* \otimes V^* \otimes V \otimes V$ and $\rho(C) \in V^* \otimes V$ are computable, so Z is computable too. But in exponential time!

Ribbon=Slice?



Gompf, Scharlemann, Thompson

Solution 2, Solvable Approximation. Work directly in $\hat{\mathcal{U}}(\mathfrak{g}_k)$, where $\mathfrak{g}_k = sl_2^k$ (or a similar algebra); everything is expressible using low-degree polynomials in a small number of variables, hence everything is poly-time computable!

Example 0. Take $\mathfrak{g}_0 = sl_2^0 = \mathbb{Q}\langle h, e, l, f\rangle$, with h central and $[f, l] = f$, $[e, l] = -e$, $[e, f] = h$. In it, using normal orderings,
$$R = \mathbb{O}\left(\exp\left(hl + \frac{e^h - 1}{h} ef\right) \mid e \otimes lf\right), \quad \text{and,}$$
$$\mathbb{O}\left(e^{\delta ef} \mid fe\right) = \mathbb{O}\left(\nu e^{\nu\delta ef} \mid ef\right) \quad \text{with } \nu = (1 + h\delta)^{-1}.$$

Example 1. Take $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$ and $\mathfrak{g}_1 = sl_2^1 = R\langle h, e, l, f\rangle$, with h central and $[f, l] = f$, $[e, l] = -e$, $[e, f] = h - 2\epsilon l$. In it,
$$\mathbb{O}\left(e^{\delta ef} \mid fe\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\delta\Lambda/2)e^{\nu\delta ef} \mid elf\right), \quad \text{where } \Lambda \text{ is}$$
$$4\nu^3\delta^2e^2f^2 + 3\nu^3\delta^3he^2f^2 + 8\nu^2\delta ef + 4\nu^2\delta^2hef + 4\nu\delta elf - 2\nu\delta h + 4l.$$

Fact. Setting $h_i = h$ (for all i) and $t = e^h$, the $\mathfrak{g}_1$ invariant of any tangle T can be written in the form
$$Z_{\mathfrak{g}_1}(T) = \mathbb{O}\left(\omega^{-1}e^{hL + \omega^{-1}Q}(1 + \epsilon\omega^{-4}P) \mid \bigotimes_i e_i l_i f_i\right),$$
where L is linear, Q quadratic, and P quartic in the $\{e_i, l_i, f_i\}$ with ω and all coefficients polynomials in t. Furthermore, everything is poly-time computable.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/McGill-1702/

Dror Bar-Natan: Talks: GWU-1612:

On Elves and Invariants

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen.

ωεβ:=http://drorbn.net/GWU-1612/

Work in Progress! Fluid! Help Needed!

Abstract. Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

Three steps to the computation of ρ_1:

1. Preparation. Given K, results

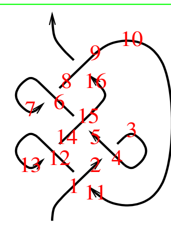
$\langle$*long word* $\|$ *simple formulas*$\rangle$.

2. Rewrite rules. Make the *word* simpler and the *formulas* more complicated, until the *word* "*elf*" is reached.

3. Readout. The invariant ρ_1 is read from the last *formulas*.

$$\begin{array}{c} \text{Knot } K \\ \downarrow \text{ preparation} \\ \langle elf\dots elf \,\|\, \omega_0; L_0; Q_0; P_0\rangle \\ \downarrow \text{ rewrite rules} \\ \langle elf \,\|\, \omega; -; -; P\rangle \\ \downarrow \text{ readout} \\ \rho_1(K) = \rho_1(\omega, P) \end{array}$$

Preparation. Draw K using a 0-framed 0-rotation planar diagram D where all crossings are pointing up. Walk along D labeling features by $1,\dots,m$ in order: over-passes, under-passes, and right-heading cups and caps ("±-cuaps"). If x is a xing, let i_x and j_x be the labels on its over/under strands, and let s_x be 0 if it right-handed and -1 otherwise. If c is a cuap, let i_c be its label and s_c be its sign. Set



$$(L;Q;P) = \sum_{x:\,(i,j,s)} (-)^s \left(l_j;\ t^s e_i f_j;\ (-t)^s e_i l_{(1+s)i-sj} f_j + l_i l_j + \frac{t^{2s} e_i^2 f_j^2}{4} \right) + \sum_{c:\,(i,s)} (0;\ 0;\ s\cdot l_i)\,.$$

This done, output $\langle e_1 l_1 f_1 e_2 l_2 f_2 \cdots e_m l_m f_m \,\|\, 1; L; Q; P\rangle$.

In *formulas*, L is always $\mathbb{Z}$-linear in $\{l_i\}$, Q is an R-linear combination of $\{e_i f_j\}$ where $R := \mathbb{Q}[t^{\pm 1}]$, and P is an R-linear combination of $\{1,\ l_i,\ l_i l_j,\ e_i f_j,\ e_i l_j f_k,\ e_i e_j f_k f_l\}$. (The key to computability!)

Rewrite Rules. Manipulate $\langle$*word* $\|$ *formulas*$\rangle$ expressions using the rewrite rules below, until you come to the form $\langle e_1 l_1 f_1 \,\|\, \omega; -; -; P\rangle$. Output (ω, P).

Rule 1, Deletions. If a letter appears in *word* but not in *formulas*, you can delete it.

Rule 2, Merges. In *word*, you can replace *adjacent* $v_i v_j$ with v_k (for $v \in \{e,l,f\}$) while making the same changes in *formulas* (provided k creates no naming clashes). E.g.,

$$\langle \dots e_i e_j \dots \,\|\, Z\rangle \to \langle \dots e_k \dots \,\|\, Z|_{e_i,e_j\to e_k}\rangle.$$

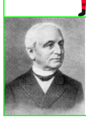
Rule 3, *le* Sorts. Provided k introduces no clashes, given $\langle \dots l_j e_i \dots \,\|\, \omega; L; Q; P\rangle$, decompose $L = \lambda l_j + L'$, $Q = \alpha e_i + Q'$, write $P = P(e_i, l_j)$ (with messy coefficients), set $q = \mathbb{e}^\gamma \beta e_k + \gamma l_k$, and output

$$\left\langle \dots e_k l_k \dots \,\|\, \omega; L|_{l_j\to l_k}; t^\lambda \alpha e_k + Q'; \mathbb{e}^{-q} P(\partial_\beta, \partial_\gamma)\mathbb{e}^q|_{\beta\to\alpha/\omega,\,\gamma\to\lambda\log t}\right\rangle.$$

Rule 4, *fl* Sorts. Provided k introduces no clashes, given $\langle \dots f_i l_j \dots \,\|\, \omega; L; Q; P\rangle$, decompose $L = \lambda l_j + L'$, $Q = \alpha f_i + Q'$, write $P = P(f_i, l_j)$ (with messy coefficients), set $q = \mathbb{e}^\gamma \beta f_k + \gamma l_k$, and output

$$\left\langle \dots l_k f_k \dots \,\|\, \omega; L|_{l_j\to l_k}; t^\lambda \alpha f_k + Q'; \mathbb{e}^{-q} P(\partial_\beta, \partial_\gamma)\mathbb{e}^q|_{\beta\to\alpha/\omega,\,\gamma\to\lambda\log t}\right\rangle.$$

Happy Birthday, Scott!



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Rule 5, *fe* Sorts. Provided k introduces no clashes, given $\langle \dots f_i e_j \dots \,\|\, \omega; L; Q; P\rangle$, decompose $Q = Q_{fe} f_i e_j + Q_f f_i + Q_e e_j + Q'$ write $P = P(f_i, e_j)$ (with messy coefficients), set $\mu = 1 + (t-1)\delta$ and $q = ((1-t)\alpha\beta + \beta e_k + \alpha f_k + \delta e_k f_k)/\mu$, and output

$$\left\langle \dots e_k f_k \dots \,\middle\|\, \begin{array}{c} \mu\omega; L; \mu\omega q + \mu Q'; \\ \omega^4 \Lambda_k + \mathbb{e}^{-q} P(\partial_\alpha, \partial_\beta)(\mathbb{e}^q) \end{array} \right\rangle\Bigg|_{\substack{\alpha\to Q_f/\omega,\,\beta\to Q_e/\omega, \\ \delta\to Q_{fe}/\omega}},$$

where Λ_k is the Λόγος, "a principle of order and knowledge":

$$\begin{aligned} \Lambda_k = \frac{t+1}{4}\Big(& -\delta(\mu+1)\left(\beta^2 e_k^2 + \alpha^2 f_k^2\right) - \delta^3(3\mu+1) e_k^2 f_k^2 \\ & - 2\,(\beta e_k + \alpha f_k)\left(\alpha\beta + 2\delta\mu + \delta^2(2\mu+1) e_k f_k + 2\delta\mu^2 l_k\right) \\ & - 4(\alpha\beta + \delta\mu)\left(\delta(\mu+1) e_k f_k + \mu^2 l_k\right) - 4\delta^2\mu^2 e_k f_k l_k \\ & + (t-1)\left(2(\alpha\beta+\delta\mu)^2 - \alpha^2\beta^2\right)\Big). \end{aligned}$$

***elf* merges,** m^{ij}_k, are defined as compositions

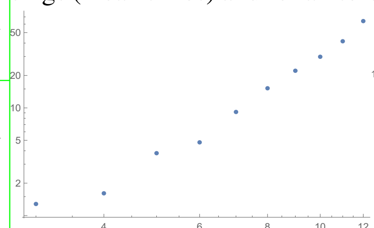
$$e_i l_i \overline{f_i e_j} l_j f_j \xrightarrow{S_x^{f_i e_j}} e_i \overline{l_i e_x}\, \overline{f_x l_j} f_j \xrightarrow{S_x^{l_i e_x} /\!/ S_x^{f_x l_j}} \overline{e_i e_x}\, \overline{l_x l_x}\, \overline{f_x f_j} \xrightarrow{i,j,x\to k} e_k l_k f_k$$

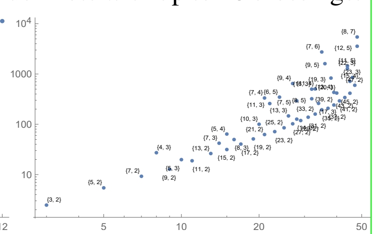
Readout. Given $\langle elf \,\|\, \omega; -; -; P\rangle$, output

$$\rho_1(K) := \frac{t(P|_{e,l,f\to 0} - t\omega'\omega^3)}{(t-1)^2\omega^2}.$$

(ω is the Alexander polynomial, L and Q are not interesting).

Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:





Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg \rho_1^+ \le 2g-1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Why Works? The Lie algebra $\mathfrak{g}_1$ (below) is a "solvable approximation of sl_2".

Theorem. The map (as defined below)

$$\langle w \,\|\, \omega; L; Q; P\rangle \mapsto \mathbb{O}\left(\omega^{-1}\mathbb{e}^{L\log t + \omega^{-1}Q}(1 + \epsilon\omega^{-4}P) :\ w\right) \in \hat{\mathcal{U}}(\mathfrak{g}_1)$$

is well defined modulo the sorting rules. It maps the initial preparation to a product of "R-matrices" and "cuap values" satisfying the usual moves for Morse knots (R3, etc.). (And hence the result is a "quantum invariant", except computed very differently; no representation theory!).

Video and more at http://www.math.toronto.edu/~drorbn/Talks/GWU-1612/

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle h, e', l, f\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with h central and with $[f,l] = f$, $[e',l] = -e'$, and $[e',f] = h - 2\epsilon l$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle h, e', f, \epsilon h, \epsilon e', \epsilon l, \epsilon f\rangle \supset \langle h, \epsilon h, \epsilon e', \epsilon l, \epsilon f\rangle \supset 0$. Pragmatics: declare $\deg(h, e', l, f, \epsilon) = (1,1,0,0,1)$ and set $t := \mathbb{e}^h$ and $e := (t-1)e'/h$.

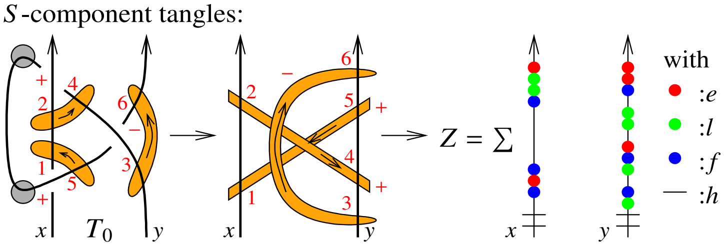
How did it arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle l, f\rangle/[f,l] = f$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (l,f) \mapsto (0, l\wedge f)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle h', e', l, f\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 1$, get $[f,l] = f$, $[f,h'] = -\epsilon f$, $[l,e'] = e'$, $[h',e'] = -\epsilon e'$, $[h',l] = 0$, and $[e',f] = h' - \epsilon l$. Now note that $h' + \epsilon l$ is central, so switch to $h := h' + \epsilon l$. This is $\mathfrak{g}_1$.

Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of $poly$ in $\hat{\mathcal{S}}(\oplus_i \mathfrak{g})$ along $\hat{\mathcal{U}}(\mathfrak{g})$ according to $specs$. E.g.,

$$\mathbb{O}\left(e_1 \mathbb{e}^{e_3} l_1^3 l_2 f_3^9 \mid f_3 l_1 e_1 e_3 l_2\right) = f^9 l^3 e \mathbb{e}^e l \in \hat{\mathcal{U}}(\mathfrak{g}).$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})$ using commutative polynomials / power series. In $\mathfrak{g}_1$, no need to specify h / t.

Algebras and Invariants. Given any unital algebra A (even better if A is Hopf; typically, $A \sim \hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R \in A \otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:



What we didn't say (more, including videos, in ωεβ/Talks).

- ρ_1 is "line" in the coloured Jones polynomial; related to Melvin-Morton-Rozansky.
- ρ_1 extends to "rotational virtual tangles" and is a projection of the universal finite type invariant of such.
- ρ_1 seems to have a better chance than anything else we know to detect a counterexample to slice=ribbon.
- ρ_1 leads to many questions and a very long to-do list. Years of work, many papers ahead. Have fun!

Demo Programs. ωεβ/Demo

Formatting (prints differ ☹)

```
CF[ℰ_] := Module[{vars = Union@Cases[ℰ, e_ | l_ | f_, ∞]},
  If[vars === {}, Factor[ℰ],
   Total[CoefficientRules[ℰ, vars] /.
     (p_ → c_) ⧴ Factor[c] Times @@ (vars^p)] ]];
CF[ℰ_𝔼] := CF /@ ℰ;
```

Preparation

```
𝔼[i_, j_, s_] := 𝔼[1, (-1)^s l_j, (-t)^s e_i f_j,
   t^s e_i l_(1+s) i-s j f_j + (-1)^s l_i l_j + (-t^2)^s e_i^2 f_j^2/4];
𝔼[i_, s_] := 𝔼[1, 0, 0, s l_i];
𝔼 /: 𝔼[1, L1_, Q1_, P1_] 𝔼[1, L2_, Q2_, P2_] :=
  𝔼[1, L1 + L2, Q1 + Q2, P1 + P2];
```

Preparing the Trefoil

```
z1 = (𝔼[1, 11, 0] 𝔼[4, 2, -1] 𝔼[15, 5, 0]
   𝔼[6, 8, -1] 𝔼[9, 16, 0] 𝔼[12, 14, -1] 𝔼[3, -1] 𝔼[7, +1]
   𝔼[10, -1] 𝔼[13, +1])
```

$$\mathbb{E}\Big[1,\ -l_2 + l_5 - l_8 + l_{11} - l_{14} + l_{16},\ -\frac{e_4 f_2}{t} + e_{15} f_5 - \frac{e_6 f_8}{t} + e_1 f_{11} - \frac{e_{12} f_{14}}{t} + e_9 f_{16},\ -\frac{e_4^2 f_2^2}{4t^2} + \frac{1}{4} e_{15}^2 f_5^2 - \frac{e_6^2 f_8^2}{4t^2} + \frac{1}{4} e_1^2 f_{11}^2 - \frac{e_{12}^2 f_{14}^2}{4t^2} + \frac{1}{4} e_9^2 f_{16}^2 + e_1 f_{11} l_1 + \frac{e_4 f_2 l_2}{t} - l_3 - l_2 l_4 + l_7 + \frac{e_6 f_8 l_8}{t} - l_6 l_8 + e_9 f_{16} l_9 - l_{10} + l_1 l_{11} + l_{13} + \frac{e_{12} f_{14} l_{14}}{t} - l_{12} l_{14} + e_{15} f_5 l_{15} + l_5 l_{15} + l_9 l_{16}\Big]$$

Differential Polynomials (Implementing $P(\partial_\alpha, \partial_\beta)(f)$)

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] :=
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

***le* and *fl* Sorts**

```
S_{l_j_ (x:e|f)_i_ →k_}[𝔼[ω_, L_, Q_, P_]] :=
 With[{λ = ∂_{l_j} L, α = ∂_{x_i} Q, q = 𝕖^γ β x_k + γ l_k}, CF[
   𝔼[ω, L /. l_j → l_k, t^λ α x_k + (Q /. x_i → 0),
    𝕖^-q DP_{l_j→D_γ, x_i→D_β}[P][𝕖^q] /. {β → α/ω, γ → λ Log[t]}] ]];
```

The Λόγος

```
Λ[k_] := ((t - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 e_k l_k f_k δ^2 μ^2 -
   δ (1 + μ) (f_k^2 α^2 + e_k^2 β^2) - e_k^2 f_k^2 δ^3 (1 + 3 μ) -
   2 (α β + 2 δ μ + e_k f_k δ^2 (1 + 2 μ) + 2 l_k δ μ^2) (f_k α + e_k β) -
   4 (l_k μ^2 + e_k f_k δ (1 + μ)) (α β + δ μ)) (1 + t) / 4;
```

***fe* Sorts**

```
S_{f_i_ e_j_ →k_}[𝔼[ω_, L_, Q_, P_]] :=
 With[{q = ((1 - t) α β + β e_k + α f_k + δ e_k f_k) / μ}, CF[
   𝔼[μ ω, L, μ ω q + μ (Q /. f_i | e_j → 0),
     μ^4 𝕖^-q DP_{f_i→D_α, e_j→D_β}[P][𝕖^q] + ω^4 Λ[k]] /. μ → 1 + (t - 1) δ /.
    {α → ω^-1 (∂_{f_i} Q /. e_j → 0), β → ω^-1 (∂_{e_j} Q /. f_i → 0),
     δ → ω^-1 ∂_{f_i,e_j} Q}]];
```

Elf Merges

```
m_{i_,j_→k_}[Z_𝔼] := Module[{x, z},
  CF[(Z // S_{f_i e_j→x} // S_{l_i e_x→x} // S_{f_x l_j→x}) /. z_i|j|x → z_k]]
```

Rewriting the Trefoil (by merging 16 elves)

```
(Do[z1 = z1 // m_{1,k→1}, {k, 2, 16}]; z1)
```

$$\mathbb{E}\Big[\frac{1-t+t^2}{t},\ 0,\ 0,\ \frac{(-1+t)\,(1-t+t^2)^2\,(1-t+2t^2)}{t^3} - \frac{2\,(1+t)\,(1-t+t^2)^3\, e_1 f_1}{t^4} - \frac{2\,(-1+t)\,(1+t)\,(1-t+t^2)^3\, l_1}{t^4}\Big]$$

Readout

```
ρ_1[𝔼[ω_, _, _, P_]] := CF[ t ((P /. e_ | l_ | f_ → 0) - t ω^3 (∂_t ω)) / ((t - 1)^2 ω^2) ]
```

$\rho_1(3_1)$

```
ρ_1[z1] // Expand
```

$$\frac{1}{t} + t$$

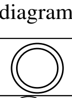
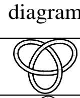
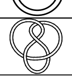
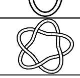
References.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

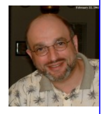
diagram	n_k^t Alexander's ω^+ / Today's / Rozansky's ρ_1^+	genus / ribbon / unknotting number / amphicheiral	diagram	n_k^t Alexander's ω^+ / Today's / Rozansky's ρ_1^+	genus / ribbon / unknotting number / amphicheiral
	0_1^a 1 0	0 / ✔ 0 / ✔		3_1^a $t-1$ t	1 / ✘ 1 / ✘
	4_1^a $3-t$ 0	1 / ✘ 1 / ✔		5_1^a t^2-t+1 $2t^3+3t$	2 / ✘ 2 / ✘

Video and more at http://www.math.toronto.edu/~drorbn/Talks/GWU-1612/

A Poly-Time Knot Polynomial Via Solvable Approximation

Work in Progress! Fluid! Help Needed!

Abstract. Rozansky [Ro2] and Overbay [Ov] described a spectacular knot polynomial that failed to attract the attention it deserved as the first poly-time-computable knot polynomial since Alexander's [Al, 1928] and (in my opinion) as the second most likely knot polynomial (after Alexander's) to carry topological information. With Roland van der Veen, I will explain how to compute the Rozansky polynomial using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.


Rozansky

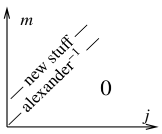

Overbay

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing


Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot A(K)(e^\hbar)=1$.

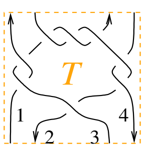


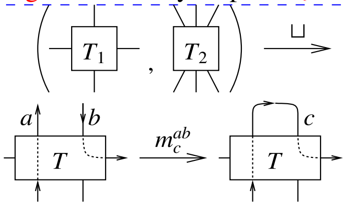
"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})A(K)(q^d)}\left(1+\sum_{k=1}^\infty \frac{(q-1)^k R_k(K)(q^d)}{A^{2k}(K)(q^d)}\right).$$

Why "spectacular"? Foremost reason: *OBVIOUSLY*. Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).
Also, will bound genus and may disprove {ribbon} = {slice}.

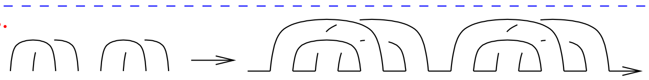
(v-)Tangles.

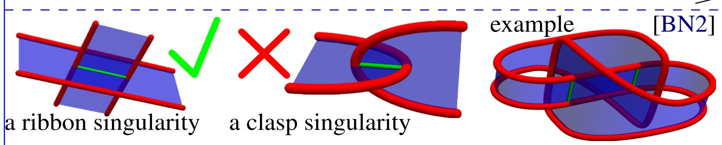




(meta-associativity: $m_x^{ab}/\!/m_y^{xc}=m_x^{bc}/\!/m_y^{ax}$) (tangles are generated by ⤧ and ⤨)

Genus.

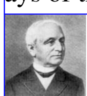




example [BN2]

a ribbon singularity a clasp singularity

A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3=\partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,
Conjecture. Some slice knots are not ribbon.
Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t)=f(t)f(1/t)$. (also for slice)



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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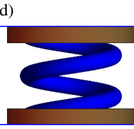
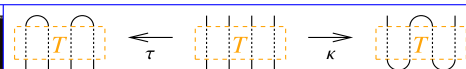
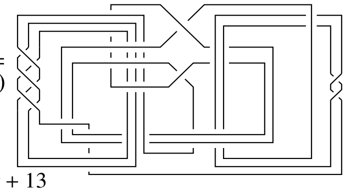




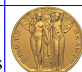
Gompf, Scharlemann, Thompson [GST]

$$\begin{array}{ccccc} & U\in\mathcal{T}_n & & 1\in\mathcal{A}_n & \\ & \nearrow^{\tau} & & \nearrow^{\tau} & \\ \mathcal{T}_{2n} & & \xrightarrow{z} & \mathcal{A}_{2n} & \text{with } \mathcal{R}:=\kappa(\tau^{-1}(1)) \\ & \searrow_{\kappa} & & \searrow_{\kappa} & \\ & \text{ribbon } K\in\mathcal{T}_1 & & z(K)\in\mathcal{R}\subseteq\mathcal{A}_1 & \end{array}$$



Faster is better, leaner is meaner!

$A^+=-t^8+2t^7-t^6-2t^4+5t^3-2t^2-7t+13$
$\rho_1^+=5t^{15}-18t^{14}+33t^{13}-32t^{12}+2t^{11}+42t^{10}-62t^9-8t^8+166t^7-242t^6+108t^5+132t^4-226t^3+148t^2-11t-36$



The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T)\in R_S\times M_{S\times S}(R_S)=\left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S:=\mathbb{Z}(\{t_a\colon a\in S\})$:

$$\left({}_a\nearrow_b,\ {}_b\nwarrow_a\right)\to\begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-t_a^{\pm1}\\ b & 0 & t_a^{\pm1}\end{array}\qquad T_1\sqcup T_2\to\begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array}\ \xrightarrow[t_a,t_b\to t_c]{m_c^{ab}}\ \left(\begin{array}{c|cc}(1-\beta)\omega & c & S\\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta}\\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

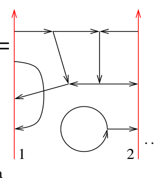
(Roland: "add to A the product of column b and row a, divide by $(1-A_{ab})$, delete column b and row a".)

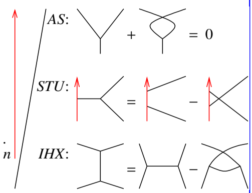
For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

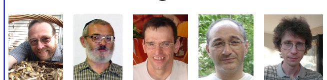
(There are also formulas for strand doubling and strand reversal).

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

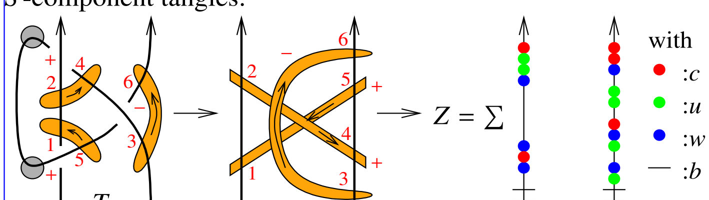
$$Z\colon\left\{\begin{array}{c}S\text{-component}\\ (v/b\text{-})\text{tangles}\end{array}\right\}\to\mathcal{A}_S^v:=$$





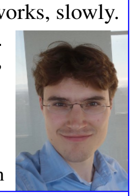

Etingof Kazhdan Haviv Enriquez Ševera

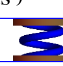
Algebras and Invariants. Given any unital algebra A (even better if A is Hopf; typically, $A\sim\hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R\in A\otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:



Good News. In theory, enough to know R, the cuaps, and stitching/multiplication $m_k^{ij}\colon A_i\otimes A_j\to A_k$.
Problem. Extract information out of Z.
Textbook Solution. Use representation theory ...works, slowly.
Today's Solution (with van der Veen). For some specific $\mathfrak{g}$'s, work in a space of "formulas of a specific type" for elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$:

$$\left\{\begin{array}{c}\text{ordered perturbed}\\ \text{Gaussian formulas}\end{array}\right\}\to\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$$


van der Veen



This is http://www.math.toronto.edu/~drorbn/Talks/MIT-1612/. Better videos at .../Indiana-1611/, .../LesDiablerets-1608/

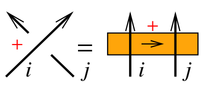
1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b,c,u,w\rangle$ over the ring $R=\mathbb{Q}[\epsilon]/(\epsilon^2=0)$, with b central and with $[w,c]=w$, $[c,u]=u$, and $[u,w]=b-2\epsilon c$, with CYBE $r_{ij}=(b_i-\epsilon c_i)c_j+u_iw_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1\supset\langle b,u,w,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset\langle b,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset 0$. (note: $\deg(b,c,u,w,\epsilon)=(1,0,1,0,1)$)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon=0$, or $\mathbb{Q}\langle b,c,u,w\rangle/([b,\cdot]=0,\ [c,u]=u,\ [c,w]=-w,\ [u,w]=b$ with $r_{ij}=b_ic_j+u_iw_j$. It is $\mathfrak{b}^*\rtimes\mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c,w\rangle$ and (b,u) is the dual basis of (c,w). For topology, it is more valuable than $\mathfrak{g}_1$ / sl_2, but topology already got by other means almost everything $\mathfrak{g}_0$ gives.

How did these arise? $sl_2=\mathfrak{b}^+\oplus\mathfrak{b}^-/\mathfrak{h}=:sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+=\langle c,w\rangle/[w,c]=w$ is a Lie bialgebra with $\delta\colon\mathfrak{b}^+\to\mathfrak{b}^+\otimes\mathfrak{b}^+$ by $\delta\colon(c,w)\mapsto(0,c\wedge w)$. Going back, $sl_2^+=\mathcal{D}(\mathfrak{b}^+)=(\mathfrak{b}^+)^*\oplus\mathfrak{b}^+=\langle b,u,c,w\rangle/\cdots$. **Idea.** Replace $\delta\to\epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1}=0)$. At $k=0$, get $\mathfrak{g}_0$. At $k=1$, get $[w,c]=w$, $[w,b']=-\epsilon w$, $[c,u]=u$, $[b',u]=-\epsilon u$, $[b',c]=0$, and $[u,w]=b'-\epsilon c$. Now note that $b'+\epsilon c$ is central, so switch to $b:=b'+\epsilon c$. This is $\mathfrak{g}_1$.

Ordering Symbols. $\mathbb{O}(poly\mid specs)$ plants the variables of $poly$ in $\mathcal{S}(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to $specs$. E.g.,
$$\mathbb{O}\left(c_1^3u_1c_2e^{u_3}w_3^9\middle|x\colon w_3c_1,\ y\colon u_1u_3c_2\right)=w^9c^3\otimes ue^uc\in\mathcal{U}(\mathfrak{g})_x\otimes\mathcal{U}(\mathfrak{g})_y$$
This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

0-Smidgen Invariants. $r=Id\in\mathfrak{b}^-\otimes\mathfrak{b}^+$ solves the CYBE $[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes3}$ and, by luck,



$=R_{ij}=e^{r_{ij}}=e^{b_ic_j+u_iw_j}\in\mathcal{U}(\mathfrak{g}_{0,i}\oplus\mathfrak{g}_{0,j})$ solves YB/R3.

Lemma. $R_{ij}=e^{b_ic_j+u_iw_j}=\mathbb{O}\left(\exp\left(b_ic_j+\frac{e^{b_i}-1}{b_i}u_iw_j\right)\middle|i\colon u_i,\ j\colon c_jw_j\right)$

Example. $Z(T_0)=$ $\qquad=\sum_{m,n}\frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!}u^n\otimes c^mw^n.$
$$\mathbb{O}\Bigg(\exp\left(b_5c_1+\frac{e^{b_5}-1}{b_5}u_5w_1+b_2c_4+\frac{e^{b_2}-1}{b_2}u_2w_4-b_3c_6+\frac{e^{-b_3}-1}{b_3}u_3w_6\right)\Bigg|$$
"ucw form"
$$x\colon c_1w_1u_2,\ y\colon u_3c_4w_4u_5c_6w_6\Bigg)=\mathbb{O}\left(\zeta\middle|x\colon u_xc_xw_x,\ y\colon u_yc_yw_y\right)$$

Goal. Write ζ as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S:=\mathbb{Q}(b_i,e^{b_i})$, and $\omega\in R_S$.

The Big $\mathfrak{g}_0$ Lemma. Under $[c,u]=u$, $[c,w]=-w$, and $[u,w]=b$:

1a. $N^{cu}:=\mathbb{O}(e^{\gamma c+\beta u}|uc)\stackrel{\rightarrow}{=}\mathbb{O}(e^{\gamma c+e^\gamma\beta u}|cu)$ (means $e^{\beta u}e^{\gamma c}=e^{\gamma c}e^{e^\gamma\beta u}$

1b. $N^{wc}:=\mathbb{O}(e^{\gamma c+\alpha w}|wc)\stackrel{\rightarrow}{=}\mathbb{O}(e^{\gamma c+e^\gamma\alpha w}|cw)$ …in the $\{ax+b\}$ group)

2. $\mathbb{O}(e^{\alpha w+\beta u}|wu)=\mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw)$ (the Weyl relations)

3. $\mathbb{O}(e^{\delta uw}|wu)e^{\beta u}=e^{\nu\beta u}\mathbb{O}(e^{\delta uw}|wu)$, with $\nu=(1+b\delta)^{-1}$

(a. expand and crunch. b. use $w=b\hat{x}$, $u=\partial_x$. c. use "scatter and glow".)

4. $\mathbb{O}(e^{\delta uw}|wu)=\mathbb{O}(\nu e^{\nu\delta uw}|uw)$ (same techniques)

5. $N^{wu}:=\mathbb{O}(e^{\beta u+\alpha w+\delta uw}|wu)\stackrel{\rightarrow}{=}\mathbb{O}(\nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\delta uw}|uw)$

6. $N_k^{c_ic_j}:=\mathbb{O}(\zeta|c_ic_j)\stackrel{\rightarrow}{=}\mathbb{O}(\zeta/(c_i,c_j\to c_k)|c_k)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition
$$u_ic_i\,\overline{w_iu_j}\,c_jw_j\xrightarrow{N_x^{w_iu_j}}u_i\,\overline{c_iu_x}\,\overline{w_xc_j}\,w_j\xrightarrow{N_x^{c_iu_x}/\!/N_x^{w_xc_j}}\overline{u_iu_x}\,\overline{c_xc_x}\,\overline{w_xw_j}\xrightarrow{i,j,x\to k}u_kc_kw_k$$

On to 1-smidgen invariants, where much is the same…

The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 6 are the same, yet

5. $\mathbb{O}\left(e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\nu(1+\epsilon\nu\Lambda)e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|ucw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, u, c, and w:
$$\begin{aligned}\Lambda=&-b\nu(\alpha^2\beta^2\nu^2+4\alpha\beta\delta\nu+2\delta^2)/2+\beta^2\delta\nu^3(b\delta+2)u^2/2\\&+\delta^3\nu^3(3b\delta+4)u^2w^2/2+\beta\delta^2\nu^3(2b\delta+3)u^2w\\&+\alpha\delta^2\nu^3(2b\delta+3)uw^2+2\delta\nu^2(b\delta+2)(\alpha\beta\nu+\delta)uw\\&+\alpha^2\delta\nu^3(b\delta+2)w^2/2+2(\alpha\beta\nu+\delta)c+2\beta\delta\nu uc+2\delta^2\nu ucw\\&+2\alpha\delta\nu cw+\beta\nu^2(\alpha\beta\nu+2\delta)u+\alpha\nu^2(\alpha\beta\nu+2\delta)w.\end{aligned}$$

Proof. A lengthy computation. (Verification: ωεβ/Big)

Problem. We now need to normal-order perturbed Gaussians!

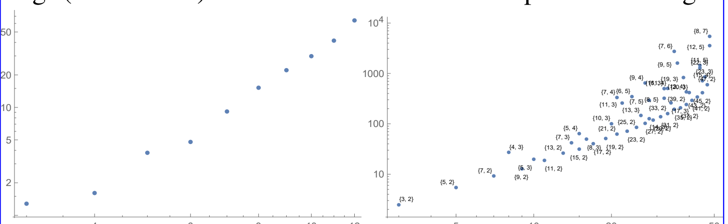
Solution. Borrow some tactics from QFT:
$$\mathbb{O}(\epsilon P(c,u)e^{\gamma c+\beta u}|uc)=\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+\beta u}|uc)=$$
and likewise $\qquad\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+e^{-\gamma}\beta u}|cu),$
$$\mathbb{O}\left(\epsilon P(u,w)e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\epsilon P(\partial_\beta,\partial_\alpha)\nu e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|ucw\right)$$

Finally, the values of the generators ⤯, ⤰, $\overrightarrow{n}$, and $\underset{\rightarrow}{u}$, are set by solving many equations, non-uniquely.

Pragmatic Simplifications. Set $t:=e^b$, work with $v:=(t-1)u/b$, and set $\mathbb{E}(\omega,L,Q,P):=\mathbb{O}\left(\omega^{-1}e^{L+Q/\omega}(1+\epsilon\omega^{-4}P)\colon(i\colon v_ic_iw_i)\right)$. Now $\omega\in R_S:=\mathbb{Z}[t_i,t_i^{-1}]$ is Laurent, $L=\sum l_{ij}\log(t_i)c_j$ with $l_{ij}\in\mathbb{Z}$, $Q=\sum q_{ij}v_iw_j$ with $q_{ij}\in R_S$, and P is a quartic polynomial in v_i, c_j, w_k with coefficients in R_S. The operations are lightly modified, and the Λόγος and the values of the generators become somewhat simpler, as in the implementation below.

Rough complexity estimate, after $t_k\to t$. n: xing number; w: width, maybe $\sim\sqrt{n}$.
$$\underbrace{n}_{A}\overbrace{\sum_{d=0}^4\underbrace{w^{4-d}}_{E}\underbrace{w^d}_{F}\underbrace{n^2}_{G}}^{B}=n^3w^4\in[n^5,n^7]$$
A: go over stitchings in order. B: multiplication ops per $N^{u_iw_j}$. d: deg of u_i,w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Conjecture (checked on the same collections). Given a knot K with Alexander polynomial A, there is a polynomial ρ_1 such that
$$P=A^2\frac{(t-1)^3\rho_1+t^2(2vw+(1-t)(1-2c))AA'}{(1-t)t}.$$
Furthermore, A and ρ_1 are symmetric under $t\to t^{-1}$, so let A^+ and ρ_1^+ be their "positive parts", so e.g., $\rho_1(t)=\rho_1^+(t)+\rho_1^+(t^{-1})-\rho_1^+(0)$.

Power. On the 250 knots with at most 10 crossings, the pair (A,ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always $\deg\rho_1^+\le 2g-1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

This is `http://www.math.toronto.edu/~drorbn/Talks/MIT-1612/`. Better videos at …/Indiana-1611/, …/LesDiablerets-1608/

Demo Programs for 0-Co. ωεβ/Demo

The R-matrices

$$R^+_{0,i\_,j\_} := \mathbb{E}\left[b_i c_j + b_i^{-1}\left(e^{b_i}-1\right) u_i w_j\right];$$
$$R^-_{0,i\_,j\_} := \mathbb{E}\left[-b_i c_j + b_i^{-1}\left(e^{-b_i}-1\right) u_i w_j\right];$$

Utilities

```
CF[ω_. 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_. 𝔼[Q1_] ≡ ω2_. 𝔼[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
```

Normal Ordering Operators

$$N_{(x:w|u)_i\_\, c_j\_\to k\_}[\omega\_.\ \mathbb{E}[Q\_]] := \mathrm{CF}\big[\omega\,\mathbb{E}[e^{\gamma}\alpha x_k + \gamma c_k + (Q\ /.\ c_j\ |\ x_i \to 0)]\ /.\ \{\gamma \to \partial_{c_j} Q,\ \alpha \to \partial_{x_i} Q\}\big];$$
$$N_{w_i\_\, u_j\_\to k\_}[\omega\_.\ \mathbb{E}[Q\_]] := \mathrm{CF}\big[\nu\,\omega\,\mathbb{E}[-b_k \nu\alpha\beta + \nu\beta u_k + \nu\alpha w_k + \nu\delta u_k w_k + (Q\ /.\ w_i\ |\ u_j \to 0)]\ /.\ \nu \to (1 + b_k\delta)^{-1}\ /.\ \{\alpha \to \partial_{w_i} Q\ /.\ u_j \to 0,\ \beta \to \partial_{u_j} Q\ /.\ w_i \to 0,\ \delta \to \partial_{w_i,u_j} Q\}\big];$$

Stitching

$$m_{i\_,j\_\to k\_}[Z\_] := \mathrm{Module}\big[\{x, z\},\ \mathrm{CF}\big[\big(Z\ //\ N_{w_i u_j\to x}\ //\ N_{c_i u_x\to x}\ //\ N_{w_x c_j\to x}\big)\ /.\ z\_{i|j|x} \to z_k\big]\big]$$

Some calculations for T_0

$$T_0 = R^+_{0,5,1}\, R^+_{0,2,4}\, R^-_{0,3,6}$$
$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5}) u_5 w_1}{b_5} + \frac{(-1+e^{b_2}) u_2 w_4}{b_2} + \frac{(-1+e^{-b_3}) u_3 w_6}{b_3}\right]$$
$$T_0\ //\ m_{1,2\to1}\ //\ m_{3,4\to3}\ //\ m_{3,5\to3}\ //\ m_{3,6\to3}$$
$$\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})}\,\mathbb{E}\Big[b_3 c_1 + b_1 c_3 - b_3 c_3 + \frac{e^{b_3}(-1+e^{b_1})(-1+e^{b_3}) u_1 w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3}) b_1} - \frac{e^{b_1}(-1+e^{b_3}) u_3 w_1}{(-1+(-1+e^{b_1})(-1+e^{b_3})) b_3} - \frac{e^{-b_3}(-1+e^{b_3}) u_3 w_3}{b_3} - \frac{e^{-b_3}(-1+e^{b_1})(-e^{b_3} b_3 u_1 + e^{b_1}(-1+e^{b_3}) b_1 u_3) w_3}{b_1 (b_3 - (-1+e^{b_1})(-1+e^{b_3})) b_3}\Big]$$

Verifying meta-associativity

```
Q0 = 𝔼[Sum[f_i c_i, {i, 3}] + Sum[f_{i,j} u_i w_j, {i, 3}, {j, 3}]]
```

$$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$

$$(Q0\ //\ m_{1,2\to1}\ //\ m_{1,3\to1}) \equiv (Q0\ //\ m_{2,3\to2}\ //\ m_{1,2\to1})$$

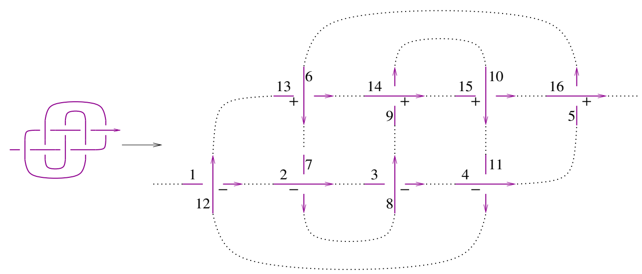
```
True
```

Testing R3

$$t1 = R^+_{0,1,2}\, R^+_{0,3,4}\, R^+_{0,5,6}\ //\ m_{3,5\to x}\ //\ m_{1,6\to y}\ //\ m_{2,4\to z}$$
$$\mathbb{E}\left[b_x c_y + b_x c_z + b_y c_z + \frac{e^{b_x}(-1+e^{b_y}) u_y w_z}{b_y} + \frac{(-1+e^{b_x}) u_x (w_y + w_z)}{b_x}\right]$$
$$t1 \equiv (R^+_{0,1,2}\, R^+_{0,3,4}\, R^+_{0,5,6}\ //\ m_{1,3\to x}\ //\ m_{2,5\to y}\ //\ m_{4,6\to z})$$

```
True
```

8_{17}



$$z1 = R^-_{0,12,1}\, R^-_{0,2,7}\, R^-_{0,8,3}\, R^-_{0,4,11}\, R^+_{0,16,5}\, R^+_{0,6,13}\, R^+_{0,14,9}\, R^+_{0,10,15};$$

```
Do[z1 = (z1 // m_{1,n→1}) /. b_ → b, {n, 2, 16}];
{CF@z1, KnotData[{8, 17}, "AlexanderPolynomial"][t]}
```

$$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11 - \frac{1}{t^3} + \frac{4}{t^2} - \frac{8}{t} - 8t + 4t^2 - t^3\right\}$$

Demo Programs for 1-Co. ωεβ/Demo

$$\Lambda[k\_] := \Big((t_k - 1)\big(2(\alpha\beta + \delta\mu)^2 - \alpha^2\beta^2\big) - 4 v_k c_k w_k \delta^2\mu^2 - \delta(1+\mu)\big(w_k^2\alpha^2 + v_k^2\beta^2\big) - v_k^2 w_k^2 \delta^3 (1 + 3\mu) - 2\big(\alpha\beta + 2\delta\mu + v_k w_k \delta^2 (1+2\mu) + 2 c_k \delta\mu^2\big)(w_k\alpha + v_k\beta) - 4\big(c_k\mu^2 + v_k w_k \delta(1+\mu)\big)(\alpha\beta + \delta\mu)\Big)(1 + t_k)/4;$$

The Λόγος

The Generators

$$R^+_{i\_,j\_} := \mathbb{E}\left[1,\ \mathrm{Log}[t_i]\, c_j,\ v_i w_j,\ v_i c_i w_j + c_i c_j + v_i^2 w_j^2/4\right];$$
$$R^-_{i\_,j\_} := \mathbb{E}\left[1,\ -\mathrm{Log}[t_i]\, c_j,\ -t_i^{-1} v_i w_j,\ t_i^{-1} v_i c_j w_j - c_i c_j - t_i^{-2} v_i^2 w_j^2/4\right];$$
$$\left(ur_{i\_} := \mathbb{E}\left[t_i^{-1/2}, 0, 0, c_i t_i^{-2}\right];\ nr_{i\_} := \mathbb{E}\left[t_i^{1/2}, 0, 0, -c_i t_i^2\right];\right)$$

Differential Polynomials

$$DP_{x\_\to D_\alpha\_,\,y\_\to D_\beta\_}[P\_][f\_] := \text{(* means } P[\partial_\alpha, \partial_\beta][f] \text{ *)}\ \mathrm{Total}[\mathrm{CoefficientRules}[P, \{x, y\}]\ /.\ (\{m\_, n\_\} \to c\_) \mapsto c\, D[f, \{\alpha, m\}, \{\beta, n\}]]$$

Utilities

```
CF[ℰ_𝔼] := Expand /@ Together /@ ℰ;
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
```

Normal Ordering Operators

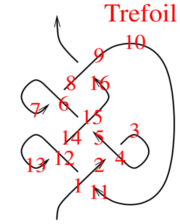
$$N_{c_j\_\,(x:v|w)_i\_\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \mathrm{With}\big[\{q = e^{\gamma}\beta x_k + \gamma c_k\},\ \mathrm{CF}\big[\mathbb{E}\big[\omega,\ \gamma c_k + (L\ /.\ c_j \to 0),\ \omega e^{\gamma}\beta x_k + (Q\ /.\ x_i \to 0),\ e^{-q} DP_{c_j\to D_\gamma,\, x_i\to D_\beta}[P][e^q]\big]\ /.\ \{\gamma \to \partial_{c_j} L,\ \beta \to \omega^{-1}\partial_{x_i} Q\}\big]\big];$$
$$N_{w_i\_\, v_j\_\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \mathrm{With}\big[\{q = ((1 - t_k)\alpha\beta + \beta v_k + \alpha w_k + \delta v_k w_k)/\mu\},\ \mathrm{CF}\big[\mathbb{E}\big[\mu\omega,\ L,\ \mu\omega q + \mu(Q\ /.\ w_i\ |\ v_j \to 0),\ \mu^4 e^{-q} DP_{w_i\to D_\alpha,\, v_j\to D_\beta}[P][e^q] + \omega^4\Lambda[k]\big]\ /.\ \mu \to 1 + (t_k - 1)\delta\ /.\ \{\alpha \to \omega^{-1}(\partial_{w_i} Q\ /.\ v_j \to 0),\ \beta \to \omega^{-1}(\partial_{v_j} Q\ /.\ w_i \to 0),\ \delta \to \omega^{-1}\partial_{w_i,v_j} Q\}\big]\big];$$

Stitching

$$m_{i\_,j\_\to k\_}[Z\_\mathbb{E}] := \mathrm{Module}\big[\{x, z\},\ \mathrm{CF}\big[\big(Z\ //\ N_{w_i v_j\to x}\ //\ N_{c_i v_x\to x}\ //\ N_{w_x c_j\to x}\big)\ /.\ z\_{i|j|x} \to z_k\big]\big]$$

The 0-Framed Trefoil

$$z2 = R^+_{1,11}\, R^-_{4,2}\, nr_3\, R^+_{15,5}\, R^-_{6,8}\, ur_7\, R^+_{9,16}\, nr_{10}\, R^-_{12,14}\, ur_{13};$$
$$(\mathrm{Do}[z2 = z2\ //\ m_{1,k\to1},\ \{k, 2, 16\}];\ z2 = z2\ /.\ a\_1 \mapsto a)$$



$$\mathbb{E}\Big[-1 + \frac{1}{t} + t,\ 0,\ 0,\ 16 + \frac{2c}{t^4} - \frac{1}{t^3} - \frac{6c}{t^3} + \frac{4}{t^2} + \frac{10c}{t^2} - \frac{10}{t} - \frac{8c}{t} - 18t + 8ct + 14t^2 - 10ct^2 - 7t^3 + 6ct^3 + 2t^4 - 2ct^4 + 2vw - \frac{2vw}{t^4} + \frac{4vw}{t^3} - \frac{6vw}{t^2} + \frac{2vw}{t} - 6tvw + 4t^2vw - 2t^3vw\Big]$$

Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality ($\mathbb{Z}$) properties? • Can everything be re-stated using integrals ($\int$)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $\mathfrak{b}^+ \leftrightarrow \mathfrak{b}^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra. • k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

This is http://www.math.toronto.edu/~drorbn/Talks/MIT-1612/. Better videos at .../Indiana-1611/, .../LesDiablerets-1608/

References.

[Al] J. W. Alexander, *Topological invariants of knots and link,* Trans. Amer. Math. Soc. **30** (1928) 275–306.

[BN1] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* ωεβ/KBH, arXiv:1308.1721.

[BN2] D. Bar-Natan, *Polynomial Time Knot Polynomial,* research proposal for the 2017 Killam Fellowship, ωεβ/K17.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras,* Adv. in Math. **197-2** (2005) 430âĂŞ-479, arXiv:math/0212325.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[Ha] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants,* Hebrew University PhD thesis, Sep. 2002, arXiv:math.QA/0211031.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

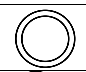
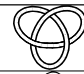
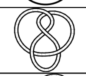
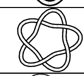
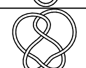
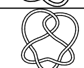
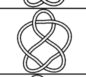
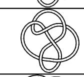
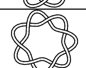
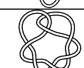
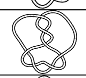
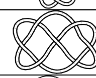
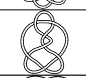
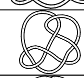
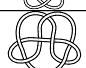
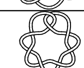
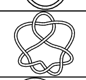
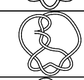
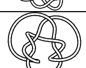
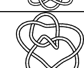
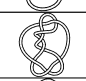
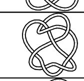
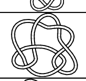
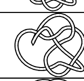
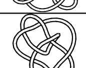
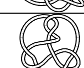
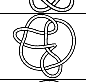
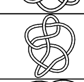
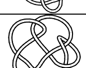
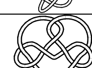
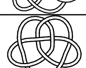
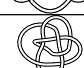
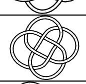
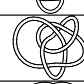
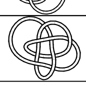
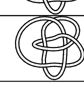
[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

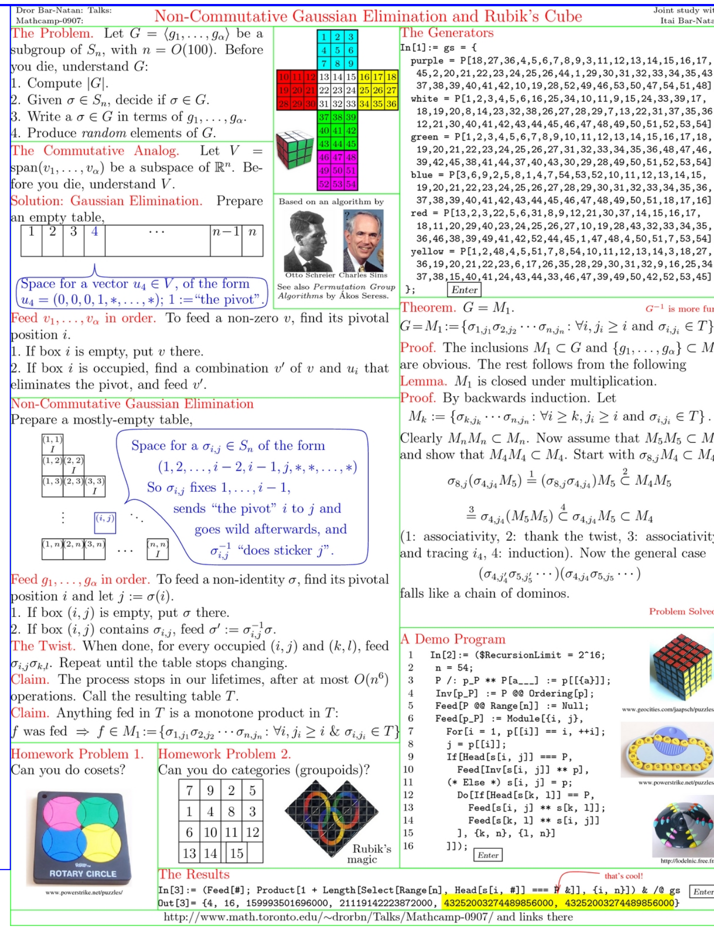
[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Se] P. Ševera, *Quantization of Lie Bialgebras Revisited,* Sel. Math., NS, to appear, arXiv:1401.6164.

diagram	n_k^t	Alexander's A_+	Today's / Rozansky's ρ_1^+	genus / ribbon	unknotting number / amphicheiral
	0_1^a	1	0	0 / ✔	0 / ✔
	3_1^a	$t-1$	t	1 / ✘	1 / ✘
	4_1^a	$3-t$	0	1 / ✘	1 / ✔
	5_1^a	t^2-t+1	$2t^3+3t$	2 / ✘	2 / ✘
	5_2^a	$2t-3$	$5t-4$	1 / ✘	1 / ✘
	6_1^a	$5-2t$	$t-4$	1 / ✔	1 / ✘
	6_2^a	$-t^2+3t-3$	t^3-4t^2+4t-4	2 / ✘	1 / ✘
	6_3^a	t^2-3t+5	0	2 / ✘	1 / ✔
	7_1^a	t^3-t^2+t-1	$3t^5+5t^3+6t$	3 / ✘	3 / ✘
	7_2^a	$3t-5$	$14t-16$	1 / ✘	1 / ✘
	7_3^a	$2t^2-3t+3$	$-9t^3+8t^2-16t+12$	2 / ✘	2 / ✘
	7_4^a	$4t-7$	$32-24t$	1 / ✘	2 / ✘
	7_5^a	$2t^2-4t+5$	$9t^3-16t^2+29t-28$	2 / ✘	2 / ✘
	7_6^a	$-t^2+5t-7$	$t^3-8t^2+19t-20$	2 / ✘	1 / ✘
	7_7^a	t^2-5t+9	$8-3t$	2 / ✘	1 / ✘
	8_1^a	$7-3t$	$5t-16$	1 / ✘	1 / ✘
	8_2^a	$-t^3+3t^2-3t+3$	$2t^5-8t^4+10t^3-12t^2+13t-12$	3 / ✘	2 / ✘
	8_3^a	$9-4t$	0	1 / ✘	2 / ✔
	8_4^a	$-2t^2+5t-5$	$3t^3-8t^2+6t-4$	2 / ✘	2 / ✘
	8_5^a	$-t^3+3t^2-4t+5$	$-2t^5+8t^4-13t^3+20t^2-22t+24$	3 / ✘	2 / ✘
	8_6^a	$-2t^2+6t-7$	$5t^3-20t^2+28t-32$	2 / ✘	2 / ✘
	8_7^a	t^3-3t^2+5t-5	$-t^5+4t^4-10t^3+12t^2-13t+12$	3 / ✘	1 / ✘
	8_8^a	$2t^2-6t+9$	$-t^3+4t^2-12t+16$	2 / ✔	2 / ✘
	8_9^a	$-t^3+3t^2-5t+7$	0	3 / ✔	1 / ✔
	8_{10}^a	t^3-3t^2+6t-7	$-t^5+4t^4-11t^3+16t^2-21t+20$	3 / ✘	2 / ✘
	8_{11}^a	$-2t^2+7t-9$	$5t^3-24t^2+39t-44$	2 / ✘	1 / ✘
	8_{12}^a	$t^2-7t+13$	0	2 / ✘	2 / ✔
	8_{13}^a	$2t^2-7t+11$	$-t^3+4t^2-14t+20$	2 / ✘	1 / ✘
	8_{14}^a	$-2t^2+8t-11$	$5t^3-28t^2+57t-68$	2 / ✘	1 / ✘
	8_{15}^a	$3t^2-8t+11$	$21t^3-64t^2+120t-140$	2 / ✘	2 / ✘
	8_{16}^a	t^3-4t^2+8t-9	$t^5-6t^4+17t^3-28t^2+35t-36$	3 / ✘	2 / ✘
	8_{17}^a	$-t^3+4t^2-8t+11$	0	3 / ✘	1 / ✔
	8_{18}^a	$-t^3+5t^2-10t+13$	0	3 / ✘	2 / ✔
	8_{19}^n	t^3-t^2+1	$-3t^5-4t^2-3t$	3 / ✘	3 / ✘
	8_{20}^n	t^2-2t+3	$4t-4$	2 / ✔	1 / ✘
	8_{21}^n	$-t^2+4t-5$	$t^3-8t^2+16t-20$	2 / ✘	1 / ✘

Abstract. What's the hardest math I've ever used in real life? Me, myself, directly - not by using a cellphone or a GPS device that somebody else designed? And in "real life" — not while studying or teaching mathematics?
I use addition and subtraction daily, adding up bills or calculating change. I use percentages often, though mostly it is just "add 15 percents". I seldom use multiplication and division: when I buy in bulk, or when I need to know how many tiles I need to replace my kitchen floor. I've used powers twice in my life, doing calculations related to mortgages. I've used a tiny bit of geometry and algebra for a tiny bit of non-math-related computer graphics I've played with. And for a long time, that was all. In my talk I will tell you how recently a math topic discovered only in the 1800s made a brief and modest appearance in my non-mathematical life. There are many books devoted to that topic and a lot of active research. Yet for all I know, nobody ever needed the actual formulas for such a simple reason before.
Hence we'll talk about the motion of movie cameras, and the fastest way to go from A to B subject to driving speed limits that depend on the locale, and the "happy segway principle" which is a the heart of the least action principle which in itself is at the heart of all of modern physics, and finally, about that funny discovery of Janos Bolyai's and Nikolai Ivanovich Lobachevsky's, that the famed axiom of parallels of the ancient Greeks need not actually be true.



Dror Bar-Natan: Talks: Mathcamp-0907:

Non-Commutative Gaussian Elimination and Rubik's Cube

Joint study with Itai Bar-Natan

The Problem. Let $G = \langle g_1, \dots, g_\alpha \rangle$ be a subgroup of S_n, with $n = O(100)$. Before you die, understand G:
1. Compute $|G|$.
2. Given $\sigma \in S_n$, decide if $\sigma \in G$.
3. Write a $\sigma \in G$ in terms of $g_1, \dots, g_\alpha$.
4. Produce *random* elements of G.

The Commutative Analog. Let $V = \operatorname{span}(v_1, \dots, v_\alpha)$ be a subspace of $\mathbb{R}^n$. Before you die, understand V.

Solution: Gaussian Elimination. Prepare an empty table,

1	2	3	4	…	$n-1$	n

Space for a vector $u_4 \in V$, of the form $u_4 = (0, 0, 0, 1, *, \dots, *)$; $1 :=$ "the pivot".

Based on an algorithm by Otto Schreier Charles Sims

See also *Permutation Group Algorithms* by Ákos Seress.

Feed $v_1, \dots, v_\alpha$ in order. To feed a non-zero v, find its pivotal position i.
1. If box i is empty, put v there.
2. If box i is occupied, find a combination v' of v and u_i that eliminates the pivot, and feed v'.

Non-Commutative Gaussian Elimination
Prepare a mostly-empty table,

(1,1) I				
(1,2)	(2,2) I			
(1,3)	(2,3)	(3,3) I		
⋮		(i,j)	⋱	
(1,n)	(2,n)	(3,n)	…	(n,n) I

Space for a $\sigma_{i,j} \in S_n$ of the form $(1, 2, \dots, i-2, i-1, j, *, *, \dots, *)$
So $\sigma_{i,j}$ fixes $1, \dots, i-1$,
sends "the pivot" i to j and
goes wild afterwards, and
$\sigma_{i,j}^{-1}$ "does sticker j".

Feed $g_1, \dots, g_\alpha$ in order. To feed a non-identity σ, find its pivotal position i and let $j := \sigma(i)$.
1. If box (i, j) is empty, put σ there.
2. If box (i, j) contains $\sigma_{i,j}$, feed $\sigma' := \sigma_{i,j}^{-1}\sigma$.

The Twist. When done, for every occupied (i, j) and (k, l), feed $\sigma_{i,j}\sigma_{k,l}$. Repeat until the table stops changing.

Claim. The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table T.

Claim. Anything fed in T is a monotone product in T:
f was fed $\Rightarrow f \in M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} : \forall i, j_i \geq i \ \&\ \sigma_{i,j_i} \in T\}$

The Generators

```
In[1]:= gs = {
  purple = P[18,27,36,4,5,6,7,8,9,3,11,12,13,14,15,16,17,
   45,2,20,21,22,23,24,25,26,44,1,29,30,31,32,33,34,35,43,
   37,38,39,40,41,42,10,19,28,52,49,46,53,50,47,54,51,48],
  white = P[1,2,3,4,5,6,16,25,34,10,11,9,15,24,33,39,17,
   18,19,20,8,14,23,32,38,26,27,28,29,7,13,22,31,37,35,36,
   12,21,30,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54],
  green = P[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,
   19,20,21,22,23,24,25,26,27,31,32,33,34,35,36,48,47,46,
   39,42,45,38,41,44,37,40,43,30,29,28,49,50,51,52,53,54],
  blue = P[3,6,9,2,5,8,1,4,7,54,53,52,10,11,12,13,14,15,
   19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,
   37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,18,17,16],
  red = P[13,2,3,22,5,6,31,8,9,12,21,30,37,14,15,16,17,
   18,11,20,29,40,23,24,25,26,27,10,19,28,43,32,33,34,35,
   36,46,38,39,49,41,42,52,44,45,1,47,48,4,50,51,7,53,54],
  yellow = P[1,2,48,4,5,51,7,8,54,10,11,12,13,14,3,18,27,
   36,19,20,21,22,23,6,17,26,35,28,29,30,31,32,9,16,25,34,
   37,38,15,40,41,24,43,44,33,46,47,39,49,50,42,52,53,45]
};   Enter
```

Theorem. $G = M_1$. G^{-1} is more fun!

$G = M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} : \forall i, j_i \geq i \text{ and } \sigma_{i,j_i} \in T\}$.

Proof. The inclusions $M_1 \subset G$ and $\{g_1, \dots, g_\alpha\} \subset M_1$ are obvious. The rest follows from the following

Lemma. M_1 is closed under multiplication.

Proof. By backwards induction. Let
$$M_k := \{\sigma_{k,j_k}\cdots\sigma_{n,j_n} : \forall i \geq k, j_i \geq i \text{ and } \sigma_{i,j_i} \in T\}.$$
Clearly $M_nM_n \subset M_n$. Now assume that $M_5M_5 \subset M_5$ and show that $M_4M_4 \subset M_4$. Start with $\sigma_{8,j}M_4 \subset M_4$:
$$\sigma_{8,j}(\sigma_{4,j_4}M_5) \overset{1}{=} (\sigma_{8,j}\sigma_{4,j_4})M_5 \overset{2}{\subset} M_4M_5$$
$$\overset{3}{=} \sigma_{4,j_4}(M_5M_5) \overset{4}{\subset} \sigma_{4,j_4}M_5 \subset M_4$$
(1: associativity, 2: thank the twist, 3: associativity and tracing i_4, 4: induction). Now the general case
$$(\sigma_{4,j_4'}\sigma_{5,j_5'}\cdots)(\sigma_{4,j_4}\sigma_{5,j_5}\cdots)$$
falls like a chain of dominos.

Problem Solved!

A Demo Program

```
In[2]:= ($RecursionLimit = 2^16;
  n = 54;
  P /: p_P ** P[a___] := p[[{a}]];
  Inv[p_P] := P @@ Ordering[p];
  Feed[P @@ Range[n]] := Null;
  Feed[p_P] := Module[{i, j},
    For[i = 1, p[[i]] == i, ++i];
    j = p[[i]];
    If[Head[s[i, j]] === P,
      Feed[Inv[s[i, j]] ** p],
    (* Else *) s[i, j] = p;
      Do[If[Head[s[k, l]] == P,
        Feed[s[i, j] ** s[k, l]];
        Feed[s[k, l] ** s[i, j]]
      ], {k, n}, {l, n}]
    ]]);   Enter
```

www.geocities.com/jaapsch/puzzles/

www.powerstrike.net/puzzles/

http://lodelnic.free.fr/

Homework Problem 1.
Can you do cosets?

ROTARY CIRCLE

www.powerstrike.net/puzzles/

Homework Problem 2.
Can you do categories (groupoids)?

7	9	2	5
1	4	8	3
6	10	11	12
13	14	15	

Rubik's magic

The Results

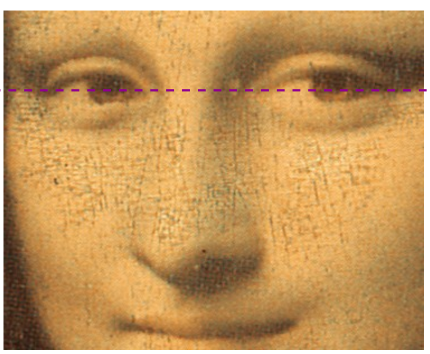
that's cool!

```
In[3]:= (Feed[#]; Product[1 + Length[Select[Range[n], Head[s[i, #]] === P &]], {i, n}]) & /@ gs   Enter
Out[3]= {4, 16, 159993501696000, 21119142223872000, 43252003274489856000, 43252003274489856000}
```

http://www.math.toronto.edu/~drorbn/Talks/Mathcamp-0907/ and links there

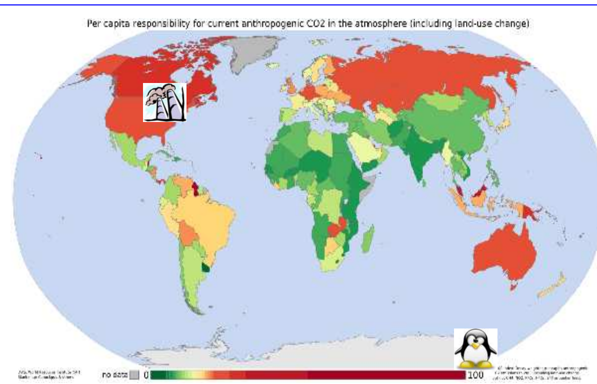
I could be a mathematician ...

...or an art historian...

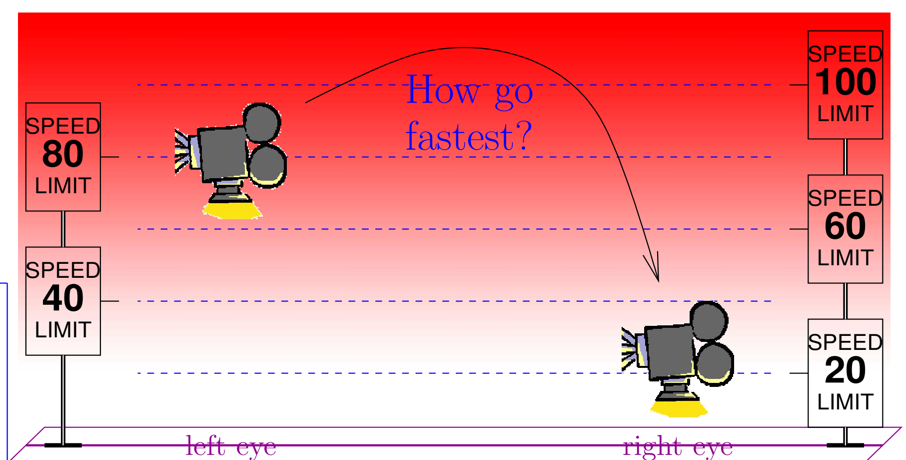


...or an environmentalist.

Al Gore in Futurama, circa 3000AD



Goal. Find the least-blur path to go from Mona's left eye to Mona's right eye in fixed time. Alternatively, fix your blur-tolerance, and find the fastest path to do the same. For fixed blur, our camera moves at a speed proportional to its distance from the image plane:



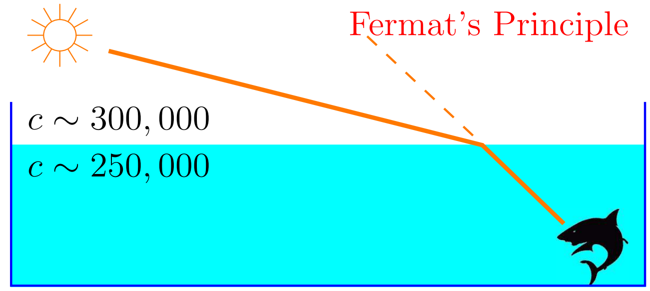
http://drorbn.net/n16

Video at http://www.math.toronto.edu/~drorbn/Talks/RCI-110213/, more at
http://www.math.toronto.edu/~drorbn/Talks/Niagara-1612/

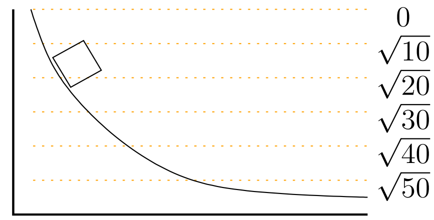
The Hardest Math I've Ever Really Used, 2

Picture credits. Mona: Leonrado; Al Gore: Futurama; Map 1: en.wikipedia.org/wiki/Greenhouse_gas; Smokestacks: gbuapcd.org/complaint.htm; Penguin: brentpabst.com/bp/2007/12/15/BrentGoesPenguin.aspx; Map 2: flightpedia.org; Segway: co2calculator.wordpress.com/2008/10; Lobachevsky: en.wikipedia.org/wiki/Nikolai_Lobachevsky; Eschers: www.josleys.com/show_gallery.php?galid=325;

Fermat's Principle



The Brachistochrone



$$mgh = \frac{1}{2}mv^2$$

Bernoulli on Newton. "I recognize the lion by his paw".

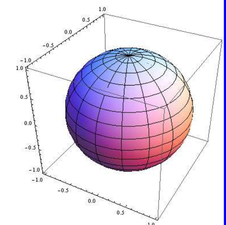
Flatlanders airline route map

576
252
167
131
112
103
100
103
112

The Least Action Principle. Everywhere in physics, a system goes from A to B along the path of least action.

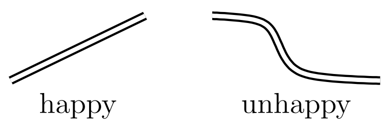
With small print for quantum mechanics.

```
ParametricPlot3D[{
  Sin[u] Cos[v],
  Sin[u] Sin[v],
  Cos[u]
 }, {u, 0, π}, {v, 0, 2π}]
```

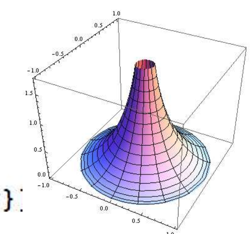


The Happy Segway Principle

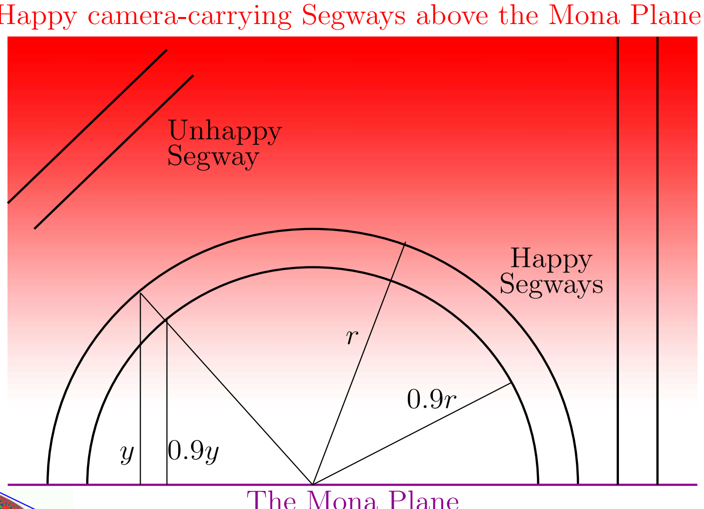
A Segway is happy iff both its wheels are



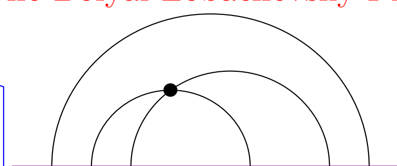

```
ParametricPlot3D[{
   Sech[u] Cos[v],
   Sech[u] Sin[v],
   u - Tanh[u]
 }, {u, 0, e}, {v, 0, 2π}]
```



Happy camera-carrying Segways above the Mona Plane

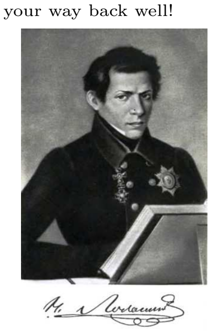


The Bolyai-Lobachevsky Plane



Two parallels through one point

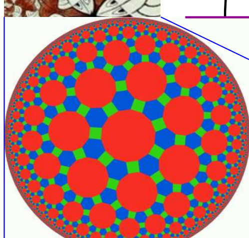
Further Fun Facts. • In small scale, $\pi^H \to \pi^E$. In large scale, $\pi^H \to \infty$. • The sum of the angles of a triangle is always less than π. In fact, sum+area= π, so the largest possible area of a triangle is π. • If your friend walks away, she'll drop out of sight before you know it. • There are so many places just a stone throw away! But you'd better remember your way back well!



The Actual Code

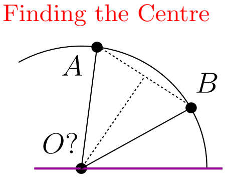
```
p3.y = p2.y + b*x3p;
x = p1.x-p2.x; y = p1.y-p2.y;
d1 = p1.d; d2 = p2.d;
norm = sqrt(x*x + y*y);
a = x/norm; b = y/norm;
x1p = a*x + b*y;
x0 = (x1p + (d1*d1-d2*d2)/x1p)/2;
r = sqrt((x1p-x0)*(x1p-x0)+d1*d1);
x1pp = (x1p-x0)/r; x2pp = -x0/r;
theta1 = acos(x1pp);
theta2 = acos(x2pp);
t1 = log(tan(theta1/2));
t2 = log(tan(theta2/2));
t3 = t1 + s*(t2-t1);
theta3 = 2*atan(exp(t3));
x3pp = cos(theta3);
d3pp = sin(theta3);
x3p = x0 + r*x3pp;
p3.d = r*d3pp;
p3.x = p2.x + a*x3p;
```

Ops used. $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, cos, sin, tan, arccos, arctan, log, exp.

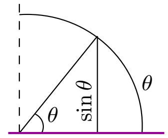


Some further basic geometry also occurs:

Finding the Centre



Parametrization



$$\theta'(t) = \sin\theta(t)$$
$$\Downarrow$$
$$\theta = 2\arctan e^t$$

Video at http://www.math.toronto.edu/~drorbn/Talks/RCI-110213/, more at http://www.math.toronto.edu/~drorbn/Talks/Niagara-1612/

Work in Progress!

The Brute and the Hidden Paradise

Abstract. There is expected to be a hidden paradise of poly-time computable knot polynomials lying just beyond the Alexander polynomial. I will describe my brute attempts to gain entry.

Why "expected"? Gauss diagram formulas [PV, GPV] show that $v_{d,f}(K) = \sum_{Y\subset X(K),\ |Y|=d} f(Y)$ finite-type invariants are all poly-time, and tempt to conjecture that there are no others. But Alexander shows it nonsense:

d	2	3	4	5	6	7	8	$\cdots$
known invts in* $O(n^d)$	1	1	∞	3	4	8	11	$\cdots$

This is an unreasonable picture! **Fresh, numerical, no cheating.*
So there ought to be further poly-time invariants.

Also. • The line above the Alexander line in the Melvin-Morton [MM, Ro] expansion of the coloured Jones polynomial. • The 2-loop contribution to the Kontsevich integral.


Rozansky

Why "paradise"? Foremost answer: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

ωεβ/K17: (extend to tangles, perhaps detect non-slice ribbon knots)

Second Answer. The second answer has to do with "Algebraic Knot Theory", so let me start with that. Somewhat informally, a "tangle" is a piece of a knot, or a "knot with endpoints" (an example is on the right). Knots can be assembled by stitching together the strands of several tangles, or the different strands of a single tangle. Some interesting classes of knots can be defined algebraically using tangles and these stitching operations. Here is the most interesting example:

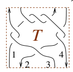


Figure 1. A tangle.

Definition 1. A "ribbon knot" is a knot K that can be presented as the boundary of a disk D which is allowed to have "ribbon singularities" but not "clasp singularities". See Figure 2.

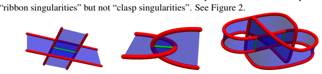
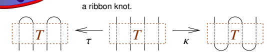


Figure 2. A ribbon singularity, a clasp singularity, and an example of a ribbon knot.

Definition 2. Let $\mathcal{T}_{2n}$ denote the set of all tangles T with $2n$ components that connect $2n$ points along a "top end" with $2n$ points along a "bottom end" inducing the identity permutation of ends (an example is the tangle in Figure 1). Given $T \in \mathcal{T}_{2n}$, let $\tau(T)$ be the result of stitching its components at the top in pairs as on the right — it is an n-component tangle all of whose ends are at the bottom, and we (somewhat loosely) denote the set of all such by $\mathcal{T}_n$, so $\tau\colon \mathcal{T}_{2n}\to\mathcal{T}_n$. Likewise let $\kappa(T)$ be the result of stitching T both at the top and at the bottom, also as on the right. So $\kappa(T)$ is a 1-component tangle, which is the same as a knot, and $\kappa\colon \mathcal{T}_{2n}\to\mathcal{T}_1$.



gives $\mathcal{T}_n \xleftarrow{\tau} \mathcal{T}_{2n} \xrightarrow{\kappa} \mathcal{T}_1$

Theorem 1 (I have not seen this theorem in the literature, yet it is not difficult to prove). The set of ribbon knots is the set of all knots K that can be written as $K=\kappa(T)$ for some tangle T for which $\tau(T)$ is the untangled (crossingless) tangle U:

$$\{\text{ribbon knots}\} = \{\kappa(T)\colon T\in\mathcal{T}_{2n} \text{ and } \tau(T)=U\in\mathcal{T}_n\}.$$

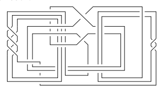
Now suppose we have an invariant $Z\colon \mathcal{T}_k\to A_k$ of tangles, which takes values in some spaces A_k. Suppose also we have operations $\tau_A\colon A_{2n}\to A_n$ and $\kappa_A\colon A_{2n}\to A_1$ such that the diagram on the right is commutative. Then

$$\begin{tikzcd} \mathcal{T}_n \arrow[d, "Z"] & \mathcal{T}_{2n} \arrow[l, "\tau"'] \arrow[r, "\kappa"] \arrow[d, "Z"] & \mathcal{T}_1 \arrow[d, "Z"] \\ A_n & A_{2n} \arrow[l, "\tau_A"'] \arrow[r, "\kappa_A"] & A_1 \end{tikzcd} \qquad (1)$$

$$Z(\{\text{ribbon knots}\}) \subseteq \mathcal{R}_A := \{\kappa_A(\zeta)\colon \zeta\in A_{2n} \text{ and } \tau_A(\zeta)=1_A\in A_n\}\subset\mathcal{A}_1, \qquad (2)$$

where $1_A := Z(U)\in A_n$. If the target spaces A_k are algebraic (polynomials, matrices, matrices of polynomials, etc.) and the operations τ_A and κ_A are algebraic maps between them (at this stage, meaning just "have simple algebraic formulas"), then $\mathcal{R}_A$ is an algebraically defined set. Hence we potentially have an algebraic way to detect non-ribbon knots: if $Z(K)\notin\mathcal{R}_A$, then K is not ribbon.

As it turns out, it is valuable to detect non-ribbon knots. Indeed the Slice-Ribbon Conjecture (Fox, 1960s) asserts that every slice knot (a knot in S^3 that can be presented as the boundary of a disk embedded in B^4) is ribbon. Gompf, Scharlemann, and Thompson [GST] describe a family of slice knots which they conjecture are not ribbon (the simplest of those is on the right). With the algebraic technology described above it may be possible to show that the [GST] knots are indeed non-ribbon, thus disproving the Slice-Ribbon Conjecture.



What would it take?

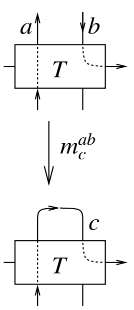
C1. An invariant Z which makes sense on tangles and for which diagram (1) commutes.
C2. Z cannot be a simple extension of the Alexander polynomial to tangles, for by Fox-Milnor [FM] the Alexander polynomial does not detect non-ribbon slice knots.
C3. Z cannot be computable from finitely many finite type invariants, for this would contradict the results of Ng [Ng].[1]
C4. Z must be computable on at least the simplest [GST] knot, which has 48 crossings.
C5. It is better if in some meaningful sense the size of the spaces A_k grows slowly in k. Indeed in (2), if A_{2n} is much bigger than A_n and A_1 then at least generically $\mathcal{R}_A$ will be the full set A_1 and our condition will be empty.

No invariant that I know now meets these criteria. Alexander and Vassiliev fail C2 and C3, respectively. Almost all quantum invariants and knot homologies pass C1-C3, but fail C4. Jones, HOMFLY-PT and Khovanov potentially pass C4, yet fail C5. We must come up with something new.

[FM] R. H. Fox and J. W. Milnor, *Singularities of 2-Spheres in 4-Space and Cobordism of Knots,* Osaka J. Math. **3** (1966) 257–267.
[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.
[Ng] K. Y. Ng, *Groups of ribbon knots,* Topology **37** (1998) 441–458, arXiv:q-alg/9502017 (with an addendum at arXiv:math.GT/0310074)

[1] A slight subtlety arises: There is no taking limits here, and C3 does not preclude the possibility that Z is computable from infinitely many finite type invariants. The Fox-Milnor condition on the Alexander polynomial of ribbon knots, for example, is expressible in terms of the full Alexander polynomial, yet not in terms of any finite type reduction thereof. Unfortunately by C2 it cannot be used here.

Moral. Need "stitching":



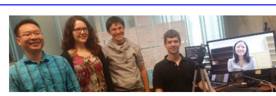
Why "brute"? Cause it's the only thing I know, for now. There may be better ways in, and it's fair to hope that sooner or later they will be found.

The Gold Standard is set by the formulas [BNS, BN] for Alexander. An S-component tangle T has $\Gamma(T)\in$ $R_S\times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a\colon a\in S\})$:

$$\left(\nearrow_a{}_b,\ {}_b\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm1} \\ b & 0 & t_a^{\pm1}\end{array} \qquad T_1\sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi\end{array} \xrightarrow[t_a,t_b\to t_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta} \\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

Help Needed! Disorganized videos of talks in a private seminar are at ωεβ/PP.

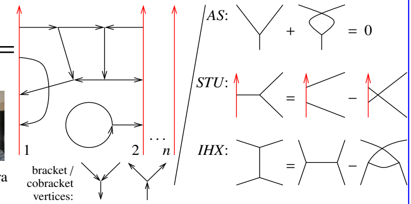

Vo, Halacheva, Dalvit, Ens, Lee (van der Veen, Schaveling)

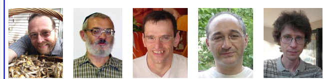
For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know!


Dunfield: 1000-crossing fast.

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z\colon \left\{\begin{matrix} S\text{-component} \\ (v/b\text{-})\text{tangles}\end{matrix}\right\} \to \mathcal{A}^v_S :=$$

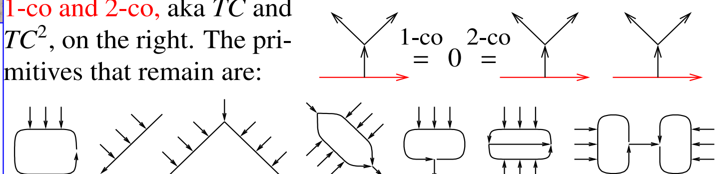



Etingof Kazhdan Haviv Enriquez Ševera

(it is enough to know Z on $\nwarrow\!\!\nearrow$ and have disjoint union and stitching formulas) ...exponential and too hard!

Idea. Look for "ideal" quotients of $\mathcal{A}^v_S$ that have poly-sized descriptions; ...specifically, limit the co-brackets.

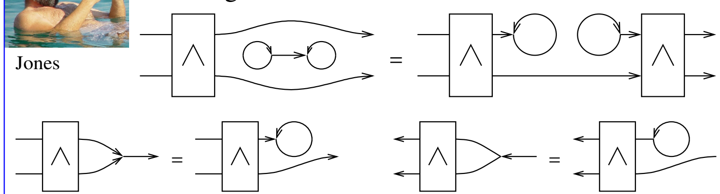
1-co and 2-co, aka TC and TC^2, on the right. The primitives that remain are:



...manageable but still exponential!


Jones

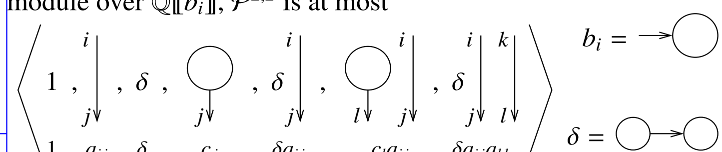
The 2D relations come from the relation with 2D Lie bialgebras:



We let $\mathcal{A}^{2,2}$ be $\mathcal{A}^v$ modulo 2-co and 2D, and $z^{2,2}$ be the projection of $\log Z$ to $\mathcal{P}^{2,2} := \pi\mathcal{P}^v$, where $\mathcal{P}^v$ are the primitives of $\mathcal{A}^v$.

Main Claim. $z^{2,2}$ is poly-time computable.

Main Point. $\mathcal{P}^{2,2}$ is poly-size, so how hard can it be? Indeed, as a module over $\mathbb{Q}[\![b_i]\!]$, $\mathcal{P}^{2,2}$ is at most



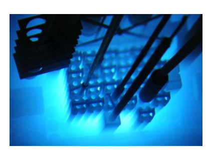
Claim. $R_{jk} = e^{a_{jk}}e^{\rho_{jk}}$ is a solution of the Yang-Baxter / R3 equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in $\exp\mathcal{P}^{2,2}$, with $\rho_{jk} :=$

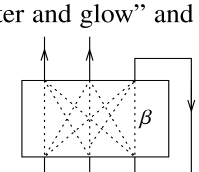
$$\psi(b_j)\left(-c_k + \frac{c_k a_{jk}}{b_j} - \frac{\delta a_{jk}a_{jk}}{b_j^2}\right) + \frac{\phi(b_j)\psi(b_k)}{b_k\phi(b_k)}\left(c_k a_{kk} - \frac{\delta a_{jk}a_{kk}}{b_j}\right),$$

and with $\phi(x) := e^{-x}-1 = -x+x^2/2-\dots$, and $\psi(x) := ((x+2)e^{-x}-2+x)/(2x) = x^2/12 - x^3/24 + \dots$. (This already gives some new (v-)braid group representations, as below).

Problem. How do we multiply in $\exp(\mathcal{P}^{2,2})$? How do we stitch? BCH is a theoretical dream. Instead, use "scatter and glow" and "feedback loops":

The Euler trick:
With $Ef := (\deg f)f$ get $Ee^x = xe^x$ and $E(e^xe^ye^z) =$ $xe^xe^ye^z + e^xye^ye^z + e^xe^yze^z$.


Cherenkov radiation



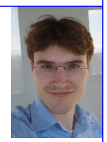
Feedback loop

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Greece-1607/

Dror Bar-Natan: Talks: Greece-1607: ωεβ:=http://drorbn.net/Greece-1607/

Work in Progress!

The Brute and the Hidden Paradise



van der Veen

Local Algebra (with van der Veen) Much can be reformulated as (non-standard) "quantum algebra" for the 4D Lie algebra $\mathfrak{g} = \langle b, c, u, w\rangle$ over $\mathbb{Q}[\epsilon]/(\epsilon^2=0)$, with b central and $[w,c]=w$, $[c,u]=u$, and $[u,w]=b-2\epsilon c$. The key: $a_{ij}=(b_i-\epsilon c_i)c_j+u_iw_j$ in $\mathcal{U}(\mathfrak{g})^{\otimes\{i,j\}}$.

Some (new) representationss of the (v-)braid groups. ωεβ/Reps

Burau (old)

```
B_{i_,j_}[ξ_] := ξ /. v_j :> (1 - t) v_i + t v_j
```

…testing R3

```
Column@{lhs = {v1, v2, v3} // B_{1,2} // B_{1,3} // B_{2,3},
  rhs = {v1, v2, v3} // B_{2,3} // B_{1,3} // B_{1,2} ,
  lhs - rhs // Expand}

{v1, (1 - t) v1 + t v2, (1 - t) v1 + t ((1 - t) v2 + t v3)}
{v1, (1 - t) v1 + t v2,
  (1 - t) ((1 - t) v1 + t v2) + t ((1 - t) v1 + t v3)}
{0, 0, 0}
```

Gassner (old)

```
G_{i_,j_}[ξ_] := ξ /. v_j :> (1 - t_i) v_i + t_i v_j
```

…Overcrossings Commute (OC):

```
Column@{lhs = {v1, v2, v3} // G_{1,2} // G_{1,3},
  Expand[lhs - ({v1, v2, v3} // G_{1,3} // G_{1,2})]}

{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 v3}
{0, 0, 0}
```

…Undercrossings Commute (UC):

```
Column@{lhs = {v1, v2, v3} // G_{1,3} // G_{2,3},
  rhs = {v1, v2, v3} // G_{2,3} // G_{1,3} ,
  lhs - rhs // Expand}

{v1, v2, (1 - t1) v1 + t1 ((1 - t2) v2 + t2 v3)}
{v1, v2, (1 - t2) v2 + t2 ((1 - t1) v1 + t1 v3)}
{0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}
```

Gassner Plus (new?)

```
GP_{i_,j_}[ξ_] := Expand[ξ /. {u_j :> (1 - t_i) u_i + t_i u_j,
    f_. v_j :> f (1 - t_i) v_i + f t_i v_j + (t_i - 1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) u_i +
      f t_i u_i }];
bas = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3,
   u1, u2, u3};
```

…R3 (left)

```
Short[lhs = bas // GP_{1,2} // GP_{1,3} // GP_{2,3}, 2]

{f[t1, t2, t3] v1, f[t1, t2, t3] t1 u1 + f[t1, t2, t3] v1 -
  f[t1, t2, t3] t1 v1 + <<6>> + t1^2 u1 f^(1,0,0)[t1, t2, t3],
 <<1>> + <<19>> + <<1>>, <<1>>, u1 - t1 u1 + t1 u2,
 u1 - t1 u1 + t1 u2 - t1 t2 u2 + t1 t2 u3}
```

…R3 (rest)

```
(bas // GP_{2,3} // GP_{1,3} // GP_{1,2}) - lhs
{0, 0, 0, 0, 0, 0}
```

…OC

```
(bas // GP_{1,2} // GP_{1,3}) - (bas // GP_{1,3} // GP_{1,2})
{0, 0, 0, 0, 0, 0}
```

Question. Does Gassner Plus factor through Gassner?

Turbo-Gassner (new!)

```
Kδ_{i_,j_} := KroneckerDelta[i, j];
TG_{i_,j_}[ξ_] := Expand[ξ /. {
    f_. v_k_ :> Plus[f v_k /. v_j -> (1 - t_i) v_i + t_i v_j,
      (1 - t_i^-1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) *
        (u_k /. u_j -> (1 - t_i) u_i + t_i u_j) * u_i w_j,
      Kδ_{k,i} f (u_j - u_i) u_i w_j],
    u_j -> (1 - t_i) u_i + t_i u_j,
    w_i -> w_i + (1 - t_i^-1) w_j, w_j -> t_i^-1 w_j}];
bas = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3,
   u1, u2, u3, w1, w2, w3};
```

Satisfies R3…

…OC

```
(bas // TG_{1,2} // TG_{1,3}) - (bas // TG_{1,3} // TG_{1,2})

{0, -f[t1, t2, t3] u1 u2 w3 + f[t1, t2, t3] t1 u1 u2 w3 +
  f[t1, t2, t3] u1 u3 w3 - f[t1, t2, t3] t1 u1 u3 w3,
 -f[t1, t2, t3] u1 u2 w2 + f[t1, t2, t3] t1 u1 u2 w2 +
  f[t1, t2, t3] u1 u3 w2 -
  f[t1, t2, t3] t1 u1 u3 w2, 0, 0, 0, 0, 0, 0}
```

Turbo-Burau (new!)

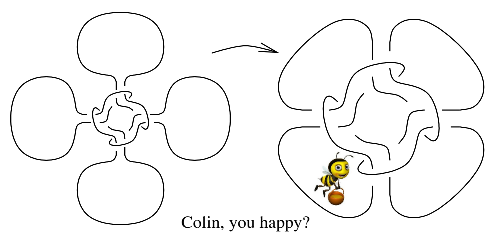
```
η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
TB_{i_,j_}[ξ_] :=
  Expand[ξ /. {
    f_. v_k_ :> Plus[f v_k /. v_j -> (1 - t - η[i]) v_i + (t + η[i]) v_j,
      (t - 1) (Coefficient[f, η[i]] - Coefficient[f, η[j]]) *
        (u_k /. u_j -> (1 - t) u_i + t u_j) * u_i w_j,
      Kδ_{k,i} (f /. _η -> 0) (u_j - u_i) u_i w_j],
    u_j -> (1 - t) u_i + t u_j,
    w_i -> w_i + (1 - t^-1) w_j, w_j -> t^-1 w_j}];
ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];
bas = {ff v1, ff v2, ff v3, u1^2 w1, u1^2 w2, u1, u2, u3, w1, w2, w3};
```

…OC

```
(bas // TB_{1,2} // TB_{1,3}) - (bas // TB_{1,3} // TB_{1,2})

{0, -f0 u1 u2 w3 + t f0 u1 u2 w3 + f0 u1 u3 w3 - t f0 u1 u3 w3,
 -f0 u1 u2 w2 + t f0 u1 u2 w2 + f0 u1 u3 w2 - t f0 u1 u3 w2,
 0, 0, 0, 0, 0, 0, 0, 0}
```

Flower Surgery Theorem. A knot is ribbon iff it is the result of n-petal flower surgery (from thin petals to wide petals) on an n-componenet unlink, for some n.



Colin, you happy?

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I, II, IV,* ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO4, arXiv:1405.1956, arXiv:1405.1955, arXiv:1511.05624.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras,* Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[Ha] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants,* Hebrew University PhD thesis, Sep. 2002, arXiv:math.QA/0211031.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

[PV] M. Polyak and O. Viro, *Gauss Diagram Formulas for Vassiliev Invariants,* Inter. Math. Res. Notices **11** (1994) 445–453.

[Ro] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Se] P. Ševera, *Quantization of Lie Bialgebras Revisited,* Sel. Math., NS, to appear, arXiv:1401.6164.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Greece-1607/

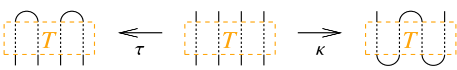
Work in Progress!

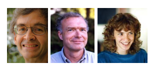
Gauss-Gassner Invariants, What?

Abstract. In a "degree d Gauss diagram formula" one produces a number by summing over all possibilities of paying very close attention to d crossings in some n-crossing knot diagram while observing the rest of the diagram only very loosely, minding only its skeleton. The result is always poly-time computable as only $\binom{n}{d}$ states need to be considered. An under-explained paper by Goussarov, Polyak, and Viro [GPV] shows that every type d knot invariant has a formula of this kind. Yet only finitely many integer invariants can be computed in this manner within any specific polynomial time bound.

I suggest to do the same as [GPV], except replacing "the skeleton" with "the Gassner invariant", which is still poly-time. One poly-time invariant that arises in this way is the Alexander polynomial (in itself it is infinitely many numerical invariants) and I believe (and have evidence to support my belief) that there are more.

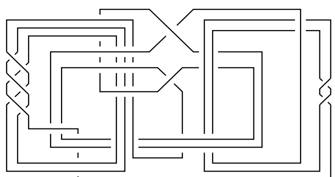
The QUILT Target. QUick Invariants of Large Tangles, for little had been found since Alexander (and if they're there, how can we not know all about them?), and for {ribbon} ≠ {slice}:





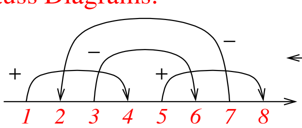
Gompf, Scharlemann, Thompson [GST]

$$U \in \mathcal{T}_n \qquad 1 \in \mathcal{A}_n$$
$$\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n}$$
$$\text{ribbon } K \in \mathcal{T}_1 \qquad z(K) \in \mathcal{A}_1$$

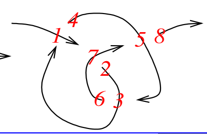


Faster is better, leaner is meaner!

Gauss Diagrams. (just QUILK, today)

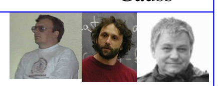


$\longleftrightarrow$

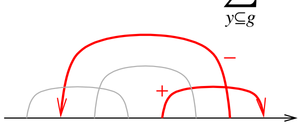


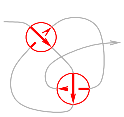
Gauss

Gauss Diagram Formulas [PV, GPV]. If g is a Gauss diagram and F an unsigned Gauss diagram, $\langle F, g\rangle_{PV} := \sum_{y\subseteq g}(-1)^y\delta(F,\bar{y})$:

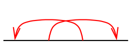


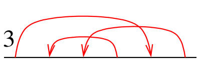
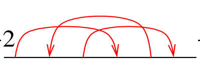
Goussarov-Polyak-Viro





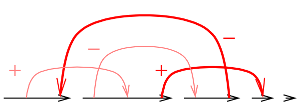
Under-Explain Theorem [GPV]. Every finite type invariant arises in this way.

$F_2 =$  $\Rightarrow \langle F_2, K\rangle = v_2(K)$

$F_3 =$ 3 +2 + rotations

$$\Rightarrow \langle F_3, K\rangle = 6v_3(K)$$

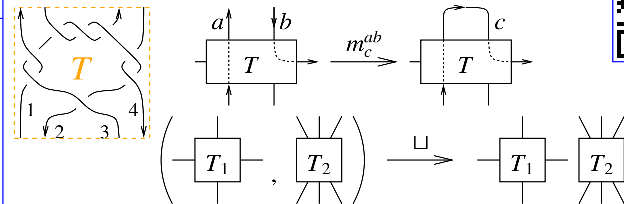
Gauss-Gassner Invariants. Want more? Increase your environmental awareness! Instead of nearly-forgetting y^c, compute its Burau/Gassner invariant (note that y^c is a tangle in a Swiss cheese; more easily, a virtual tangle):

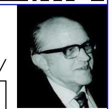


$$GG_{k,F}(g) = \sum_{y\subseteq g,\ |y|\le k} \bar{F}\left(y, z(y^c)\right) = \sum_{y\subseteq g,\ |y|\le k} F\left(y, z(g \text{ cut near } y)\right),$$

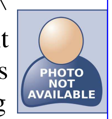
where k is fixed and $F(y,\gamma)$ is a function of a list of arrows y and a square matrix γ of side $|y|+1 \le k+1$.

The (Burau-)Gassner Invariant.





Burau



Gassner

Theorem 1. ∃! an invariant z: {pure framed S-component tangles} $\to \Gamma(S) := M_{S\times S}(R_S)$, where $R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c} & S_1 \\ \hline S_1 & A_1\end{array}, \begin{array}{c|c} & S_2 \\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc} & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc} & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[T_a,T_b\to T_c,\ \mu:=1-\beta]{m^{ab}_c} \left(\begin{array}{c|cc} & c & S \\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu \\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu \end{array}\right),$$

and satisfying $\left(|_a;\ {}_a\!\nearrow\!\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\!\nearrow_a\right) \xrightarrow{z} \left(\begin{array}{c|c} & a \\ \hline a & 1\end{array};\ \begin{array}{c|cc} & a & b \\ \hline a & 1 & 1-T_a^{\pm1} \\ b & 0 & T_a^{\pm1}\end{array}\right).$

See also [LD, KLW, CT, BNS].

Theorem 2. With $k=1$ and F_A defined by

$$F_A(\xrightarrow{s},\gamma) = s\left.\frac{\gamma_{22}\gamma_{33}-\gamma_{23}\gamma_{32}}{\gamma_{33}+\gamma_{13}\gamma_{32}-\gamma_{12}\gamma_{33}}\right|_{T_a\to T},$$

$$F_A(\xleftarrow{s},\gamma) = s\left.\frac{\gamma_{13}\gamma_{32}-\gamma_{12}\gamma_{33}}{\gamma_{32}-\gamma_{23}\gamma_{32}+\gamma_{22}\gamma_{33}}\right|_{T_a\to T},$$

$GG_{1,F_A}(K)$ is a regular isotopy invariant. Unfortunately, for every knot K, $GG_{1,F_A}(K) - T\frac{d}{dT}\log A(K)(T) \in \mathbb{Z}$, where $A(K)$ is the Alexander polynomial of K.

Expectation. Higher Gauss-Gassner invariants exist . . .
(though right now I can reach for them only wearing my exoskeleton)

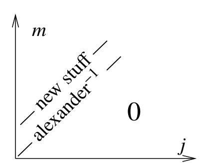
Jones, Melvin, Morton, Rozansky, Garoufalidis

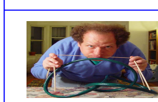
. . . and they are the "higher diagonals" in the MMR expansion of the coloured Jones polynomial J_λ.

Theorem ([BNG], conjectured [MM], elucidated [Ro]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of $sl(2)$. Writing

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\ge0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot A(K)(e^\hbar)=1$.





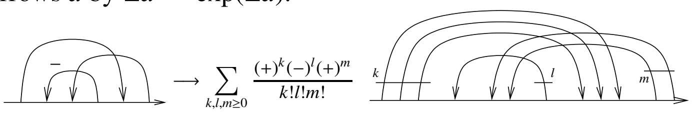
Help Needed!

I'm slow and feeble-minded.

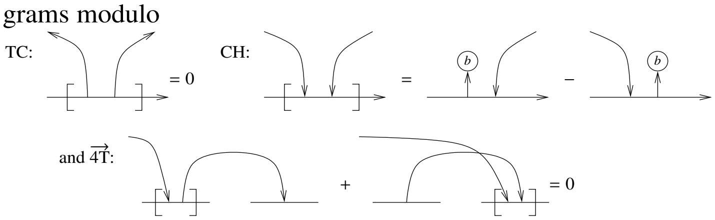
Video and more at http://www.math.toronto.edu/~drorbn/Talks/NCSU-1604/

Warning. Conventions on this page change randomly from line to line.

$Z^{w/2}$. The GGA story is about $Z^{w/2}\colon \mathcal{K} \to \mathcal{A}^{w/2}$, defined on arrows a by $\pm a \mapsto \exp(\pm a)$:



Where the target space $\mathcal{A}^{w/2}$ is the space of unsigned arrow diagrams modulo

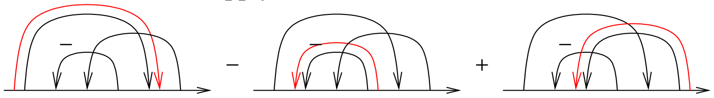


($Z^{w/2}$ is a reduction of the much-studied Z^w [BND, BN]).

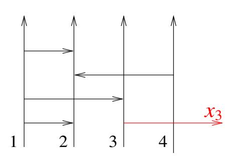
The Euler Trick. How best do non-commutative algebra with exponentials? Logarithms are from hell as $e^f e^g = e^{\text{bch}(f,g)}$, but Euler's from heaven: Let E be the derivation $Ef := (\deg f)f$ $(= xf'$, in $\mathbb{Q}[\![x]\!]$) and let $\tilde{E}Z := Z^{-1}EZ$ $(= x(\log Z)'$ in same). If $\deg x = 1$ then $\tilde{E}e^x = x$ and if $F = e^f$ and $G = e^g$, then $\tilde{E}(FG)$ is

$$(FG)^{-1}((EF)G + F(EG)) = G^{-1}(\tilde{E}F)G + \tilde{E}G = e^{-\operatorname{ad} g}(\tilde{E}F) + \tilde{E}G.$$

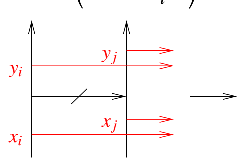
Scatter and Glow. Apply $\tilde{E}$ to $Z(K)$. EZ is shown:



Tail scattering. The algebra $\mathbb{Q}[\![b_i]\!]\langle a_{ij}\rangle$ modulo $[a_{ij}, a_{kl}] = 0$ (loc), $[a_{ij}, a_{ik}] = 0$ (TC), and $[a_{ik}, a_{jk}] = -[a_{ij}, a_{jk}] = b_j a_{ik} - b_i a_{jk}$ (CH and $\overrightarrow{4T}$), acts on $V = \mathbb{Q}[\![b_i]\!]\langle x_i = a_{i\infty}\rangle$ by $[a_{ij}, x_i] = 0$, $[a_{ij}, x_j] = b_i x_j - b_j x_i$. Hence $e^{\operatorname{ad} a_{ij}} x_i = x_i$, $e^{\operatorname{ad} a_{ij}} x_j = e^{b_i} x_j + \frac{b_j}{b_i}(1 - e^{b_i}) x_i$. Renaming $\bar{x}_i = x_i/b_i$, $T_i = e^{b_i}$, get $[e^{\operatorname{ad} a_{ij}}]_{\bar{x}_i, \bar{x}_j} = \begin{pmatrix} 1 & 1 - T_i \\ 0 & T_i \end{pmatrix}$. Alternatively,



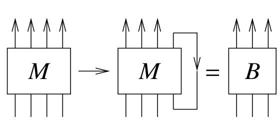
$a_{12}a_{13}a_{42}a_{12}$



$$\bar{y}_i = \bar{x}_i$$
$$\bar{y}_j = (1 - T_i)\bar{x}_i + T_i \bar{x}_j$$

Linear Control Theory.

If $\begin{pmatrix} y \\ y_n \end{pmatrix} = \begin{pmatrix} \Xi & \phi \\ \theta & \alpha \end{pmatrix} \begin{pmatrix} x \\ x_n \end{pmatrix}$, and we further impose $x_n = y_n$, then $y = Bx$ where $B = \Xi + \dfrac{\phi\theta}{1 - \alpha}$. This fully explains the Gassner formulas and the GGA formula!



All that remains now is to replace TC by something more interesting: with $\epsilon^2 = 0$,

$$[a_{ij}, a_{ik}] = \epsilon(c_j a_{ik} - c_k a_{ij}).$$

Many further changes are also necessary, and the algebra is a lot more complicated and revolves around "quantization of Lie bialgebras" [EK, En]. But the spirit is right.

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I, II, IV,* ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO4, arXiv:1405.1956, arXiv:1405.1955, arXiv:1511.05624.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[CT] D. Cimasoni and V. Turaev, *A Lagrangian Representation of Tangles,* Topology **44** (2005) 747–767, arXiv:math.GT/0406269.

[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras,* Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links,* Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.

[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner,* Comment. Math. Helv. **67** (1992) 306–315.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

[PV] M. Polyak and O. Viro, *Gauss Diagram Formulas for Vassiliev Invariants,* Inter. Math. Res. Notices **11** (1994) 445–453.

[Ro] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

Loading KnotTheory`

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<< KnotTheory`
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  of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

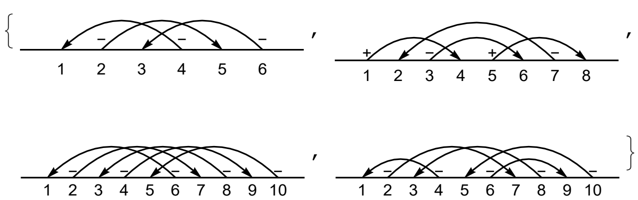
Gauss Diagram Utilities

```
GD[g_GD] := g;
GD[L_] := GD @@ PD[L] /.
   X[i_, j_, k_, l_] :> If[PositiveQ@X[i, j, k, l],
     Ap_{l,i}, Am_{j,i}];
Draw[g_GD] := Module[{n = Max@Cases[g, _Integer, ∞]},
  Graphics[{
    Line[{{0, 0}, {n + 1, 0}}],
    List @@ g /. (ah_)_{i_,j_} :> {
       Arrow[BezierCurve[{{i, 0}, {i + j, Abs[j - i]} / 2,
          {j, 0}}]],
       Text[ah /. {Ap -> "+", Am -> "-"}, {i, 0.3}]},
    Table[Text[i, {i, -0.5}], {i, n}]}]]
```

Some Gauss Diagrams

```
Draw /@ GD /@ AllKnots@{3, 5}
```

KnotTheory::loading : Loading precomputed data in PD4Knots`.



Some Gauss Diagrams, 2

```
GD /@ AllKnots@{3, 5}
```

```
{GD[Am_{4,1}, Am_{6,3}, Am_{2,5}], GD[Ap_{1,4}, Ap_{5,8}, Am_{3,6}, Am_{7,2}],
 GD[Am_{6,1}, Am_{8,3}, Am_{10,5}, Am_{2,7}, Am_{4,9}],
 GD[Am_{4,1}, Am_{8,3}, Am_{10,5}, Am_{6,9}, Am_{2,7}]}
```

V_2 Definition

```
CF[g_GD] := Sort[
   g /. Thread[Sort@Cases[g, _Integer, ∞] ->
      Range[2 Length[g]]]];
PV[F_GD, g_GD] /; Length[F] > Length[g] := 0;
PV[F_GD, g_GD] /; Length[F] < Length[g] := Sum[
   PV[F, γ], {γ, Subsets[g, {Length[F]}]}];
PV[F_GD, g_GD] /; Length[F] == Length[g] := If[
   CF[F] === CF[g /. Ap | Am -> A], (-1)^Count[g, Am__], 0];
V2[g_] := V2[g] = PV[GD[A_{3,1}, A_{2,4}], GD[g]];
```

Computing V_2

```
Format[Knot[n_, k_]] := n_k;
Table[K -> V2[K], {K, AllKnots@{3, 7}}]
```

$\{3_1\to 1,\ 4_1\to -1,\ 5_1\to 3,\ 5_2\to 2,\ 6_1\to -2,\ 6_2\to -1,\ 6_3\to 1,\ 7_1\to 6,\ 7_2\to 3,\ 7_3\to 5,\ 7_4\to 4,\ 7_5\to 4,\ 7_6\to 1,\ 7_7\to -1\}$

V_3 Definition

```
PV[F1_ + F2_, g_] := PV[F1, g] + PV[F2, g];
PV[c_ * F_GD, g_] := c PV[F, g];
ρ_k[g_] := g /. i_Integer :> Mod[i - k, 2 Length@g, 1];
F3 = Sum_{k=0}^{5} (3 ρ_k@GD[A_{1,5}, A_{4,2}, A_{6,3}] + 2 ρ_k@GD[A_{1,4}, A_{5,2}, A_{3,6}]);
V3[K_] := V3[K] = PV[F3, GD@K] / 6;
```

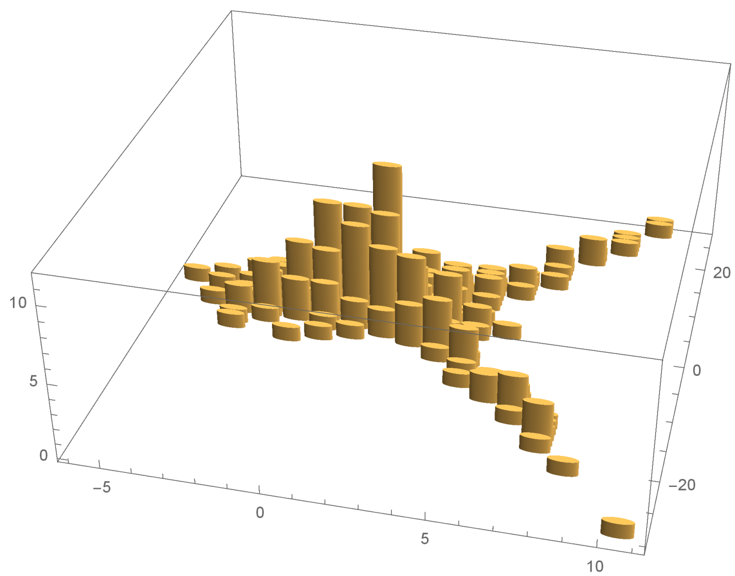
Computing V_3

```
Table[K -> V3[K], {K, AllKnots@{3, 7}}]
```

$\{3_1\to -1,\ 4_1\to 0,\ 5_1\to -5,\ 5_2\to -3,\ 6_1\to 1,\ 6_2\to 1,\ 6_3\to 0,\ 7_1\to -14,\ 7_2\to -6,\ 7_3\to 11,\ 7_4\to 8,\ 7_5\to -8,\ 7_6\to -2,\ 7_7\to -1\}$

Willerton's Fish

```
Histogram3D[
 Table[{V2[K], V3[K]}, {K, AllKnots@{3, 10}}],
 {1}]
```



Gassner Utilities

```
G[λ_]_{a_,b_} := ∂_{t_a,h_b} λ;
G /: Factor[G[λ_]] :=
  G[Collect[λ, h_, Collect[#, t_, Factor] &]];
Format@γ_G := Module[{S = Union@Cases[γ, (h | t)_a_ :> a, ∞]},
   Table[γ_{a,b}, {a, S}, {b, S}] // MatrixForm];
```

The Gassner Program

```
G /: G[λ1_] G[λ2_] := G[λ1 + λ2];
m_{a_,b_->c_}[G[λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  ( α β θ          ( ∂_{t_a,h_a} λ  ∂_{t_a,h_b} λ  ∂_{t_a} λ
    γ δ ϵ      =     ∂_{t_b,h_a} λ  ∂_{t_b,h_b} λ  ∂_{t_b} λ
    ϕ ψ Ξ )          ∂_{h_a} λ      ∂_{h_b} λ      λ        ) /. (t | h)_{a|b} -> 0;
  μ = 1 - β;
  G[Tr[(t_c, 1)^T . ( γ + α δ / μ   ϵ + δ θ / μ
                      ϕ + α ψ / μ   Ξ + ψ θ / μ ) . (h_c, 1)]] /. T_{a|b} -> T_c //
   Factor];
Rp_{a_,b_} := G[Tr[(t_a, t_b)^T . (1  1 - T_a
                                   0  T_a    ) . (h_a, h_b)]];
Rm_{a_,b_} := Rp_{a,b} /. T_a -> 1 / T_a;
```

The Gauss-Gassner-Program

```
GG[g_GD, k_, F_, BB_] :=
  Module[{n = 2 Length@g + Length@BB, y, cuts, rr, γ0, γ},
   γ0 = G[t_{n+1} h_{n+1}] Times @@ g /. {Ap -> Rp, Am -> Rm};
   γ0 *= G[Sum[β_{a,b} t_a h_b, {a, BB}, {b, BB}]];
   Sum[ γ = γ0;
    cuts = Cases[y, _Integer, ∞] ⋃ {n + 1};
    rr = Thread[cuts -> Range[Length@cuts]];
    Do[If[! MemberQ[cuts, j], γ = γ // m_{j,j+1->j+1}], {j, n}];
    F[y /. rr, γ /. (v_)_a_ :> v_{a/.rr}],
    (*over*) {y, Subsets[List @@ g, k]}]];
GG[g_GD, k_, F_] := GG[g, k, F, {}];
```

```
GG[GD@Knot[4, 1], {1}, F]
```

Example: 4_1

$$F\left[\{Am_{1,2}\},\begin{pmatrix} -\frac{-1+T_2-T_1T_2+T_3-T_1T_3-T_2T_3+T_1T_2T_3}{T_1T_3} & \frac{(-1+T_1)(1-T_2+T_1T_2)(-1+T_3)}{T_1T_3} & -\frac{(-1+T_1)(-1+T_2)}{T_1} \\ -\frac{(-1+T_2)(-1+T_3)}{T_1T_3} & \frac{-1+T_1+T_2-T_1T_2+T_3-T_2T_3+T_1T_2T_3}{T_1T_3} & -\frac{-1+T_2}{T_1} \\ \frac{T_2(-1+T_3)}{T_3} & -\frac{(-1+T_1)T_2(-1+T_3)}{T_3} & T_2 \end{pmatrix}\right]+$$

$$F\left[\{Am_{2,1}\},\begin{pmatrix} \frac{1}{T_2} & \frac{-1+T_1}{-T_1-T_2+T_1T_2} & -\frac{(-1+T_1)(-1+T_2)^2}{T_2(-T_1-T_2+T_1T_2)} \\ \frac{-1+T_2}{T_2} & \frac{1-2T_1-T_2+T_1T_2}{-T_1-T_2+T_1T_2} & -\frac{(-1+T_2)(-1+T_1+T_2-2T_1T_2-T_2^2+T_1T_2^2)}{T_2(-T_1-T_2+T_1T_2)} \\ 0 & 0 & T_2 \end{pmatrix}\right]+$$

$$F\left[\{Ap_{1,2}\},\begin{pmatrix} -\frac{1-2T_1-T_2+T_1T_2}{-1+T_1+T_2} & \frac{(-1+T_1)^2(-1+T_2)}{-1+T_1+T_2} & 0 \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_1(1-T_1-2T_2+T_1T_2)}{-1+T_1+T_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]+F\left[\{Ap_{1,2}\},\begin{pmatrix} 1 & \frac{(-1+T_1)(1-2T_2-T_3+T_2T_3)}{-1+T_2+T_3} & -\frac{(-1+T_1)(-1+T_2)}{-1+T_2+T_3} \\ 0 & -\frac{T_1(1-2T_2-T_3+T_2T_3)}{-1+T_2+T_3} & \frac{T_1(-1+T_2)}{-1+T_2+T_3} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}\right]$$

The Alexander Functional

```
FA[{x_}, γ_] := Simplify[
  Switch[x, Ap__, 1, Am__, -1] *
   Switch[x, _1,2, (γ2,2 γ3,3 - γ2,3 γ3,2)/(γ3,3 + γ1,3 γ3,2 - γ1,2 γ3,3),
    _2,1, (γ1,3 γ3,2 - γ1,2 γ3,3)/(γ3,2 - γ2,3 γ3,2 + γ2,2 γ3,3)] /. T_ → T];
GGA[K_, bb___] := GG[GD@K, {1}, FA, bb];
```

Example: 4_1

```
Simplify@With[{K = Knot[4, 1]},
  {GGA[K], Alexander[K][T], T ∂T Log[Alexander[K][T]]}]
```

$\left\{\frac{T(-3+2T)}{1-3T+T^2},\ 3-\frac{1}{T}-T,\ \frac{-1+T^2}{1-3T+T^2}\right\}$

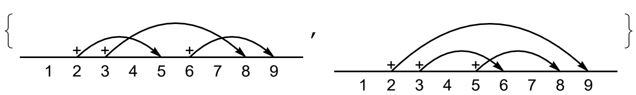
Testing for up to 7 crossings

```
Table[
 K → Simplify[GGA[K] - T ∂T Log[Alexander[K][T]]],
 {K, AllKnots@{3, 7}}]
```

$\{3_1\to -1,\ 4_1\to 1,\ 5_1\to -2,\ 5_2\to -2,\ 6_1\to 0,\ 6_2\to 0,\ 6_3\to 0,$
$7_1\to -3,\ 7_2\to -3,\ 7_3\to 4,\ 7_4\to 4,\ 7_5\to -3,\ 7_6\to -1,\ 7_7\to 2\}$

Invariance

```
Draw /@ {R3L = GD[Ap2,5, Ap3,8, Ap6,9],
  R3R = GD[Ap5,8, Ap2,9, Ap3,6]}
```




```
Simplify[
 GGA[R3L, {1, 4, 7, 10}] == GGA[R3R, {1, 4, 7, 10}] /.
  β10,b_ :→ 1 - β1,b - β4,b - β7,b]
```

True

```
GG[GD@Knot[4, 1], {1, 2}, F] /. F[y_List, γ_G] :→ F[Column@y, γ]
```

Example: Degree 2 Gauss-Gassner for 4_1

$$F\left[Am_{1,2},\begin{pmatrix} -\frac{-1+T_2-T_1T_2+T_3-T_1T_3-T_2T_3+T_1T_2T_3}{T_1T_3} & \frac{(-1+T_1)(1-T_2+T_1T_2)(-1+T_3)}{T_1T_3} & -\frac{(-1+T_1)(-1+T_2)}{T_1} \\ -\frac{(-1+T_2)(-1+T_3)}{T_1T_3} & \frac{-1+T_1+T_2-T_1T_2+T_3-T_2T_3+T_1T_2T_3}{T_1T_3} & -\frac{-1+T_2}{T_1} \\ \frac{T_2(-1+T_3)}{T_3} & -\frac{(-1+T_1)T_2(-1+T_3)}{T_3} & T_2 \end{pmatrix}\right]+$$

$$F\left[Am_{2,1},\begin{pmatrix} \frac{1}{T_2} & \frac{-1+T_1}{-T_1-T_2+T_1T_2} & -\frac{(-1+T_1)(-1+T_2)^2}{T_2(-T_1-T_2+T_1T_2)} \\ \frac{-1+T_2}{T_2} & \frac{1-2T_1-T_2+T_1T_2}{-T_1-T_2+T_1T_2} & -\frac{(-1+T_2)(-1+T_1+T_2-2T_1T_2-T_2^2+T_1T_2^2)}{T_2(-T_1-T_2+T_1T_2)} \\ 0 & 0 & T_2 \end{pmatrix}\right]+F\left[Ap_{1,2},\begin{pmatrix} -\frac{1-2T_1-T_2+T_1T_2}{-1+T_1+T_2} & \frac{(-1+T_1)^2(-1+T_2)}{-1+T_1+T_2} & 0 \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_1(1-T_1-2T_2+T_1T_2)}{-1+T_1+T_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]+$$

$$F\left[Ap_{1,2},\begin{pmatrix} 1 & \frac{(-1+T_1)(1-2T_2-T_3+T_2T_3)}{-1+T_2+T_3} & -\frac{(-1+T_1)(-1+T_2)}{-1+T_2+T_3} \\ 0 & -\frac{T_1(1-2T_2-T_3+T_2T_3)}{-1+T_2+T_3} & \frac{T_1(-1+T_2)}{-1+T_2+T_3} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}\right]+F\left[\begin{matrix}Am_{2,3}\\Am_{4,1}\end{matrix},\begin{pmatrix} \frac{1}{T_4} & 0 & -\frac{-1+T_1}{T_4} & 0 & 0 \\ 0 & 1 & \frac{T_1(-1+T_2)}{T_2} & 0 & -\frac{(-1+T_2)(-1+T_3)}{T_2} \\ 0 & 0 & \frac{T_1}{T_2} & 0 & -\frac{-1+T_3}{T_2} \\ \frac{-1+T_4}{T_4} & 0 & -\frac{(-1+T_1)(-1+T_4)}{T_4} & 1 & 0 \\ 0 & 0 & 0 & 0 & T_3 \end{pmatrix}\right]+F\left[\begin{matrix}Ap_{1,2}\\Ap_{3,4}\end{matrix},\begin{pmatrix} 1 & -\frac{-1+T_1}{T_4} & 0 & -\frac{(-1+T_1)(-1+T_2)}{T_2} & 0 \\ 0 & \frac{T_1}{T_4} & 0 & \frac{T_1(-1+T_2)}{T_2} & 0 \\ 0 & -\frac{(-1+T_3)(-1+T_4)}{T_4} & 1 & -\frac{-1+T_3}{T_2} & 0 \\ 0 & \frac{T_3(-1+T_4)}{T_4} & 0 & \frac{T_3}{T_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]+$$

$$F\left[\begin{matrix}Ap_{1,3}\\Am_{2,4}\end{matrix},\begin{pmatrix} 1 & 0 & 1-T_1 & 0 & 0 \\ 0 & -\frac{-1+T_4-T_2T_4+T_5-T_2T_5-T_4T_5+T_2T_4T_5}{T_2T_5} & 0 & \frac{-1+T_2}{T_2} & -\frac{(-1+T_2)(-1+T_4)}{T_2} \\ 0 & 0 & T_1 & 0 & 0 \\ 0 & -\frac{(-1+T_4)(-1+T_5)}{T_2T_5} & 0 & \frac{1}{T_2} & -\frac{-1+T_4}{T_2} \\ 0 & \frac{T_4(-1+T_5)}{T_5} & 0 & 0 & T_4 \end{pmatrix}\right]+F\left[\begin{matrix}Ap_{1,3}\\Am_{4,2}\end{matrix},\begin{pmatrix} 1 & 0 & 1-T_1 & -\frac{(-1+T_1)(-1+T_3)}{T_3} & \frac{(-1+T_1)(-1+T_3)(-1+T_4)}{T_3} \\ 0 & \frac{1}{T_4} & 0 & 0 & 0 \\ 0 & 0 & T_1 & \frac{T_1(-1+T_3)}{T_3} & -\frac{T_1(-1+T_3)(-1+T_4)}{T_3} \\ 0 & \frac{-1+T_4}{T_4} & 0 & \frac{1}{T_3} & -\frac{-1+T_4}{T_3} \\ 0 & 0 & 0 & 0 & T_4 \end{pmatrix}\right]+$$

$$F\left[\begin{matrix}Ap_{2,4}\\Am_{1,3}\end{matrix},\begin{pmatrix} \frac{1}{T_4} & -\frac{-1+T_1}{T_4} & \frac{-1+T_1}{T_1} & 0 & 0 \\ -\frac{(-1+T_2)(-1+T_4)}{T_4} & \frac{-1+T_1+T_2-T_1T_2+T_4-T_2T_4+T_1T_2T_4}{T_4} & 0 & 1-T_2 & 0 \\ 0 & 0 & \frac{1}{T_1} & 0 & 0 \\ \frac{T_2(-1+T_4)}{T_4} & -\frac{(-1+T_1)T_2(-1+T_4)}{T_4} & 0 & T_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]+F\left[\begin{matrix}Ap_{2,4}\\Am_{3,1}\end{matrix},\begin{pmatrix} \frac{1}{T_3} & -\frac{-1+T_1}{T_3} & -\frac{(-1+T_1)(-1+T_2)}{T_2T_3} & 0 & 0 \\ 0 & T_1 & \frac{T_1(-1+T_2)}{T_2} & 1-T_2 & 0 \\ \frac{-1+T_3}{T_3} & -\frac{(-1+T_1)(-1+T_3)}{T_3} & -\frac{-1+T_1+T_2-T_1T_2-T_1T_3-T_2T_3+T_1T_2T_3}{T_2T_3} & 0 & 0 \\ 0 & 0 & 0 & T_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/NCSU-1604/

The Kashiwara-Vergne Problem and Topology

Handout, video, and links at ω/

Abstract. I will describe the general “expansions” machine whose inputs are topics in topology (and more) and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978 ω/KV, solved Alekseev-Meinrenken 2006 ω/AM, elucidated Alekseev-Torossian 2008-2012 ω/AT), a problem about convolutions on Lie groups and Lie algebras.

The Kashiwara-Vergne Conjecture. There exist two series F and G in the completed free Lie algebra FL in generators x and y so that

$$x+y-\log e^y e^x = (1-e^{-\operatorname{ad} x})F + (e^{\operatorname{ad} y}-1)G \quad \text{in } FL$$

and so that with $z=\log e^x e^y$,

$$\operatorname{tr}(\operatorname{ad} x)\partial_x F + \operatorname{tr}(\operatorname{ad} y)\partial_y G \quad \text{in cyclic words}$$
$$= \frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1} - 1\right)$$

Implies the loosely-stated convolutions statement: Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra.


Kashiwara


Vergne


Alekseev


Meinrenken


Torossian

The Machine. Let G be a group, $\mathcal{K}=\mathbb{Q}G=\{\sum a_i g_i\colon a_i\in\mathbb{Q},\ g_i\in G\}$ its group-ring, $\mathcal{I}=\{\sum a_i g_i\colon \sum a_i=0\}\subset\mathcal{K}$ its augmentation ideal. Let

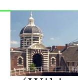
$$\mathcal{A}=\operatorname{gr}\mathcal{K}:=\widehat{\bigoplus_{m\geq 0}}\mathcal{I}^m/\mathcal{I}^{m+1}.$$

P.S. $(\mathcal{K}/\mathcal{I}^{m+1})^*$ is Vassiliev / finite-type / polynomial invariants.

Note that $\mathcal{A}$ inherits a product from G.

Definition. A linear $Z\colon\mathcal{K}\to\mathcal{A}$ is an “expansion” if for any $\gamma\in\mathcal{I}^m$, $Z(\gamma)=(0,\dots,0,\gamma/\mathcal{I}^{m+1},*,\dots)$, and a “homomorphic expansion” if in addition it preserves the product.

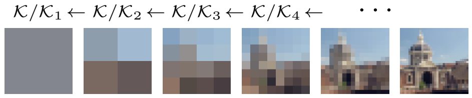
Example. Let $\mathcal{K}=C^\infty(\mathbb{R}^n)$ and $\mathcal{I}=\{f\colon f(0)=0\}$. Then $\mathcal{I}^m=\{f\colon f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ is degree m homogeneous polynomials and $\mathcal{A}=\{\text{power series}\}$. The Taylor series is a homomorphic expansion!

Just for fun. $\mathcal{K}=\left\{ \right.$  $\left.\right\} = \left(\begin{array}{l}\text{The set of all 2D}\\ \text{projections of reality}\end{array}\ (=\mathbb{Q}^3\mathbb{R}^2)\right)$

(Wikimedia Commons image, ω/WM)

$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow \quad\cdots$

Rotate
Colour Correct
Adjoin



An expansion Z is a choice of a “progressive scan” algorithm.

Rotate
Colour Correct
Adjoin

$\mathcal{K}/\mathcal{K}_1\oplus\mathcal{K}_1/\mathcal{K}_2\oplus\mathcal{K}_2/\mathcal{K}_3\oplus\mathcal{K}_3/\mathcal{K}_4\oplus\mathcal{K}_4/\mathcal{K}_5\oplus\mathcal{K}_5/\mathcal{K}_6\oplus\quad\cdots$

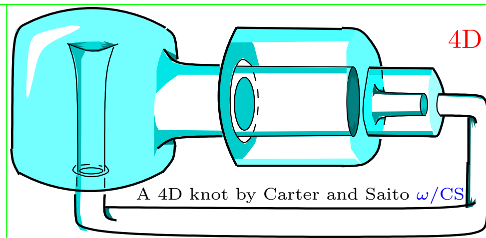
$\mathcal{K}/\mathcal{K}_1 = \mathbb{Q}^3$, $\quad \mathcal{K}_3/\mathcal{K}_4 = \ker(\mathcal{K}/\mathcal{K}_4\to\mathcal{K}/\mathcal{K}_3)$

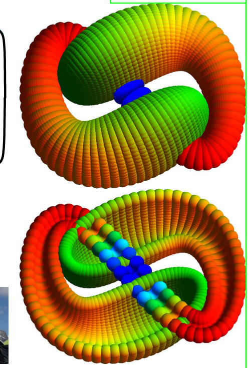
In the finitely presented case, finding Z amounts to solving a system of equations in a graded space.

Theorem (with Zsuzsanna Dancso, ω/WKO). There is a bijection between the set of homomorphic expansions for $w\mathcal{K}$ and the set of solutions of the Kashiwara-Vergne problem. This is the tip of a major iceberg!


Dancso, ω/ZD

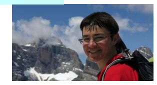
4D Knots.


A 4D knot by Carter and Saito ω/CS

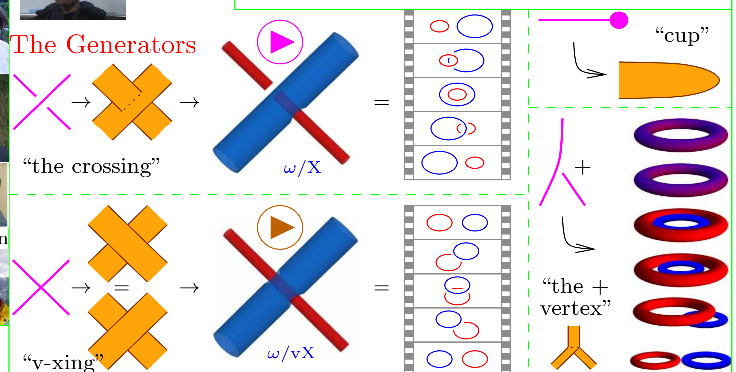



ω/F

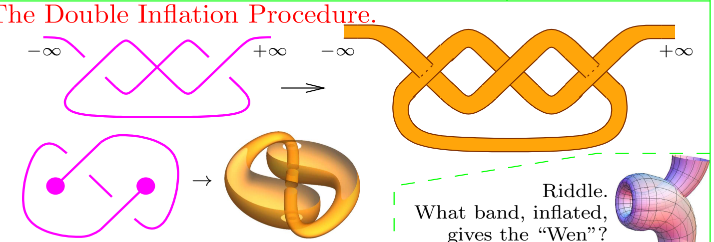

Satoh


Dalvit ω/Dal

The Generators



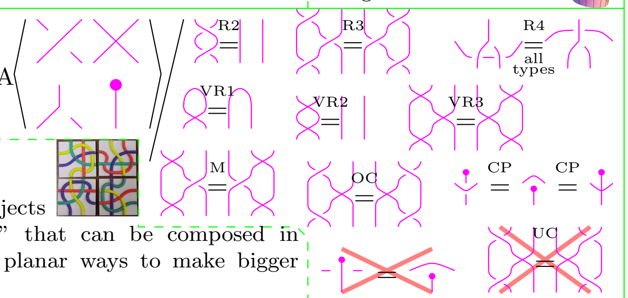
The Double Inflation Procedure.



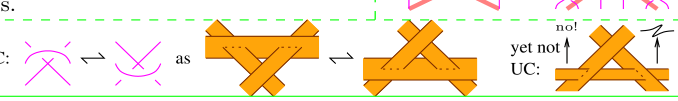
Riddle. What band, inflated, gives the “Wen”?

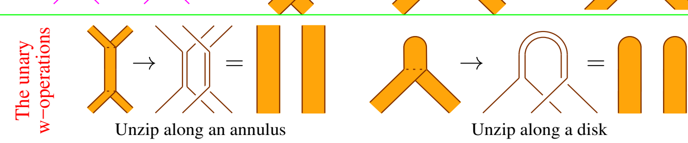
wKO.

$w\mathcal{K}:=$PA

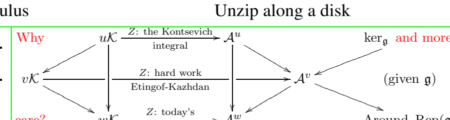


“Planar Algebra”: The objects are “tiles” that can be composed in arbitrary planar ways to make bigger tiles.





The Machine generalizes to arbitrary algebraic structures!



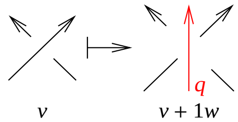
“God created the knots, all else in topology is the work of mortals.”
Leopold Kronecker (modified) www.katlas.org

Crossing the Crossings

Abstract. The subject will be very close to Manturov's representation of vB_n into $\mathrm{Aut}(FG_{n+1})$ — I'll describe how I think about it in terms of a very simple minded map $\mathcal{K}$ from n-component v-tangles to $(n+1)$-component w-tangles. It is possible that you all know this already. Possibly my talk will be very short — it will be as long as it is necessary to describe $\mathcal{K}$ and say a few more words, and if this is little, so be it.

All you need is $\mathcal{K}$... • What is its domain? • What is its target? • Why should one care?

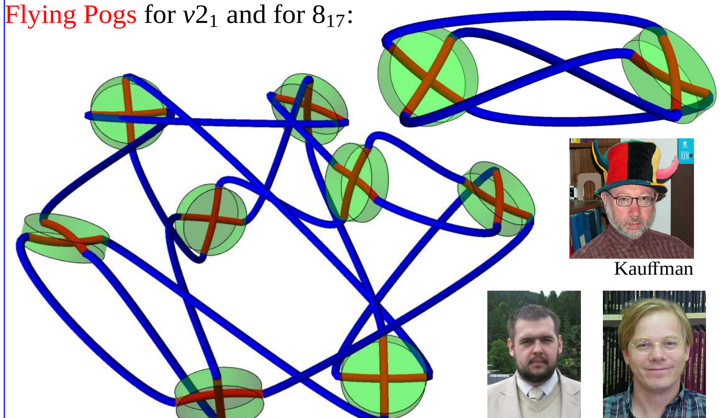
$\mathcal{K}$:



Virtual Knots. Virtual knots are the algebraic structure underlying the Reidemeister presentation of ordinary knots, without the topology. Locally they are knot diagrams modulo the Reidemeister relations; globally, who cares? So,

$vT = \mathrm{CA}\langle \nearrow\!\!\!\nwarrow, \nwarrow\!\!\!\nearrow : R1, R2, R3\rangle$ CA = "Circuit Algebra"

Flying Pogs for $v2_1$ and for 8_{17}:

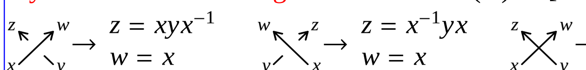


No! Note that also (with PA ="Planar Algebra")

$vT = \mathrm{PA}\langle \nearrow\!\!\!\nwarrow, \nwarrow\!\!\!\nearrow, \times : R1, R2, R3, VR1, VR2, VR3, M\rangle,$

but I have a prejudice, or a deeply held belief, that this is morally wrong!

My moment of reckoning. Manturov's $VG(K)$: [Ma, BGHNW]



$\to z = xyx^{-1},\ w = x$ $\to z = x^{-1}yx,\ w = x$ $\to z = q^{-1}yq,\ w = qxq^{-1}$

Manturov's $\mu\colon vB_n \to \mathrm{Aut}(F(x_1,\dots,x_n,q))$: [Ma, BGHNW]

$$\sigma_i = \nearrow\!\!\!\nwarrow_i \mapsto \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} \mapsto x_i \end{cases} \qquad \tau_i = \times_i \mapsto \begin{cases} x_i \mapsto q x_{i+1} q^{-1} \\ x_{i+1} \mapsto q^{-1} x_i q \end{cases}.$$

Easy resolution. Setting $y_i := q^i x_i q^{-i}$, we find that μ is equivalent to

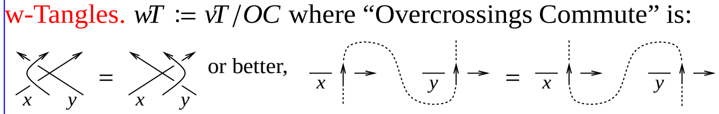
$$\nearrow\!\!\!\nwarrow_i \mapsto \begin{cases} y_i \mapsto y_i q^{-1} y_{i+1} q y_i^{-1} \\ y_{i+1} \mapsto q y_i q^{-1} \end{cases} \qquad \times_i \mapsto \begin{cases} y_i \mapsto y_{i+1} \\ y_{i+1} \mapsto y_i \end{cases},$$

and to me, virtual braids are anyways always pure. So really,

$$\sigma_{ij} \mapsto \begin{cases} y_i \mapsto q y_i q^{-1} \\ y_j \mapsto y_i^{-1} q^{-1} y_j q y_i \end{cases}.$$

But why does it exist? Especially, wherefore $vB_n \to wB_{n+1}$?

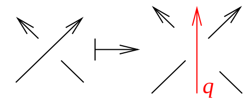
w-Tangles. $wT := vT/OC$ where "Overcrossings Commute" is:



π_1 is defined on wT; Artin's representation ϕ is defined on wB_n.

Back to $\mathcal{K}$. The "crossing the crossings" map $\mathcal{K}\colon vT_n \to wT_{n+1}$ is defined by the picture below. Equally well, it is $\mathcal{K}\colon vB_n \to wB_{n+1}$. Better, it is $\mathcal{K}\colon vT_n \to (nv+1w)T$ or $\mathcal{K}\colon vB_n \to (nv+1w)B$.

Claims.

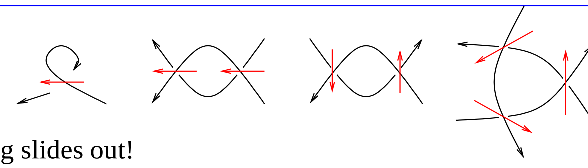


1. $\mathcal{K}$ is well defined.
2. On u-links, $\mathcal{K}$ "factors".
3. $\mathcal{K}$ does not respect OC.
4. $\mathcal{K}$ recovers Manturov's VG and μ: $VG(K) = \pi_1(\mathcal{K}(K))$, $\mu = \mathcal{K}\circ\phi = \phi /\!\!/ \mathcal{K}$.

Even better, $\mathcal{K}$ pulls back *any* invariant of 2-component w-knots to an invariant of virtual knots. in particular, there is a wheel-valued "non-commutative" invariant ω as in [BN] and DBN: Talks: Hamilton-1412 (next page).

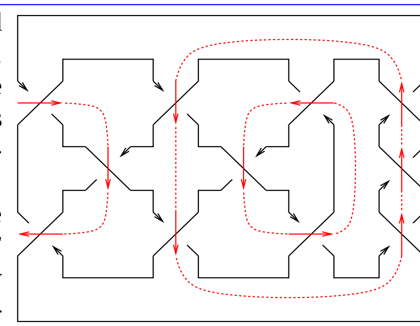
Likely, the various "2-variable Alexander polynomials" for virtual knots arise in this way.

Proof of 1.



Everything slides out!

Proof of 2. The net "red flow" into every face is 0, so the red arrows can be paired. They form cycles that can hover off the picture.



No proof of 3. Well, there simply is no proof that OC is respected, and it's easy to come up with counterexamples.

Proof of 4. A simple verification, except my conventions are off...

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* Acta Mathematica Vietnamica **40-2** (2015) 271–329, arXiv:1308.1721.

[BGHNW] H. U. Boden, A. I. Gaudreau, E. Harper, A. J. Nicas, and L. White, *Virtual Knot Groups and Almost Classical Knots*, arXiv:1506.01726.

[Ma] V. O. Manturov, *On Invariants of Virtual Links*, Acta Applicandae Mathematica **72-3** (2002) 295–309.

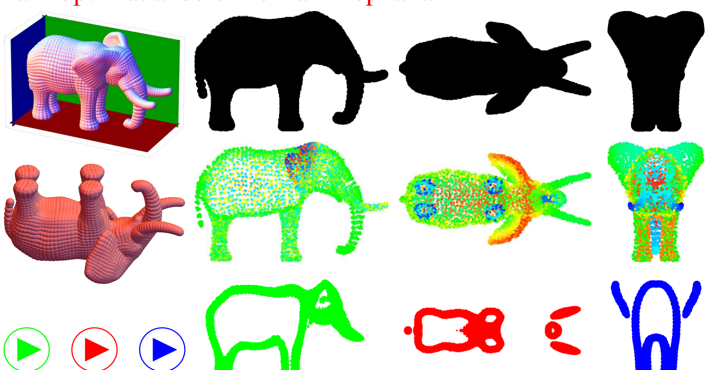
Prejudices should always be re-evaluated!

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-1511/

Knots in Three and Four Dimensions

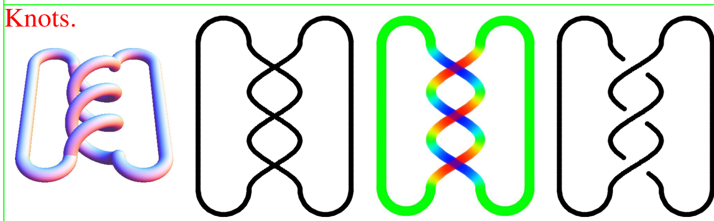
Abstract. Much as we can understand 3-dimensional objects by staring at their pictures and x-ray images and slices in 2-dimensions, so can we understand 4-dimensional objects by staring at their pictures and x-ray images and slices in 3-dimensions, capitalizing on the fact that we understand 3-dimensions pretty well. So we will spend some time staring at and understanding various 2-dimensional views of a 3-dimensional elephant, and then even more simply, various 2-dimensional views of some 3-dimensional knots. This achieved, we'll take the leap and visualize some 4-dimensional knots by their various traces in 3-dimensional space, and if we'll still have time, we'll prove that these knots are really knotted.

Warmup: Flatlanders View an Elephant.

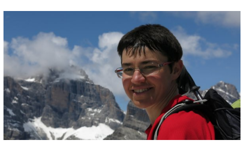


ω/g ω/r ω/b coords from ω/Jeff2207

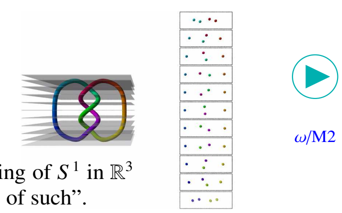
Knots.



"broken curve diagram"



with Ester Dalvit ω/Dal



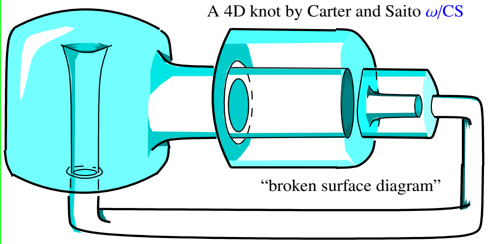
ω/M2

Formally, "a differentiable embedding of S^1 in $\mathbb{R}^3$ modulo differentiable deformations of such".

2-Knots / 4D Knots.

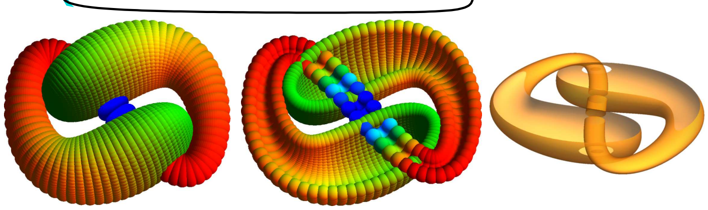
Formally, "a differentiable embedding of S^2 in $\mathbb{R}^4$ modulo differentiable deformations of such".

A 4D knot by Carter and Saito ω/CS

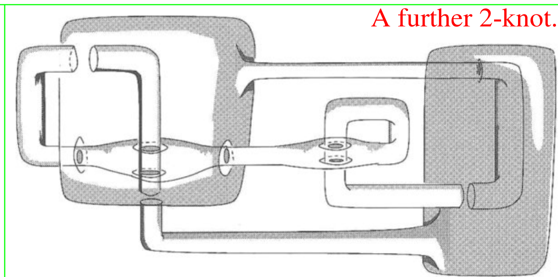


"broken surface diagram"

Carter, Banach, Saito

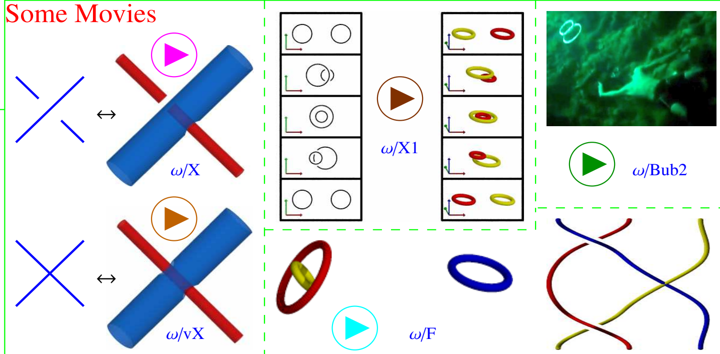


A further 2-knot.



ω/CS

Some Movies

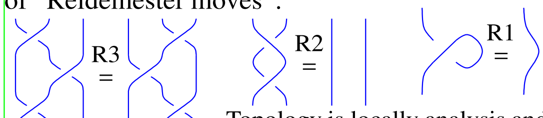


Some Unknots

Thistlethwaite's unknot Scharein's relaxation Haken's unknot

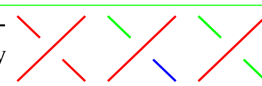
Reidemeister's Theorem. (a) Every knot has a "broken curve diagram", made only of curves and "crossings" like ⤫. (b) Two knot diagrams represent the same 3D knot iff they differ by a sequence of "Reidemester moves":

Kurt Reidemeister

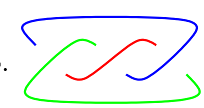


Topology is locally analysis and globally algebra

3-Colourings. Colour the arcs of a broken arc diagram in RGB so that every crossing is either mono-chromatic or tri-chromatic. Let $\lambda(K)$ be the number of such 3-colourings that K has.



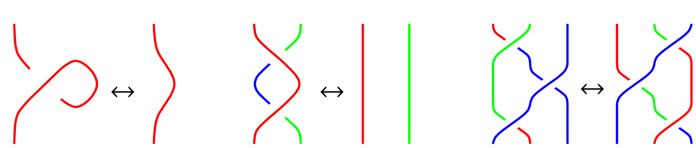
good good bad

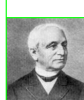


Example. $\lambda(\bigcirc) = 3$ while $\lambda(♲) = 9$; so $\bigcirc \neq$ ♲.

Riddle. Is $\lambda(K)$ always a power of 3?

Proof sketch. It is enough to show that for each Reidemeister move, there is an end-colours-preserving bijection between the colourings of the two sides. E.g.:





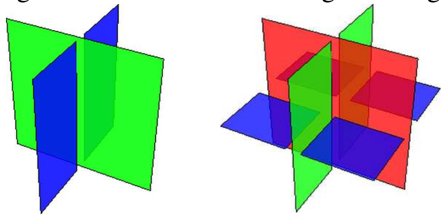
"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

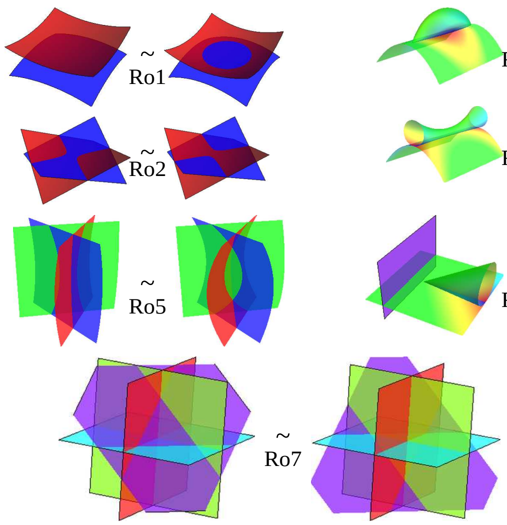
www.katlas.org

Knots in Three and Four Dimensions, 2

Theorem. Every 2-knot can be represented by a “broken surface diagram” made of the following basic ingredients,

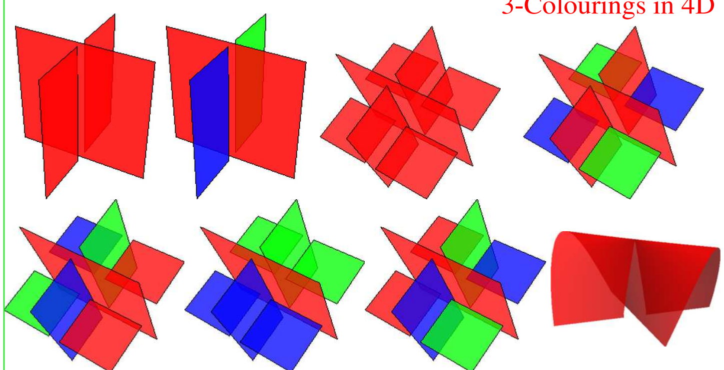


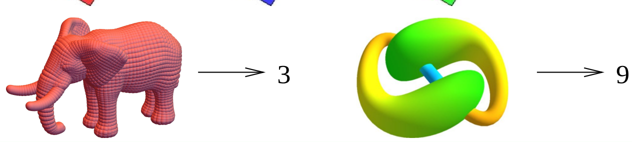
...and any two representations of the same knot differ by a sequence of the following “Roseman moves”:



D. Roseman

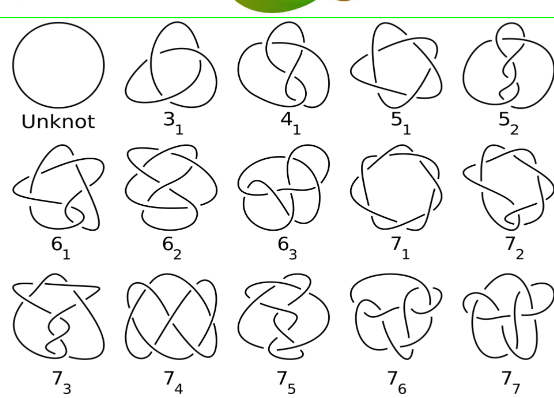
3-Colourings in 4D





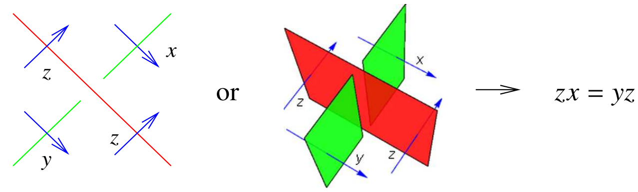
A Knot Table

There are many more!



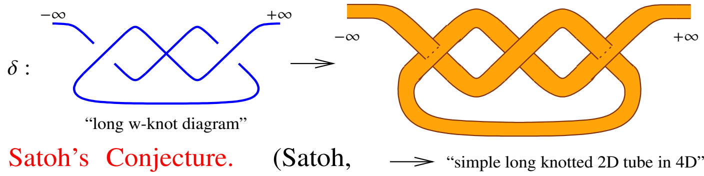
ω/KT

A Stronger Invariant. There is an assigment of groups to knots / 2-knots as follows. Put an arrow “under” every un-broken curve / surface in a broken curve / surface diagram and label it with the name of a group generator. Then mod out by relations as below.



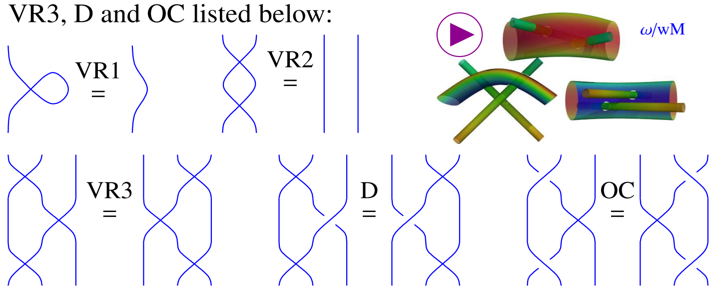
Facts. The resulting “Fundamental group” $\pi_1(K)$ of a knot / 2-knot K is a very strong but not very computable invariant of K. Though it has computable projections; e.g., for any finite G, count the homomorphisms from $\pi_1(K)$ to G.

Exercise. Show that $|\mathrm{Hom}(\pi_1(K) \to S_3)| = \lambda(K) + 3$.



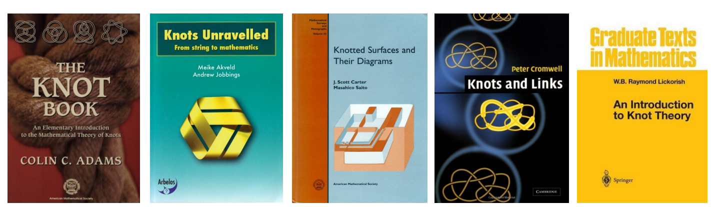
Satoh’s Conjecture. (Satoh, *Virtual Knot Presentations of Ribbon Torus-Knots,* J. Knot Theory and its Ramifications **9** (2000) 531–542). Two long w-knot diagrams represent via the map δ the same simple long 2D knotted tube in 4D iff they differ by a sequence of R-moves as above and the “w-moves” VR1–VR3, D and OC listed below:

Shin Satoh



Some knot theory books.

- Colin C. Adams, *The Knot Book, an Elementary Introduction to the Mathematical Theory of Knots,* American Mathematical Society, 2004.
- Meike Akveld and Andrew Jobbings, *Knots Unravelled, from Strings to Mathematics,* Arbelos 2011.
- J. Scott Carter and Masahico Saito, *Knotted Surfaces and Their Diagrams,* American Mathematical Society, 1997.
- Peter Cromwell, *Knots and Links,* Cambridge University Press, 2004.
- W.B. Raymond Lickorish, *An Introduction to Knot Theory,* Springer 1997.



Video and more at http://www.math.toronto.edu/~drorbn/Talks/Cornell-150925/. Similar talks at .../CUMC-1307/, .../CUMC-1307/

Dror Bar-Natan: Talks: LesDiablerets-1508:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/

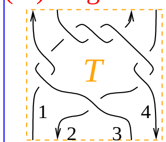
Work in Progress on

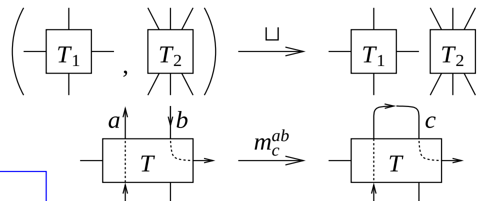
Polynomial Time Knot Polynomials, A

Abstract. The value of things is inversely correlated with their computational complexity. “Real time” machines, such as our brains, only run linear time algorithms, and there’s still a lot we don’t know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.

I will explain some things I know about polynomial time knot polynomials and explain where there’s more, within reach.

(v-)Tangles.

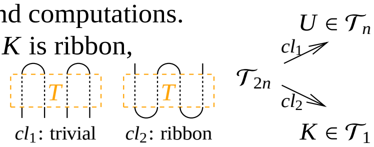




(meta-associativity: $m_a^{ab} /\!/ m_a^{ac} = m_b^{bc} /\!/ m_a^{ab}$)

Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- “Algebraic Knot Theory”: If K is ribbon, $z(K) \in \{cl_2(\zeta)\colon cl_1(\zeta) = 1\}$.



(Genus and crossing number are also definable properties).

A blackboard aside on genus?

Faster is better, leaner is meaner!

Theorem 1. ∃! an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\ \sqcup\ } \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\substack{T_a, T_b \to T_c\\ \mu := 1-\beta}]{m_c^{ab}} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma + \alpha\delta/\mu & \epsilon + \delta\theta/\mu\\ S & \phi + \alpha\psi/\mu & \Xi + \psi\theta/\mu\end{array}\right),$$

and satisfying $\left(\uparrow_a;\ {}_a\!\nearrow\!\!\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\!\!\nearrow_a\right) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right)$.

In Addition

- The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].
- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega \det'(A - I)/(1 - T')$ is the MVA, mod units.
- The fastest Alexander algorithm I know.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

M. Polyak & T. Ohtsuki
@ Heian Shrine, Kyoto

Implementation key idea:

ωεβ/Demo

$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)a_ :> a, ∞];
  M = Outer[Factor[∂_{h_a1 t_a2} λ] &, S, S];
  M = Prepend[M, t# & /@ S] // Transpose;
  M = Prepend[M, Prepend[h# & /@ S, ω]];
  M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_ b_ → c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  ( α β θ ; γ δ ϵ ; ϕ ψ Ξ ) = ( ∂_{t_a,h_a}λ ∂_{t_a,h_b}λ ∂_{t_a}λ ; ∂_{t_b,h_a}λ ∂_{t_b,h_b}λ ∂_{t_b}λ ; ∂_{h_a}λ ∂_{h_b}λ λ ) /. (t | h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.( γ+αδ/μ ϵ+δθ/μ ; ϕ+αψ/μ Ξ+ψθ/μ ).{h_c, 1}]
   /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_ b_} := Γ[1, {t_a, t_b}.( 1 1-T_a ; 0 T_a ).{h_a, h_b}];
Rm_{a_ b_} := Rp_{ab} /. T_a → 1 / T_a;
```

Meta-Associativity

Runs.

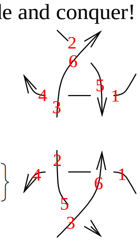
```
ζ = Γ[ω, {t1, t2, t3, tS}.( α11 α12 α13 θ1 ; α21 α22 α23 θ2 ; α31 α32 α33 θ3 ; ϕ1 ϕ2 ϕ3 Ξ ).{h1, h2, h3, hS}];
(ζ // m12→1 // m13→1) == (ζ // m23→2 // m12→1)
True
```

R3

. . . divide and conquer!

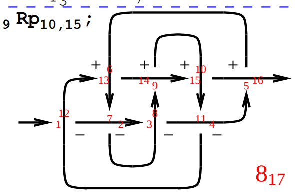
```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
  Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$



```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$

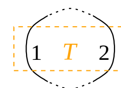


8_{17}

Closed Components.

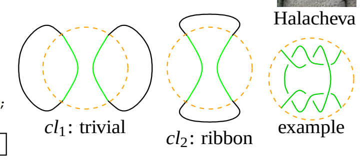
The Halacheva trace tr_c satisfies $m_c^{ab} /\!/ \mathrm{tr}_c = m_c^{ba} /\!/ \mathrm{tr}_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

$$\begin{array}{c|cc}\omega & c & S\\ \hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu := 1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c}\mu\omega & S\\ \hline S & \Xi + \psi\theta/\mu\end{array}$$



Halacheva

```
trc_[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  ( α θ ; ψ Ξ ) = ( ∂_{t_c,h_c}λ ∂_{t_c}λ ; ∂_{h_c}λ λ ) /. (t | h)_c → 0;
  Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ζ // m12→1 // tr1) == (ζ // m21→1 // tr1)
True
```

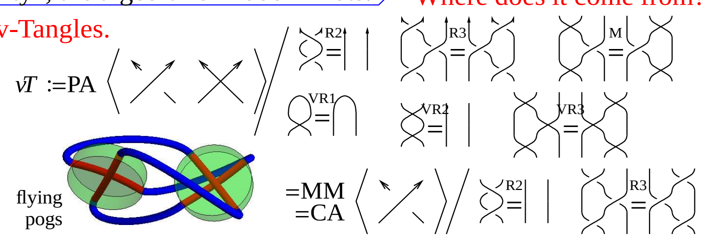


Weaknesses.

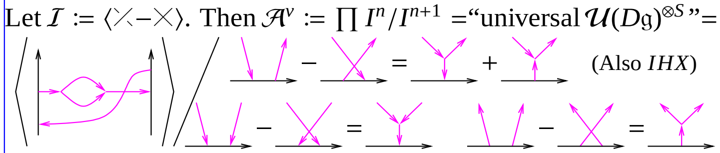
- m_c^{ab} and tr_c are non-linear.
- The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions.
- I still don’t understand tr_c, “unitarity”, the algebra for ribbon knots.

Where does it come from?

v-Tangles.



Let $\mathcal{I} := \langle \times - \times \rangle$. Then $\mathcal{A}^v := \prod I^n/I^{n+1}$ =“universal $\mathcal{U}(D\mathfrak{g})^{\otimes S}$”=

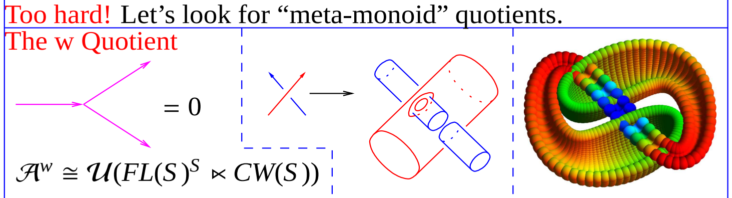


Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

Likely Theorem. [EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z\colon v\mathcal{T} \to \mathcal{A}^v$. (issues suppressed)

Too hard! Let’s look for “meta-monoid” quotients.

The w Quotient

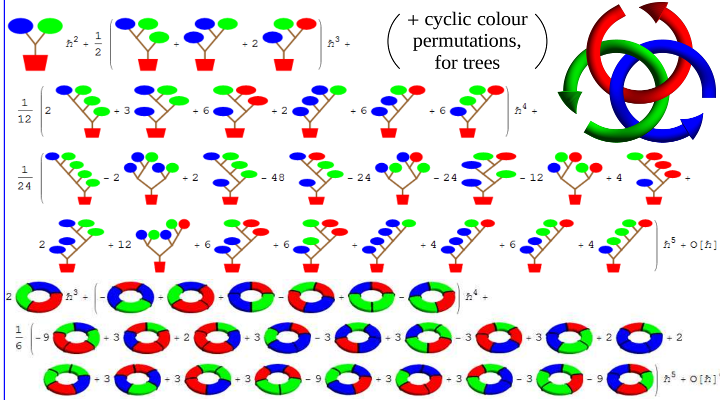


$\mathcal{A}^w \cong \mathcal{U}(FL(S)^S \ltimes CW(S))$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/

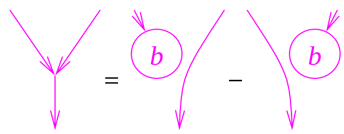
Theorem 2 [BND]. $\exists!$ a homomorphic expansion, aka a homomorphic universal finite type invariant Z^w of pure w-tangles. $z^w := \log Z^w$ takes values in $FL(S)^S \times CW(S)$.

z is computable. z of the Borromean tangle, to degree 5 [BN]:



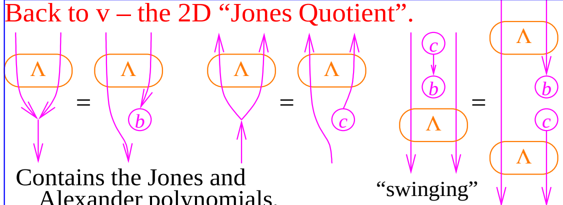
(I have a fancy free-Lie calculator!) (ωεβ/FLD) Nice, but too hard!

Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, z^w reduces to z_0.

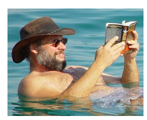


$[u, v] = b_u v - b_v u$

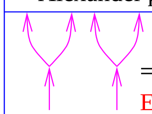
Back to v – the 2D "Jones Quotient".



Contains the Jones and Alexander polynomials, "swinging" ...yet still too hard!



V. Jones



The OneCo Quotient. Likely related to [ADO]

$= 0$, only one co-bracket is allowed.

Everything should work, and everything is being worked!

References.
[ADO] Y. Akutsu, T. Deguchi, and T. Ohtsuki, *Invariants of Colored Links*, J. of Knot Theory and its Ramifications **1-2** (1992) 161–184.
[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.
[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.
[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.
[CT] D. Cimasoni and V. Turaev, *A Lagrangian Representation of Tangles*, Topology **44** (2005) 747–767, arXiv:math.GT/0406269.
[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras*, Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.
[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I*, Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.
[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures*, Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.
[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.
[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

Help Needed!

I'm slow and feeble-minded.

Definition. (Compare [BNS, BN]) A meta-monoid is a functor M: (finite sets, injections)→(sets) (think "$M(S)$ is quantum G^S", for G a group) along with natural operations $*: M(S_1) \times M(S_2) \to M(S_1 \sqcup S_2)$ whenever $S_1 \cap S_2 = \emptyset$ and $m^{ab}_c : M(S) \to M((S \setminus \{a, b\}) \sqcup \{c\})$ whenever $a \neq b \in S$ and $c \notin S \setminus \{a, b\}$, such that

The Abstract Context

$$\text{meta-associativity:}\quad m^{ab}_a /\!\!/ m^{ac}_a = m^{bc}_b /\!\!/ m^{ab}_a$$

$$\text{meta-locality:}\quad m^{ab}_c /\!\!/ m^{de}_f = m^{de}_f /\!\!/ m^{ab}_c$$

and, with $\epsilon_b = M(S \hookrightarrow S \sqcup \{b\})$,

$$\text{meta-unit:}\quad \epsilon_b /\!\!/ m^{ab}_a = Id = \epsilon_b /\!\!/ m^{ba}_a.$$

Claim. Pure virtual tangles $\mathcal{P}\!v\!\mathcal{T}$ form a meta-monoid.

Theorem. $S \mapsto \Gamma_0(S)$ is a meta-monoid and $z_0 : \mathcal{P}\!v\!\mathcal{T} \to \Gamma_0$ is a morphism of meta-monoids.

Strong Conviction. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S) = \Gamma_0(S) \times \Gamma_1(S)$, along with an extension of z_0 to $z_{01} : \mathcal{P}\!v\!\mathcal{T} \to \Gamma_{01}$, with

$$\Gamma_1(S) = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \mathcal{S}^2(V)^{\otimes 2} \qquad (\text{with } V := R_S\langle S\rangle).$$

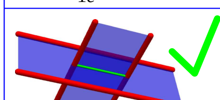
Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

Furthermore, Γ_{01} is given using a "meta-2-cocycle ρ^{ab}_c over Γ_0": In addition to $m^{ab}_c \to m^{ab}_{0c}$, there are R_S-linear $m^{ab}_{1c} : \Gamma_1(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$, a meta-right-action $\alpha^{ab} : \Gamma_1(S) \times \Gamma_0(S) \to \Gamma_1(S)$ R_S-linear in the first variable, and a first order differential operator (over R_S) $\rho^{ab}_c : \Gamma_0(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$ such that

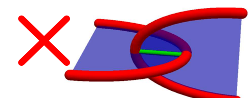
$$(\zeta_0, \zeta_1) /\!\!/ m^{ab}_c = \left(\zeta_0 /\!\!/ m^{ab}_{0c}, (\zeta_1, \zeta_0) /\!\!/ \alpha^{ab} /\!\!/ m^{ab}_{1c} + \zeta_0 /\!\!/ \rho^{ab}_c\right)$$

What's done? The braid part, with still-ugly formulas.

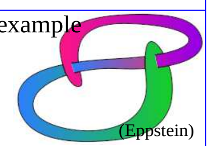
What's missing? A lot of concept- and detail-sensitive work towards m^{ab}_{1c}, α^{ab}, and ρ^{ab}_c. The "ribbon element".



a ribbon singularity



a clasp singularity



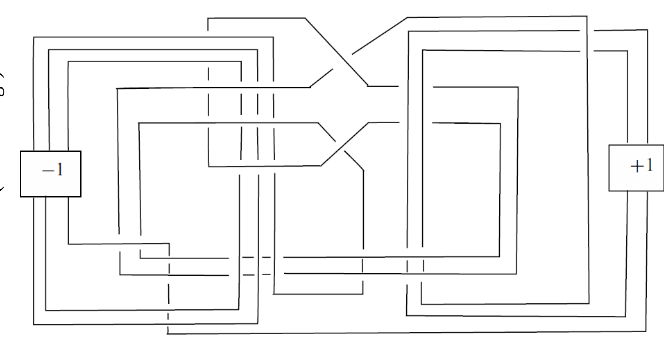
example (Eppstein)

A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

[GST]: a slice knot that might not be ribbon (48 crossings).



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

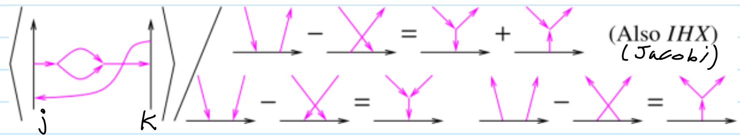
Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/

PolyPoly Extras

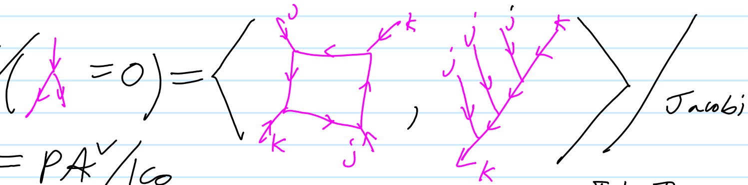
Monday, August 24, 2015 3:10 AM

Dror Bar-Natan, Les Diablerets, August 2015
http://drorbn.net/LD

$\mathcal{A}^v =$

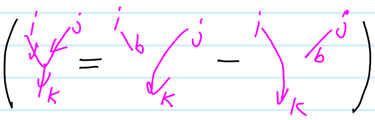


$P\mathcal{A}^v / (\ \ =0) =$



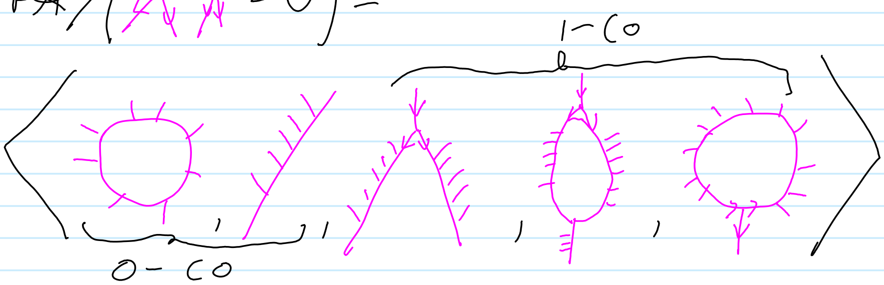
$P\mathcal{A}^w = P\mathcal{A}^v / 1co$

So

$P\mathcal{A}^w(\uparrow_S)/\Big($  $\Big) = \hat{R}_S \oplus M_{S\times S}(\hat{R}_S)$, with $\otimes[b_i]$ over $\hat{R}_S$

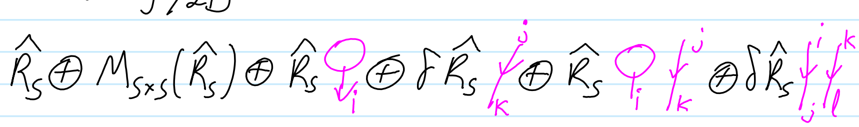
and the rest is (hard?) calculations, which lead to a simple rational-function result.

$P\mathcal{A}^v / (\ \ =0) =$



So with $b_i :=$, $c_j :=$, $\delta :=$

$(P\mathcal{A}^v/2co)/_{2D} \subset$



$\hat{R}_S \oplus M_{S\times S}(\hat{R}_S) \oplus \hat{R}_S \ \oplus \delta\hat{R}_S \ \oplus \hat{R}_S \ \oplus \delta\hat{R}_S$

$= V_S + V_S^{\otimes 2} + V_S + V_S^{\otimes 2} + V_S^{\otimes 3} + (S^2(V_S))^{\otimes 2}$

[The product law is awful, but experience shows that things simplify....]

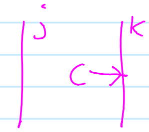
Stitching is clearly possible, but I still don't have explicit formulas.

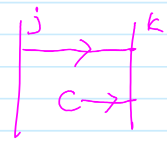
Proposition The element R_{ij} given below solves the YB equation

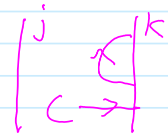
$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

in $\mathcal{A}^v/2co/2D$!

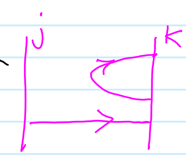
$R_{jk} = e^{j\longrightarrow k} e^{\rho}$, with

$\rho = -\phi_2(b_j)$ 

$+\dfrac{\phi_2(b_j)}{b_j}$ 

$+\dfrac{\phi_1(b_j)\phi_2(b_k)}{b_k\phi_1(b_k)}$ 

$-\dfrac{\phi_2(b_j)}{b_j^2}\delta$ 

$-\dfrac{\phi_1(b_j)\phi_2(b_k)}{b_j b_k\phi_1(b_k)}\delta$ 

where $\phi_1(x) = e^{-x} - 1$

and $\phi_2(x) = \dfrac{(x+2)e^{-x} - 2 + x}{2x}$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/

Loading, initializing variables, setting default degree to 6.

(The *Mathematica* packages `FreeLie`` and `AwCalculus`` are at ωεβ/WKO4).

```
path = "C:/drorbn/AcademicPensieve/";
SetDirectory[path <> "2015-08/LesDiablerets-1508"];
Get[path <> "Projects/WKO4/FreeLie.m"];
Get[path <> "Projects/WKO4/AwCalculus.m"];
x = LW@"x"; y = LW@"y"; u = LW@"u";
$SeriesShowDegree = 6;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence,
  BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, DegreeScale,
  DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA,
  LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism,
  New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t,
  tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌒, ⌢}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends
 {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hΔ, hη,
  hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.
```

```
BCH[x, y] (* Can raise degree to 22 *)
```

$$\mathrm{LS}\Big[\overline{x}+\overline{y},\ \frac{\overline{xy}}{2},\ \frac{1}{12}\overline{x\,\overline{xy}}+\frac{1}{12}\overline{\overline{xy}\,y},\ \frac{1}{24}\overline{x\,\overline{\overline{xy}\,y}},\ -\frac{1}{720}\overline{xxxxy}+\frac{1}{180}\overline{xxxyy}+\frac{1}{180}\overline{xxyyy}+\frac{1}{120}\overline{xy\,xyy}+\frac{1}{360}\overline{xxy\,xy}-\frac{1}{720}\overline{xyyyy},\ -\frac{\overline{xxxxyy}}{1440}+\frac{1}{360}\overline{xxxyyy}+\frac{1}{240}\overline{xxy\,xyy}+\frac{1}{720}\overline{xxxy\,xy}-\frac{\overline{xxyyyy}}{1440},\ \dots\Big]$$

KV Direct.

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1/2;
SeriesSolve[{F, G},
  ℏ^-1 (LS[x + y] - BCH[y, x] ≡ F - G - Ad[-x][F] + Ad[y][G]) ⋀
   divx[F] + divy[G] ≡
    1/2 tru[adSeries[ad/(e^ad - 1), x][u] + adSeries[ad/(e^ad - 1), y][u] -
      adSeries[ad/(e^ad - 1), BCH[x, y]][u]]];
{F, G} (* Can raise degree to 13 *)
```

$$\Big\{\mathrm{LS}\Big[\frac{\overline{y}}{2},\ \frac{\overline{xy}}{6},\ \frac{1}{24}\overline{xyy},\ -\frac{1}{180}\overline{xxxy}+\frac{1}{80}\overline{xxyy}+\frac{1}{360}\overline{xyyy},\ -\frac{1}{720}\overline{xxxyy}+\frac{1}{240}\overline{xxyyy}+\frac{1}{240}\overline{xy\,xyy}+\frac{1}{720}\overline{xxy\,xy}-\frac{\overline{xyyyy}}{1440},\ \frac{\overline{xxxxxy}}{5040}-\frac{\overline{xxxxyy}}{1344}+\frac{13\,\overline{xxxyyy}}{15120}+\frac{1}{840}\overline{xxy\,xyy}+\frac{\overline{xxxy\,xy}}{3360}+\frac{\overline{xxyyyy}}{6720}+\frac{\overline{xy\,xyyy}}{1260}+\frac{\overline{xxyy\,xy}}{1680}-\frac{\overline{xyyyyy}}{10080},\ \dots\Big],$$
$$\mathrm{LS}\Big[0,\ \frac{\overline{xy}}{12},\ \frac{1}{24}\overline{xyy},\ -\frac{1}{360}\overline{xxxy}+\frac{1}{120}\overline{xxyy}+\frac{1}{180}\overline{xyyy},\ -\frac{1}{720}\overline{xxxyy}+\frac{1}{240}\overline{xxyyy}+\frac{1}{240}\overline{xy\,xyy}+\frac{1}{720}\overline{xxy\,xy}-\frac{\overline{xyyyy}}{1440},\ \frac{\overline{xxxxxy}}{10080}-\frac{\overline{xxxxyy}}{2016}+\frac{\overline{xxxyyy}}{1890}+\frac{\overline{xxy\,xyy}}{1120}+\frac{\overline{xxxy\,xy}}{5040}+\frac{\overline{xxyyyy}}{2520}+\frac{1}{840}\overline{xy\,xyyy}+\frac{\overline{xxyy\,xy}}{1260}-\frac{\overline{xyyyyy}}{5040},\ \dots\Big]\Big\}$$

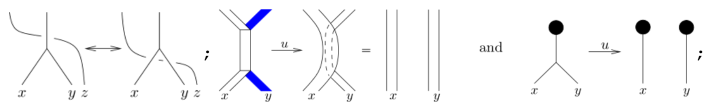
Meaningless calculations.

```
{b[F, G], trx[F]}
```

$$\Big\{\mathrm{LS}\Big[0,\ 0,\ -\frac{1}{24}\overline{xyy},\ -\frac{1}{48}\overline{xyyy},\ \frac{1}{720}\overline{xxxyy}-\frac{1}{240}\overline{xxyyy}-\frac{\overline{xy\,xyy}}{1440}-\frac{1}{720}\overline{xxy\,xy}-\frac{1}{360}\overline{xyyyy},\ \frac{\overline{xxxyyy}}{1440}-\frac{1}{480}\overline{xxyyyy}-\frac{1}{288}\overline{xy\,xyyy}-\frac{7\,\overline{xxyy\,xy}}{2880}+\frac{\overline{xyyyyy}}{2880},\ \dots\Big],$$
$$\mathrm{CWS}\Big[-\frac{\overline{y}}{6},\ \frac{\overline{yy}}{24},\ \frac{\overline{xxy}}{180}+\frac{\overline{xyy}}{80}-\frac{\overline{yyy}}{360},\ -\frac{\overline{xxxy}}{180}+\frac{\overline{xyxy}}{240}-\frac{\overline{xyyy}}{240}-\frac{\overline{yyyy}}{1440},\ -\frac{\overline{xxxxy}}{5040}+\frac{\overline{xxxyy}}{6720}-\frac{\overline{xxyxy}}{1120}+\frac{2\,\overline{xxyyy}}{945}-\frac{\overline{xyxyy}}{336}+\frac{\overline{xyyyy}}{6720}+\frac{\overline{yyyyy}}{10080},\ \frac{\overline{xxxxyy}}{3360}-\frac{\overline{xxxyxy}}{1344}-\frac{\overline{xxxyyy}}{2240}+\frac{\overline{xxyxxy}}{2016}+\frac{13\,\overline{xxyxyy}}{10080}+\frac{\overline{xxyyxy}}{1680}-\frac{\overline{xxyyyy}}{3780}-\frac{\overline{xyxyxy}}{840}+\frac{\overline{xyxyyy}}{5040}+\frac{\overline{xyyxyy}}{2240}+\frac{\overline{xyyyyy}}{6720}+\frac{\overline{yyyyyy}}{60480},\ \dots\Big]\Big\}$$

(Also implemented: ∂_λ and derivations in general, tb, e^{∂_λ} and morphisms in general, div, j, Drinfel'd-Kohno, etc.)

The [BND] "vertex" equations.




```
α = LS[{x, y}, αs]; β = LS[{x, y}, βs];
γ = CWS[{x, y}, γs];
V = Es[⟨x → α, y → β⟩, γ];
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R4Eqn = V ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V;
UnitarityEqn =
  (V ** (V // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]));
βs["x"] = 1/2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ},
  (ℏ^-1 R4Eqn) ⋀ UnitarityEqn ⋀ CapEqn];
{V, κ}
```

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.
General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. >>

$$\Big\{\mathrm{Es}\Big[\Big\langle \overline{x}\to\mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ 0,\ \frac{7\,\overline{xxxy}}{5760}-\frac{7\,\overline{xxyy}}{5760}+\frac{\overline{xyyy}}{1440},\ 0,\ -\frac{31\,\overline{xxxxxy}}{967680}+\frac{31\,\overline{xxxxyy}}{483840}-\frac{83\,\overline{xxxyyy}}{967680}-\frac{31\,\overline{xxy\,xyy}}{725760}-\frac{31\,\overline{xxxy\,xy}}{645120}+\frac{13\,\overline{xxyyyy}}{241920}+\frac{101\,\overline{xy\,xyyy}}{1451520}+\frac{527\,\overline{xxyy\,xy}}{5806080}-\frac{\overline{xyyyyy}}{60480},\ \dots\Big],$$
$$\overline{y}\to\mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ 0,\ \frac{\overline{xxxy}}{5760}-\frac{1}{720}\overline{xxyy}+\frac{1}{720}\overline{xyyy},\ -\frac{\overline{xxxxy}}{7680}+\frac{\overline{xxxyy}}{3840}-\frac{\overline{xy\,xy\,y}}{6912},\ -\frac{\overline{xxxxxy}}{645120}+\frac{23\,\overline{xxxxyy}}{483840}-\frac{13\,\overline{xxxyyy}}{161280}-\frac{\overline{xxy\,xyy}}{22680}-\frac{41\,\overline{xxxy\,xy}}{580608}+\frac{\overline{xxyyyy}}{15120}+\frac{\overline{xy\,xyyy}}{12096}+\frac{71\,\overline{xxyy\,xy}}{483840}-\frac{\overline{xyyyyy}}{30240},\ \dots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xy}}{48},\ 0,\ \frac{\overline{xxxy}}{2880}+\frac{\overline{xxyy}}{2880}+\frac{\overline{xyxy}}{5760}+\frac{\overline{xyyy}}{2880},\ 0,\ -\frac{\overline{xxxxxy}}{120960}-\frac{\overline{xxxxyy}}{120960}-\frac{\overline{xxxyxy}}{120960}-\frac{\overline{xxxyyy}}{120960}-\frac{\overline{xxyxxy}}{241920}-\frac{\overline{xxyxyy}}{120960}-\frac{\overline{xxyyxy}}{120960}-\frac{\overline{xxyyyy}}{120960}-\frac{\overline{xyxyxy}}{362880}-\frac{\overline{xyxyyy}}{120960}-\frac{\overline{xyyxyy}}{241920}-\frac{\overline{xyyyyy}}{120960},\ \dots\Big]\Big],$$
$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xx}}{96},\ 0,\ \frac{\overline{xxxx}}{11520},\ 0,\ -\frac{\overline{xxxxxx}}{725760},\ \dots\Big]\Big\}$$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/

From V to F to KV following [AT].

```
logF = Λ[V][[1]] // dσ[{x, y} → {y, x}];
logF // EulerE // adSeries[(e^ad - 1)/ad, logF, tb]
```

⟨x → LS[y/2, xy/6, 1/24 xyy, −1/180 xxxy + 1/80 xxyy + 1/360 xyyy, −1/720 xxxyy + 1/240 xxyyy + 1/240 xyxyy + 1/720 xxyxy − xyyyy/1440, xxxxy/5040 − xxxxy/1344 + 13 xxxyy/15120 + 1/840 xxyxyy + xxxyxy/3360 + xxyyyy/6720 + xyxyyy/1260 + xxyyxy/1680 − xyyyyy/10080, …],

y → LS[0, xy/12, 1/24 xyy, −1/360 xxxy + 1/120 xxyy + 1/180 xyyy, −1/720 xxxyy + 1/240 xxyyy + 1/240 xyxyy + 1/720 xxyxy − xyyyy/1440, xxxxy/10080 − xxxxy/2016 + xxxyyy/1890 + xxyxyy/1120 + xxxyxy/5040 + xxyyyy/2520 + 1/840 xyxyyy + xxyyxy/1260 − xyyyyy/5040, …]⟩

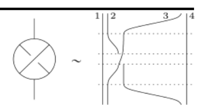
Solving for an associator Φ.

```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0;
Φs[3, 1, 2] = 1/24; Φ = DKS[3, Φs];
SeriesSolve[Φ,
  (Φ^σ[3,2,1] ≡ −Φ) ∧
   (Φ ** Φ^σ[1,23,4] ** Φ^σ[2,3,4] ≡ Φ^σ[12,3,4] ** Φ^σ[1,2,34])];
Φ (* Can raise degree to 10 *)
```

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

DKS[0, 1/24 t13 t23, 0, −7 t13 t23 t23 t23/5760 + 7 t13 t13 t23 t23/5760 − t13 t13 t13 t23/1440, 0, 31 t13 t23 t23 t23 t23 t23/967680 − 157 t13 t13 t23 t23 t13 t23/1935360 − 31 t13 t23 t13 t23 t23 t23/387072 − 31 t13 t13 t23 t23 t23 t23/483840 + 11 t13 t13 t13 t23 t13 t23/290304 + 31 t13 t13 t23 t13 t23 t23/725760 + 83 t13 t13 t13 t23 t23 t23/967680 − 13 t13 t13 t13 t13 t23 t23/241920 + t13 t13 t13 t13 t13 t23/60480, …]

The "buckle" Z_B, from Φ.




```
R = DKS[t[1, 2]/2];
Z_B = (−Φ)^σ[13,2,4] ** Φ^σ[1,3,2] ** R^σ[2,3] ** (−Φ)^σ[1,2,3] **
   Φ^σ[12,3,4];
Z_B@{4}
```

DKS[t23/2, −1/12 t13 t23 − 1/24 t14 t24 + 1/24 t14 t34 + 1/12 t24 t34, 0, t13 t23 t23 t23/5760 + 7 t14 t24 t24 t24/5760 + t14 t34 t24 t24/1920 − t14 t34 t34 t24/1920 − 7 t14 t34 t34 t34/5760 − t24 t34 t34 t34/5760 + t14 t24 t34 t24/1920 + t14 t24 t14 t34/1920 − t14 t34 t24 t34/1920 − 1/720 t13 t13 t23 t23 + 1/720 t13 t13 t13 t23 − 7 t14 t14 t24 t24/5760 + 7 t14 t14 t34 t34/5760 − t14 t24 t34 t34/5760 + t14 t14 t14 t24/1440 − t14 t14 t14 t34/1440 − 1/960 t14 t14 t24 t34 + t14 t24 t24 t34/5760 − 1/960 t24 t24 t34 t34 − t24 t24 t24 t34/5760, …]

V from Z_B, following [AET, BND].

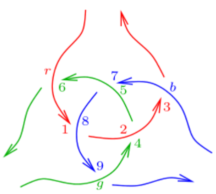
```
(El[Z_B // αMap[1, 2, 3, 4], CWS[0]] // Γ // tη^1 // tη^3 //
    hη^2 // hη^4 // hσ[{3} → {2}] // tσ[{2, 4} → {1, 2}])[[
  1]]
```

⟨1 → LS[0, −12/24, 0, 71112/5760 − 71122/5760 + 1222/1440, 0, −31 111112/967680 + 31 111122/483840 − 83 111222/967680 − 31 112122/725760 − 31 112112/645120 + 13 12222/241920 + 101 121222/1451520 + 527 112212/5806080 − 122222/60480, …],

2 → LS[1/2, −12/12, 0, 1112/5760 − 1/720 1122 + 1/720 1222, −11112/7680 + 11122/3840 − 11212/6912, −111112/645120 + 23 111122/483840 − 13 111222/161280 − 112122/22680 − 41 112112/580608 + 11222 2/15120 + 121222/12096 + 71 112212/483840 − 122222/30240, …]⟩

The Borromean tangle.

```
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → −LS[LW@a]⟩, CWS[0]];
ξ = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
Do[ξ = ξ // dm[r, k, r], {k, 1, 3}];
Do[ξ = ξ // dm[g, k, g], {k, 4, 6}];
Do[ξ = ξ // dm[b, k, b], {k, 7, 9}];
{ξ[[1]]_r@{5}, ξ[[2]]@{5}} // Print
```



;

{LS[0, bg, 1/2 bbg + bgr + 1/2 bgg, 1/6 bbbg + 1/2 bbgr + 1/2 bggr + 1/4 bbgg + 1/2 bgrr + 1/6 bggg, 1/24 bbbbg + 1/6 bbbgr + 1/4 bbggr + 1/12 bbbgg + 1/4 bbgrr + 1/6 bgggr + 1/4 bggrr − bbgrg + 1/12 bbggg − 2 bbrgg + 1/6 bgrrr + 1/2 bgbgr − bgbrg − 1/12 bbgbg − 1/2 bgrgr + 1/24 bgggg, …],
CWS[0, 0, 2 bgr, bbgr − bgbr + bggr − bgrg + bgrr − brgr, bbbgr/3 − bbgbr/2 + bbggr/2 + bbgrg/2 + bbgrr/2 + bbrbg/2 − 3 bbrgr/2 + bgbrr/2 − 3 bggbr/2 + bgggr/3 − bggrg/2 + bggrr/2 + bgrgg/2 − 3 bgrrg/2 + bgrrr/3 + brggr/2 − brgrr/2 + brrgr/2, …]}

References.

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* Annals of Mathematics **175** (2012) 415–463, arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld's associators, braid groups and an explicit solution of the Kashiwara-Vergne equations,* Publications Mathématiques de L'IHÉS, **112-1** (2010) 143–189, arXiv:0903.4067

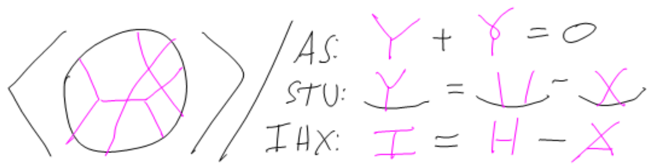
[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-IV,* ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO3, ωεβ/WKO4, and arXiv:1405.1956, arXiv:1405.1955, arXiv:not.yet×2.

Warning. Fidgety!

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/`

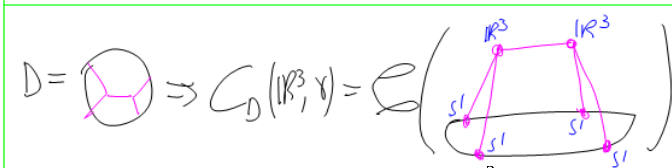
Day 3: Chern-Simons, Gaussian Integration, Feynman Diagrams

Recall. $\mathcal{K} = \{\text{knots}\}$, $\mathcal{A} := \text{gr}\mathcal{A} = \mathcal{D}/\text{rels} =$



Seek $Z\colon \mathcal{K} \to \hat{\mathcal{A}}$ such that if K is n-singular, $Z(K) = D_k + \ldots$

$$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z\text{: high algebra}} \mathcal{A} := \text{gr}\mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a "Lie" algebra } \mathfrak{g}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$



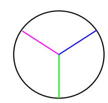
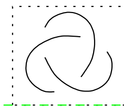
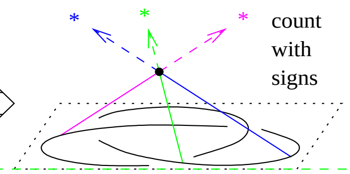
Theorem. Given a parametrized knot γ in $\mathbb{R}^3$, up to renormalizing the "framing anomaly",

$$Z(\gamma) = \sum_{D\in\mathcal{D}} \frac{C(D)D}{|\text{Aut}(D)|} \int_{C_D(\mathbb{R}^3,\gamma)} \bigwedge_{e\in E(D)} \phi_e^*\omega \in \mathcal{A}$$

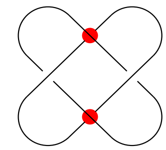
is an expansion. Here $\mathcal{D}$ is the set of all "Feynman diagrams", $E(D)$ is the set of internal edges (and chords) of D, $C_D(\mathbb{R}^3,\gamma)$ is the configuration space of placements of D on/around γ, $\phi\colon C_D(\mathbb{R}^3,\gamma) \to (S^2)^{E(D)}$ is the "direction of the edges" map, and ω is a volume form on S^2.

Cosmic Coincidences

$\langle D, K\rangle_{\text{ⲡ}} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}$: count with signs

$D=$  $K=$  $\Longrightarrow$ 

The Gaussian linking number $lk(\bigcirc\!\bigcirc) = \sum_{\text{vertical chopsticks}} (\text{signs}) = \langle \bigcirc\!\!-\!\!\bigcirc, \bigcirc\!\bigcirc \rangle_{\text{ⲡ}}$



C.F. Gauss

The generating function of all cosmic coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{\text{3-valent } D} \frac{\langle D, K\rangle_{\text{ⲡ}} D}{2^c c! \binom{N}{e}} \in \mathcal{A}$$

D. Thurston

Claim. It all comes from the Chern-Simons-Witten theory,

$$\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\, \text{tr}_R hol_\gamma(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \text{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right],$$

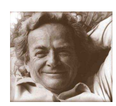
E. Witten

where $\Omega^1(\mathbb{R}^3,\mathfrak{g})$ is the space of all $\mathfrak{g}$-valued 1-forms on $\mathbb{R}^3$ (really, connections), k is some large constant, R is some representation of $\mathfrak{g}$ and tr_R is trace in R, and $hol_\gamma(A)$ is the holonomy of A along γ.

References. Witten's *Quantum field theory and the Jones polynomial*, Axelrod-Singer's *Chern-Simons perturbation theory I-II*, D. Thurston's arXiv:math.QA/9901110, Polyak's arXiv:math.GT/0406251, and my videotaped 2014 class ω/AKT.

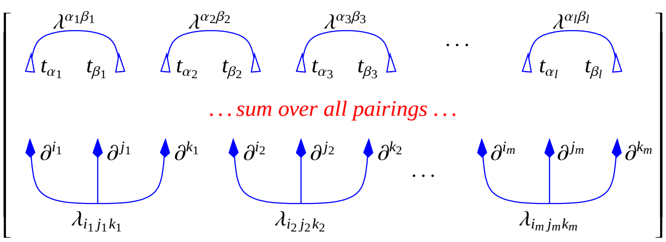
Gaussian Integration. (λ_{ij}) is a symmetic positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

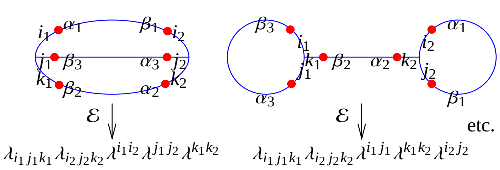
$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^ix^j + \frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k} = \sum_{m\geq 0} \frac{\epsilon^m}{6^m m!} \int_{\mathbb{R}^n} (\lambda_{ijk}x^ix^jx^k)^m e^{-\frac{1}{2}\lambda_{ij}x^ix^j}$$



Feynman

$$= \sum_{m\geq 0} \frac{C\epsilon^m}{6^m m!} (\lambda_{ijk}\partial^i\partial^j\partial^k)^m e^{\frac{1}{2}\lambda^{\alpha\beta}t_\alpha t_\beta}\Big|_{t_\alpha=0} = \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C\epsilon^m}{6^m m! 2^l l!} \left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m \left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C\epsilon^m}{6^m m! 2^l l!}$$ 

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C\epsilon^m}{6^m m! 2^l l!} \sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}} \mathcal{E}(D)$$ 

$$= C \sum_{\substack{\text{unmarked Feynman}\\ \text{diagrams } D}} \frac{\epsilon^{m(D)}\mathcal{E}(D)}{|\text{Aut}(D)|}.$$

Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m! 2^l l!}{|\text{Aut}(D)|}$.

Proof of the Claim. The group $G_{m,l} := [(S_3)^m \rtimes S_m] \times [(S_2)^l \rtimes S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D, and the stabilizer of any given P is $\text{Aut}(D)$. □

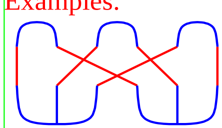
The Fourier Transform.

$$(F\colon V\to\mathbb{C}) \Rightarrow (\tilde{f}\colon V^*\to\mathbb{C})$$

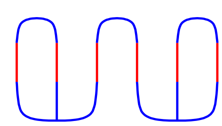
via $\tilde{F}(\varphi) := \int_V f(v)e^{-i\langle\varphi,v\rangle}dv$. Some facts:

- $\tilde{f}(0) = \int_V f(v)dv$.
- $\frac{\partial}{\partial\varphi_i}\tilde{f} \sim \widetilde{v^i f}$.
- $\widetilde{(e^{Q/2})} \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v\rangle$ for $L\colon V\to V^*$, and $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$. (This is the key point in the proof of the Fourier inversion formula!)

Examples.



$|\text{Aut}(D)| = 12$



$|\text{Aut}(D)| = 8$

Monsters left to Slay.

- Convergence.
- Proof of invariance.
- The framing anomaly.
- Universallity.
- d^{-1} doesn't really exist, Faddeev-Popov, determinants, ghosts, Berezin integration.
- Assembly.

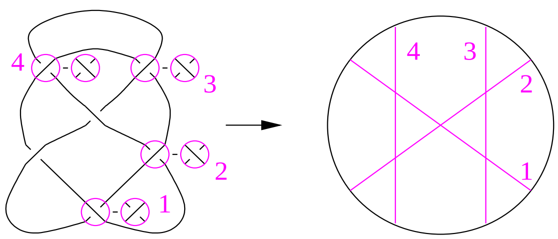
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506/

Dror Bar-Natan: Talks: Louvain-1506:

Knotted Trivalent Graphs, Tetrahedra and Associators

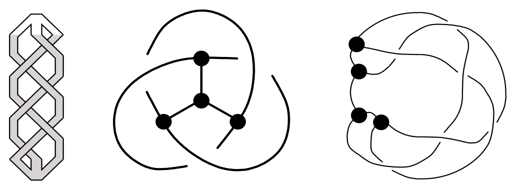
ω :=http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506 Handout, video, and links at ω/

Goal: Z:{knots}−>{chord diagrams}/4T so that

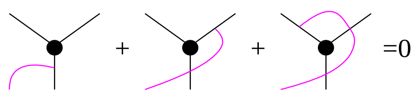


The Miller Institute knot

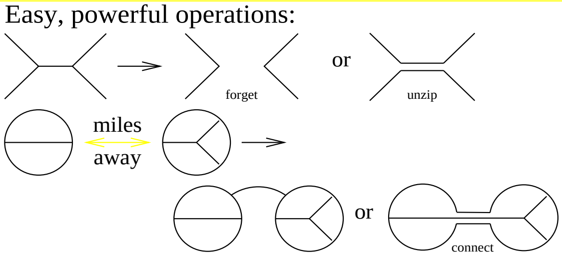
Extend to Knotted Trivalent Graphs (KTG's):

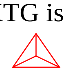


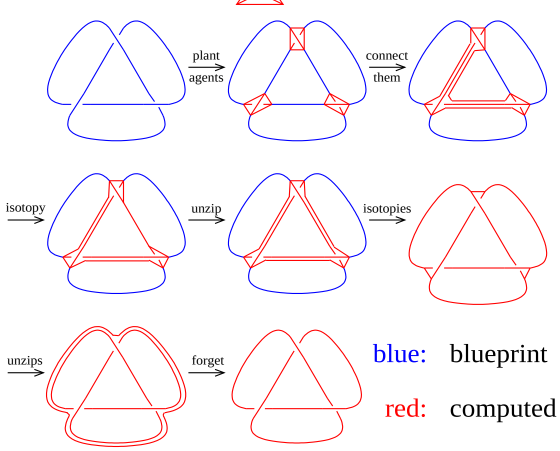
Need a new relation:



Easy, powerful operations:



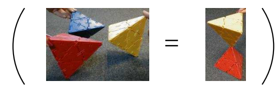
Using operations, KTG is generated by ribbon twists and the tetrahedron  :

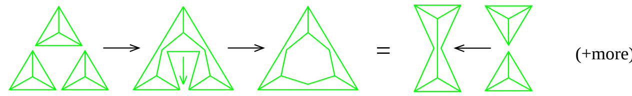


blue: blueprint

red: computed

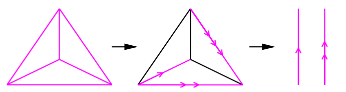
Modulo the relation(s):



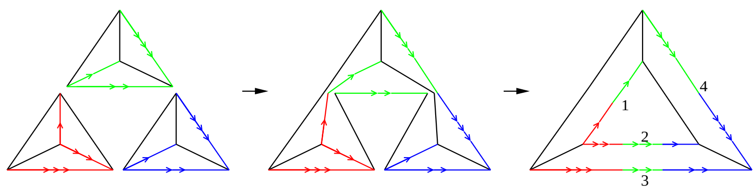


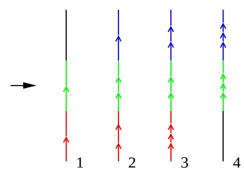
Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi Hopf algebras.

Proof.

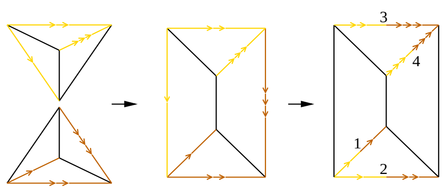


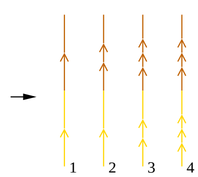
$:= \Phi \in \mathcal{A}(\uparrow_3)$





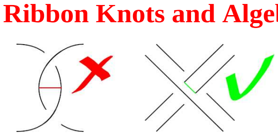
$= (\Phi \otimes 1)(1 \otimes \Delta \otimes 1)(\Phi)(1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$



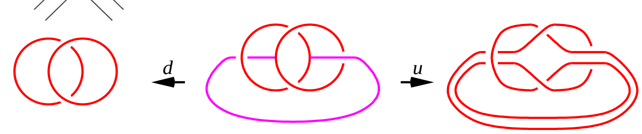


$= (\Delta \otimes 1 \otimes 1)(\Phi)(1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$

Ribbon Knots and Algebraic Knot Theory.



$\mathcal{R} \sim \{u\gamma \colon \gamma \in \mathcal{K}(\circ\!\circ),\ d\gamma = U_2\}$



Video and more at http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506/

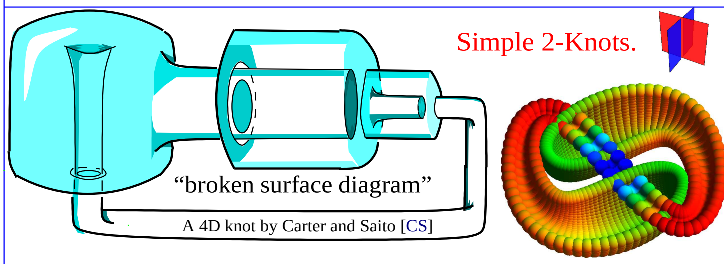
Expansions Day 5: Back to 4D

Abstract. We will repeat the 3D story of the previous 3 talks one dimension up, in 4D. Surprisingly, there's more room in 4D, and things get easier, at least when we restrict our attention to "w-knots", or to "simply-knotted 2-knots". But even then there are intricacies, and we try to go beyond simply-knotted, we are completely confused.

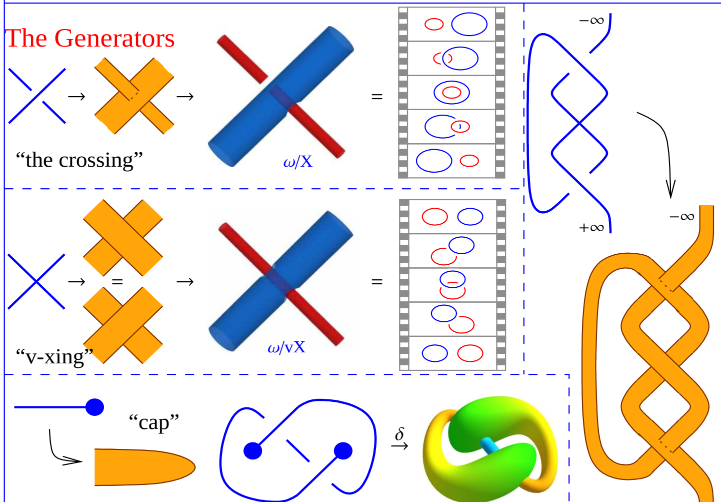
Recall.

$\mathcal{K}$ Topology $\xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z\text{: high algebra}}$ $\mathcal{A} := \operatorname{gr}\mathcal{K}$ Combinatorics $\xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a “Lie” algebra } \mathfrak{g}}$ "$\mathcal{U}(\mathfrak{g})$"

Simple 2-Knots.

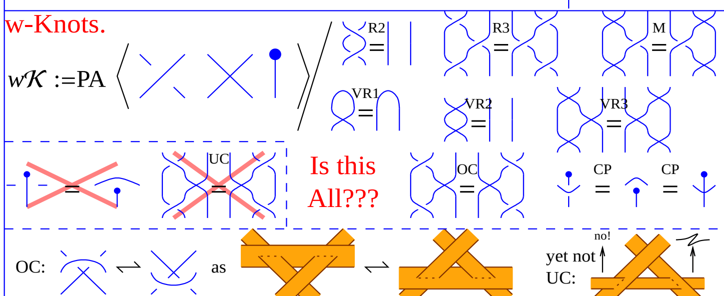


The Generators



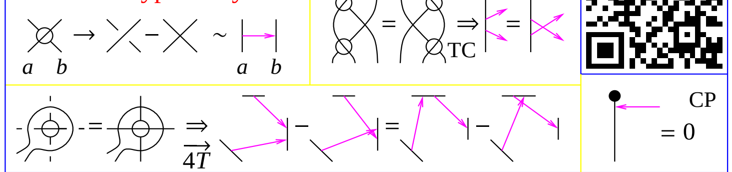
The Double Inflation Procedure δ.

w-Knots.

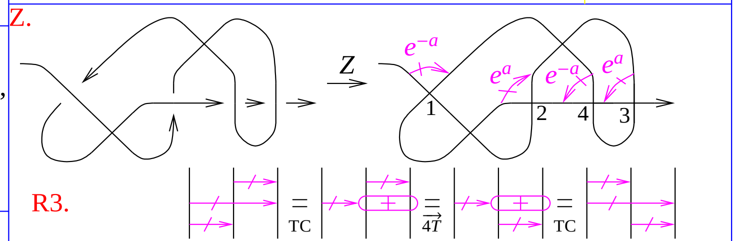


A Big Open Problem. δ maps w-knots onto simple 2-knots. To what extent is it a bijection? What other relations are required? In other words, **find a simple description of simple 2-knots**. Kawauchi [Ka] may already know the answer.

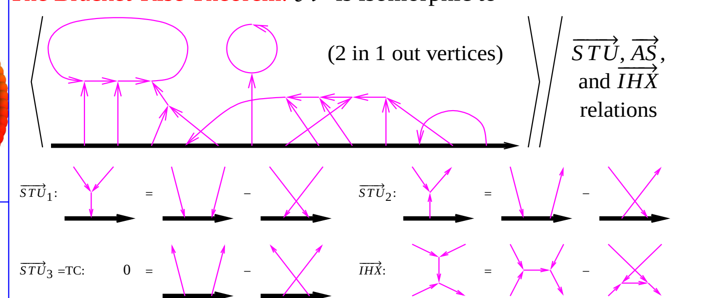
The Finite Type Story.



Z.



The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to



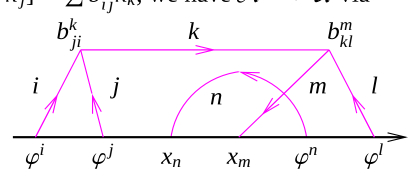
Corollaries. (1) Only wheels and isolated arrows persist:

$$\mathcal{A}^w(\uparrow_n) \cong \mathcal{U}(FL(n)^n_{tb} \ltimes CW(n)) \quad \text{and} \quad \zeta := \log Z \in FL(n)^n \times CW(n)$$

has completely explicit formulas using natural FL/CW operations [BN].
(2) Related to f.d. Lie algebras!

Low Algebra.

With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij} x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



Penrose Cvitanovic

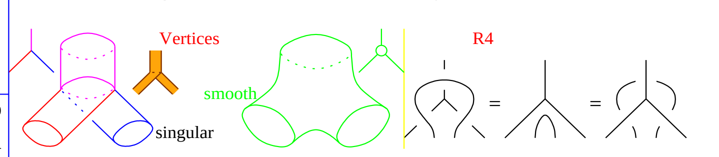
$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij} b^m_{kl} \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$

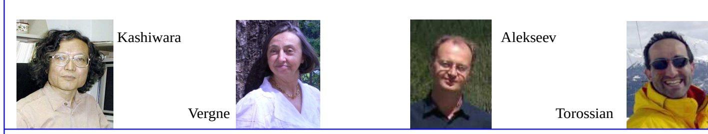
Differential Ops.

We can also interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on Fun($\mathfrak{g}$): $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator, and $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of ad x: $(x\varphi)(y) := \varphi([x, y])$.

Too easy so far!

Yet once you add “foam vertices”, it gets related to the Kashiwara-Vergne problem [KV] as told by Alekseev-Torossian [AT]:

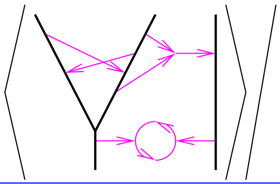




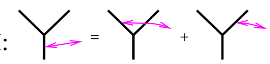
Kashiwara Vergne Alekseev Torossian

Expansions Day 5: Back to 4D — Side 2

w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is

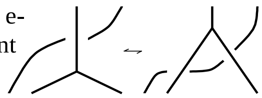


same relations, plus

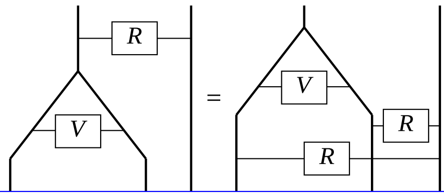
VI: 

deg$=\frac{1}{2}$#{vertices}=6

Knot-Theoretic statement (simplified). There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$.



Diagrammatic statement (simplified). Let $R = \exp \overset{\rightarrow}{\vdash\!\dashv} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that:



Algebraic statement (simplified). With $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

Unitary statement (simplified). There exists a unitary tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that $V\widehat{e^{x+y}} = \widehat{e^x e^y} V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

Group-Algebra statement (simplified). For every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$:

$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)e^x e^y.$$

(shhh, this is Duflo)

Unitary $\Longrightarrow$ Group-Algebra. $\iint e^{x+y}\phi(x)\psi(y) = \langle 1, e^{x+y}\phi(x)\psi(y)\rangle = \langle V1, Ve^{x+y}\phi(x)\psi(y)\rangle = \langle 1, e^x e^y V\phi(x)\psi(y)\rangle = \langle 1, e^x e^y \phi(x)\psi(y)\rangle = \iint e^x e^y \phi(x)\psi(y)$.

Convolutions statement (Kashiwara-Vergne, simplified). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f) \star \Phi(g) = \Phi(f \star g)$.

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G), \star) \to (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

$$\begin{array}{ccc} (\mathfrak{g},+) \ni x & \xrightarrow{\tau_0=\exp_S} & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \exp_G & \searrow \exp_{\mathcal{U}} & \downarrow \chi \\ (G,\cdot) \ni e^x & \xrightarrow{\tau_1} & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) \end{array} \quad \text{so} \quad \begin{array}{ccc} \mathrm{Fun}(\mathfrak{g}) & \xrightarrow{L_0} & \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \Phi^{-1} & & \downarrow \chi \\ \mathrm{Fun}(G) & \xrightarrow{L_1} & \hat{\mathcal{U}}(\mathfrak{g}) \end{array}$$

with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$:

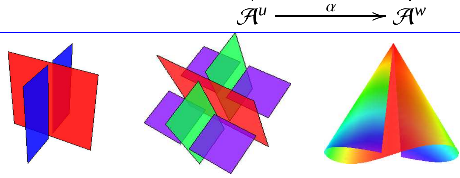
$$\star \text{ in } G: \iint \psi_1(x)\psi_2(y)e^x e^y \qquad \star \text{ in } \mathfrak{g}: \iint \psi_1(x)\psi_2(y)e^{x+y}$$

$u \leftrightarrow w$ The diagram on the right explains the relationship between associators and solutions of the Kashiwara-Vergne problem.

$$\begin{array}{ccc} \mathcal{K}TG & \xrightarrow{a} & w\mathcal{T}F \\ \downarrow Z^u & & \downarrow Z^w \\ \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^w \end{array}$$

The Full

2-Knot Story



Question. Does it all extend to arbitrary 2-knots (not necessarily "simple")? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A, B) := \int_M \langle B, F_A \rangle.$$

Rossi

With $\kappa\colon (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

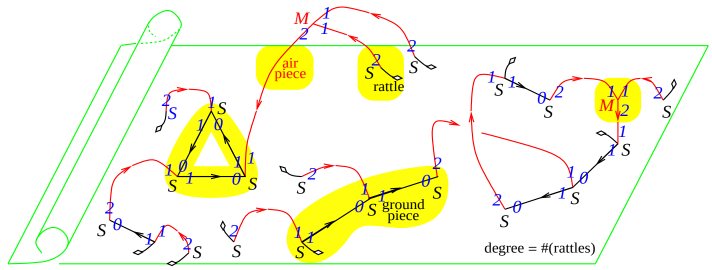
$$\mathcal{O}(A, B, \kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle \beta, d_{\kappa^* A}\alpha + \kappa^* B\rangle\right).$$

Cattaneo

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then

$$Z_{BF} = \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4} \cdots \int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

(modulo some IHX-like relations). See also [Wa]



Issues. • Signs don't quite work out, and BF seems to reproduce only "half" of the wheels invariant on simple 2-knots.
• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
• I don't know how to define / analyze "finite type" for general 2-knots.
• I don't know how to reduce Z_{BF} to combinatorics / algebra.

References.

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, Annals of Mathematics **175** (2012) 415–463, arXiv:0802.4300.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ω/KBH, arXiv:1308.1721.

[BND1] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: W-Knots and the Alexander Polynomial*, ω/WKO1, arXiv:1405.1956.

[BND2] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects II: Tangles and the Kashiwara-Vergne Problem*, ω/WKO2, arXiv:1405.1955.

[CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Math. Surv. and Mono. **55**, Amer. Math. Soc., Providence 1998.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions*, Invent. Math. **47** (1978) 249–272.

[Ka] A. Kawauchi, *A Chord Diagram of a Ribbon Surface-Link*, http://www.sci.osaka-cu.ac.jp/~kawauchi/.

[Wa] T. Watanabe, *Configuration Space Integrals for Long n-Knots, the Alexander Polynomial and Knot Space Cohomology*, Alg. and Geom. Top. **7** (2007) 47–92, arXiv:math/0609742.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506/

Commutators

Abstract. The commutator of two elements x and y in a group G is $xyx^{-1}y^{-1}$. That is, x followed by y followed by the inverse of x followed by the inverse of y. In my talk I will tell you how commutators are related to the following four riddles:

1. Can you send a secure message to a person you have never communicated with before (neither privately nor publicly), using a messenger you do not trust?
2. Can you hang a picture on a string on the wall using n nails, so that if you remove any one of them, the picture will fall?
3. Can you draw an n-component link (a knot made of n non-intersecting circles) so that if you remove any one of those n components, the remaining $(n-1)$ will fall apart?
4. Can you solve the quintic in radicals? Is there a formula for the zeros of a degree 5 polynomial in terms of its coefficients, using only the operations on a scientific calculator?

Definition. The commutator of two elements x and y in a group G is $[x,y] := xyx^{-1}y^{-1}$.
Example 1. In S_3, $[(12),(23)] = (12)(23)(12)^{-1}(23)^{-1} = (123)$ and in general in $S_{\geq 3}$,

$$[(ij),(jk)] = (ijk).$$

Example 2. In $S_{\geq 4}$,

$$[(ijk),(jkl)] = (ijk)(jkl)(ijk)^{-1}(jkl)^{-1} = (il)(jk).$$

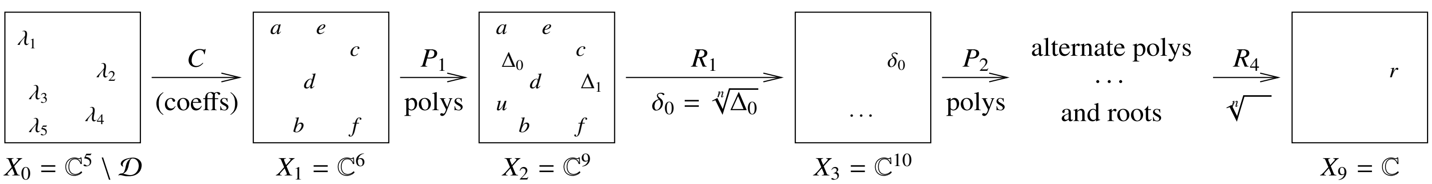
Example 3. In $S_{\geq 5}$,

$$[(ijk),(klm)] = (ijk)(klm)(ijk)^{-1}(klm)^{-1} = (jkm).$$

Example 4. So, in fact, in S_5, $(123) = [(412),(253)] = [[(341),(152)],[(125),(543)]] = [[[(234),(451)],[(315),(542)]],[[(312),(245)],[(154),(423)]]] = [\ [[[(123),(354)],[(245),(531)]],[[(231),(145)],[(154),(432)]]], [[[(431),(152)],[(124),(435)]],[[(215),(534)],[(142),(253)]]]\ ]$.

Solving the Quadratic, $ax^2+bx+c=0$: $\delta = \sqrt{\Delta}$; $\Delta = b^2-4ac$; $r = \frac{\delta-b}{2a}$.
Solving the Cubic, $ax^3+bx^2+cx+d=0$: $\Delta = 27a^2d^2-18abcd+4ac^3+4b^3d-b^2c^2$; $\delta = \sqrt{\Delta}$; $\Gamma = 27a^2d-9abc+3\sqrt{3}a\delta+2b^3$; $\gamma = \sqrt[3]{\frac{\Gamma}{2}}$; $r = -\frac{\frac{b^2-3ac}{\gamma}+b+\gamma}{3a}$.
Solving the Quartic, $ax^4+bx^3+cx^2+dx+e=0$: $\Delta_0 = 12ae-3bd+c^2$; $\Delta_1 = -72ace+27ad^2+27b^2e-9bcd+2c^3$; $\Delta_2 = \frac{1}{27}\left(\Delta_1^2-4\Delta_0^3\right)$; $u = \frac{8ac-3b^2}{8a^2}$; $v = \frac{8a^2d-4abc+b^3}{8a^3}$; $\delta_2 = \sqrt{\Delta_2}$; $Q = \frac{1}{2}\left(3\sqrt{3}\delta_2+\Delta_1\right)$; $q = \sqrt[3]{Q}$; $S = \frac{\frac{\Delta_0}{q}+q}{12a} - \frac{u}{6}$; $s = \sqrt{S}$; $\Gamma = -\frac{v}{s}-4S-2u$; $\gamma = \sqrt{\Gamma}$; $r = -\frac{b}{4a}+\frac{\gamma}{2}+s$.

Theorem. The is no general formula, using only the basic arithmetic operations and taking roots, for the solution of the quintic equation $ax^5+bx^4+cx^3+dx^2+ex+f=0$.
Key Point. The "persistent root" of a closed path (path lift, in topological language) may not be closed, yet the persistent root of a commutators of closed paths is always closed.
Proof. Suppose there was a formula, and consider the corresponding "composition of machines" picture:

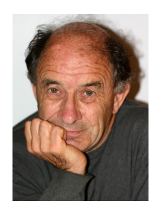


Now if $\gamma_1^{(1)}, \gamma_2^{(1)}, \ldots, \gamma_{16}^{(1)}$, are paths in X_0 that induce permutations of the roots and we set $\gamma_1^{(2)} := [\gamma_1^{(1)},\gamma_2^{(1)}]$, $\gamma_2^{(2)} := [\gamma_3^{(1)},\gamma_4^{(1)}]$, ..., $\gamma_8^{(2)} := [\gamma_{15}^{(1)},\gamma_{16}^{(1)}]$, $\gamma_1^{(3)} := [\gamma_1^{(2)},\gamma_2^{(2)}]$, ..., $\gamma_4^{(3)} := [\gamma_7^{(2)},\gamma_8^{(2)}]$, $\gamma_1^{(4)} := [\gamma_1^{(3)},\gamma_2^{(3)}]$, $\gamma_2^{(4)} := [\gamma_3^{(3)},\gamma_4^{(3)}]$, and finally $\gamma^{(5)} := [\gamma_1^{(4)},\gamma_2^{(4)}]$ (all of those, commutators of "long paths"; I don't know the word "homotopy"), then $\gamma^{(5)} /\!\!/ C /\!\!/ P_1 /\!\!/ R_1 /\!\!/ \cdots /\!\!/ R_4$ is a closed path. Indeed,

- In X_0, none of the paths is necessarily closed.
- After C, all of the paths are closed.
- After P_1, all of the paths are still closed.
- After R_1, the $\gamma^{(1)}$'s may open up, but the $\gamma^{(2)}$'s remain closed.

...

- At the end, after R_4, $\gamma^{(4)}$'s may open up, but $\gamma^{(5)}$ remains closed.

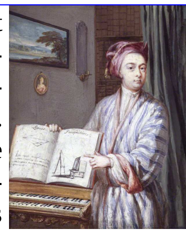

V.I. Arnold

But if the paths are chosen as in Example 4, $\gamma^{(5)} /\!\!/ C /\!\!/ P_1 /\!\!/ R_1 /\!\!/ \cdots /\!\!/ R_4$ is not a closed path. □

References. V.I. Arnold, 1960s, hard to locate.
V.B. Alekseev, *Abel's Theorem in Problems and Solutions, Based on the Lecture of Professor V.I. Arnold,* Kluwer 2004.
A. Khovanskii, *Topological Galois Theory, Solvability and Unsolvability of Equations in Finite Terms,* Springer 2014.
B. Katz, *Short Proof of Abel's Theorem that 5th Degree Polynomial Equations Cannot be Solved,* YouTube video, http://youtu.be/RhpVSV6iCko.

When does a group have a Taylor expansion?

Abstract. It is insufficiently well known that the good old Taylor expansion has a completely algebraic characterization, which generalizes to arbitrary groups (and even far beyond). Thus one may ask: Does the braid group have a Taylor expansion? (Yes, using iterated integrals and/or associators). Do braids on a torus ("elliptic braids") have Taylor expansions? (Yes, using more sophisticated iterated integrals / associators). Do virtual braids have Taylor expansions? (No, yet for nearby objects the deep answer is Probably Yes). Do groups of flying rings (braid groups one dimension up) have Taylor expansions? (Yes, easily, yet the link to TQFT is yet to be fully explored).



Brook Taylor

Disclaimer. I'm asked to talk in a meeting on "iterated integrals", and that's my best. Many of you may think it all trivial. Sorry.

Expansions for Groups. Let G be a group, $\mathcal{K} = \mathbb{Q}G = \{\sum a_i g_i\colon a_i \in \mathbb{Q},\ g_i \in G\}$ its group-ring, $\mathcal{I} = \{\sum a_i g_i\colon \sum a_i = 0\}$ its augmentation ideal. Let

$$\mathcal{A} = \operatorname{gr}\mathcal{K} := \widehat{\bigoplus}_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}.$$

P.S. $(\mathcal{K}/\mathcal{I}^{m+1})^*$ is Vassiliev / finite-type / polynomial invariants.

Note that $\mathcal{A}$ inherits a product from G.

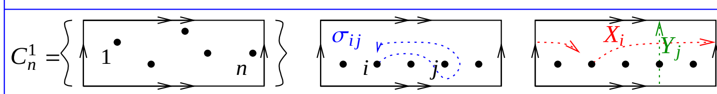
Definition. A linear $Z\colon \mathcal{K} \to \mathcal{A}$ is an "expansion" if for any $\gamma \in \mathcal{I}^m$, $Z(\gamma) = (0,\dots,0,\gamma/\mathcal{I}^{m+1},*,\dots)$, a "multiplicative expansion" if in addition it preserves the product, and a "Taylor expansion" if it also preserves the co-product, induced from the diagonal map $G \to G\times G$.

Malcev

Quillen

Example. Let $\mathcal{K} = C^\infty(\mathbb{R}^n)$ and $\mathcal{I} = \{f\colon f(0) = 0\}$. Then $\mathcal{I}^m = \{f\colon f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ degree m homogeneous polynomials and $\mathcal{A} = \{\text{power series}\}$. The Taylor series is the unique Taylor expansion!

Comment. Unlike lower central series constructions, this generalizes effortlessly to arbitrary algebraic structures.

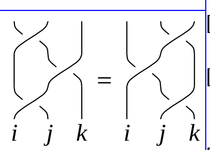


Elliptic Braids. $PB_n^1 := \pi_1(C_n^1)$ is generated by σ_{ij}, X_i, Y_j, with PB_n relations and $(X_i,X_j) = 1 = (Y_i,Y_j)$, $(X_i,Y_j) = \sigma_{ij}^{-1}$, $(X_iX_j,\sigma_{ij}) = 1 = (Y_iY_j,\sigma_{ij})$, and $\prod X_i$ and $\prod Y_j$ are central. [Bez] implies $\mathcal{A}(PB_n^1) = \left\langle x_i, y_j\right\rangle / \big([x_i,x_j] = [y_i,y_j] = [x_i+x_j,[x_i,y_j]] = [y_i+y_j,[x_i,y_j]] = [x_i,\sum y_j] = [y_j,\sum x_i] = 0,\ [x_i,y_j] = [x_j,y_i]\big)$, and [CEE] construct a Taylor expansion using *sophisticated* iterated integrals. [En2] relates this to *Elliptic Associators*.

Virtual Braids. PvB_n is given by the "braids for dummies" presentation:

$\left\langle \sigma_{ij} \mid \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij},\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij}\right\rangle$

(every quantum invariant extends to PvB_n!).



By [Lee], $\mathcal{A}(PvB_n)$ is

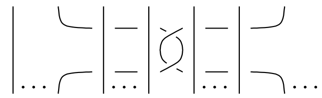
$\left\langle a_{ij} \mid [a_{ij},a_{ik}] + [a_{ij},a_{jk}] + [a_{ik},a_{jk}] = 0 = [a_{ij},a_{kl}]\right\rangle$

Peter Lee

Theorem [Lee]. While quadratic, PvB_n does not have a Taylor expansion.

Comment. By the tough theory of quantization of solutions of the classical Young-Baxter equation [EK, En1], PvT_n does have a Taylor expansion. But PvT_n is not a group.

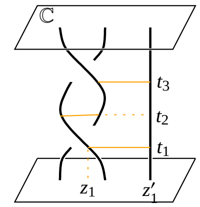
Pure Braids. Take $G = PB_n = \pi_1(C_n = \mathbb{C}^n\backslash\text{diags})$. It is generated by the love-behind-the-bars braids σ_{ij}, modulo "Reidemeister moves". $\mathcal{I}$ is generated by $\{\sigma_{ij}-1\}$ and $\mathcal{A}$ by $\{t_{ij}\}$, the classes of the $\sigma_{ij}-1$ in $\mathcal{A}_1 = \mathcal{I}/\mathcal{I}^2$. Reidemeister becomes $[t_{ij}+t_{ik},t_{jk}] = 0$ and $[t_{ij},t_{kl}] = 0$.



Theorem. For $\gamma\colon [0,1] \to C_n$, with z_i its ith coordinate, the iterated integral formula on the right defines a Taylor expansion for PB_n.

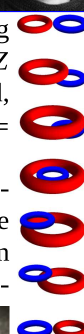
$$Z(\gamma) = \sum_{m\geq 0} \int_{\substack{0<t_1<\ldots<t_m<1 \\ 1\leq i_1<j_1,i_2<j_2,\ldots,i_m<j_m\leq n}} \prod_{\alpha=1}^m \frac{t_{i_\alpha j_\alpha}}{2\pi i} d\log(z_{i_\alpha}-z_{j_\alpha}),$$

Comments. • I don't know a combinatorial/algebraic proof that PB_n has a Taylor expansion. • Generic "partial expansion" do not extend! • This is the seed for the Drinfel'd theory of associators! • Confession: I don't know a clean derivation of a presentation of PB_n.

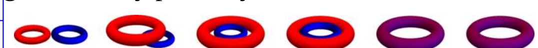


Knizhnik
Zamolodchikov
Kohno
Drinfel'd
Kontsevich

Flying Rings. $PwB_n = PvB_n/(\sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij})$ is π_1(flying rings in $\mathbb{R}^3$). $\mathcal{A}(PwB_n) = \mathcal{A}(PvB_n)/[a_{ij},a_{ik}] = 0$, and Z is as easy as it gets: $Z(\sigma_{ij}) = e^{a_{ij}}$ [BP, BND]. Indeed, $Z(\sigma_{ij}\sigma_{ik}\sigma_{jk}) = e^{a_{ij}}e^{a_{ik}}e^{a_{jk}} = e^{a_{ij}+a_{ik}}e^{a_{jk}} = e^{a_{ij}+a_{ik}+a_{jk}} = Z(\sigma_{jk}\sigma_{ik}\sigma_{ij})$.



Comments. • Extends to PwT and generalizes the Alexander polynomial, and even to $PwTT$ and interprets the Kashiwara-Vergne problem [BND]. • I don't know an iterated-integral derivation, or any TQFT derivation, though BF theory probably comes close [CR].



The "Vertex" in TT.

Dancso

References. Paper in Progress: ωεβ/ExQu

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: Braids, Knots and the Alexander Polynomial*, ωεβ/WKO1, arXiv:1405.1956; and *II: Tangles and the Kashiwara-Vergne Problem*, ωεβ/WKO2, arXiv:1405.1955.

[BP] B. Berceanu and S. Papadima, *Universal Representations of Braid and Braid-Permutation Groups*, J. of Knot Theory and its Ramifications **18-7** (2009) 973–983, arXiv:0708.0634.

[Bez] R. Bezrukavnikov, *Koszul DG-Algebras Arising from Configuration Spaces*, Geom. Func. Anal. **4-2** (1994) 119–135.

[CEE] D. Calaque, B. Enriquez, and P. Etingof, *Universal KZB Equations I: The Elliptic Case*, Prog. in Math. **269** (2009) 165–266, arXiv:math/0702670.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

[En1] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras*, Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.

[En2] B. Enriquez, *Elliptic Associators*, Selecta Mathematica **20** (2014) 491–584, arXiv:1003.1012.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I*, Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[Lee] P. Lee, *The Pure Virtual Braid Group Is Quadratic*, Selecta Mathematica **19-2** (2013) 461–508, arXiv:1110.2356.

Tangles, Wheels, Balloons

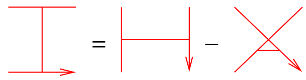
Abstract. I will describe a computable, non-commutative invariant of tangles with values in wheels, almost generalize it to some balloons, and then tell you why I care. Spoilers: tangles are you know what, wheels are linear combinations of cyclic words in some alphabet, balloons are 2-knots, and one reason I care is because quantum field theory predicts more than I can actually get (but also less).

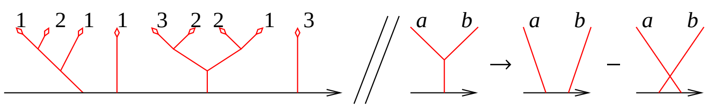
Why I like "non-commutative"? With $FA(x_i)$ the free associative non-commutative algebra,

$$\dim \mathbb{Q}[x,y]_d \sim d \ll 2^d \sim \dim FA(x,y)_d.$$

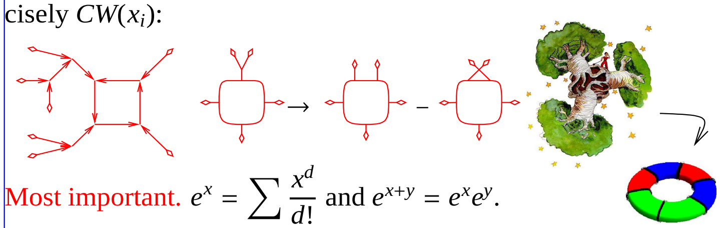
Why I like "computable"?
- Because I'm weird.
- Note that π_1 isn't computable.

Preliminaries from Algebra. $FL(x_i)$ denotes the free Lie algebra in (x_i); $FL(x_i)$ = (binary trees with AS vertices and coloured leafs)/(IHX relations). There an obvious map $FA(FL(x_i)) \to FA(x_i)$ defined by $[a,b] \to ab - ba$, which in itself, is IHX.



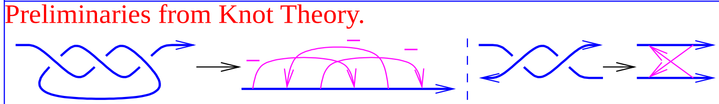


$CW(x_i)$ denotes the vector space of cyclic words in (x_i): $CW(x_i) = FA(x_i)/(x_i w = w x_i)$. There an obvious map $CW(FL(x_i)) \to CW(x_i)$. In fact, connected uni-trivalent 2-in-1-out graphs with univalents with colours in $\{1,\ldots,n\}$, modulo AS and IHX, is precisely $CW(x_i)$:

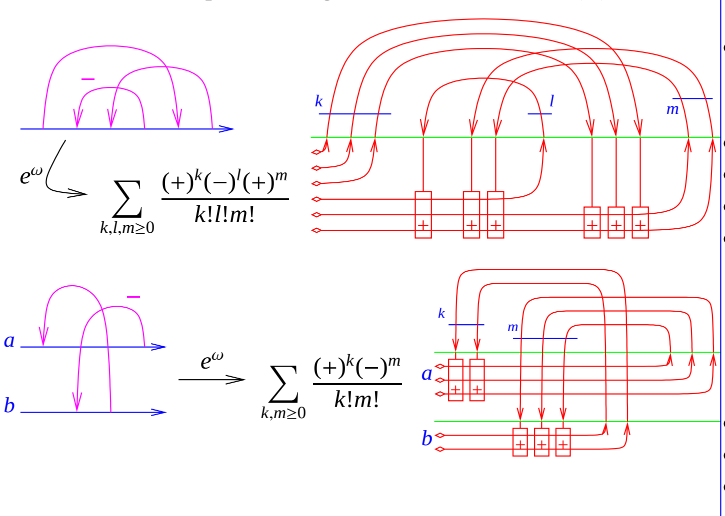


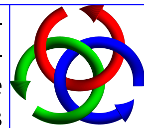
Most important. $e^x = \sum \frac{x^d}{d!}$ and $e^{x+y} = e^x e^y$.

Preliminaries from Knot Theory.

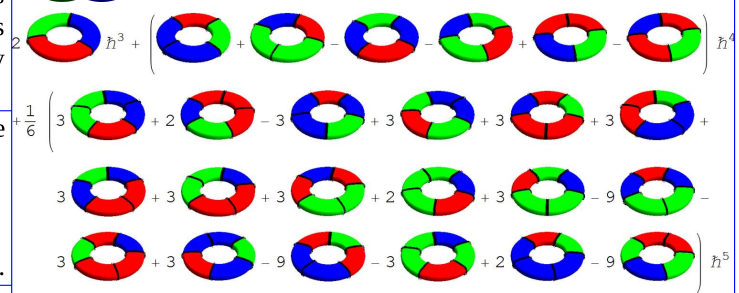


Theorem. ω, the connected part of the procedure below, is an invariant of S-component tangles with values in $CW(S)$:



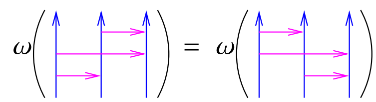


ω is practically computable! For the Borromean tangle, to degree 5, the result is: (see [BN])

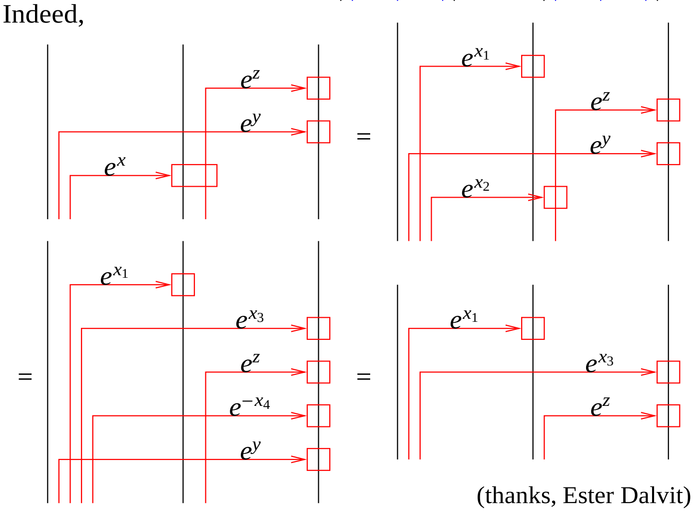


Proof of Invariance.

Need to show:



Indeed,

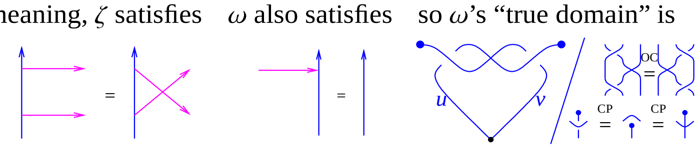


(thanks, Ester Dalvit)

Further Facts

- ω is really the second part of a (trees,wheels)-valued invariant $\zeta = (\lambda, \omega)$. The tree part λ is just a repackaging of the Milnor μ-invariants.
- On u-tangles, ζ is equivalent to the trees&wheels part of the Kontsevich integral, except it is computable and is defined with no need for a choice of parenthesization.
- On long/round u-knots, ω is equivalent to the Alexander polynomial.
- The multivariable Alexander polynomial (and Levine's factorization thereof [Le]) is contained in the Abelianization of ζ [BNS].
- ω vanishes on braids.
- Related to / extends Farber's [Fa]?
- Should be summed and categorified.
- Extends to v and descends to w: meaning, ζ satisfies ω also satisfies so ω's "true domain" is

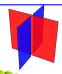
Does ω extend to balloons?

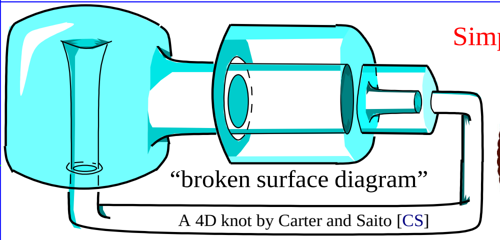


- Agrees with BN-Dancso [BND1, BND2] and with [BN].
- ζ, ω are universal finite type invariants.
- Using ⋊K: $v\mathcal{K}_n \to w\mathcal{K}_{n+1}$, defines a strong invariant of v-tangles / long v-knots. (⋊K in LaTeX: ωεβ/zhe)

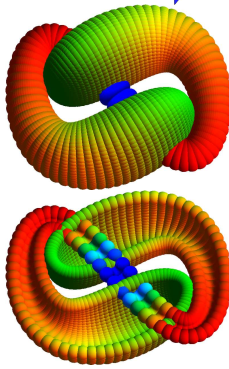
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Hamilton-1412/

Simple 2-Knots.





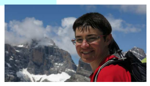
A 4D knot by Carter and Saito [CS]



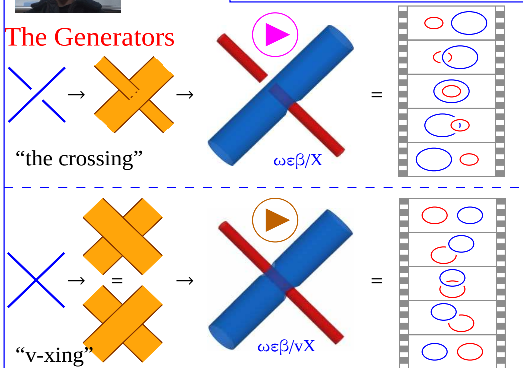
ωεβ/F

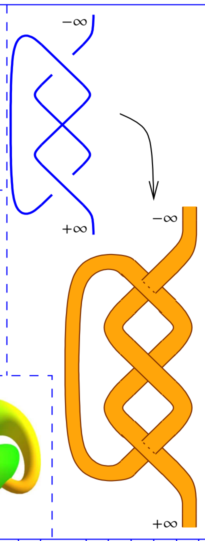
Satoh

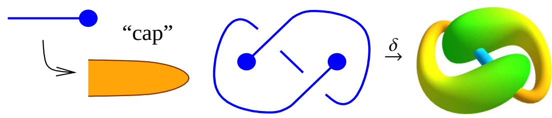
Dalvit
ωεβ/Dal



The Generators

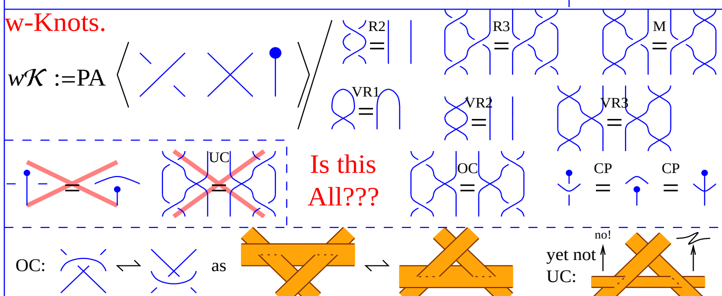






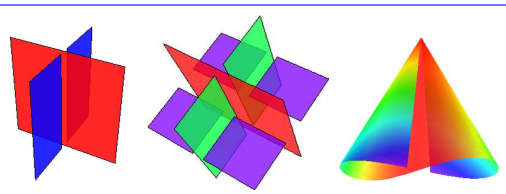
The Double Inflation Procedure δ.

w-Knots.

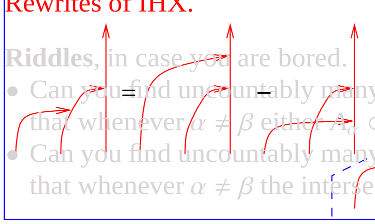


A Big Open Problem. δ maps w-knots onto simple 2-knots. To what extent is it a bijection? What other relations are required? In other words, **find a simple description of simple 2-knots**.

The Full 2-Knot Story



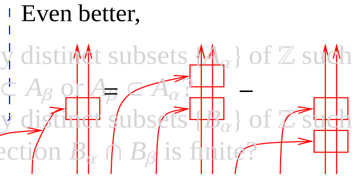
Rewrites of IHX.



Riddles, in case you are bored.

- Can you find uncountably many distinct subsets $\{A_\alpha\}$ of $\mathbb{Z}$ such that whenever $\alpha \neq \beta$ either $A_\alpha \subset A_\beta$ or $A_\beta \subset A_\alpha$?
- Can you find uncountably many distinct subsets $\{B_\alpha\}$ of $\mathbb{Z}$ such that whenever $\alpha \neq \beta$ the intersection $B_\alpha \cap B_\beta$ is finite?

Even better,



Question. Does it all extend to arbitrary 2-knots (not necessarily "simple")? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A,B) := \int_M \langle B, F_A \rangle.$$

Rossi

With $\kappa\colon (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

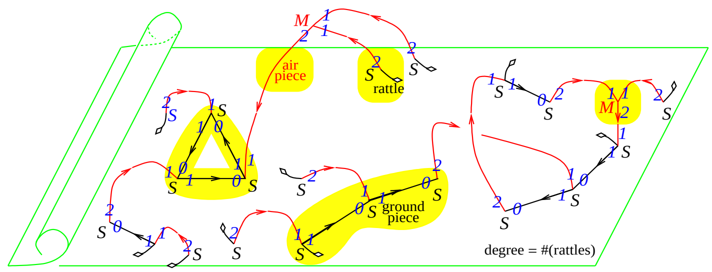
$$\mathcal{O}(A,B,\kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle \beta, d_{\kappa^* A}\alpha + \kappa^* B\rangle\right).$$

Cattaneo

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then

$$Z_{BF} = \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

(modulo some IHX-like relations). See also [Wa]



Issues. • Signs don't quite work out, and BF seems to reproduce only "half" of the wheels invariant on simple 2-knots.
• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
• I don't know how to define / analyze "finite type" for general 2-knots.
• I don't know how to reduce Z_{BF} to combinatorics / algebra.

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.

[BND1] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: W-Knots and the Alexander Polynomial*, ωεβ/WKO1, arXiv:1405.1956.

[BND2] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects II: Tangles and the Kashiwara-Vergne Problem*, ωεβ/WKO2, arXiv:1405.1955.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Math. Surv. and Mono. **55**, Amer. Math. Soc., Providence 1998.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

[Fa] M. Farber, *Noncommutative Rational Functions and Boundary Links*, Math. Ann. **293** (1992) 543–568.

[Le] J. Levine, *A Factorization of the Conway Polynomial*, Comment. Math. Helv. **74** (1999) 27–53, arXiv:q-alg/9711007.

[Wa] T. Watanabe, *Configuration Space Integrals for Long n-Knots, the Alexander Polynomial and Knot Space Cohomology*, Alg. and Geom. Top. **7** (2007) 47–92, arXiv:math/0609742.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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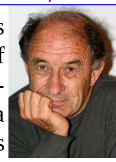
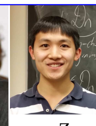
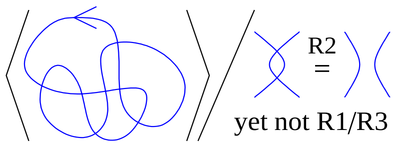
Finite Type Invariants of Doodles, 1

Abstract. I will describe my former student's Jonathan Zung work on finite type invariants of "doodles", plane curves modulo the second Reidemeister move but not modulo the third. We use a definition of "finite type" different from Arnold's and more along the lines of Goussarov's "Interdependent Modifications", and come to a conjectural combinatorial description of the set of all such invariants. We then describe how to construct many such invariants (though perhaps not all) using a certain class of 2-dimensional "configuration space integrals". An unfinished project!


Arnold


Zung

Doodles. $\mathcal{K} = \mathcal{K}_0 = \mathbb{Q}$



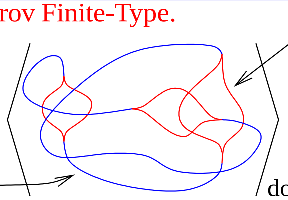
Easy to classify!

Prior Art. Arnold [Ar] first studied doodles within his study of plane curves and the "strangeness" *St* invariant. Vassiliev [Va1, Va2] defined finite type invariants in a different way, and Merkov [Me] proved that they separate doodles.

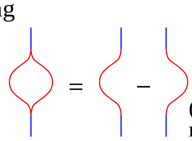

Merkov and Vassiliev

Goussarov Finite-Type.

$\mathcal{K}_n =$



doodles and detours (dnd's)

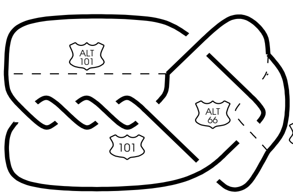



Goussarov

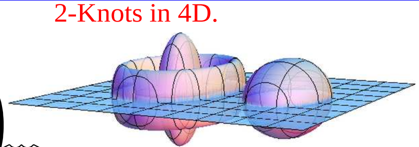
(equal rotation numbers)

Def. V is of type n if it vanishes on $\mathcal{K}_{n+1}$. $(\mathcal{K}_0/\mathcal{K}_{n+1})^\star \leftrightsquigarrow \mathcal{K}_n/\mathcal{K}_{n+1}$

Knots in 3D.



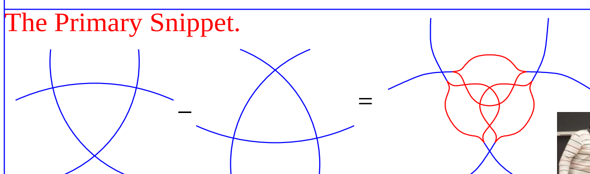
2-Knots in 4D.



The reason I care!

Goals. • Describe $\mathcal{A}_n := \mathcal{K}_n/\mathcal{K}_{n+1}$ using diagrams/relations. • Get many or all finite type invariants of doodles using configurations space integrals. • Do these come from a TQFT? • See if $\mathcal{A}_n$ has a "Lie theoretic" (tensors/relations) meaning. • See if/how Arnold's *St* and the Merkov invariants integrate in.

The Primary Snippet.



(Arnold's "strangeness" is type 3)

Goryunov, here

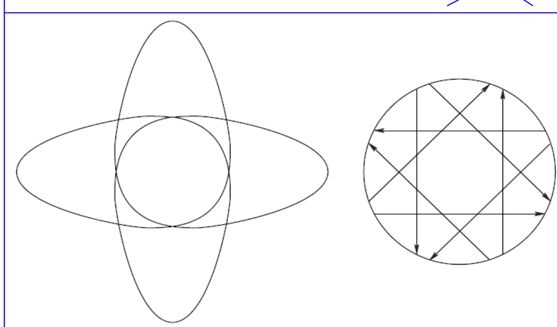


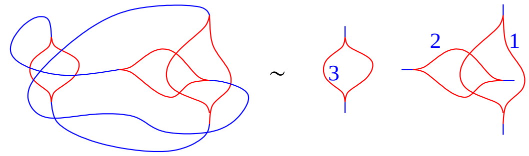
Figure 3. A non-trivial 1-doodle and its arrow diagram

From Merkov's [Me]

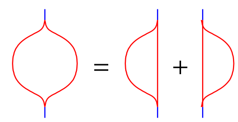
Chord Diagrams and an Upper Bound on $\mathcal{K}_n/\mathcal{K}_{n+1}$

The Rayman Principle. In $\mathcal{K}_n/\mathcal{K}_{n+1}$, "joins are irrelevant"

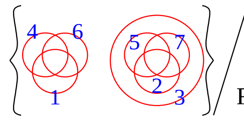

Rayman by Ubisoft



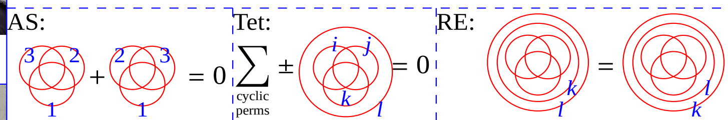
The Subdivision Relations. In $\mathcal{K}_n/\mathcal{K}_{n+1}$,



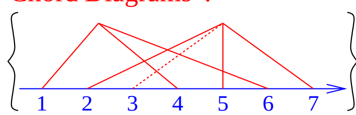
$\Rightarrow$ Rings can be subdivided until each one participates in just one "feature". $\Rightarrow$



Anti-Symmetry (AS)
Tetrahedron (Tet) $\longrightarrow\!\!\!\!\!\rightarrow \mathcal{K}_n/\mathcal{K}_{n+1}$
Ring Exchange (RE)

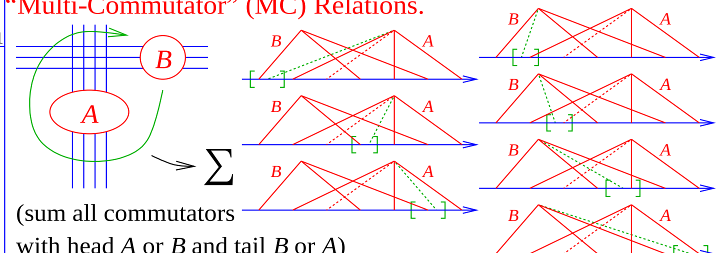


"Chord Diagrams".

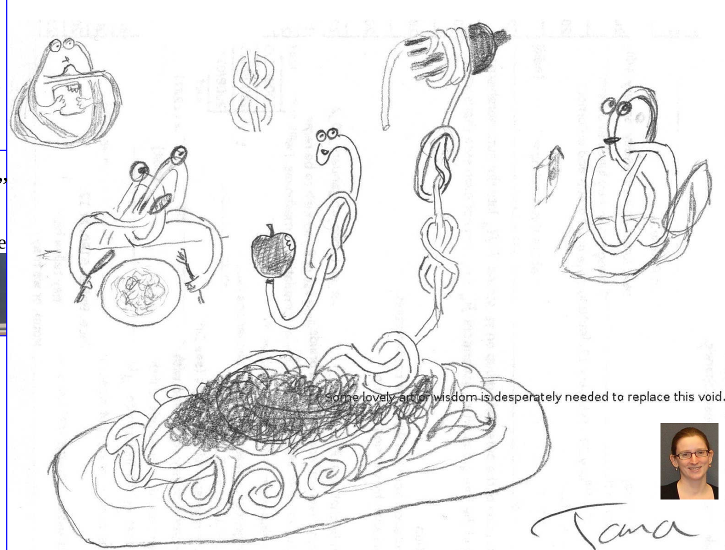


AS, Tet, RE, MC $\xrightarrow{\pi} \mathcal{K}_n/\mathcal{K}_{n+1}$

"Multi-Commutator" (MC) Relations.



(sum all commutators with head A or B and tail B or A)



Doodles by my former student Jana Archibald

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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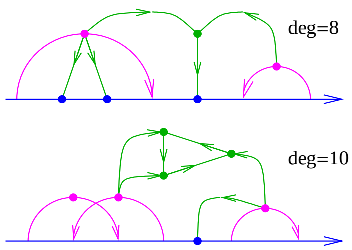
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Fields-1411/

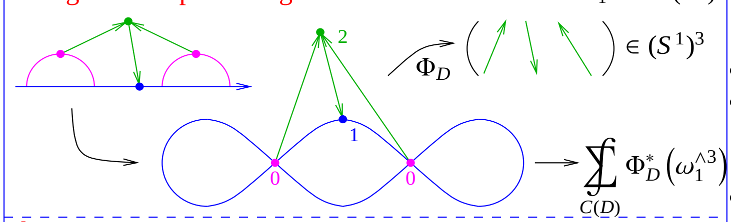
Finite Type Invariants of Doodles, 2

Feynman Diagrams and a Lower Bound on $(\mathcal{K}_0/\mathcal{K}_{n+1})^*$.



Feynman Diagrams. A blue "skeleton line" at the bottom. A magenta "arrow diagram" (directed pairing of skeleton points) on top, with a magenta dot at the middle of each arrow. A green directed graph on top, with 2-in 1-out antisymmetric green vertices, with arbitrary number of green edges starting at the magenta dots, and with some green edges terminating at distinct blue skeleton points. The *degree* is the total valency of the magenta dots.

Configuration Space Integrals. $\omega_1:\ \mathrm{vol.}(S^1)$



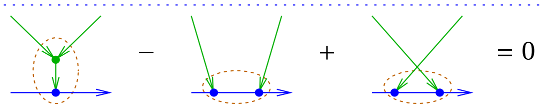
The "Partition Function" Z.

$$K \mapsto Z(K) := \sum_{\substack{\text{Feynman}\\ \text{diagrams}}} \sum_{C(D)} \Phi_D^*\left(\omega_1^{\wedge e(D)}\right) \in \mathcal{A}^t := \langle D\rangle/(\partial\text{-relations}).$$

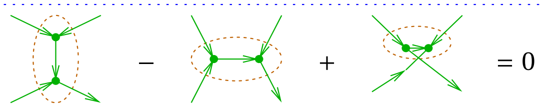
Theorem (90%). Z is an invariant of doodles.

∂-relations. STU, IHX, Foot Swap (FS), Arrow Exchange (AE), and Combinatorial R2 (CR2):

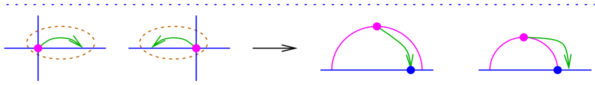
STU:



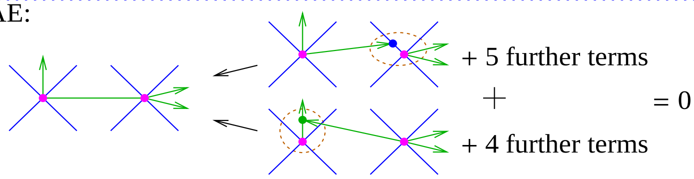
IHX:



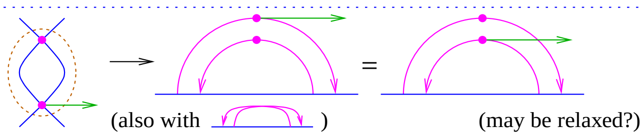
FS:



AE:



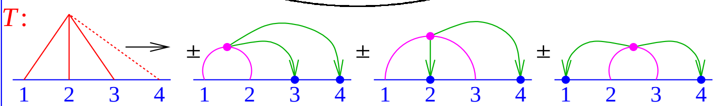
CR2:



Summary Diagram.

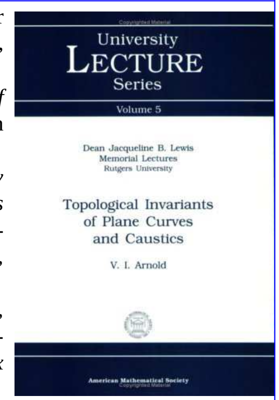
$$\begin{tikzcd} & \mathcal{K} \arrow[d, Rightarrow] \arrow[dr, "Z"] & \\ \mathcal{D}^c/MC =: \mathcal{A}^c \arrow[r, two heads, "\pi"] \arrow[rr, bend right, "T:\ 1\text{-}1?"'] & \mathrm{gr}\mathcal{K} \arrow[r, "\mathrm{gr}Z"] & \mathcal{A}^t := \mathcal{D}^t/FDR \end{tikzcd}$$

T:



An unfinished project!

- Nothing is written up.
- We don't know if T is injective (meaning, if our upper and lower bounds agree).
- We don't know if all of $\mathcal{A}^t$ is necessary — it is very possible that it is enough to restrict to the green-less part of $\mathcal{A}^t$ — to "Gauss Diagram Formulas".
- We haven't clarified the relationship with Merkov's [Me].
- A few further configuration space integrals can be written beyond those that we have used. We don't know what to do with those, if anything.
- We don't know the relationship, if any, with algebra.
- We don't know the relationship, if any, with quantum field theory.
- We don't know how to do similar things with 2-knots.



References. The root, of course, is [Ar]. Further references on doodles include [Kh, FT, Me, Ta, Va1, Va2]. On Goussarov finite-type: [Go, BN].

[Ar] V.I. Arnold, *Topological Invariants of Plane Curves and Caustics*, American Mathematical Society, 1994.

[BN] D. Bar-Natan, *Bracelets and the Goussarov filtration of the space of knots, Invariants of knots and 3-manifolds (Kyoto 2001)*, Geometry and Topology Monographs **4** 1–12, arXiv:math.GT/0111267.

[FT] R. Fenn and P. Taylor, *Introducing Doodles*, in *Topology of Low-Dimensional Manifolds, Proceedings of the Second Sussex Conference, 1977*, Springer 1979.

[Go] M. Goussarov, *Interdependent modifications of links and invariants of finite degree*, Topology **37-3** (1998) 595–602.

[Kh] M. Khovanov, *Doodle Groups*, Trans. Amer. Math. Soc. **349-6** (1997) 2297–2315.

[Me] A.B. Merkov, *Vassiliev Invariants Classify Plane Curves and Doodles*, Sbornik: Mathematics **194-9** (2003) 1301.

[Ta] S. Tabachnikov, *Invariants of Smooth Triple Point Free Plane Curves*, Jour. of Knot Theory and its Ramifications **5-4** (1996) 531–552.

[Va1] V.A. Vassiliev, *On Finite Order Invariants of Triple Point Free Plane Curves*, 1999 preprint, arXiv:1407.7227.

[Va2] V.A. Vassiliev, *Invariants of Ornaments*, Adv. in Soviet Math. **21** (1994) 225–262.

open space for your doodles

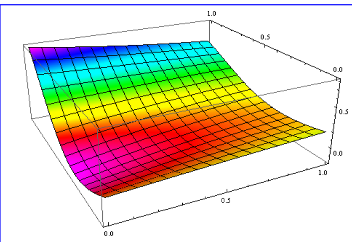
Dessert: Hilbert's 13th Problem, in Full Colour

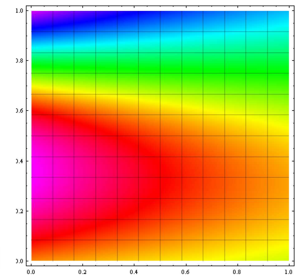
Dror Bar-Natan, Toronto November 2014. More at http://www.math.toronto.edu/~drorbn/Talks/Fields-1411

Abstract. To break a week of deep thinking with a nice colourful light dessert, we will present the Kolmogorov-Arnold solution of Hilbert's 13th problem with lots of computer-generated rainbow-painted 3D pictures.

In short, Hilbert asked if a certain specific function of three variables can be written as a multiple (yet finite) composition of continuous functions of just two variables. Kolmogorov and Arnold showed him silly (ok, it took about 60 years, so it was a bit tricky) by showing that **any** continuous function f of any finite number of variables is a finite composition of continuous functions of a single variable and several instances of the binary function "+" (addition). For $f(x,y)=xy$, this may be $xy=\exp(\log x+\log y)$. For $f(x,y,z)=x^y/z$, this may be $\exp(\exp(\log y+\log\log x)+(-\log z))$. What might it be for (say) the real part of the Riemann zeta function?

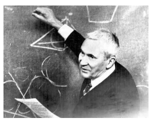
The only original material in this talk will be the pictures; the math was known since around 1957.





$\frac{1}{3}\mathrm{Re}(\zeta(x+iy))$ on $[0,1]\times[13,17]$

Hilbert



Kolmogorov

Arnold (by Moser)

Fix an irrational $\lambda>0$, say $\lambda=(\sqrt{5}-1)/2$. All functions are continuous.

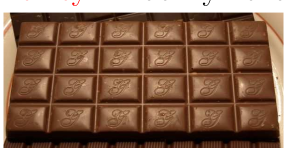
Theorem. There exist five $\phi_i:[0,1]\to[0,1]$ ($1\le i\le 5$) so that for every $f:[0,1]\times[0,1]\to\mathbb{R}$ there exists a $g:[0,1+\lambda]\to\mathbb{R}$ so that

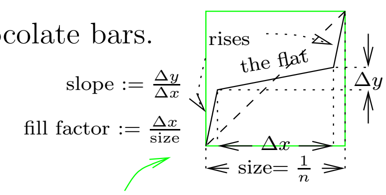
$$f(x,y)=\sum_{i=1}^{5} g(\phi_i(x)+\lambda\phi_i(y))$$

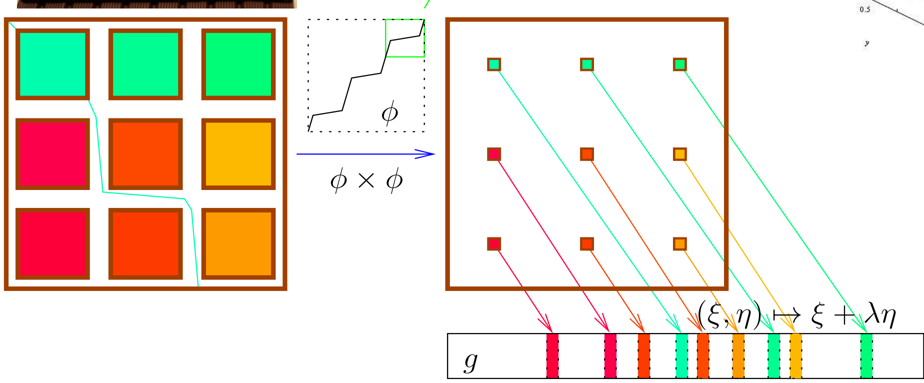
for every $x,y\in[0,1]$.

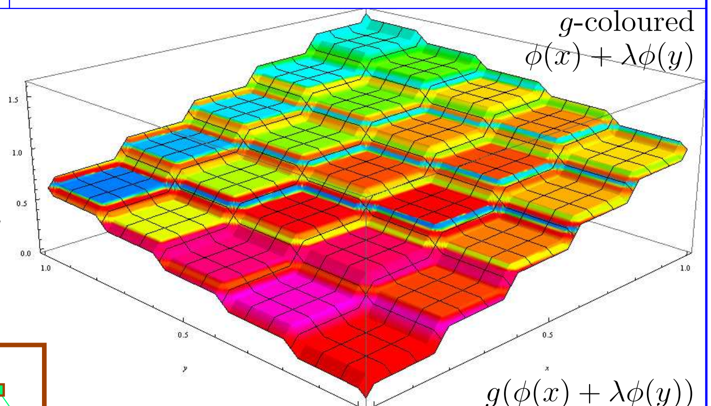
Step 1. If $\epsilon>0$ and $f:[0,1]\times[0,1]\to\mathbb{R}$, then there exists $\phi:[0,1]\to[0,1]$ and $g:[0,1+\lambda]\to\mathbb{R}$ so that $|f(x,y)-g(\phi(x)+\lambda\phi(y))|<\epsilon$ on at least 98% of the area of $[0,1]\times[0,1]$.

The key. "Poorify" chocolate bars.

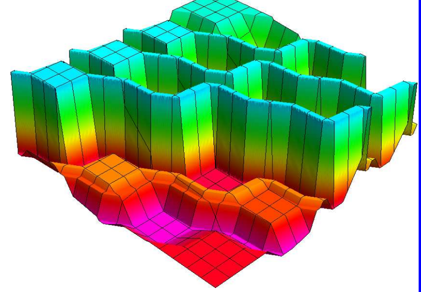




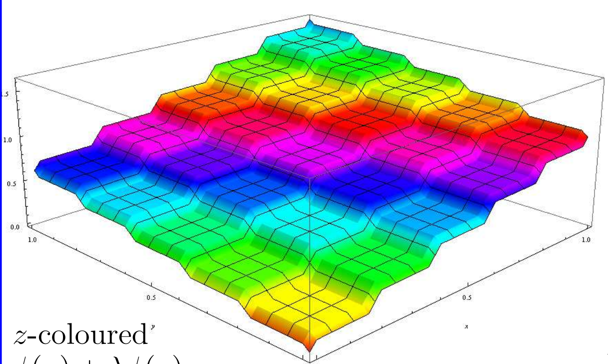




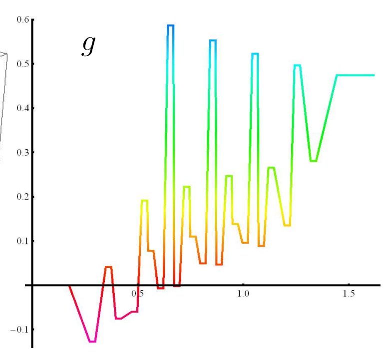
g-coloured $\phi(x)+\lambda\phi(y)$

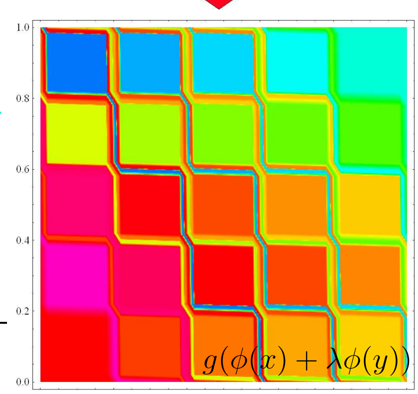


$g(\phi(x)+\lambda\phi(y))$



z-coloured $\phi(x)+\lambda\phi(y)$





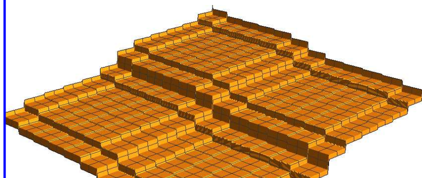
$g(\phi(x)+\lambda\phi(y))$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Fields-1411/

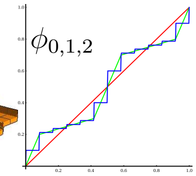
Dessert: Hilbert's 13th Problem, in Full Colour (Page 2)

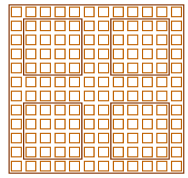
Step 2. There exists $\phi : [0,1] \to [0,1]$ so that for every $\epsilon > 0$ and every $f : [0,1]\times[0,1] \to \mathbb{R}$ there exists a $g : [0, 1+\lambda] \to \mathbb{R}$ so that $|f(x,y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on a set of area at least $1-\epsilon$ in $[0,1]\times[0,1]$.

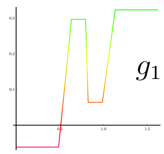
The key. "Iterated poorification".

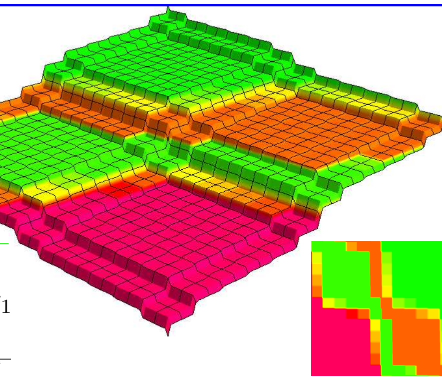


$\phi(x) + \lambda\phi(y)$

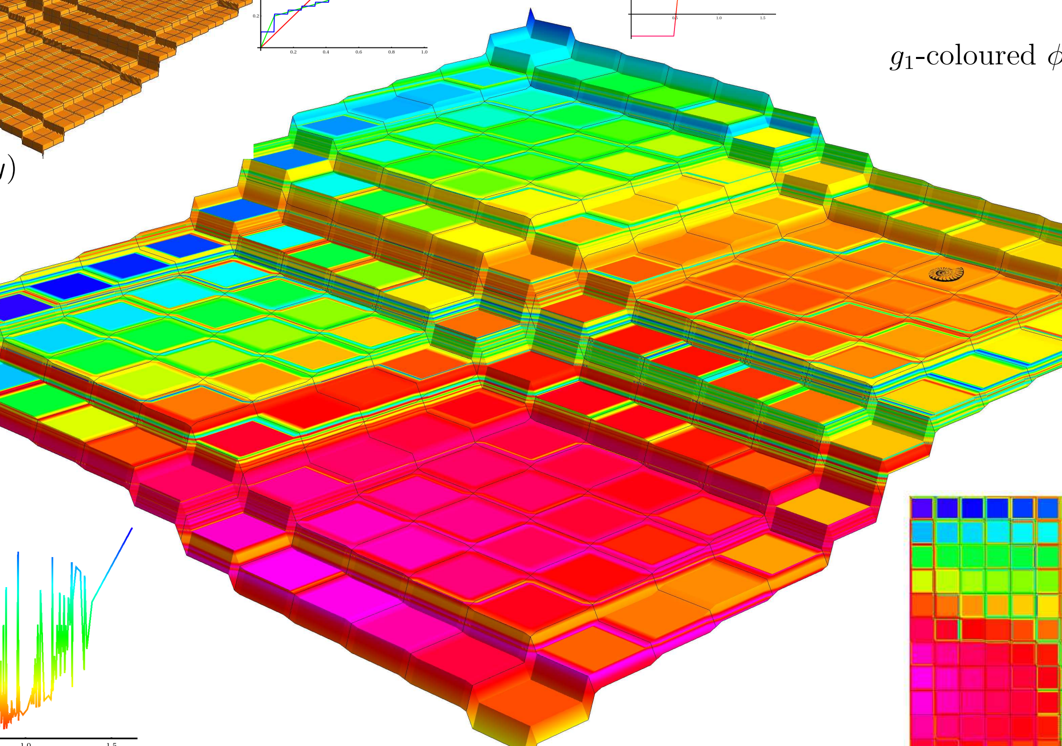


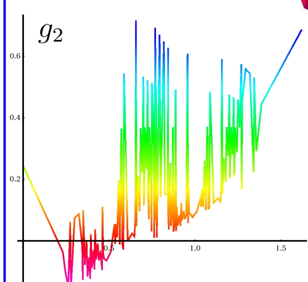




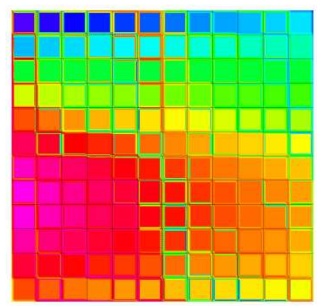


g_1-coloured $\phi(x) + \lambda\phi(y)$





g_2-coloured $\phi(x) + \lambda\phi(y)$

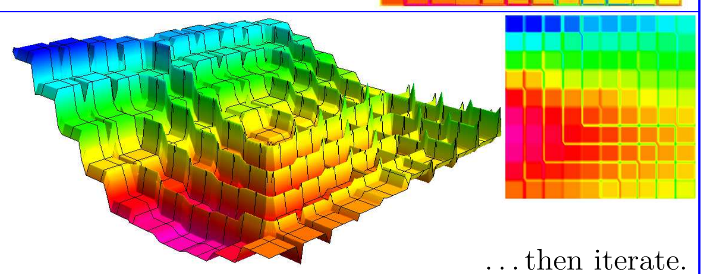


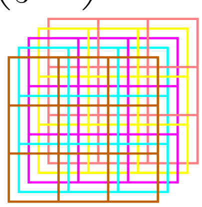
Step 3. There exist $\phi_i : [0,1] \to [0,1]$ $(1 \le i \le 5)$ so that for every $\epsilon > 0$ and every $f : [0,1]\times[0,1] \to \mathbb{R}$ there exists a $g : [0, 1+\lambda] \to \mathbb{R}$ so that

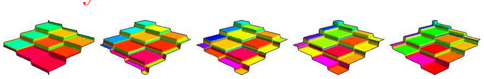
$$\left|f(x,y) - \sum_{i=1}^{5} g(\phi_i(x) + \lambda\phi_i(y))\right| < \left(\frac{2}{3} + \epsilon\right) \|f\|_\infty$$

for every $x, y \in [0,1]$.

The key. "Shift the chocolates"...







...then iterate.

Step 4. We are done.

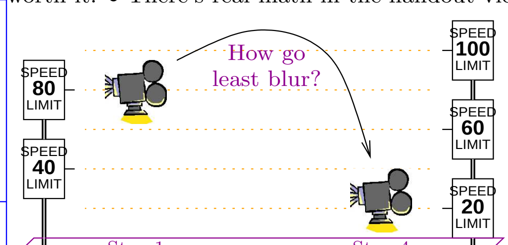
The key. Learn from the artillery!

Set $Tg := \sum_{i=1}^{5} g(\phi_i(x) + \lambda\phi_i(y))$, $f_1 := f$, $M := \|f\|$, and iterate "shooting and adjusting". Find g_1 with $\|g_1\| \le M$ and $\|f_2 := f_1 - Tg_1\| \le \frac{3}{4}M$. Find g_2 with $\|g_2\| \le \frac{3}{4}M$ and $\|f_3 := f_2 - Tg_2\| \le (\frac{3}{4})^2 M$. Find g_3 with $\|g_3\| \le (\frac{3}{4})^2 M$ and $\|f_4 := f_3 - Tg_3\| \le (\frac{3}{4})^3 M$. Continue to eternity. When done, set $g = \sum g_k$ and note that $f = Tg$ as required.

Exercise 1. Do the m-dimensional case.

Exercise 2. Do $\mathbb{R}^m$ instead of just I^m.

Propaganda. I love handouts! • I have nothing to hide and you can take what you want, forwards, backwards, here and at home. • What doesn't fit on one sheet can't be done in one hour. • It takes learning and many hours and a few pennies. The audience's worth it! • There's real math in the handout viewer!



Video and more at http://www.math.toronto.edu/~drorbn/Talks/Fields-1411/

The 17 Tiling Patterns: Gotta catch 'em all!

Video, handout, links at *http://drorbn.net/Treehouse*

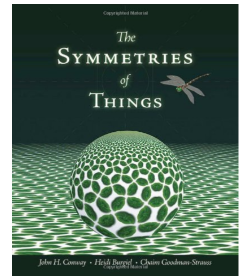
Treehouse Talks, Friday October 17, 2014, Beeton Auditorium, Toronto Reference Library, 789 Yonge Street, 6:30PM

Abstract. My goal is to get you hooked, captured and unreleased until you find all 17 in real life, around you.

We all know know that the plane can be filled in different periodic manners: floor tiles are often square but sometimes hexagonal, bricks are often laid in an interlaced pattern, fabrics often carry interesting patterns. A little less known is that there are precisely 17 symmetry patterns for tiling the plane; not one more, not one less. It is even less known how easy these 17 are to identify in the patterns around you, how fun it is, how common some are, and how rare some others seem to be.

Gotta catch 'em all!

Reading. An excellent book on the subject is *The Symmetries of Things* by J. H. Conway, H. Burgiel, and C. Goodman-Strauss, CRC Press, 2008. Another nice text is *Classical Tessellations and Three-Manifolds* by J. M. Montesinos, Springer-Verlag, 1987.



Question. In what ways can you make \$2 change, using coins denominated $\frac12$, $\frac23$, $\frac34$, $\frac45$, $\frac56$, etc.?

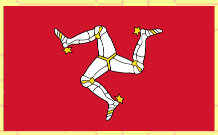
Answer. $2=\frac12+\frac12+\frac12+\frac12=\frac23+\frac23+\frac23=\frac34+\frac34+\frac12=\frac56+\frac23+\frac12$, and that's it.

Video, handout, links at *drorbn.net/Treehouse*

The Basic Features.

Gotta catch 'em all!

3



rotation only

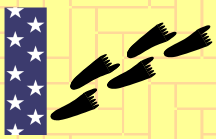
~~3~~

rotation-reflection

M

free mirror-reflection

G

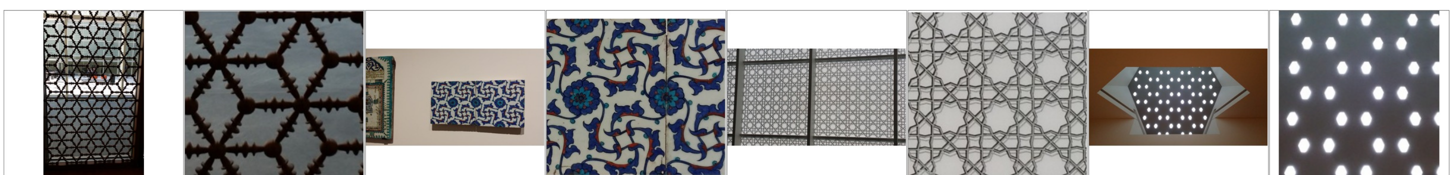


free glide-reflection

Theorem. There are precisely 17 patterns with which to tile the plane, no more, no less. They are all made of combinations of the 10 basic features, 2, 3, 4, 6, ~~2~~, ~~3~~, ~~4~~, ~~6~~, M, and G, as follows:

✓	Dror's	Conway's	crystallo-graphic	✓	Dror's	Conway's	crystallo-graphic
	2222	2222	p2		3~~3~~	3*3	p31m
	333	333	p3		2~~22~~	2*22	cmm
	442	442	p4		22M	22*	pmg
	632	632	p6		MM	**	pm
	~~2222~~	*2222	pmm		MG	*o	cm
	~~333~~	*333	p3m1		GG	oo	pg
	~~442~~	*442	p4m		22G	22o	pgg
	~~632~~	*632	p6m		∅	0	p1
	4~~2~~	4*2	p4g				

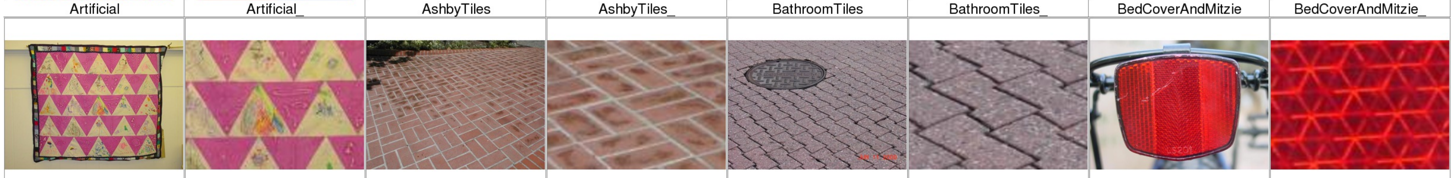
Tilings worksheet. Classify the following pictures according to the following possibilities: 2222=2222, 333=333, 442=442, 632=632, ~~2222~~=*2222, ~~333~~=*333, ~~442~~=*442, ~~632~~=*632, 4~~2~~=4*2, 3~~3~~=3*3, 2~~22~~=2*22, 22M=22*, MM=**, MG=*o, GG=oo, 22G=22o, and ∅=0 (the pictures come in {context, pattern} pairs).



AgaKhan-3 | AgaKhan-3_ | AgaKhan-4 | AgaKhan-4_ | AgaKhan-5 | AgaKhan-5_ | AgaKhan-6 | AgaKhan-6_

AgaKhan-7 | AgaKhan-7_ | Alhambra | Alhambra_ | AnteaterCage | AnteaterCage_ | ArchStreetFence | ArchStreetFence_

Artificial | Artificial_ | AshbyTiles | AshbyTiles_ | BathroomTiles | BathroomTiles_ | BedCoverAndMitzie | BedCoverAndMitzie_



BethElQuilt | BethElQuilt_ | BethElSidewalk | BethElSidewalk_ | BethlehemRoadTiles | BethlehemRoadTiles_ | BicycleReflector | BicycleReflector_

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Treehouse-1410/

The 17 Worlds of Planar Ants

Goal. Get you hooked!

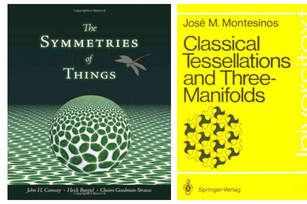
Video, handout, links at ω/

Abstract. Back in early 2000, I got my first digital camera and set out to take pictures of my kids and of symmetric patterns in the plane (ω/Tilings). There are exactly 17 of those, no more, no less. It is an addicting challenge to walk around looking at buildings, brick walls, people's ties, fabrics, what's not, and to try figure out which of the 17 is each one.

- What would history look like if we were living on Venus?
- What do the ants on Lou Kauffman's tie think?

Lou Kauffman's Tie

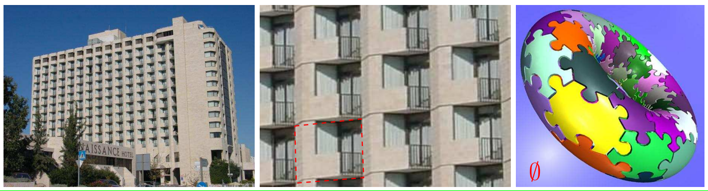
Books.



- J. H. Conway, H. Burgiel, and C. Goodman-Strauss, *The Symmetries of Things*, CRC Press, 2008.
- J. M. Montesinos, *Classical Tessellations and Three-Manifolds*, Springer-Verlag, 1987.

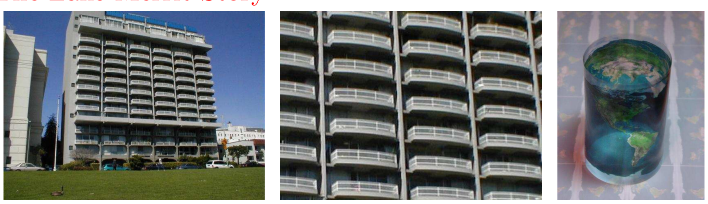
The Renaissance Story

ω/Longtin



The Venus Story

The Lake Merrit Story

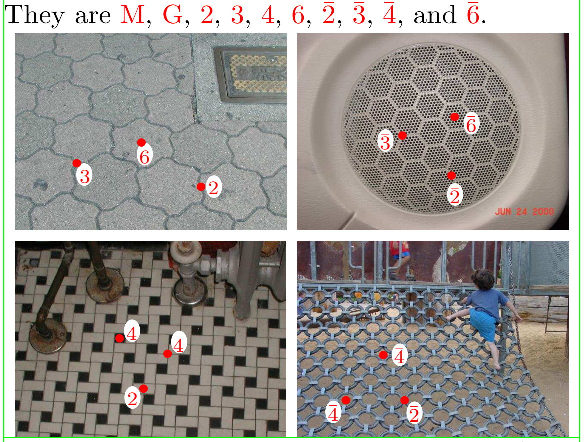


The Racha Cafe Story

M. C. Escher, 1963

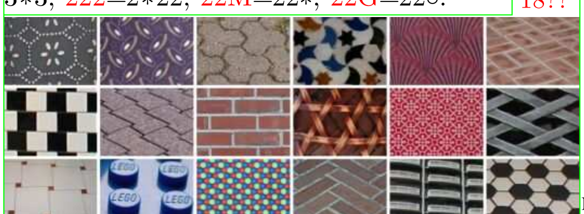
Tie@Fry's

Claim. Exactly 10 "features" are possible. They are M, G, 2, 3, 4, 6, $\bar{2}$, $\bar{3}$, $\bar{4}$, and $\bar{6}$.



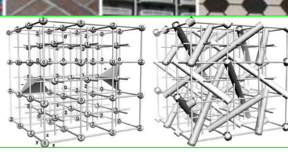
Theorem. There are exactly 17 "tilings" of the plane: $\emptyset$=0, MM=$**$, MG=$*\circ$, GG=$\circ\circ$, 2222=2222, 333=333, 442=442, 632=632, $\bar{2}\bar{2}\bar{2}\bar{2}$=$*2222$, $\bar{3}\bar{3}\bar{3}$=$*333$, $\bar{4}\bar{4}\bar{2}$=$*442$, $\bar{6}\bar{3}\bar{2}$=$*632$, $4\bar{2}$=$4*2$, $3\bar{3}$=$3*3$, $2\bar{2}\bar{2}$=$2*22$, 22M=$22*$, 22G=$22\circ$.

18??



The 230 Worlds of Spacial Monkeys (The 219 worlds of Monkeys that Can't Tell their Left from their Right)

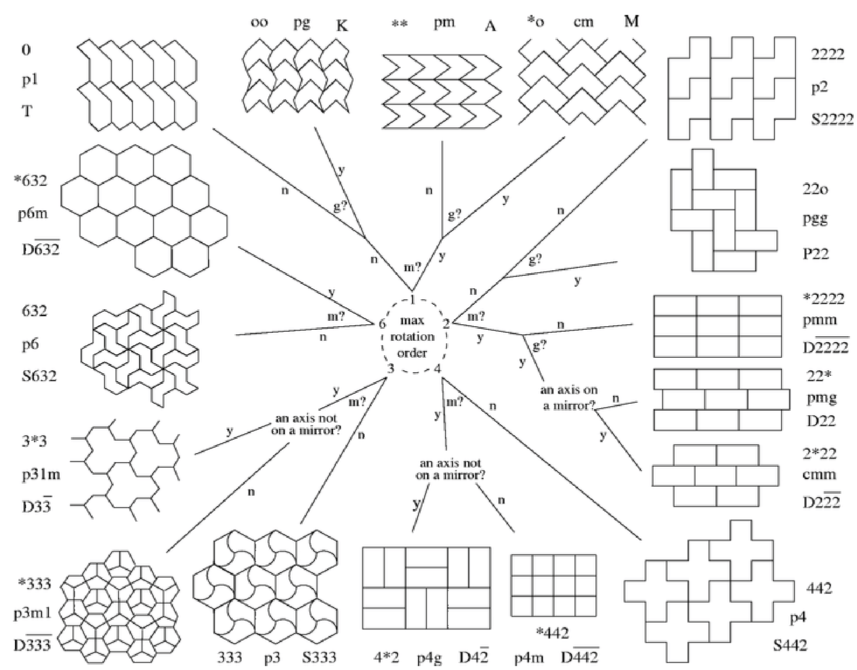
ω/Crys, ω/CFHT



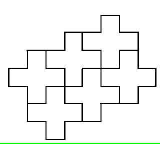
ω/Sanderson

Brian Sanderson's Pattern Recognition Algorithm

Is the maximum rotation order 1,2,3,4 or 6? Is there a mirror (m)? Is there an indecomposable glide reflection (g)? Is there a rotation axis on a mirror? Is there a rotation axis not on a mirror?



Note: Every pattern is identified according to three systems of notation, as in the example below:



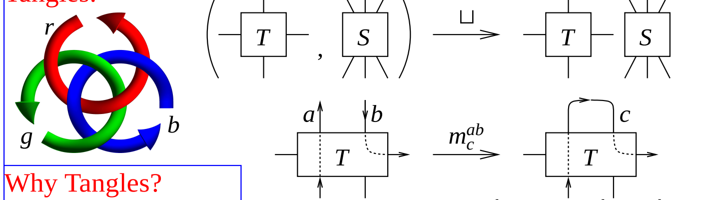
442: The Conway-Thurston notation, as used in my tilings page.
p4: The International Union of Crystallography notation.
S442: The Montesinos notation, as in his book Classical Tesselations and Three Manifolds

Video and more at http://www.math.toronto.edu/~drorbn/Talks/ClassroomAdventures-1408/

Some very good formulas for the Alexander polynomial, 1

Abstract. I will describe some very good formulas for a *(matrix plus scalar)*-valued extension of the Alexander polynomial to tangles, then say that everything extends to virtual tangles, then roughly to simply knotted balloons and hoops in 4D, then the target space extends to *(free Lie algebras plus cyclic words)*, and the result is a universal finite type of the knotted objects in its domain. Taking a cue from the BF topological quantum field theory, everything should extend (with some modifications) to arbitrary codimension-2 knots in arbitrary dimension and in particular, to arbitrary 2-knots in 4D. But what is really going on is still a mystery.

Tangles.



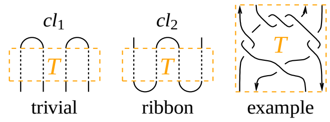
(meta-associativity: $m_a^{ab}/\!/m_a^{ac} = m_b^{bc}/\!/m_a^{ab}$)

Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon,

$Z(K) \in \{cl_2(Z) \colon cl_1(Z) = 1\}$.

(Genus and crossing number are also definable properties).



Theorem 1.

$\exists!$ an invariant γ: {pure framed S-component tangles} $\to R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

1. $$\left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

2. $$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\beta]{m_c^{ab}} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu\end{array}\right)_{T_a,T_b\to T_c},$$

and satisfying $\left(|_a;\ {}_a\nearrow\!\!\!\nwarrow_b,\ {}_b\nwarrow\!\!\!\nearrow_a\right) \xrightarrow{\gamma} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right)$.

In Addition,

- This is really "just" a stitching formula for Burau/Gassner [LD, KLW, CT].
- $L \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega\det'(A - I)/(1 - T')$ is the MVA, mod units.
- The "fastest" Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

Implementation key idea:

ωεβ/Demo

$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab}t_ah_b)$

```
ΓCollect[Γ[ω_, λ_]] := Γ[Factor[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)a_ ⧴ a, ∞];
   M = Outer[Factor[∂h#1 t#2 λ] &, S, S];
   M = Prepend[M, t# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h# & /@ S, ω]];
   M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1*ω2, λ1+λ2];
ma_ b_→c_[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂ta,ha λ  ∂ta,hb λ  ∂ta λ;
                          ∂tb,ha λ  ∂tb,hb λ  ∂tb λ;
                          ∂ha λ     ∂hb λ     λ) /. (t | h)a|b → 0;
  Γ[(μ = 1 - β) ω, {tc, 1}.(γ + α δ/μ  ϵ + δ θ/μ; ϕ + α ψ/μ  Ξ + ψ θ/μ).{hc, 1}]
   /. {Ta → Tc, Tb → Tc} // ΓCollect];
Rpa_ b_ := Γ[1, {ta, tb}.(1  1 - Ta; 0  Ta).{ha, hb}];
Rma_ b_ := Rpab /. Ta → 1/Ta;
```

Meta-Associativity

Runs.

```
γ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];
(γ // m12→1 // m13→1) == (γ // m23→2 // m12→1)
```

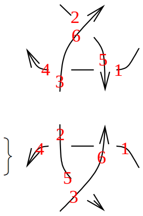
```
True
```

R3

. . . divide and conquer!

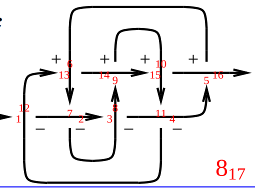
```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$




```
γ = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[γ = γ // m1k→1, {k, 2, 16}];
γ
```

$$\begin{pmatrix}-\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^3} & h_1\\ t_1 & 1\end{pmatrix}$$

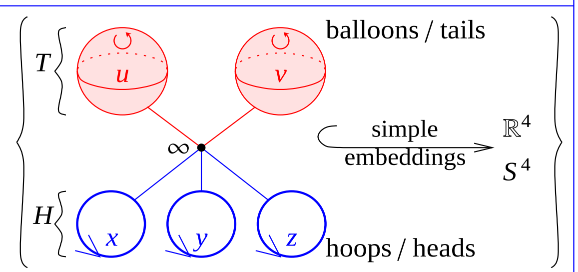


Weaknesses,

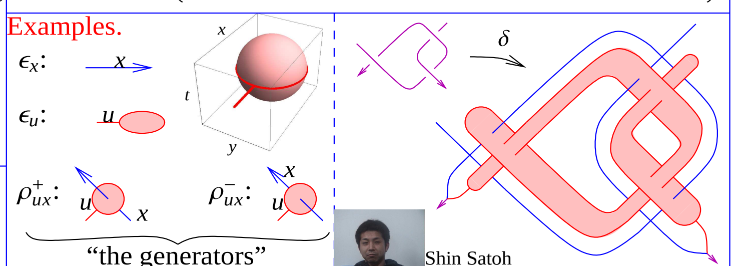
- m_c^{ab} is non-linear.
- The product ωA is always Laurent, but proving this takes induction with exponentially many conditions.

$\mathcal{K}^{bh}(H;T)$.

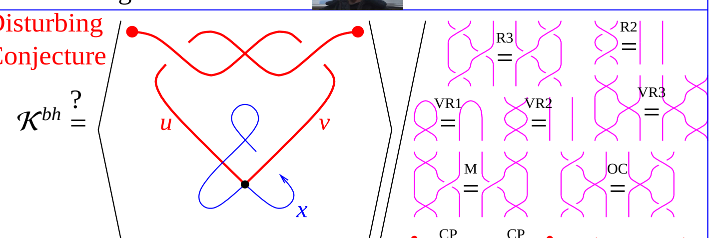
"Simply-knotted balloons and hoops"



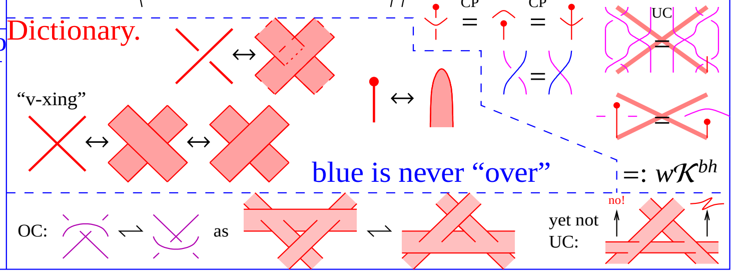
Examples.



Disturbing Conjecture

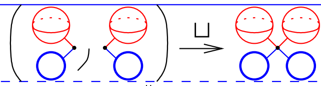


Dictionary.



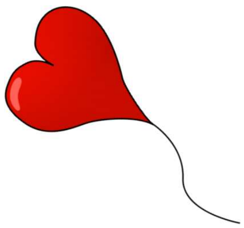
Some very good formulas for the Alexander polynomial, 2

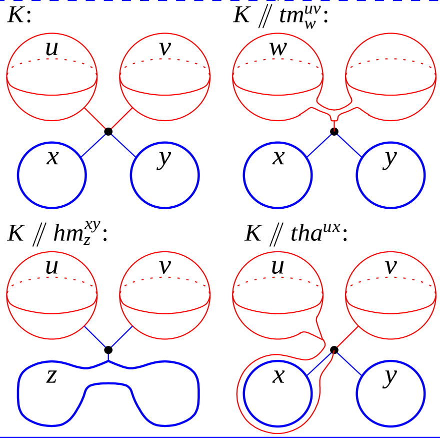
Operations
Punctures & Cuts | Connected Sums.



If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

Proposition. The generators generate.





Definition. l_{xu} is the linking number of hoop x with balloon u. For $x \in H$, $\sigma_x := \prod_{u\in T} T_u^{l_{xu}} \in R = R_T = \mathbb{Z}((T_a)_{a\in T})$, the ring of rational functions in T variables.

Theorem 2 [BNS]. $\exists!$ an invariant $\beta\colon w\mathcal{K}^{bh}(H;T) \to R \times M_{T\times H}(R)$, intertwining

1. $$\left(\begin{array}{c|c}\omega_1 & H_1\\\hline T_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & H_2\\\hline T_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\\hline T_1 & A_1 & 0\\ T_2 & 0 & A_2\end{array},$$

2. $$\begin{array}{c|c}\omega & H\\\hline u & \alpha\\ v & \beta\\ T & \Xi\end{array} \xrightarrow{tm^{uv}_w} \left(\begin{array}{c|c}\omega & H\\\hline w & \alpha+\beta\\ T & \Xi\end{array}\right)_{T_u,T_v\to T_w},$$

3. $$\begin{array}{c|ccc}\omega & x & y & H\\\hline T & \alpha & \beta & \Xi\end{array} \xrightarrow{hm^{xy}_z} \begin{array}{c|cc}\omega & z & H\\\hline T & \alpha+\sigma_x\beta & \Xi\end{array},$$

4. $$\begin{array}{c|cc}\omega & x & H\\\hline u & \alpha & \theta\\ T & \phi & \Xi\end{array} \xrightarrow[\nu:=1+\alpha]{tha^{ux}} \begin{array}{c|cc}\nu\omega & x & H\\\hline u & \sigma_x\alpha/\nu & \sigma_x\theta/\nu\\ T & \phi/\nu & \Xi-\phi\theta/\nu\end{array},$$

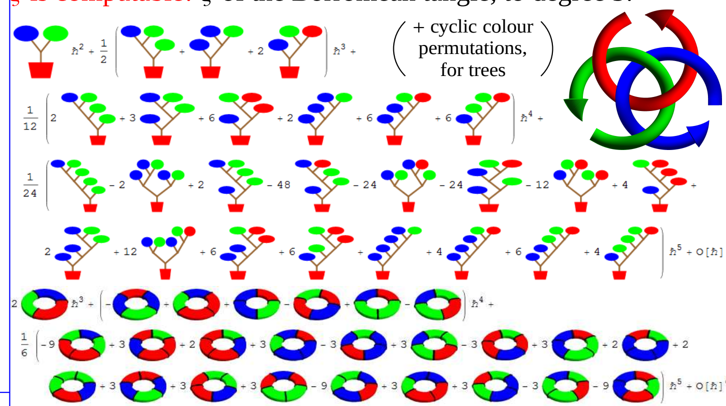
and satisfying $(\epsilon_x;\epsilon_u;\rho^{\pm}_{ux}) \xrightarrow{\beta} \left(\begin{array}{c|c}1 & x\\\hline & \end{array}; \begin{array}{c|c}1 & \\\hline u & \end{array}; \begin{array}{c|c}1 & x\\\hline u & T_u^{\pm1}-1\end{array}\right).$

Proposition. If T is a u-tangle and $\beta(\delta T) = (\omega, A)$, then $\gamma(T) = (\omega, \sigma - A)$, where $\sigma = \mathrm{diag}(\sigma_a)_{a\in S}$. Under this, $m^{ab}_c \leftrightarrow tha^{ab}/\!\!/tm^{ab}_c/\!\!/hm^{ab}_c$.

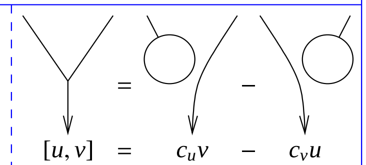
References.
[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.
[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.
[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.
[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.
[CT] D. Cimasoni and V. Turaev, *A Lagrangian Representation of Tangles*, Topology **44** (2005) 747–767, arXiv:math.GT/0406269.
[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.
[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

Theorem 3 [BND, BN]. $\exists!$ a homomorphic expansion, aka a homomorphic universal finite type invariant Z of w-knotted balloons and hoops. $\zeta := \log Z$ takes values in $FL(T)^H \times CW(T)$.

ζ is computable! ζ of the Borromean tangle, to degree 5:

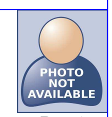


Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, ζ reduces to β and the KBH operations on ζ reduce to the formulas in Theorem 2.



A Big Question. Does it all extend to arbitrary 2-knots (not necessarily "simple")? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,
$$S(A,B) := \int_M \langle B, F_A\rangle.$$

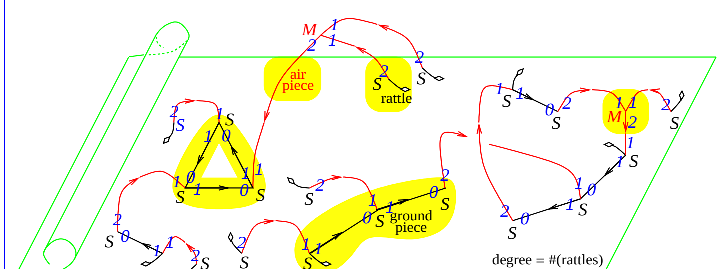


Rossi

With $\kappa\colon (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S,\mathfrak{g})$, $\alpha \in \Omega^1(S,\mathfrak{g}^*)$, set
$$O(A,B,\kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle\beta, d_{\kappa^*A}\alpha + \kappa^*B\rangle\right).$$


Cattaneo

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then
$$Z_{BF} = \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$
(modulo some STU- and IHX-like relations).



Issues. • Signs don't quite work out, and BF seems to reproduce only "half" of the wheels invariant.
• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
• I don't know how to define "finite type" for arbitrary 2-knots.

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

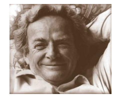
www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Oberwolfach-1405/

Gaussian Integration, Determinants, Feynman Diagrams

Gaussian Integration. (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

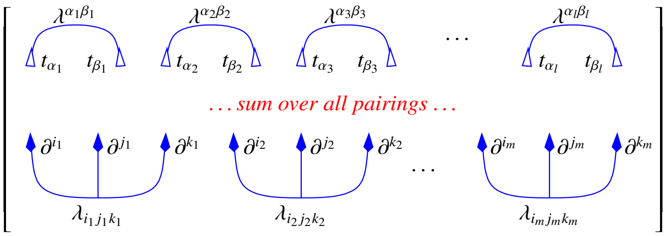
$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k} = \sum_{m\geq 0}\frac{\epsilon^m}{6^m m!}\int_{\mathbb{R}^n}(\lambda_{ijk}x^ix^jx^k)^m e^{-\frac{1}{2}\lambda_{ij}x^ix^j}$$



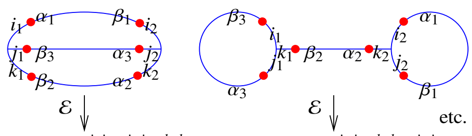
Feynman

$$= \sum_{m\geq 0}\frac{C\epsilon^m}{6^m m!}(\lambda_{ijk}\partial^i\partial^j\partial^k)^m e^{\frac{1}{2}\lambda^{\alpha\beta}t_\alpha t_\beta}\Big|_{t_\alpha=0} = \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}$$



$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}}\mathcal{E}(D)$$



$$= C\sum_{\substack{\text{unmarked Feynman}\\ \text{diagrams } D}}\frac{\epsilon^{m(D)}\mathcal{E}(D)}{|\mathrm{Aut}(D)|}.$$

Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m!2^l l!}{|\mathrm{Aut}(D)|}$.

Proof of the Claim. The group $G_{m,l} := [(S_3)^m \rtimes S_m]\times[(S_2)^l \rtimes S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D, and the stabilizer of any given P is $\mathrm{Aut}(D)$. □

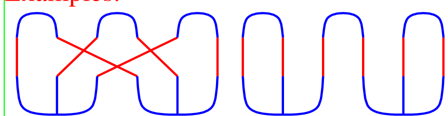
The Fourier Transform.

$$(F\colon V\to\mathbb{C}) \Rightarrow (\tilde{f}\colon V^*\to\mathbb{C})$$

via $\tilde{F}(\varphi) := \int_V f(v)e^{-i\langle\varphi,v\rangle}dv$. Some facts:

- $\tilde{f}(0) = \int_V f(v)dv$.
- $\frac{\partial}{\partial\varphi_i}\tilde{f} \sim \widetilde{v^if}$.
- $\widetilde{(e^{Q/2})} \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v\rangle$ for $L\colon V\to V^*$, and $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$. (This is the key point in the proof of the Fourier inversion formula!)

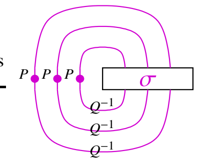
Examples.



$|\mathrm{Aut}(D)| = 12$ $|\mathrm{Aut}(D)| = 8$

Perturbing Determinants. If Q and P are matrices and Q is invertible,

$$|Q|^{-1}|Q+\epsilon P| = |I+\epsilon Q^{-1}P|$$
$$= \sum_{k\geq 0}\epsilon^k\mathrm{tr}\left(\bigwedge^k Q^{-1}P\right)$$
$$= \sum_{k\geq 0,\,\sigma\in S_k}\frac{\epsilon^k(-)^\sigma}{k!}\mathrm{tr}\left(\sigma(Q^{-1}P)^{\otimes k}\right)$$
$$= \sum_{k\geq 0,\,\sigma\in S_k}\frac{(-\epsilon)^k(-)^{\#\text{cycles}}}{k!}$$

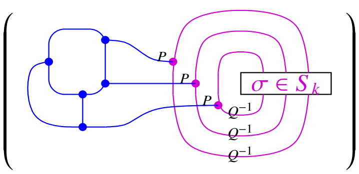


Determinants. Now suppose Q and P_i $(1\leq i\leq n)$ are $d\times d$ matrices and Q is invertible. Then

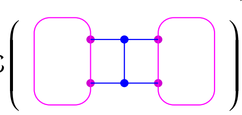
$$|Q|^{-1}I_{\epsilon,\lambda_{ij},\lambda_{ijk},Q,P_i} = |Q|^{-1}\int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k}\det(Q+\epsilon x^iP_i)$$

$$= \sum_{m,k\geq 0,\,\sigma\in S_k}\frac{C\epsilon^{m+k}(-)^\sigma}{6^m m!k!}\int_{\mathbb{R}^n}(\lambda_{ijk}x^ix^jx^k)^m\mathrm{tr}\left(\sigma(x^iQ^{-1}P_i)^{\otimes k}\right)e^{-\frac{1}{2}\lambda_{ij}x^ix^j}$$

$$= \sum_{\substack{\text{fully marked}\\ \text{Feynman diagrams}}}\frac{C\epsilon^{m+k}(-)^\sigma}{6^m m!k!}\mathcal{E}$$



$$= \sum_{\text{Feynman diagrams}} C\epsilon^{m+k}(-)^k(-)^l\mathcal{E}$$



where l is the number of purple ("Fermion") loops.

Ghosts. Or else, introduce "ghosts" $\bar{c}_a$ and c^b, write

$$I_{\epsilon,\lambda_{ij},\lambda_{ijk},Q,P_i} = \int_{\mathbb{R}^n}dx\int_{\mathbb{R}^d\times\mathbb{R}^d}e^{-\frac{1}{2}\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k+\bar{c}_a(Q^a_b+\epsilon x^iP^a_{ib})c^b}$$

and use "ordinary" perturbation theory.

$\bar{c}$ and c

The Berezin Integral (physics / math language, formulas from Wikipedia:Grassmann integral).

Berezin

The *Berezin Integral* is linear on functions of anti-commuting variables, and satisfies $\int\theta d\theta = 1$, and $\int 1 d\theta = 0$, so that $\int\frac{\partial f(\theta)}{\partial\theta}d\theta = 0$.

Let V be a vector space, $\theta\in V$, $d\theta\in V^*$ s.t. $\langle d\theta,\theta\rangle = 1$. Then $f\mapsto\int fd\theta$ is the interior multiplication map $\bigwedge V\to\bigwedge V$: $\int fd\theta := i_{d\theta}(f)\left(=\frac{\partial f}{\partial\theta}\right)$.

Multiple integration via "Fubini": $\int f_1(\theta_1)\cdots f_n(\theta_n)d\theta_1\dots d\theta_n := \left(\int f_1d\theta_1\right)\cdots\left(\int f_nd\theta_n\right)$. $\int fd\theta_1\dots d\theta_n := f /\!/ i_{d\theta_1} /\!/ \cdots /\!/ i_{d\theta_n}$.

Change of variables. If $\theta_i = \theta_i(\xi_j)$, both θ_i and ξ_j are odd, and $J_{ij} := \partial\theta_i/\partial\xi_j$, then

$$\int f(\theta_i)d\theta = \int f(\theta_i(\xi_j))\det(J_{ij})^{-1}d\xi.$$

Given vector spaces V_{θ_i} and W_{ξ_j}, $d\theta = \bigwedge d\theta_i \in \bigwedge^{\mathrm{top}}(V^*)$, $d\xi = \bigwedge d\xi_i \in \bigwedge^{\mathrm{top}}(W^*)$, and $T\colon V\to\bigwedge^{\mathrm{odd}}(W)$. Then T induces a map $T_*\colon\bigwedge V\to\bigwedge W$ and then

$$\int fd\theta = \int(T_*f)\det\left(\frac{\partial(T\theta_i)}{\partial\xi_j}\right)^{-1}d\xi.$$

Gaussian integration. For an even matrix A and odd vectors θ, η,

$$\int e^{\theta^TA\eta}d\theta d\eta = \det(A), \qquad \int e^{\theta^TA\eta+\theta^TJ+K^T\eta}d\theta d\eta = \det(A)e^{-K^TA^{-1}J}.$$

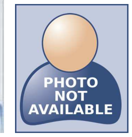
Video and more at http://drorbn.net/?title=AKT-14 (Feb 28 and March 21 classes)

A Partial Reduction of BF Theory to Combinatorics, 1

Abstract. I will describe a semi-rigorous reduction of perturbative BF theory (Cattaneo-Rossi [CR]) to computable combinatorics, in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. Weak this result is, and at least partially already known (Watanabe [Wa]). Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kashiwara-Vergne problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting. (good news in highlight)

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then for a 2-link $(\kappa_t)_{t\in T}$,


Cattaneo



Rossi

$$\zeta = \log \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

is an invariant in $CW(FL(T)) \to CW(T)/\sim$, "symmetrized cyclic words in T".

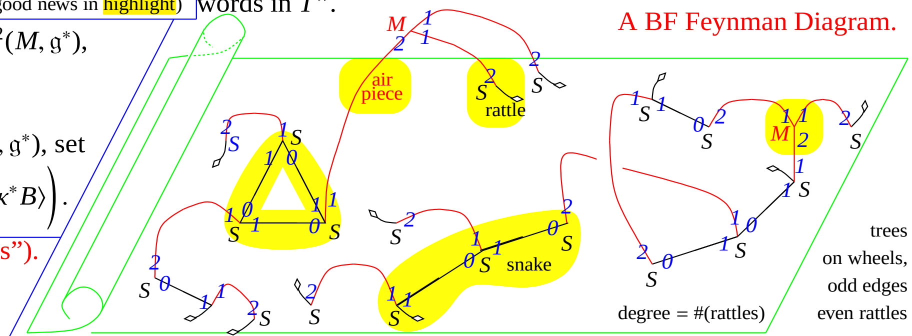
BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A, B) := \int_M \langle B, F_A\rangle.$$

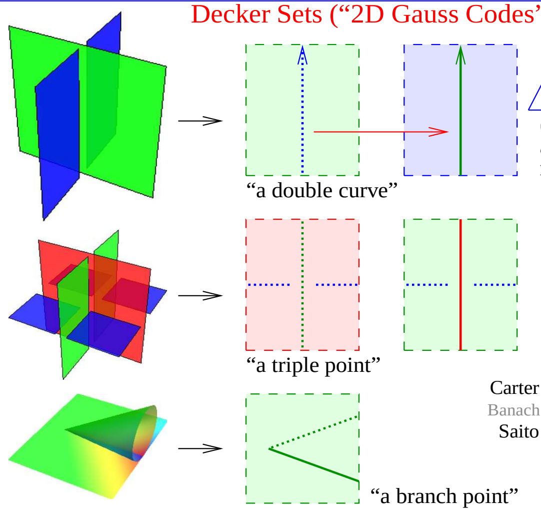
With $\kappa\colon (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

$$O(A, B, \kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle\beta, d_{\kappa^*A}\alpha + \kappa^*B\rangle\right).$$

A BF Feynman Diagram.

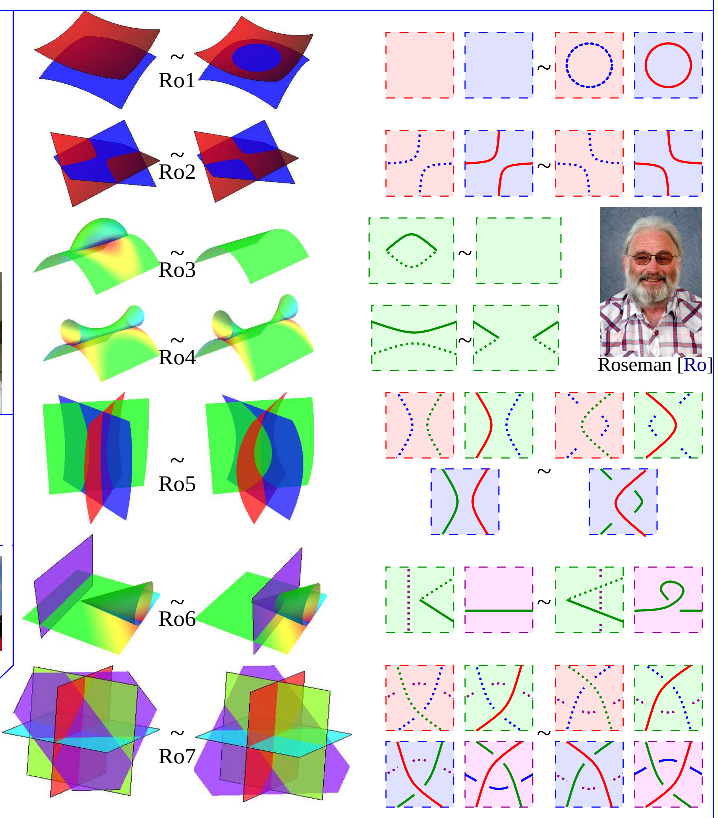


Decker Sets ("2D Gauss Codes").



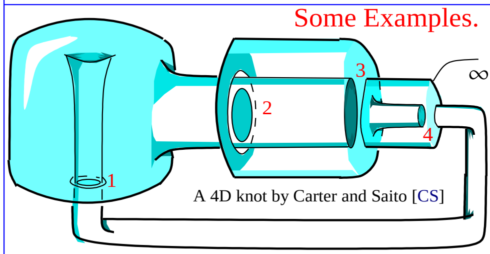
(only double curves are allowed in ribbon 2-knots)


Carter
Banach
Saito

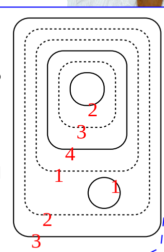


Roseman [Ro]

Some Examples.

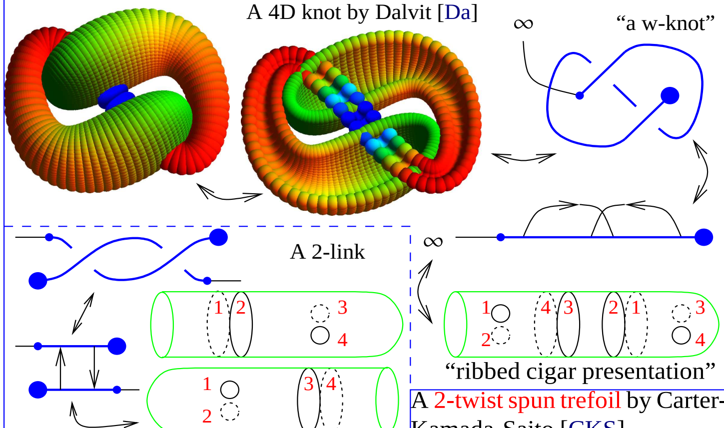


A 4D knot by Carter and Saito [CS]

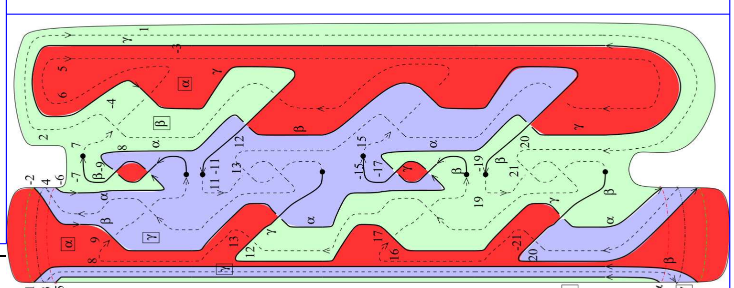



Dalvit

A 4D knot by Dalvit [Da]



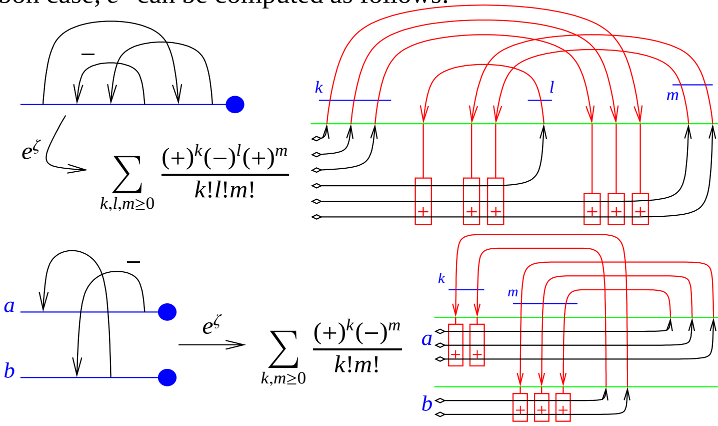
"ribbed cigar presentation"

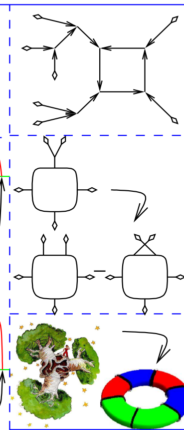

A 2-twist spun trefoil by Carter-Kamada-Saito [CKS].

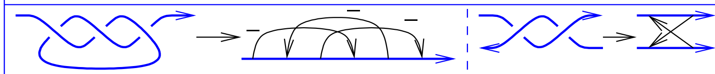
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Vienna-1402/

A Partial Reduction of BF Theory to Combinatorics, 2

Theorem 1 (with Cattaneo, Dalvit (credit, no blame)). In the ribbon case, e^ζ can be computed as follows:

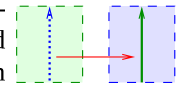


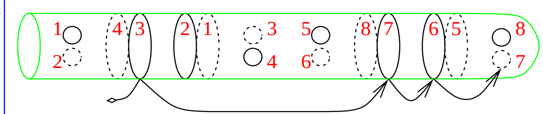




Theorem 2. Using Gauss diagrams to represent knots and T-component pure tangles, the above formulas define an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".
• Agrees with BN-Dancso [BND] and with [BN2]. • In-practice computable! • Vanishes on braids. • Extends to w. • Contains Alexander. • The "missing factor" in Levine's factorization [Le] (the rest of [Le] also fits, hence contains the MVA). • Related to / extends Farber's [Fa]? • Should be summed and categorified.

Sketch of Proof. In $4D$ axial gauge, only "drop down" red propagators, hence in the ribbon case, no M-trivalent vertices. S integrals are ± 1 iff "ground pieces" run on nested curves as below, and exponentials arise when several propagators compete for the same double curve. And then the combinatorics is obvious...

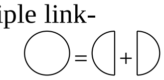


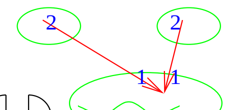


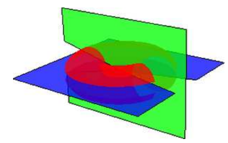
Musings

Chern-Simons. When the domain of BF is restricted to ribbon knots, and the target of Chern-Simons is restricted to trees and wheels, they agree. Why?

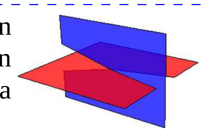
Is this all? What about the ∨-invariant? (the "true" triple linking number)



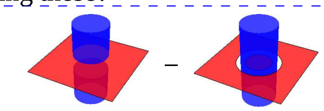




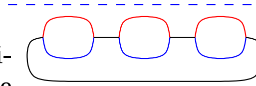
Gnots. In 3D, a generic immersion of S^1 is an embedding, a knot. In 4D, a generic immersion of a surface has finitely-many double points (a gnot?). Perhaps we should be studying these?



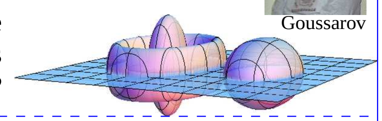
Finite type. What are finite-type invariants for 2-knots? What would be "chord diagrams"?



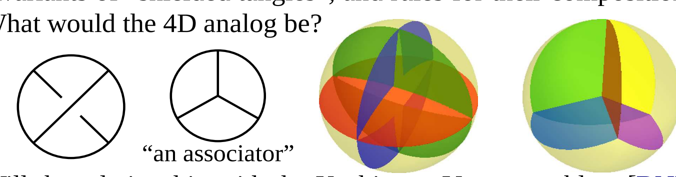
Bubble-wrap-finite-type. There's an alternative definition of finite type in 3D, due to Goussarov (see [BN1]). The obvious parallel in 4D involves "bubble wraps". Is it any good?




Goussarov

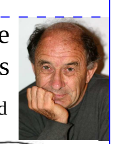


Shielded tangles. In 3D, one can't zoom in and compute "the Chern-Simons invariant of a tangle". Yet there are well-defined invariants of "shielded tangles", and rules for their compositions. What would the 4D analog be?

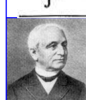

"an associator"

Will the relationship with the Kashiwara-Vergne problem [BND] necessarily arise here?

Plane curves. Shouldn't we understand integral / finite type invariants of plane curves, in the style of Arnold's J^+, J^-, and St [Ar], a bit better?


Arnold

	a(⋇)	a(⤫)	a(⤫)	∞	○	◎			⋯
St	1	0	0	0	0	1	2	3	⋯
J^+	0	2	0	0	0	−2	−4	−6	⋯
J^-	0	0	−2	−1	0	−3	−6	−9	⋯



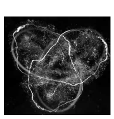
"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

References.
[Ar] V. I. Arnold, *Topological Invariants of Plane Curves and Caustics*, University Lecture Series **5**, American Mathematical Society 1994.
[BN1] D. Bar-Natan, *Bracelets and the Goussarov filtration of the space of knots, Invariants of knots and 3-manifolds (Kyoto 2001)*, Geometry and Topology Monographs **4** 1–12, arXiv:math.GT/0111267.
[BN2] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, http://www.math.toronto.edu/~drorbn/papers/KBH/, arXiv:1308.1721.
[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne*, http://www.math.toronto.edu/~drorbn/papers/WKO/.
[CKS] J. S. Carter, S. Kamada, and M. Saito, *Diagrammatic Computations for Quandles and Cocycle Knot Invariants*, Contemp. Math. **318** (2003) 51–74.
[CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Mathematical Surveys and Monographs **55**, American Mathematical Society, Providence 1998.
[Da] E. Dalvit, http://science.unitn.it/~dalvit/.
[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.
[Fa] M. Farber, *Noncommutative Rational Functions and Boundary Links*, Math. Ann. **293** (1992) 543–568.
[Le] J. Levine, *A Factorization of the Conway Polynomial*, Comment. Math. Helv. **74** (1999) 27–53, arXiv:q-alg/9711007.
[Ro] D. Roseman, *Reidemeister-Type Moves for Surfaces in Four-Dimensional Space*, Knot Theory, Banach Center Publications **42** (1998) 347–380.
[Wa] T. Watanabe, *Configuration Space Integrals for Long n-Knots, the Alexander Polynomial and Knot Space Cohomology*, Alg. and Geom. Top. **7** (2007) 47-92, arXiv:math/0609742.

Continuing Joost Slingerland...


http://youtu.be/YCA0VIExVhge


http://youtu.be/mHyTOcfF99o

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Vienna-1402/

What happens to a quantum particle on a pendulum at $T = \frac{\pi}{2}$?

Abstract. This subject is the best one-hour introduction I know for the mathematical techniques that appear in quantum mechanics — in one short lecture we start with a meaningful question, visit Schrödinger's equation, operators and exponentiation of operators, Fourier analysis, path integrals, the least action principle, and Gaussian integration, and at the end we land with a meaningful and interesting answer.

Based a lecture given by the author in the "trivial notions" seminar in Harvard on April 29, 1989. This edition, January 10, 2014.

1. The Question

Let the complex valued function $\psi = \psi(t, x)$ be a solution of the Schrödinger equation

$$\frac{\partial\psi}{\partial t} = -i\left(-\frac{1}{2}\Delta_x + \frac{1}{2}x^2\right)\psi \qquad \text{with} \qquad \psi|_{t=0} = \psi_0.$$

What is $\psi|_{t=T=\frac{\pi}{2}}$?

In fact, the major part of our discussion will work just as well for the general Schrödinger equation,

$$\frac{\partial\psi}{\partial t} = -iH\psi, \qquad H = -\frac{1}{2}\Delta_x + V(x),$$

$$\psi|_{t=0} = \psi_0, \qquad \text{arbitrary } T,$$

where,

- ψ is the "wave function", with $|\psi(t, x)|^2$ representing the probability of finding our particle at time t in position x.
- H is the "energy", or the "Hamiltonian".
- $-\frac{1}{2}\Delta_x$ is the "kinetic energy".
- $V(x)$ is the "potential energy at x".

2. The Solution

The equation $\frac{\partial\psi}{\partial t} = -iH\psi$ with $\psi|_{t=0} = \psi_0$ formally implies

$$\psi(T, x) = \left(e^{-iTH}\psi_0\right)(x) = \left(e^{i\frac{T}{2}\Delta - iTV}\psi_0\right)(x).$$

By Lemma 3.1 with $n = 10^{58}+17$ and setting $x_n = x$ we find that $\psi(T, x)$ is

$$\left(e^{i\frac{T}{2n}\Delta}e^{-i\frac{T}{n}V}e^{i\frac{T}{2n}\Delta}e^{-i\frac{T}{n}V}\dots e^{i\frac{T}{2n}\Delta}e^{-i\frac{T}{n}V}\psi_0\right)(x_n).$$

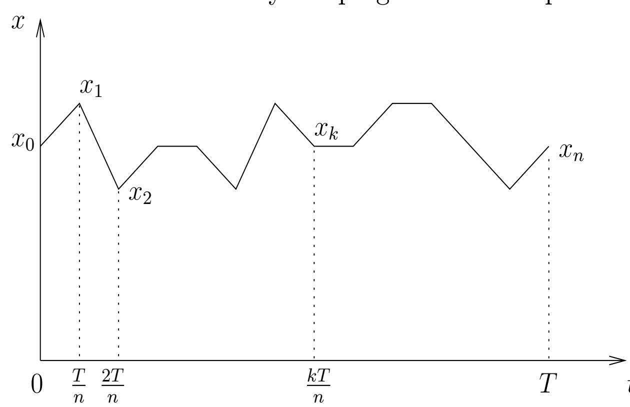
Now using Lemmas 3.2 and 3.3 we find that this is: (c denotes the ever-changing universal fixed numerical constant)

$$c\int dx_{n-1}e^{i\frac{(x_n-x_{n-1})^2}{2T/n}}e^{-i\frac{T}{N}V(x_{n-1})}\dots$$
$$\int dx_1 e^{i\frac{(x_2-x_1)^2}{2T/n}}e^{-i\frac{T}{N}V(x_1)}$$
$$\int dx_0 e^{i\frac{(x_1-x_0)^2}{2T/n}}e^{-i\frac{T}{N}V(x_0)}\psi_0(x_0).$$

Repackaging, we get

$$c\int dx_0\dots dx_{n-1}$$
$$\exp\left(i\frac{T}{2n}\sum_{k=1}^{n}\left(\frac{x_k-x_{k-1}}{T/n}\right)^2 - i\frac{T}{n}\sum_{k=0}^{n-1}V(x_k)\right)$$
$$\psi_0(x_0).$$

Now comes the novelty. keeping in mind the picture



and replacing Riemann sums by integrals, we can write

$$\psi(T, x) = c\int dx_0 \int_{W_{x_0x_n}} \mathcal{D}x$$
$$\exp\left(i\int_0^T dt\left(\frac{1}{2}\dot{x}^2(t) - V(x(t))\right)\right)\psi_0(x_0),$$

where $W_{x_0x_n}$ denotes the space of paths that begin at x_0 and end at x_n,

$$W_{x_0x_n} = \left\{x : [0, T] \to \mathbb{R} :\ x(0) = x_0,\ x(T) = x_n\right\},$$

and $\mathcal{D}x$ is the formal "path integral measure".

This is a good time to introduce the "action" $\mathcal{L}$:

$$\mathcal{L}(x) := \int_0^T dt\left(\frac{1}{2}\dot{x}^2(t) - V(x(t))\right).$$

With this notation,

$$\psi(T, x) = c\int dx_0\psi_0(x_0)\int_{W_{x_0x_n}}\mathcal{D}x e^{i\mathcal{L}(x)}.$$

Let x_c denote the path on which $\mathcal{L}(x)$ attains its minimum value, write $x = x_c + x_q$ with $x_q \in W_{00}$, and get

$$\psi(T,x) = c\int dx_0\psi_0(x_0)\int_{W_{00}} \mathcal{D}x_q e^{i\mathcal{L}(x_c+x_q)}.$$

In our particular case $\mathcal{L}$ is quadratic in x, and therefore $\mathcal{L}(x_c+x_q) = \mathcal{L}(x_c) + \mathcal{L}(x_q)$ (this uses the fact that x_c is an extremal of $\mathcal{L}$, of course). Plugging this into what we already have, we get

$$\begin{aligned}\psi(T,x) &= c\int dx_0\psi_0(x_0)\int_{W_{00}} \mathcal{D}x_q e^{i\mathcal{L}(x_c)+i\mathcal{L}(x_q)}\\ &= c\int dx_0\psi_0(x_0)e^{i\mathcal{L}(x_c)}\int_{W_{00}} \mathcal{D}x_q e^{i\mathcal{L}(x_q)}.\end{aligned}$$

Now this is excellent news, because the remaining path integral over W_{00} does not depend on x_0 or x_n, and hence it is a constant! Allowing c to change its value from line to line, we get

$$\psi(T,x) = c\int dx_0\psi_0(x_0)e^{i\mathcal{L}(x_c)}.$$

Lemma 3.4 now shows us that $x_c(t) = x_0\cos t + x_n \sin t$. An easy explicit computation gives $\mathcal{L}(x_c) = -x_0x_n$, and we arrive at our final result,

$$\psi(\frac{\pi}{2},x) = c\int dx_0\psi_0(x_0)e^{-ix_0x_n}.$$

Notice that this is precisely the formula for the Fourier transform of ψ_0! That is, the answer to the question in the title of this document is "the particle gets Fourier transformed", whatever that may mean.

3. The Lemmas

Lemma 3.1. *For any two matrices A and B,*

$$e^{A+B} = \lim_{n\to\infty}\left(e^{A/n}e^{B/n}\right)^n.$$

Proof. (sketch) Using Taylor expansions, we see that $e^{\frac{A+B}{n}}$ and $e^{A/n}e^{B/n}$ differ by terms at most proportional to c/n^2. Raising to the nth power, the two sides differ by at most $O(1/n)$, and thus

$$e^{A+B} = \lim_{n\to\infty}\left(e^{\frac{A+B}{n}}\right)^n = \lim_{n\to\infty}\left(e^{A/n}e^{B/n}\right)^n,$$

as required. □

Lemma 3.2.

$$\left(e^{itV}\psi_0\right)(x) = e^{itV(x)}\psi_0(x).$$

□

Lemma 3.3.

$$\left(e^{i\frac{t}{2}\Delta}\psi_0\right)(x) = c\int dx' e^{i\frac{(x-x')^2}{2t}}\psi_0(x').$$

Proof. In fact, the left hand side of this equality is just a solution $\psi(t,x)$ of Schrödinger's equation with $V=0$:

$$\frac{\partial\psi}{\partial t} = \frac{i}{2}\Delta_x\psi, \qquad \psi|_{t=0} = \psi_0.$$

Taking the Fourier transform $\tilde\psi(t,p) = \frac{1}{\sqrt{2\pi}}\int e^{-ipx}\psi(t,x)dx$, we get the equation

$$\frac{\partial\tilde\psi}{\partial t} = -i\frac{p^2}{2}\tilde\psi, \qquad \tilde\psi|_{t=0} = \tilde\psi_0.$$

For a fixed p, this is a simple first order linear differential equation with respect to t, and thus,

$$\tilde\psi(t,p) = e^{-i\frac{tp^2}{2}}\tilde\psi_0(p).$$

Taking the inverse Fourier transform, which takes products to convolutions and Gaussians to other Gaussians, we get what we wanted to prove. □

Lemma 3.4. *With the notation of Section 2 and at the specific case of $V(x) = \frac{1}{2}x^2$ and $T = \frac{\pi}{2}$, we have*

$$x_c(t) = x_0\cos t + x_n\sin t.$$

Proof. If x_c is a critical point of $\mathcal{L}$ on $W_{x_0x_n}$, then for any $x_q \in W_{00}$ there should be no term in $\mathcal{L}(x_c+\epsilon x_q)$ which is linear in ϵ. Now recall that

$$\mathcal{L}(x) = \int_0^T dt\left(\frac{1}{2}\dot x^2(t) - V(x(t))\right),$$

so using $V(x_c+\epsilon x_q) \sim V(x_c) + \epsilon x_q V'(x_c)$ we find that the linear term in ϵ in $\mathcal{L}(x_c+\epsilon x_q)$ is

$$\int_0^T dt\left(\dot x_c\dot x_q - V'(x_c)x_q\right).$$

Integrating by parts and using $x_q(0) = x_q(T) = 0$, this becomes

$$\int_0^T dt\left(-\ddot x_c - V'(x_c)\right)x_q.$$

For this integral to vanish independently of x_q, we must have $-\ddot x_c - V'(x_c) \equiv 0$, or

$$\ddot x_c = -V'(x_c).$$

(This is the famous $F = ma$ of Newton's, and we have just rediscovered the principle of least action!)

In our particular case this boils down to the equation

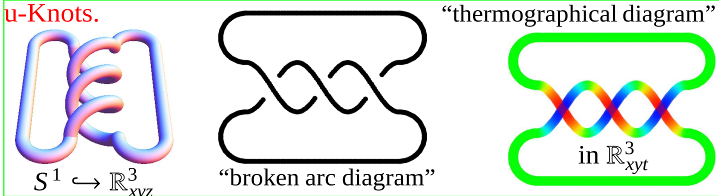
$$\ddot x_c = -x_c, \qquad x_c(0) = x_0, \qquad x_c(\pi/2) = x_n,$$

whose unique solution is displayed in the statement of this lemma. □

Knots in Four Dimensions and the Simplest Open Problem About Them

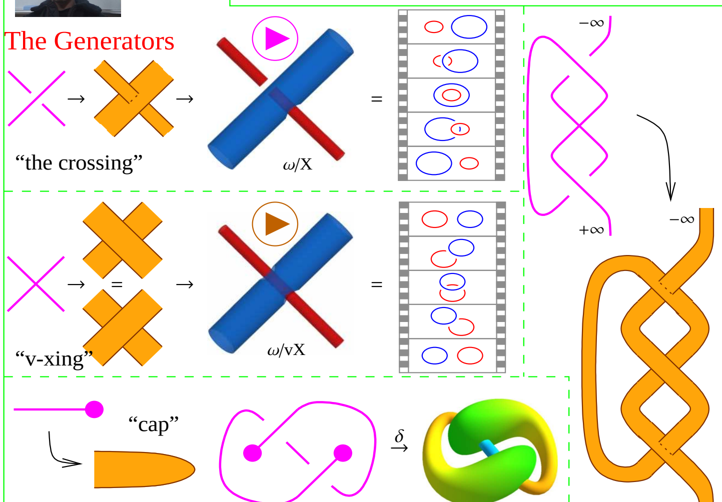
Abstract. I will describe a few 2-dimensional knots in 4 dimensional space in detail, then tell you how to make many more, then tell you that I don't really understand my way of making them, yet I can tell at least some of them apart in a colourful way.

u-Knots.



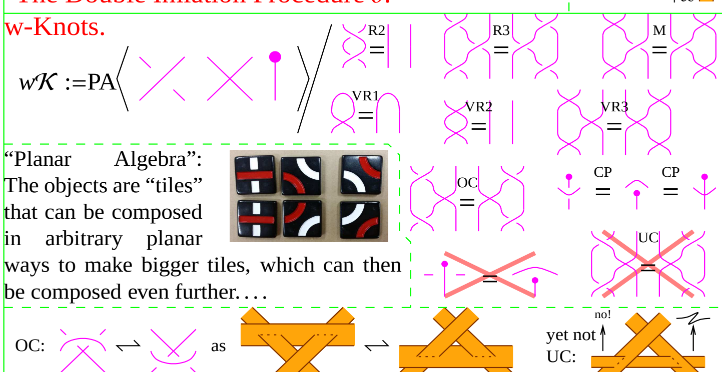
2-Knots.

The Generators



The Double Inflation Procedure δ.

w-Knots.



"Planar Algebra": The objects are "tiles" that can be composed in arbitrary planar ways to make bigger tiles, which can then be composed even further....

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

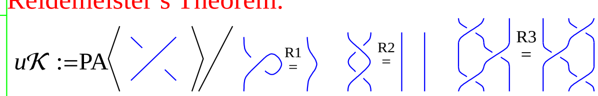
Satoh's Conjecture.

(ω/Sat) The "kernel" of the double inflation map δ, mapping w-knot diagrams in the plane to knotted 2D tubes and spheres in 4D, is precisely the moves R2-3, VR1-3, M, CP and OC listed above. In other words, two w-knot diagrams represent via δ the same 2D knot in 4D iff they differ by a sequence of the said moves.

First Isomorphism Thm: $\delta\colon G \to H \;\Rightarrow\; \operatorname{im}\delta \cong G/\ker(\delta)$

δ is a map from algebra to topology. So a thing in "hard" topology ($\operatorname{im}\delta$) is the same as a thing in "easy" algebra ($w\mathcal{K}$).

Reidemeister's Theorem.

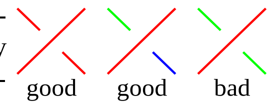


Proof by a genericity / "shaking" argument

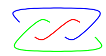
Kurt Reidemeister

3-Colourings.

Colour the arcs of a broken arc diagram in RGB so that every crossing is either mono-chromatic or tri-chromatic; $\lambda(K) := |\{\text{3-colourings}\}|$.

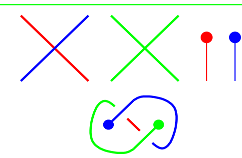


Example. $\lambda(\bigcirc) = 3$ while $\lambda(\text{trefoil}) = 9$; so $\bigcirc \neq$ trefoil.

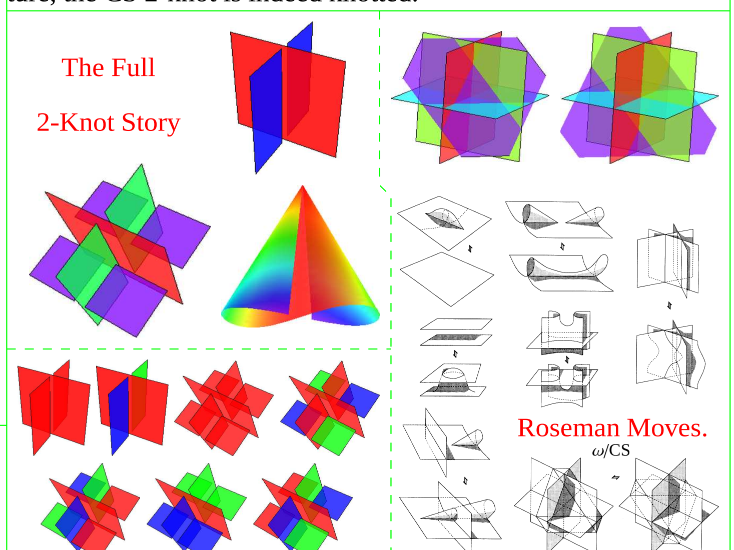


Exercise. Show that the set of colourings of K is a vector space over $\mathbb{F}_3$ hence $\lambda(K)$ is always a power of 3.

Extend λ to $w\mathcal{K}$ by declaring that arcs "don't see" v-xings, and that caps are always "kosher". Then $\lambda(\bullet\!\!-\!\!\bullet) = 3 \neq 9 = \lambda(\text{CS 2-knot})$, so assuming Conjecture, the CS 2-knot is indeed knotted.



The Full 2-Knot Story



Expansions.

Given a "ring" K and an ideal $I \subset K$, set

$$A := I^0/I^1 \oplus I^1/I^2 \oplus I^2/I^3 \oplus \cdots.$$

A homomorphic expansion is a multiplicative $Z\colon K \to A$ such that if $\gamma \in I^m$, then $Z(\gamma) = (0, 0, \ldots, 0, \gamma/I^{m+1}, *, *, \ldots)$.

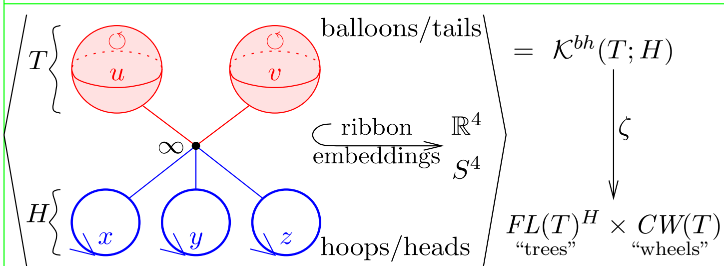
Example. Let $K = C^\infty(\mathbb{R}^n)$ be smooth functions on $\mathbb{R}^n$, and $I := \{f \in K\colon f(0) = 0\}$. Then $I^m = \{f\colon f \text{ vanishes as } |x|^m\}$ and I^m/I^{m+1} is $\{$homogeneous polynomials of degree $m\}$ and A is the set of power series. So Z is "a Taylor expansion".

Hence Taylor expansions are vastly general; even knots can be Taylor expanded!

More at http://www.math.toronto.edu/~drorbn/Talks/HUJI-140101/

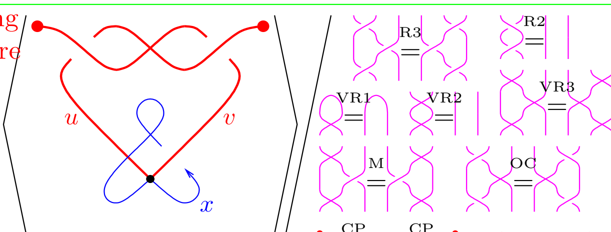
Finite Type Invariants of Ribbon Knotted Balloons and Hoops

Abstract. On my September 17 Geneva talk (ω/sep) I described a certain trees-and-wheels-valued invariant ζ of ribbon knotted loops and 2-spheres in 4-space, and my October 8 Geneva talk (ω/oct) describes its reduction to the Alexander polynomial. Today I will explain how that same invariant arises completely naturally within the theory of finite type invariants of ribbon knotted loops and 2-spheres in 4-space.



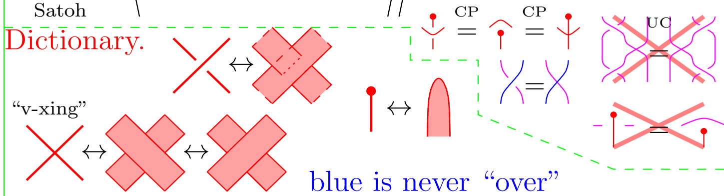
My goal is to tell you why such an invariant is expected, yet not to derive the computable formulas.

Disturbing Conjecture

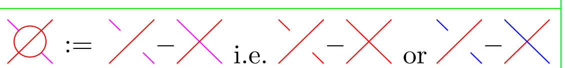


Satoh

Dictionary.



Expansions the semi-virtual



Let $\mathcal{I}^n := \langle$pictures with $\geq n$ semi-virts$\rangle \subset \mathcal{K}^{bh}$.
We seek an "expansion"

$$Z\colon \mathcal{K}^{bh} \to \operatorname{gr}\mathcal{K}^{bh} = \widehat{\bigoplus}\mathcal{I}^n/\mathcal{I}^{n+1} =: \mathcal{A}^{bh}$$

satisfying "property U": if $\gamma \in \mathcal{I}^n$, then

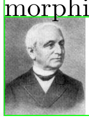
$$Z(\gamma) = (0,\ldots,0,\gamma/\mathcal{I}^{n+1},*,*,\ldots).$$

X.-S. Lin

Why? • Just because, and this is vastly more general.
• $\left(\mathcal{K}^{bh}/\mathcal{I}^{n+1}\right)^\star$ is "finite-type/polynomial invariants".
• The Taylor example: Take $\mathcal{K} = C^\infty(\mathbb{R}^n)$, $\mathcal{I} = \{f \in \mathcal{K}\colon f(0)=0\}$. Then $\mathcal{I}^n = \{f\colon f$ vanishes like $|x|^n\}$ so $\mathcal{I}^n/\mathcal{I}^{n+1}$ is homogeneous polynomials of degree n and Z is a "Taylor expansion"! (So Taylor expansions are vastly more general than you'd think).

Plan. We'll construct a graded $\tilde{\mathcal{A}}^{bh}$, a surjective graded $\pi\colon \tilde{\mathcal{A}}^{bh} \to \mathcal{A}^{bh}$, and a filtered $\tilde{Z}\colon \mathcal{K}^{bh} \to \mathcal{A}^{bh}$ so that $\pi \,/\!\!/\, \operatorname{gr}\tilde{Z} = Id$ (property U: if $\deg D = n$, $\tilde{Z}(\pi(D)) = \pi(D) + (\deg \geq n)$). Hence • π is an isomorphism. • $Z := \tilde{Z} \,/\!\!/\, \pi$ is an expansion.

$$\begin{tikzcd} & \tilde{\mathcal{A}}^{bh} \arrow[d, shift right, "\pi"'] \\ \mathcal{K}^{bh} \arrow[ur, "\tilde{Z}"] \arrow[r, "Z"'] & \mathcal{A}^{bh} \arrow[u, shift right, "\operatorname{gr}\tilde{Z}"'] \end{tikzcd}$$



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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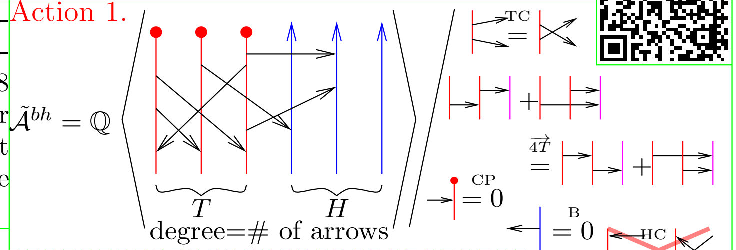
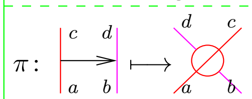
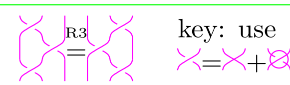
Action 1.



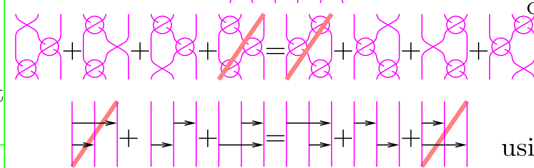


(then connect using xings or v-xings)

Deriving $\overrightarrow{4T}$. Start from



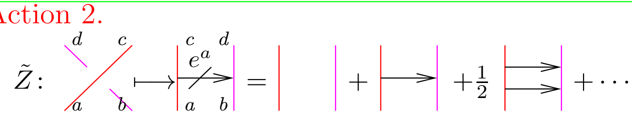
Goussarov-Polyak-Viro



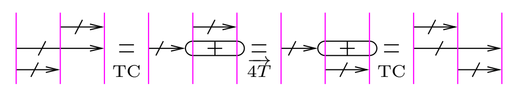
in $\mathcal{I}^n/\mathcal{I}^{n+1}$

using TC $\square$

Action 2.

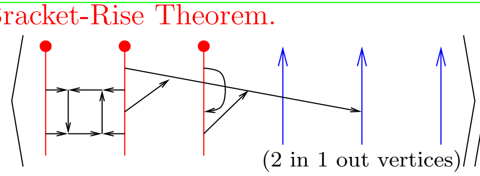


R3.

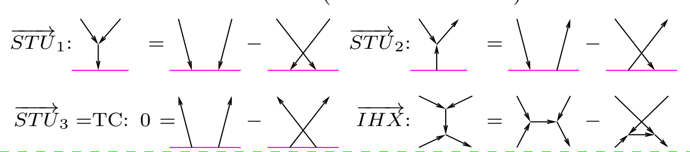


Exercise. Prove property U.

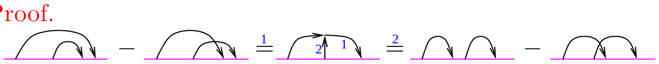
The Bracket-Rise Theorem.



CP, B, $\overrightarrow{STU}$, $\overrightarrow{AS}$, and $\overrightarrow{IHX}$ relations



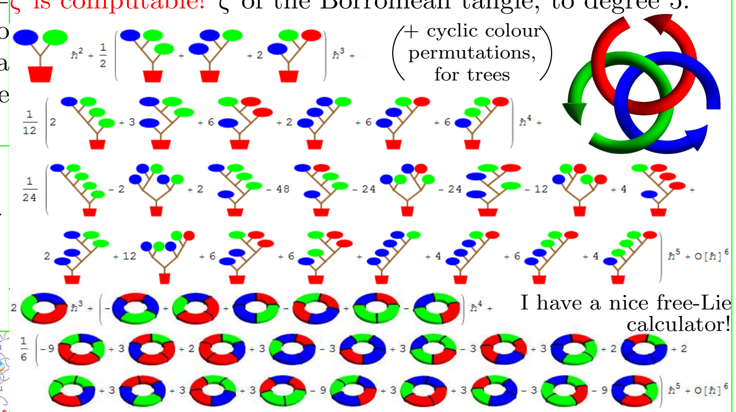
Proof.



Corollaries. (1) Related to Lie algebras! (2) Only trees and wheels persist.

Theorem. $\mathcal{A}^{bh}$ is a bi-algebra. The space of its primitives is $FL(T)^H \times CW(T)$, and $\zeta = \log Z$.

ζ is computable! ζ of the Borromean tangle, to degree 5:



I have a nice free-Lie calculator!

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Geneva-131024/

Trees and Wheels and Balloons and Hoops

Dror Bar-Natan, Zurich, September 2013

ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Zurich-130919

15 Minutes on Algebra

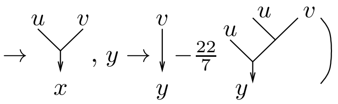
Let T be a finite set of "tail labels" and H a finite set of "head labels". Set

$$M_{1/2}(T;H) := FL(T)^H,$$

"H-labeled lists of elements of the degree-completed free Lie algebra generated by T".

$$FL(T) = \left\{2t_2 - \frac{1}{2}[t_1,[t_1,t_2]] + \dots\right\} \Big/ \left(\begin{array}{c}\text{anti-symmetry}\\ \text{Jacobi}\end{array}\right)$$

... with the obvious bracket.

$M_{1/2}(u,v;x,y) = \Big\{\lambda = \big(x \to$  $\big)\dots\Big\}$

Operations $M_{1/2} \to M_{1/2}$. newspeak!

Tail Multiply tm^{uv}_w is $\lambda \mapsto \lambda /\!\!/ (u,v \to w)$, satisfies "meta-associativity", $tm^{uv}_u /\!\!/ tm^{uw}_u = tm^{vw}_v /\!\!/ tm^{uv}_u$.

Head Multiply hm^{xy}_z is $\lambda \mapsto (\lambda\setminus\{x,y\}) \cup (z \to \mathrm{bch}(\lambda_x,\lambda_y))$, where

$$\mathrm{bch}(\alpha,\beta) := \log(e^\alpha e^\beta) = \alpha+\beta+\frac{[\alpha,\beta]}{2}+\frac{[\alpha,[\alpha,\beta]]+[[\alpha,\beta],\beta]}{12}+\dots$$

satisfies $\mathrm{bch}(\mathrm{bch}(\alpha,\beta),\gamma) = \log(e^\alpha e^\beta e^\gamma) = \mathrm{bch}(\alpha,\mathrm{bch}(\beta,\gamma))$ and hence meta-associativity, $hm^{xy}_x /\!\!/ hm^{xz}_x = hm^{yz}_y /\!\!/ hm^{xy}_x$.

Tail by Head Action tha^{ux} is $\lambda \mapsto \lambda /\!\!/ RC^{\lambda_x}_u$, where $C^{-\gamma}_u\colon FL \to FL$ is the substitution $u \to e^{-\gamma}ue^{\gamma}$, or more precisely,

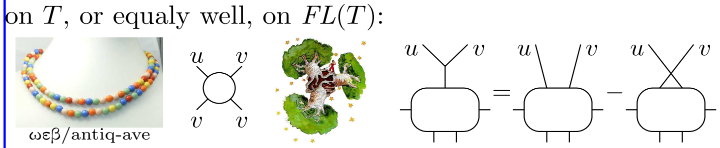
$$C^{-\gamma}_u\colon u \to e^{-\operatorname{ad}\gamma}(u) = u - [\gamma,u] + \frac{1}{2}[\gamma,[\gamma,u]] - \dots,$$

and $RC^\gamma_u = (C^{-\gamma}_u)^{-1}$. Then $C^{\mathrm{bch}(\alpha,\beta)}_u = C^{\alpha/\!\!/RC^{-\beta}_u}_u /\!\!/ C^\beta_u$ hence $RC^{\mathrm{bch}(\alpha,\beta)}_u = RC^\alpha_u /\!\!/ RC^{\beta/\!\!/RC^\alpha_u}_u$ hence "meta $u^{xy} = (u^x)^y$",

$$hm^{xy}_z /\!\!/ tha^{uz} = tha^{ux} /\!\!/ tha^{uy} /\!\!/ hm^{xy}_z,$$

and $tm^{uv}_w /\!\!/ C^{\gamma/\!\!/tm^{uv}_w}_w = C^{\gamma/\!\!/RC^{-\gamma}_v}_u /\!\!/ C^\gamma_v /\!\!/ tm^{uv}_w$ and hence "meta $(uv)^x = u^xv^x$", $tm^{uv}_w /\!\!/ tha^{wx} = tha^{ux} /\!\!/ tha^{vx} /\!\!/ tm^{uv}_w$.

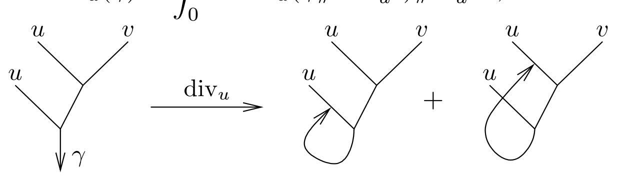
Wheels. Let $M(T;H) := M_{1/2}(T;H) \times CW(T)$, where $CW(T)$ is the (completed graded) vector space of cyclic words on T, or equaly well, on $FL(T)$:



Operations. On $M(T;H)$, define tm^{uv}_w and hm^{xy}_z as before, and tha^{ux} by adding some J-spice:

$$(\lambda;\omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) /\!\!/ RC^{\lambda_x}_u,$$

where $J_u(\gamma) := \int_0^1 ds\ \mathrm{div}_u(\gamma /\!\!/ RC^{s\gamma}_u) /\!\!/ C^{-s\gamma}_u$, and



Alekseev

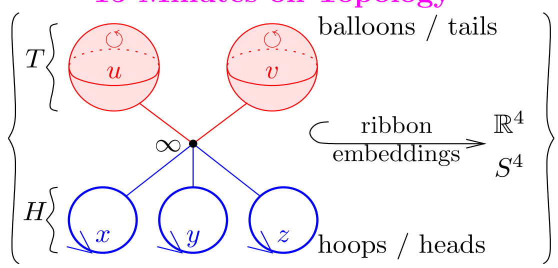
Torossian

Theorem Blue. All blue identities still hold.

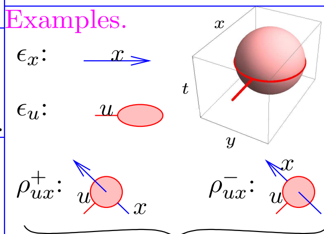
Merge Operation. $(\lambda_1;\omega_1)*(\lambda_2;\omega_2) := (\lambda_1\cup\lambda_2;\omega_1+\omega_2)$.

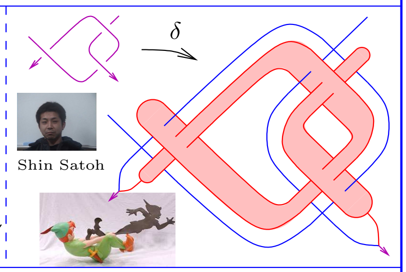
$\mathcal{K}^{bh}(T;H)$. 15 Minutes on Topology

"Ribbon-knotted balloons and hoops"

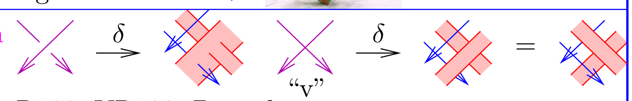


Examples.

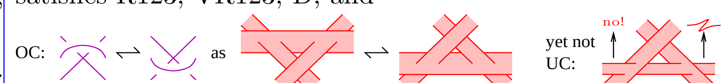




More on δ

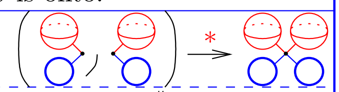


satisfies R123, VR123, D, and



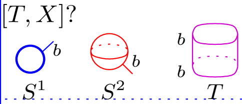
- δ injects u-knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
- δ maps v-tangles to $\mathcal{K}^{bh}$; the kernel contains the above and conjecturally (Satoh), that's all.
- Allowing punctures and cuts, δ is onto.

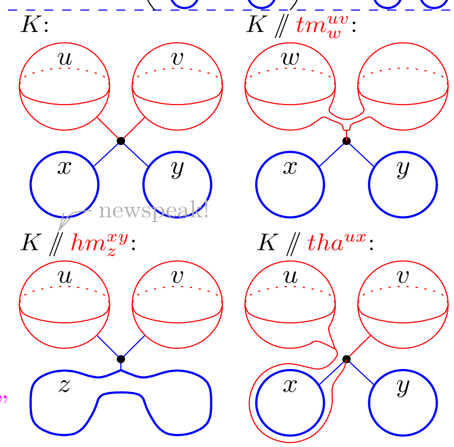
Operations Punctures & Cuts. Connected Sums.



If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

Riddle. People often study $\pi_1(X) = [S^1, X]$ and $\pi_2(X) = [S^2, X]$. Why not $\pi_T(X) := [T, X]$?



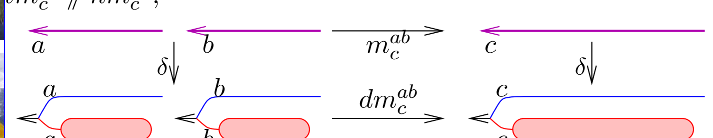


"Meta-Group-Action"

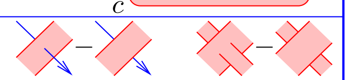
Properties.

- Associativities: $m^{ab}_a /\!\!/ m^{ac}_a = m^{bc}_b /\!\!/ m^{ab}_a$, for $m = tm, hm$.
- "$(uv)^x = u^xv^x$": $tm^{uv}_w /\!\!/ tha^{wx} = tha^{ux} /\!\!/ tha^{vx} /\!\!/ tm^{uv}_w$,
- "$u^{(xy)} = (u^x)^y$": $hm^{xy}_z /\!\!/ tha^{uz} = tha^{ux} /\!\!/ tha^{uy} /\!\!/ hm^{xy}_z$.

Tangle concatenations $\to \pi_1 \ltimes \pi_2$. With $dm^{ab}_c := tha^{ab} /\!\!/ tm^{ab}_c /\!\!/ hm^{ab}_c$,



Finite type invariants make sense in the usual way, and "algebra" is (the primitive part of) "gr" of "topology".

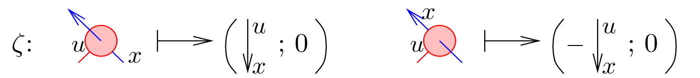


Video and more at http://www.math.toronto.edu/~drorbn/Talks/Geneva-130917/, .../Toronto-1303/ and at .../Chicago-1303/

Trees and Wheels and Balloons and Hoops: Why I Care

Moral. To construct an M-valued invariant ζ of (v-)tangles, and nearly an invariant on $\mathcal{K}^{bh}$, it is enough to declare ζ on the generators, and verify the relations that δ satisfies.

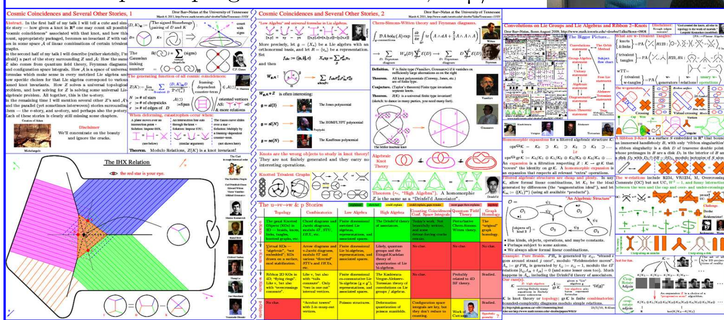
The Invariant ζ. Set $\zeta(\epsilon_x) = (x \to 0; 0)$, $\zeta(\epsilon_u) = ((); 0)$, and



Theorem. ζ is (log of) the unique homomorphic universal finite type invariant on $\mathcal{K}^{bh}$.

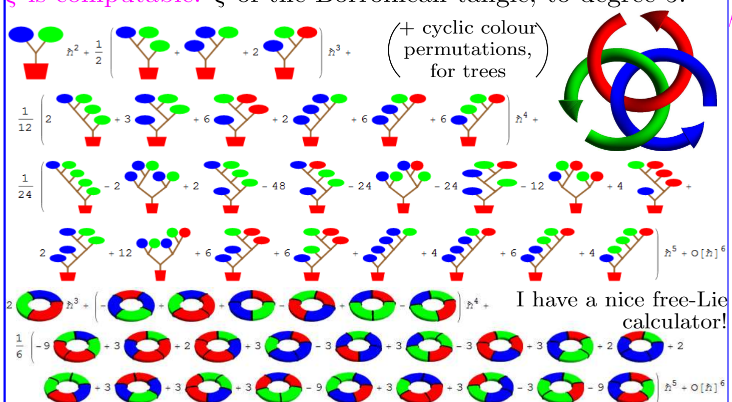
(… and is the tip of an iceberg)

Paper in progress with Dancso, ωεβ/wko



See also ωεβ/tenn, ωεβ/bonn, ωεβ/swiss, ωεβ/portfolio

ζ is computable! ζ of the Borromean tangle, to degree 5:



Tensorial Interpretation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \mathfrak{g})$ and $\tau : CW(T) \to \mathrm{Fun}(\oplus_T \mathfrak{g})$. Together, $\tau : M(T;H) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \oplus_H \mathfrak{g})$, and hence

$$e^\tau : M(T;H) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

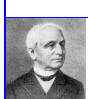
ζ and BF Theory. (See Cattaneo-Rossi, arXiv:math-ph/0210037) Let A denote a $\mathfrak{g}$-connection on S^4 with curvature F_A, and B a $\mathfrak{g}^*$-valued 2-form on S^4. For a hoop γ_x, let $\mathrm{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x. For a ball γ_u, let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be (roughly) the integral of B (transported via A to ∞) on γ_u.

Cattaneo

Loose Conjecture. For $\gamma \in \mathcal{K}(T;H)$,

$$\int \mathcal{D}A\mathcal{D}B e^{\int B \wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B))} \bigotimes_x \mathrm{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

That is, ζ is a complete evaluation of the BF TQFT.

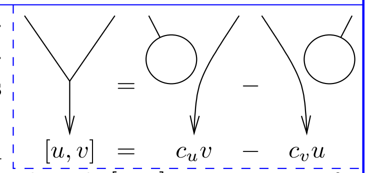


"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

The β quotient is M divided by all relations that universally hold when when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra. Let $R = \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta := R \otimes T$ with central R and with $[u,v] = c_u v - c_v u$ for $u, v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,



$$\mu \to ((\lambda_x); \omega) \quad \text{with } \lambda_x = \sum_{u\in T} \lambda_{ux} ux, \quad \lambda_{ux}, \omega \in R,$$

$$\mathrm{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u+c_v} - 1}\left(\frac{e^{c_u}-1}{c_u}u + e^{c_u}\frac{e^{c_v}-1}{c_v}v\right),$$

if $\gamma = \sum \gamma_v v$ then with $c_\gamma := \sum \gamma_v c_v$,

$$u /\!\!/ RC_u^\gamma = \left(1 + c_u\gamma_u\frac{e^{c_\gamma}-1}{c_\gamma}\right)^{-1}\left(e^{c_\gamma}u - c_u\frac{e^{c_\gamma}-1}{c_\gamma}\sum_{v\neq u}\gamma_v v\right),$$

$\mathrm{div}_u\,\gamma = c_u\gamma_u$, and $J_u(\gamma) = \log\left(1 + \frac{e^{c_\gamma}-1}{c_\gamma}c_u\gamma_u\right)$, so ζ is formula-computable to all orders! Can we simplify?

Repackaging. Given $((x \to \lambda_{ux}); \omega)$, set $c_x := \sum_v c_v\lambda_{vx}$, replace $\lambda_{ux} \to \alpha_{ux} := c_u\lambda_{ux}\frac{e^{c_x}-1}{c_x}$ and $\omega \to e^\omega$, use $t_u = e^{c_u}$, and write α_{ux} as a matrix. Get "β calculus".

β Calculus. Let $\beta(T;H)$ be

$$\left\{\begin{array}{c|ccc|l} \omega & x & y & \cdots & \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot & \omega \text{ and the } \alpha_{ux}\text{'s are rational functions in variables } t_u, \text{ one for each } u \in T. \\ v & \alpha_{vx} & \alpha_{vy} & \cdot & \\ \vdots & \cdot & \cdot & \cdot & \end{array}\right\},$$

With Selmani, ωεβ/meta

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} * \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|cccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha + \beta + \langle\alpha\rangle\beta & \gamma \end{array},$$

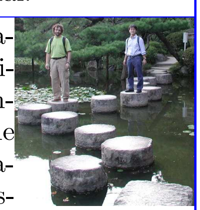
$$tha^{ux} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta - \gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1 + \alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u}\gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

On long knots, ω is the Alexander polynomial!

Why happy? An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination). *If there should be an Alexander invariant with a computable algebraic categorification, it is this one!*



See also ωεβ/regina, ωεβ/caen, ωεβ/newton.

May class: ωεβ/aarhus   Class next year: ωεβ/1350

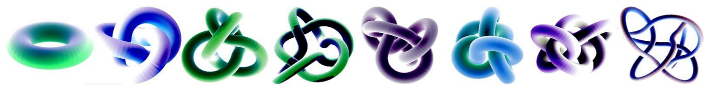
Paper: ωεβ/kbh

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Geneva-130917/, .../Toronto-1303/ and at .../Chicago-1303/

Meta–Groups, Meta–Bicrossed–Products, and the Alexander Polynomial, 1

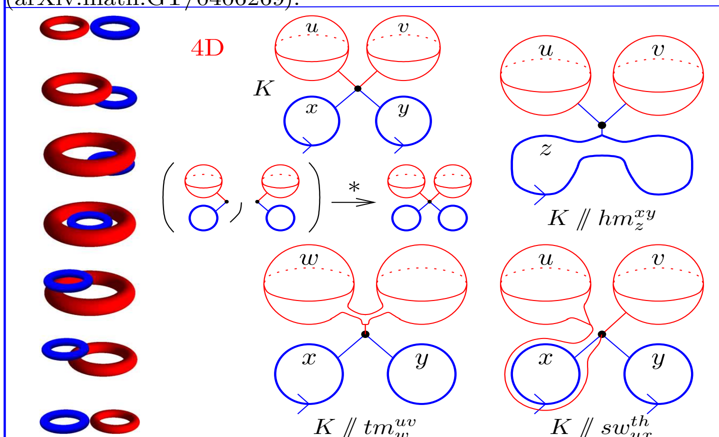
Dror Bar–Natan at Sheffield, February 2013.

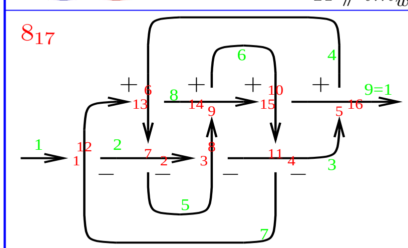
http://www.math.toronto.edu/~drorbn/Talks/Sheffield-130206/

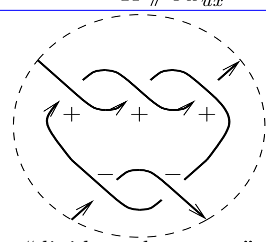


Abstract. I will define "meta-groups" and explain how one specific meta-group, which in itself is a "meta-bicrossed-product", gives rise to an "ultimate Alexander invariant" of tangles, that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, that's a wonderful playground.

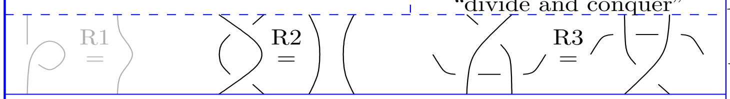
This work is closely related to work by Le Dimet (Comment. Math. Helv. **67** (1992) 306–315), Kirk, Livingston and Wang (arXiv:math/9806035) and Cimasoni and Turaev (arXiv:math.GT/0406269).



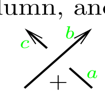




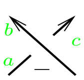
"divide and conquer"



A Standard Alexander Formula. Label the arcs 1 through $(n+1)=1$, make an $n\times n$ matrix as below, delete one row and one column, and compute the determinant:



	a	b	c
c	-1	$1-X$	X



	a	b	c
c	$-X$	$X-1$	1

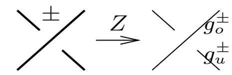
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & X-1 & 0 & -X \\ -1 & X & 0 & 0 & 0 & 0 & 1-X & 0 \\ 0 & -1 & X & 0 & 1-X & 0 & 0 & 0 \\ X-1 & 0 & -X & 1 & 0 & 0 & 0 & 0 \\ 0 & 1-X & 0 & -1 & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -X & 1 & 0 & X-1 \\ 0 & 0 & 1-X & 0 & 0 & -1 & X & 0 \\ 0 & 0 & 0 & X-1 & 0 & 0 & -X & 1 \end{pmatrix}$$

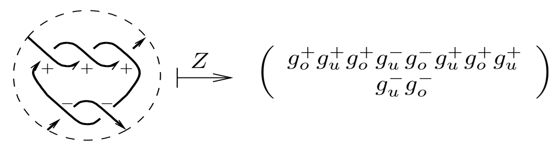
```
[[1 ;; 7, 1 ;; 7]] // Det
```

$-1+4X-8X^2+11X^3-8X^4+4X^5-X^6$

Alexander Issues.
- Quick to compute, but computation departs from topology.
- Extends to tangles, but at an exponential cost.
- Hard to categorify.

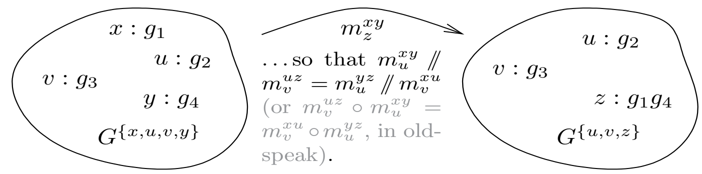
Idea. Given a group G and two "YB" pairs $R^\pm=(g_o^\pm,g_u^\pm)\in G^2$, map them to xings and "multiply along", so that





This Fails! R2 implies that $g_o^\pm g_o^\mp=e=g_u^\pm g_u^\mp$ and then R3 implies that g_o^+ and g_u^+ commute, so the result is a simple counting invariant.

A Group Computer. Given G, can store group elements and perform operations on them:



Also has S_x for inversion, e_x for unit insertion, d_x for register deletion, Δ^z_{xy} for element cloning, ρ^x_y for renamings, and $(D_1,D_2)\mapsto D_1\cup D_2$ for merging, and many obvious composition axioms relating those. $P=\{x:g_1,y:g_2\}\Rightarrow P=\{d_yP\}\cup\{d_xP\}$

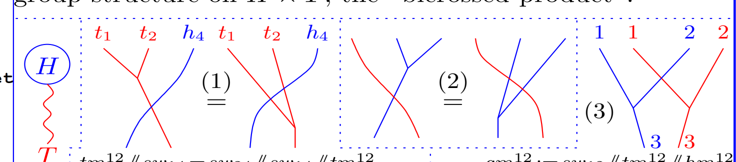
A Meta-Group. Is a similar "computer", only its internal structure is unknown to us. Namely it is a collection of sets $\{G_\gamma\}$ indexed by all finite sets γ, and a collection of operations m^{xy}_z, S_x, e_x, d_x, Δ^z_{xy} (sometimes), ρ^x_y, and $\cup$, satisfying the exact same *linear* properties.

Example 0. The non-meta example, $G_\gamma:=G^\gamma$.

Example 1. $G_\gamma:=M_{\gamma\times\gamma}(\mathbb{Z})$, with simultaneous row and column operations, and "block diagonal" merges. Here if $P=\begin{pmatrix} x: & a & b \\ y: & c & d\end{pmatrix}$ then $d_yP=(x:\ a)$ and $d_xP=(y:\ d)$ so $\{d_yP\}\cup\{d_xP\}=\begin{pmatrix} x: & a & 0 \\ y: & 0 & d\end{pmatrix}\neq P$. So this G is truly meta.

Claim. From a meta-group G and YB elements $R^\pm\in G_2$ we can construct a knot/tangle invariant.

Bicrossed Products. If $G=HT$ is a group presented as a product of two of its subgroups, with $H\cap T=\{e\}$, then also $G=TH$ and G is determined by H, T, and the "swap" map $sw^{th}:(t,h)\mapsto(h',t')$ defined by $th=h't'$. The map sw satisfies (1) and (2) below; conversely, if $sw:T\times H\to H\times T$ satisfies (1) and (2) (+ lesser conditions), then (3) defines a group structure on $H\times T$, the "bicrossed product".



$tm^{12}_1 \,/\!\!/\, sw_{14}=sw_{24}\,/\!\!/\, sw_{14}\,/\!\!/\, tm^{12}_1$

$gm^{12}_3:=sw_{12}\,/\!\!/\, tm^{12}_3\,/\!\!/\, hm^{12}_3$

$h_1t_1h_2t_2=h_1(h_2't_1')t_2=(h_1h_2')(t_1't_2)=h_3t_3$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Newton-1301/, see also .../Sheffield-130206/ and .../Regina-1206/

Meta-Groups, Meta-Bicrossed-Products, and the Alexander Polynomial, 2

A Meta-Bicrossed-Product is a collection of sets $\beta(\eta,\tau)$ and operations tm^{uv}_w, hm^{xy}_z and sw^{th}_{ux} (and lesser ones), such that tm and hm are "associative" and (1) and (2) hold (+ lesser conditions). A meta-bicrossed-product defines a meta-group with $G_\gamma := \beta(\gamma,\gamma)$ and gm as in (3).

Example. Take $\beta(\eta,\tau) = M_{\tau\times\eta}(\mathbb{Z})$ with row operations for the tails, column operations for the heads, and a trivial swap.

β Calculus. Let $\beta(\eta,\tau)$ be

$$\left\{\begin{array}{c|ccc} \omega & h_1 & h_2 & \cdots \\ \hline t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \middle| \begin{array}{l} h_j\in\eta,\ t_i\in\tau, \text{ and } \omega \text{ and} \\ \text{the } \alpha_{ij} \text{ are rational func-} \\ \text{tions in a variable } X \end{array}\right\},$$

$$tm^{uv}_w:\ \begin{array}{c|c} \omega & \cdots \\ \hline t_u & \alpha \\ t_v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline t_w & \alpha+\beta \\ \vdots & \gamma \end{array},\qquad \begin{array}{c|c} \omega_1 & \eta_1 \\ \hline \tau_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & \eta_2 \\ \hline \tau_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & \eta_1 & \eta_2 \\ \hline \tau_1 & \alpha_1 & 0 \\ \tau_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z:\ \begin{array}{c|ccc} \omega & h_x & h_y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$sw^{th}_{ux}:\ \begin{array}{c|cc} \omega & h_x & \cdots \\ \hline t_u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_x & \cdots \\ \hline t_u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1+\alpha$ and $\langle c\rangle := \sum_i c_i$, and let

$$R^p_{ab} := \begin{array}{c|cc} 1 & h_a & h_b \\ \hline t_a & 0 & X-1 \\ t_b & 0 & 0 \end{array}\qquad R^m_{ab} := \begin{array}{c|cc} 1 & h_a & h_b \\ \hline t_a & 0 & X^{-1}-1 \\ t_b & 0 & 0 \end{array}.$$

Theorem. Z^β is a tangle invariant (and more). Restricted to knots, the ω part is the Alexander polynomial. On braids, it is equivalent to the Burau representation. A variant for links contains the multivariable Alexander polynomial.

Why Happy? • Applications to w-knots.
• Everything that I know about the Alexander polynomial can be expressed cleanly in this language (even if without proof), except HF, but including genus, ribbonness, cabling, v-knots, knotted graphs, etc., and there's potential for vast generalizations.
• The least wasteful "Alexander for tangles" I'm aware of.

• Every step along the computation is the invariant of something.
• Fits on one sheet, including implementation & propaganda.

Further meta-monoids. Π (and variants), $\mathcal{A}$ (and quotients), vT, ...

Further meta-bicrossed-products. Π (and variants), $\overrightarrow{\mathcal{A}}$ (and quotients), M_0, M, $\mathcal{K}^{bh}$, $\mathcal{K}^{rbh}$, ...

Meta-Lie-algebras. $\mathcal{A}$ (and quotients), $\mathcal{S}$, ...

Meta-Lie-bialgebras. $\overrightarrow{\mathcal{A}}$ (and quotients), ...

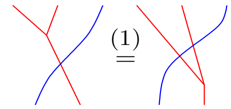
I don't understand the relationship between gr and H, as it appears, for example, in braid theory.

I mean business!

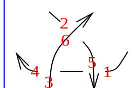
```
⟨μ_⟩ := μ /. t_ → 1;
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
   ts = Union[Cases[B[ω, Λ], t_u_ :> u, Infinity]];
   hs = Union[Cases[B[ω, Λ], h_x_ :> x, Infinity]];
   M = Outer[βSimp[Coefficient[Λ, h_#1 t_#2]] &, hs, ts];
   PrependTo[M, t_# & /@ ts];
   M = Prepend[Transpose[M], Prepend[h_# & /@ hs, ω]];
   MatrixForm[M]];
βForm[else_] := else /. β_B :> βForm[β];
Format[β_B, StandardForm] := βForm[β];
tm_{u_,v_→w_}[β_] := βCollect[β /. t_{u|v} → t_w];
hm_{x_,y_→z_}[B[ω_, Λ_]] := Module[
   {α = D[Λ, h_x], β = D[Λ, h_y], γ = Λ /. h_{x|y} → 0},
   B[ω, (α + (1 + ⟨α⟩) β) h_z + γ] // βCollect];
sw_{u_,x_}[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
   α = Coefficient[Λ, h_x t_u]; β = D[Λ, t_u] /. h_x → 0;
   γ = D[Λ, h_x] /. t_u → 0;   δ = Λ /. h_x | t_u → 0;
   ϵ = 1 + α;
   B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h_x t_u + β (1 + ⟨γ⟩ / ϵ) t_u
       + γ / ϵ h_x               + δ - γ * β / ϵ
   ] // βCollect];
gm_{a_,b_→c_}[β_] := β // sw_{ab} // hm_{ab→c} // tm_{ab→c};
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
Rp_{a_,b_} := B[1, (X - 1) t_a h_b];
Rm_{a_,b_} := B[1, (X^-1 - 1) t_a h_b];
```

```
{β = B[ω, Sum[α_{10 i+j} t_i h_j, {i, {1, 2, 3}}, {j, {4, 5}}]],
 (β // tm_{12→1} // sw_{14}) == (β // sw_{24} // sw_{14} // tm_{12→1})}
```

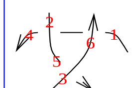
$$\left\{\begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_{14} & \alpha_{15} \\ t_2 & \alpha_{24} & \alpha_{25} \\ t_3 & \alpha_{34} & \alpha_{35} \end{pmatrix}, \text{True}\right\}$$



Some testing




```
{Rm_{51} Rm_{62} Rp_{34} // gm_{14→1} // gm_{25→2} // gm_{36→3},
 Rp_{61} Rm_{24} Rm_{35} // gm_{14→1} // gm_{25→2} // gm_{36→3}}
```



$$\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix}\right\}$$

... divide and conquer!

```
β = Rm_{12,1} Rm_{27} Rm_{83} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15}
```

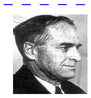
8_{17}

$$\begin{pmatrix} 1 & h_1 & h_3 & h_5 & h_7 & h_9 & h_{11} & h_{13} & h_{15} \\ t_2 & 0 & 0 & 0 & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 0 & 0 & -\frac{-1+X}{X} & 0 & 0 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 0 & -1+X & 0 \\ t_8 & 0 & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+X \\ t_{12} & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{14} & 0 & 0 & 0 & 0 & -1+X & 0 & 0 & 0 \\ t_{16} & 0 & 0 & -1+X & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Do[β = β // gm_{1k→1}, {k, 2, 10}]; β
```

8_{17}, cont.

$$\begin{pmatrix} \frac{1}{X} & h_1 & h_{11} & h_{13} & h_{15} \\ t_1 & -\frac{(-1+X)(1+X)}{X} & -(-1+X)(1-X+X^2) & (-1+X)(1-X+X^2) & -1+X \\ t_{12} & -\frac{-1+X}{X} & 0 & 0 & 0 \\ t_{14} & -1+X & \frac{(-1+X)^2(1-X+X^2)}{X} & -\frac{(-1+X)^2(1-X+X^2)}{X} & 0 \\ t_{16} & \frac{-1+X}{X} & (-1+X)^2 & -\frac{(-1+X)^3}{X} & 0 \end{pmatrix}$$

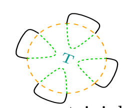


James Waddell Alexander

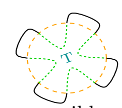
```
Do[β = β // gm_{1k→1}, {k, 11, 16}]; β
```

$$\left(-\frac{1-4X+8X^2-11X^3+8X^4-4X^5+X^6}{X^3}\right)$$

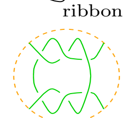
A Partial To Do List. 1. Where does it *more simply* come from?
2. Remove all the denominators.
3. How do determinants arise in this context?
4. Understand links ("meta-conjugacy classes").
5. Find the "reality condition".
6. Do some "Algebraic Knot Theory".
7. Categorify.
8. Do the same in other natural quotients of the v/w-story.



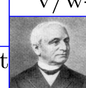
trivial



ribbon



example

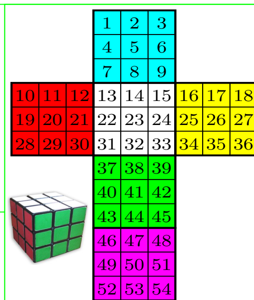


"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

Non-Commutative Gaussian Elimination and Rubik's Cube

The Problem. Let $G = \langle g_1, \dots, g_\alpha \rangle$ be a subgroup of S_n, with $n = O(100)$. Before you die, understand G:
1. Compute $|G|$.
2. Given $\sigma \in S_n$, decide if $\sigma \in G$.
3. Write a $\sigma \in G$ in terms of $g_1, \dots, g_\alpha$.
4. Produce *random* elements of G.



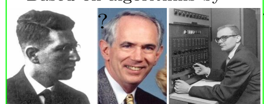
The Commutative Analog. Let $V = \operatorname{span}(v_1, \dots, v_\alpha)$ be a subspace of $\mathbb{R}^n$. Before you die, understand V.

Solution: Gaussian Elimination. Prepare an empty table,

1	2	3	4	$\cdots$	$n-1$	n

Space for a vector $u_4 \in V$, of the form $u_4 = (0, 0, 0, 1, *, \dots, *)$; $1 :=$"the pivot".

Based on algorithms by



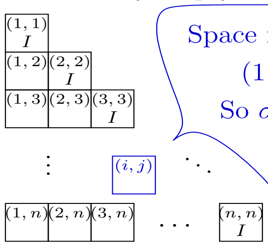
O. Schreier C. Sims D. Knuth

See also *Permutation Group Algorithms* by Á. Seress, *Efficient Representation of Perm Groups* by D. Knuth.

Feed $v_1, \dots, v_\alpha$ in order. To feed a non-zero v, find its pivotal position i.
1. If box i is empty, put v there.
2. If box i is occupied, find a combination v' of v and u_i that eliminates the pivot, and feed v'.

Non-Commutative Gaussian Elimination

Prepare a mostly-empty table,



Space for a $\sigma_{i,j} \in S_n$ of the form
$(1, 2, \dots, i-2, i-1, j, *, *, \dots, *)$
So $\sigma_{i,j}$ fixes $1, \dots, i-1$,
sends "the pivot" i to j and
goes wild afterwards, and
$\sigma_{i,j}^{-1}$ "does sticker j".

Feed $g_1, \dots, g_\alpha$ in order. To feed a non-identity σ, find its pivotal position i and let $j := \sigma(i)$.
1. If box (i, j) is empty, put σ there.
2. If box (i, j) contains $\sigma_{i,j}$, feed $\sigma' := \sigma_{i,j}^{-1}\sigma$.

The Twist. When done, for every occupied (i, j) and (k, l), feed $\sigma_{i,j}\sigma_{k,l}$. Repeat until the table stops changing.

Claim 1. The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table T.

Claim 2. Every $\sigma_{i,j}$ in T is in G.

Claim 3. Anything fed in T is now a monotone product in T:
f was fed $\Rightarrow f \in M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} : \forall i, j_i \geq i \ \&\ \sigma_{i,j_i} \in T\}$

The Generators

```
n = 54;
g1 = Cycles[{{1, 18, 45, 28}, {2, 27, 44, 19}, {3, 36, 43, 10}, {46, 52, 54, 48},
    {47, 49, 53, 51}}];
g2 = Cycles[{{7, 16, 39, 30}, {8, 25, 38, 21}, {9, 34, 37, 12}, {13, 15, 33, 31},
    {14, 24, 32, 22}}];
g3 = Cycles[{{28, 31, 34, 48}, {29, 32, 35, 47}, {30, 33, 36, 46}, {37, 39, 45, 43},
    {38, 42, 44, 40}}];
g4 = Cycles[{{1, 3, 9, 7}, {2, 6, 8, 4}, {10, 54, 16, 13}, {11, 53, 17, 14},
    {12, 52, 18, 15}}];
g5 = Cycles[{{1, 13, 37, 46}, {4, 22, 40, 49}, {7, 31, 43, 52}, {10, 12, 30, 28},
    {11, 21, 29, 19}}];
g6 = Cycles[{{3, 48, 39, 15}, {6, 51, 42, 24}, {9, 54, 45, 33}, {16, 18, 36, 34},
    {17, 27, 35, 25}}];
```
Enter

Claim 4. If two monotone products are equal,
$$\sigma_{1,j_1}\cdots\sigma_{n,j_n} = \sigma_{1,j'_1}\cdots\sigma_{n,j'_n},$$
then all the indices that appear in them are equal, $\forall i,\ j_i = j'_i$.

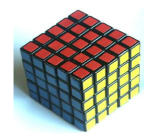
Claim 5. Let M_k denote the set of monotone products in T starting in column k:
$$M_k := \{\sigma_{k,j_k}\cdots\sigma_{n,j_n} : \forall i \geq k, j_i \geq i \text{ and } \sigma_{i,j_i} \in T\}.$$
then for every k, $M_kM_k \subset M_k$ (and so each M_k is a subgroup of G).

Proof. By backwards induction. Clearly $M_nM_n \subset M_n$. Now assume that $M_5M_5 \subset M_5$ and show that $M_4M_4 \subset M_4$. Start with $\sigma_{8,j}M_4 \subset M_4$:
$$\sigma_{8,j}(\sigma_{4,j_4}M_5) \overset{1}{=} (\sigma_{8,j}\sigma_{4,j_4})M_5 \overset{2}{\subset} M_4M_5$$
$$\overset{3}{=} \cup_j\sigma_{4,j}(M_5M_5) \overset{4}{\subset} \cup_j\sigma_{4,j}M_5 \subset M_4$$
(1: associativity, 2: thank the twist, 3: associativity and tracing i_4, 4: induction). Now the general case
$$(\sigma_{4,j'_4}\sigma_{5,j'_5}\cdots)(\sigma_{4,j_4}\sigma_{5,j_5}\cdots)$$
falls like a chain of dominos.

Theorem. $G = M_1$ and we have achieved our goals.

A Demo Program

```
σ_ ∘ τ_ := PermutationProduct[τ, σ];
Feed[Cycles[{}]] := Null;
Feed[τ_] := Module[{i, j, k, l},
   i = Min[PermutationSupport[τ]];
   j = PermutationReplace[i, τ];
   If[Head[σi,j] === Cycles,
    Feed[InversePermutation[σi,j] ∘ τ],
    (*Else*) σi,j = τ;
    For[k = 1, k < n, ++k,
     For[l = k + 1, l ≤ n, ++l,
      If[Head[σk,l] === Cycles,
       Feed[σi,j ∘ σk,l]; Feed[σk,l ∘ σi,j]]
     ]]
   ]];
$RecursionLimit = ∞;
```
Enter



www.geocities.com/jaapsch/puzzles/

www.powerstrike.net/puzzles/

http://lodelnic.free.fr/

Homework Problem 1.

Can you do cosets?

www.powerstrike.net/puzzles/

The Results

```
Table[Feed[gα]; ∏_{i=1}^n (1 + Count[Range[n], j_ /; Head[σi,j] == Cycles]), {α, 6}]
```
Enter

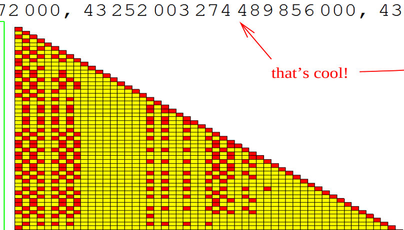
{4, 16, 159 993 501 696 000, 21 119 142 223 872 000, 43 252 003 274 489 856 000, 43 252 003 274 489 856 000}

Homework Problem 2.

Can you do categories (groupoids)?

7	9	2	5
1	4	8	3
6	10	11	12
13	14	15	

Rubik's magic



that's cool!

$43, 252, 003, 274, 489, 856, 000$

$$= \frac{8!\,3^8\,12!\,2^{12}}{12}$$

In Inuit:

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Cambridge-1301/, see also http://www.math.toronto.edu/~drorbn/Talks/Mathcamp-0907/

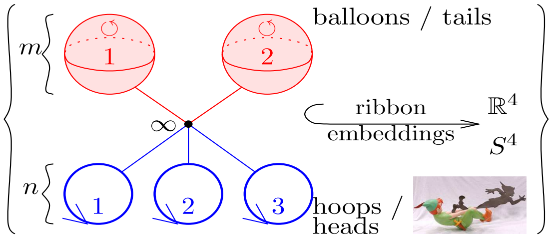
Balloons and Hoops and their Universal Finite−Type Invariant, BF Theory, and an Ultimate Alexander Invariant

Dror Bar−Natan in Oxford, January 2013
$\omega\epsilon\beta$:=http://www.math.toronto.edu/~drorbn/Talks/Oxford-130121

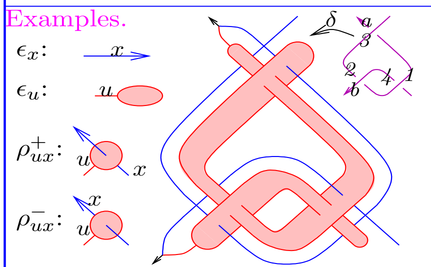
Scheme. • Balloons and hoops in $\mathbb{R}^4$, algebraic structure and relations with 3D.
• An ansatz for a "homomorphic" invariant: computable, related to finite-type and to BF.
• Reduction to an "ultimate Alexander invariant".

$\mathcal{K}^{bh}(m,n)$.


$\omega\epsilon\beta$/decorationideas



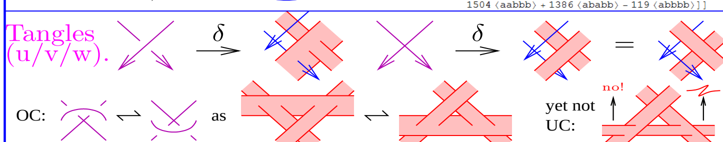
Examples. I mean business!




```
T0 = Rm[3, a] Rp[b, 2] Rp[1, 4];
ζ = T0 // dm[2, 1, 1] // dm[4, b, b] //
    dm[1, a, a] // dm[3, a, a];
ζ[{5}] /. {w_LW :→ (Deg[w] + 1) ! w,
  w_CW :→ Deg[w] ! w}

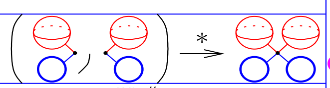
μ[CWS[-[a], -2 [ab], -3 [aab] - 3 [abb],
  -4 [aaab] + 42 [aabb] - 60 [abab] - 4 [abbb],
  -5 [aaaab] + 110 [aaabb] - 180 [aabab] +
   110 [aabbb] - 180 [ababb] - 5 [abbbb]],
 h[b] LS[2 ⟨a⟩, 0, -24 ⟨aab⟩,
    -60 ⟨aaab⟩ + 60 ⟨aabb⟩, -120 ⟨aaaab⟩ +
     900 ⟨aaabb⟩ + 360 ⟨aabab⟩ - 120 ⟨aabbb⟩] +
  h[a] LS[-2 ⟨a⟩ + 2 ⟨b⟩, 9 ⟨ab⟩, 26 ⟨aab⟩ -
     26 ⟨abb⟩, 60 ⟨aaab⟩ - 255 ⟨aabb⟩ + 60 ⟨abbb⟩,
    119 ⟨aaaab⟩ - 1504 ⟨aaabb⟩ + 118 ⟨aabab⟩ +
     1504 ⟨aabbb⟩ + 1386 ⟨ababb⟩ - 119 ⟨abbbb⟩]]
```

Tangles (u/v/w).



• δ injects u-Knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
• δ maps v/w-tangles map to $\mathcal{K}^{bh}$; the kernel contains Reidemeister moves and the "overcrossings commute" relation, and conjecturally, that's all. Allowing punctures and cuts, δ is onto.

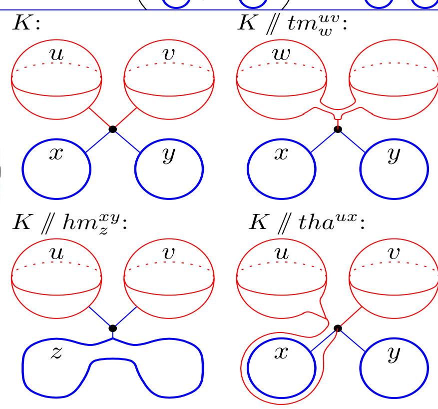
Operations Punctures & Cuts | Connected Sums.



Meta-Group-Action. If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

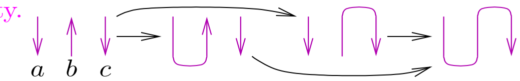
"MGA"

("$/\!\!/$" is newspeak for "apply an operator" and for "composition left to right")

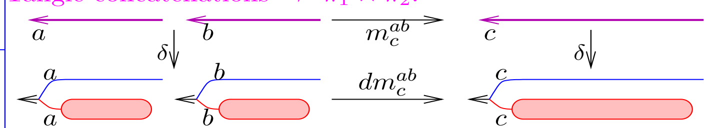


Properties.
• Associativities: $m^{ab}_a /\!\!/ m^{ac}_a = m^{bc}_b /\!\!/ m^{ab}_a$, for $m = tm, hm$.
• Action axiom t: $tm^{uv}_w /\!\!/ tha^{wx} = tha^{ux} /\!\!/ tha^{vx} /\!\!/ tm^{uv}_w$.
• Action axiom h: $hm^{xy}_z /\!\!/ tha^{uz} = tha^{ux} /\!\!/ tha^{uy} /\!\!/ hm^{xy}_z$.
• SD Product: $dm^{ab}_c := tha^{ab} /\!\!/ tm^{ab}_c /\!\!/ hm^{ab}_c$ is associative.

Meta-associativity. $m^{ab}_a /\!\!/ m^{ac}_a = m^{bc}_b /\!\!/ m^{ab}_a$.

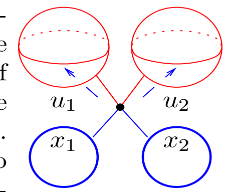


Tangle concatenations $\to$ $\pi_1 \ltimes \pi_2$.



Thus we seek homomorphic invariants of $\mathcal{K}^{bh}$!

Invariant #0. With Π_1 denoting "honest π_1", map $\gamma \in \mathcal{K}^{bh}(m,n)$ to the triple $(\Pi_1(\gamma^c), (u_i), (x_j))$, where the meridian of the balls u_i normally generate Π_1, and the "longtitudes" x_j are some elements of Π_1. $*$ acts like $*$, tm acts by "merging" two meridians/generators, hm acts by multiplying two longtitudes, and tha^{ux} acts by "conjugating a meridian by a longtitude":



Not computable! (but nearly)

$$(\Pi, (u, \dots), (x, \dots)) \mapsto (\Pi * \langle \bar{u} \rangle / (u = x\bar{u}x^{-1}), (\bar{u}, \dots), (x, \dots))$$

Failure #0. Can we write the x's as free words in the u's? If $x = uv$, compute $x /\!\!/ tha^{ux}$:

$$x = uv \to \bar{u}v = u^x v = u^{\bar{u}v}v = u^{u^x v}v = u^{u^{u^x v}v}v = \cdots$$

The Meta-Group-Action M. Let T be a set of "tail labels" ("balloon colours"), and H a set of "head labels" ("hoop colours"). Let $FL = FL(T)$ and $FA = FA(T)$ be the (completed graded) free Lie and free associative algebras on generators T and let $CW = CW(T)$ be the (completed graded) vector space of cyclic words on T, so there's $\mathrm{tr} : FA \to CW$. Let $M(T, H) := \{(\bar{\lambda} = (x : \lambda_x)_{x \in H}; \omega) : \lambda_x \in FL, \omega \in CW\}$

$$= \left\{ \left(x : \begin{smallmatrix} u & v \\ \curlyvee \end{smallmatrix},\ y : \left| -\tfrac{22}{7} \right. \begin{smallmatrix} & u & v \\ \curlyvee \end{smallmatrix};\ \bigcirc \right) \dots \right\}$$


$\omega\epsilon\beta$/antiq-ave

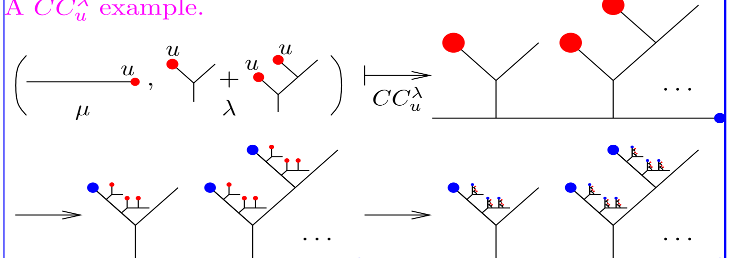
Operations. Set $(\bar{\lambda}_1; \omega_1) * (\bar{\lambda}_2; \omega_2) := (\bar{\lambda}_1 \cup \bar{\lambda}_2; \omega_1 + \omega_2)$ and with $\mu = (\bar{\lambda}; \omega)$ define

$$tm^{uv}_w : \mu \mapsto \mu /\!\!/ (u, v \mapsto w),$$

$$hm^{xy}_z : \mu \mapsto \left(\left(\dots, \widehat{x : \lambda_x}, \widehat{y : \lambda_y}, \dots, z : \mathrm{bch}(\lambda_x, \lambda_y) \right); \omega \right)$$

$$tha^{ux} : \mu \mapsto \underbrace{\overbrace{\mu \ /\!\!/\!\!/}^{\text{"stable apply"}} (u \mapsto e^{\mathrm{ad}\,\lambda_x}(\bar{u})) /\!\!/ (\bar{u} \mapsto u)}_{\mu /\!\!/ CC^{\lambda_x}_u} + \underbrace{(0; J_u(\lambda_x))}_{\text{the "J-spice"}}$$

A CC^λ_u example.



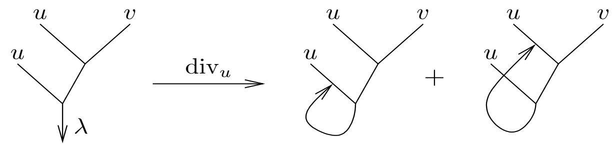
Balloons and Hoops and their Universal Finite−Type Invariant, 2

The Meta-Cocycle J. Set $J_u(\lambda) := J(1)$ where

$$J(0) = 0, \qquad \lambda_s = \lambda \mathbin{/\!\!/} CC_u^{s\lambda},$$

$$\frac{dJ(s)}{ds} = (J(s) \mathbin{/\!\!/} \operatorname{der}(u \mapsto [\lambda_s, u])) + \operatorname{div}_u \lambda_s,$$

and where $\operatorname{div}_u \lambda := \operatorname{tr}(u\sigma_u(\lambda))$, $\sigma_u(v) := \delta_{uv}$, $\sigma_u([\lambda_1, \lambda_2]) := \iota(\lambda_1)\sigma_u(\lambda_2) - \iota(\lambda_2)\sigma_u(\lambda_1)$ and ι is the inclusion $FL \hookrightarrow FA$:



Claim. $CC_u^{\operatorname{bch}(\lambda_1,\lambda_2)} = CC_u^{\lambda_1} \mathbin{/\!\!/} CC_u^{\lambda_2 /\!\!/ CC_u^{\lambda_1}}$ and

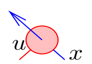
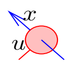
$$J_u(\operatorname{bch}(\lambda_1, \lambda_2)) = J_u(\lambda_1) \mathbin{/\!\!/} CC_u^{\lambda_2 /\!\!/ CC_u^{\lambda_1}} + J_u(\lambda_2 \mathbin{/\!\!/} CC_u^{\lambda_1}),$$

and hence tm, hm, and tha form a meta-group-action.

Why ODEs? **Q.** Find f s.t. $f(x+y) = f(x)f(y)$. **A.** $\frac{df(s)}{ds} = \frac{d}{d\epsilon}f(s+\epsilon) = \frac{d}{d\epsilon}f(s)f(\epsilon) = f(s)C$. Now solve this ODE using Picard's theorem or power series.

Alekseev Torossian

The Invariant ζ. Set $\zeta(\rho^\pm) = (\pm u_x; 0)$. This at least defines an invariant of u/v/w-tangles, and if the topologists will deliver a "Reidemeister" theorem, it is well defined on $\mathcal{K}^{bh}$.

ζ:  $\longmapsto \left(x : +\Big|^{u} ; 0\right)$  $\longmapsto \left(x : -\Big|^{u} ; 0\right)$

Theorem. ζ is (the log of) a universal finite type invariant (a homomorphic expansion) of w-tangles.

Tensorial Interpretation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \to \operatorname{Fun}(\oplus_T \mathfrak{g} \to \mathfrak{g})$ and $\tau : CW(T) \to \operatorname{Fun}(\oplus_T \mathfrak{g})$. Together, $\tau : M(T,H) \to \operatorname{Fun}(\oplus_T \mathfrak{g} \to \oplus_H \mathfrak{g})$, and hence

$$e^\tau : M(T,H) \to \operatorname{Fun}(\oplus_T \mathfrak{g} \to \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

ζ and BF Theory. Let A denote a $\mathfrak{g}$-connection on S^4 with curvature F_A, and B a $\mathfrak{g}^*$-valued 2-form on S^4. For a hoop γ_x, let $\operatorname{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x. For a ball γ_u, let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be the integral of B (transported via A to ∞) on γ_u.

Cattaneo

Loose Conjecture. For $\gamma \in \mathcal{K}(T, H)$,

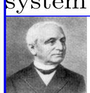
$$\int \mathcal{D}A\mathcal{D}B e^{\int B \wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B))} \bigotimes_x \operatorname{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

That is, ζ is a complete evaluation of the BF TQFT.

Issues. How exactly is B transported via A to ∞? How does the ribbon condition arise? Or if it doesn't, could it be that ζ can be generalized??

The β quotient, 1. • Arises when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra.

• Arises when reducing by relations satisfied by the weight system of the Alexander polynomial.



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

The Knot Atlas

Paper in progress: $\omega\epsilon\beta$/kbh

The β quotient, 2. Let $R = \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta := R \otimes T$ with central R and with $[u,v] = c_u v - c_v u$ for $u, v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,

$$\mu \to (\bar\lambda; \omega) \quad \text{with } \bar\lambda = \sum_{x\in H,\, u\in T} \lambda_{ux} ux, \quad \lambda_{ux}, \omega \in R,$$

$$\operatorname{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u+c_v} - 1}\left(\frac{e^{c_u}-1}{c_u}u + e^{c_u}\frac{e^{c_v}-1}{c_v}v\right),$$

if $\lambda = \sum \lambda_v v$ then with $c_\lambda := \sum \lambda_v c_v$,

$$u \mathbin{/\!\!/} CC_u^\lambda = \left(1 + c_u\lambda_u\frac{e^{c_\lambda}-1}{c_\lambda}\right)^{-1}\left(e^{c_\lambda}u - c_u\frac{e^{c_\lambda}-1}{c_\lambda}\sum_{v\neq u}\lambda_v v\right),$$

$\operatorname{div}_u \lambda = c_u \lambda_u$, and the ODE for J integrates to

$$J_u(\lambda) = \log\left(1 + \frac{e^{c_\lambda}-1}{c_\lambda}c_u\lambda_u\right),$$

so ζ is formula-computable to all orders! Can we simplify?

Repackaging. Given $((x : \lambda_{ux}); \omega)$, set $c_x := \sum_v c_v \lambda_{vx}$, replace $\lambda_{ux} \to \alpha_{ux} := c_u\lambda_{ux}\frac{e^{c_x}-1}{c_x}$ and $\omega \to \log\omega$, use $t_u = e^{c_u}$, and write α_{ux} as a matrix. Get "β calculus".

β Calculus. Let $\beta(H,T)$ be

$$\left\{\begin{array}{c|ccc|l} \omega & x & y & \cdots & \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot & \\ v & \alpha_{vx} & \alpha_{vy} & \cdot & \\ \vdots & \cdot & \cdot & \cdot & \end{array}\ \begin{array}{l}\omega \text{ and the } \alpha_{ux}\text{'s are} \\ \text{rational functions in} \\ \text{variables } t_u, \text{ one for} \\ \text{each } u \in T.\end{array}\right\},$$

In preparation, Selmani & B–N.

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha + \beta + \langle\alpha\rangle\beta & \gamma \end{array},$$

$$tha^{ux} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta - \gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1 + \alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u}\gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

On long knots, ω is the Alexander polynomial!

Why bother? (1) An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination!). *If there should be an Alexander invariant to have an algebraic categorification, it is this one!*

See also $\omega\epsilon\beta$/regina, $\omega\epsilon\beta$/gwu.

Why bother? (2) Related to A-T, K-V, and E-K, should have vast generalization beyond w-knots and the Alexander polynomial. See also $\omega\epsilon\beta$/wko, $\omega\epsilon\beta$/caen, $\omega\epsilon\beta$/swiss.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Oxford-130121/ and at http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

Braids and the Grothendieck−Teichmuller Group

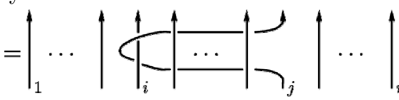
Dror Bar−Natan, the Newton Institute, January 2013, http://www.math.toronto.edu/~drorbn/Talks/Newton−1301/

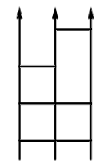
Abstract. The "Grothendieck-Teichmuller Group" (**GT**) appears as a "depth certificate" in many recent works — "we do A to B, apply the result to C, and get something related to **GT**, therefore it must be interesting". Interesting or not, in my talk I will explain how **GT** arose first, in Drinfel'd's work on associators, and how it can be used to show that "every bounded-degree associator extends", that "rational associators exist", and that "the pentagon implies the hexagon"⋆.

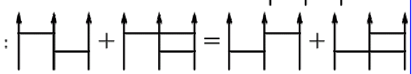
In a nutshell: the filtered tower of braid groups (with bells and whistles attached) is isomorphic to its associated graded, but the isomorphism is neither canonical nor unique — such an isomorphism is precisely the thing called "an associator". But the set of isomorphisms between two isomorphic objects **always** has two groups acting simply transitively on it — the group of automorphisms of the first object acting on the right, and the group of automorphisms of the second object acting on the left. In the case of associators, that first group is what Drinfel'd calls the Grothendieck-Teichmuller group **GT**, and the second group, isomorphic but not canonically to the first and denoted **GRT**, is the one several recent works seem to refer to.

Almost everything I will talk about is in my old paper "On Associators and the Grothendieck-Teichmuller Group I", also at arXiv:q-alg/9606021.

Baby(?) Example. PB_n: pure braids; $I\subset\mathbb{Q}PB_n$ the augmentation ideal; $B^{(m)}=\mathbb{Q}PB_n/I^{m+1}$ (filtered!); $\hat{B}=\varprojlim B^{(m)}$ (filtered!). Then $\operatorname{gr}B^{(m)}=C^{(m)}$ and then $\operatorname{gr}\hat{B}=\hat{C}$ where $C=\langle t^{ij}=t^{ji}\colon [t^{ij},t^{kl}]=[t^{ij},t^{ik}+t^{jk}]=0\rangle$, so $B^{(m)}$ and $\hat{B}$ are isomorphic to $C^{(m)}$ and $\hat{C}$, but not canonically. Me not know that the groups **GT** and **GRT** here have been analyzed.

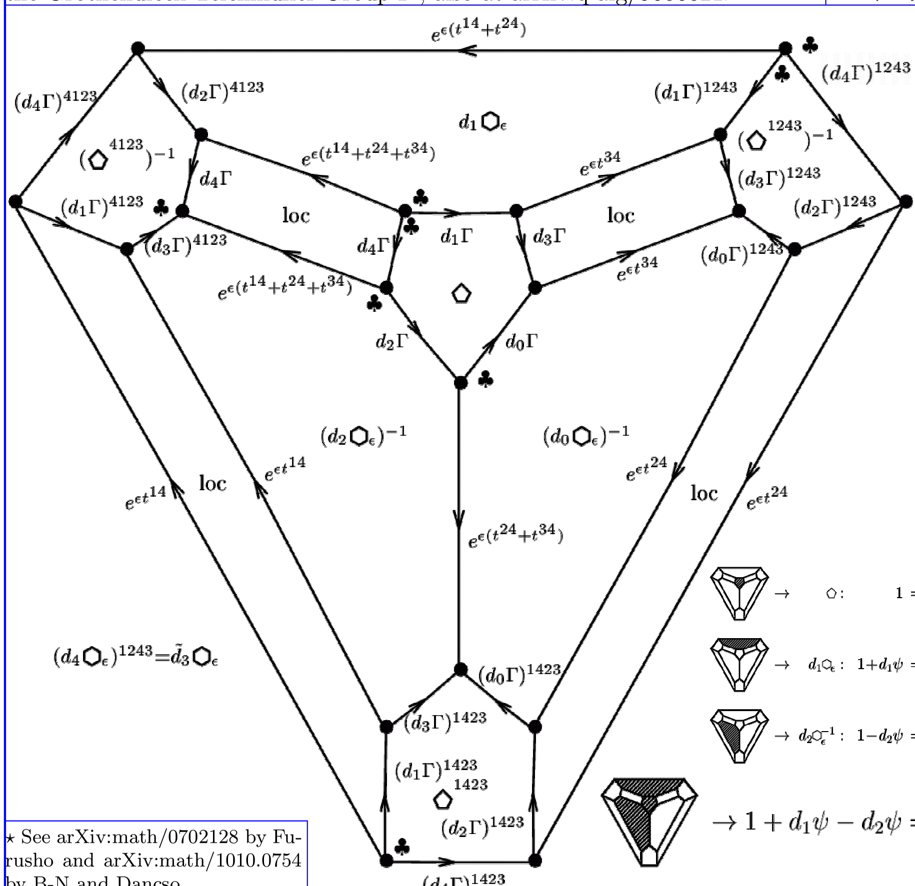
$\sigma^{ij}=$ 

$t^{13}t^{13}t^{12}t^{23}\longleftrightarrow$ 

$4T:$ 

Main Theorem. The projection $\mathbf{ASSO}^{(m)}\to\mathbf{ASSO}^{(m-1)}$ is surjective. (yey!)

Sketch. Given $\mathbf{ASSO}^{(m)}\neq\emptyset$ (hard, analytic), sufficient is surjectivity of $\mathbf{GRT}^{(m)}\to\mathbf{GRT}^{(m-1)}$, enough is surjectivity of $\mathfrak{grt}^{(m)}\to\mathfrak{grt}^{(m-1)}$, polyhedron on left use, little homological algebra too.



⋆ See arXiv:math/0702128 by Furusho and arXiv:math/1010.0754 by B-N and Dancso.

by successive approximations presents no problems. For this we introduce the following modification $\mathrm{GRT}(k)$ of the group $\mathrm{GT}(k)$. We denote by $\mathrm{GRT}_1(k)$ the set of all $g\in\mathrm{Fr}_k(A,B)$ such that

$$g(B,A)=g(A,B)^{-1},\tag{5.12}$$

$$g(C,A)g(B,C)g(A,B)=1\quad\text{for } A+B+C=0,\tag{5.13}$$

$$A+g(A,B)^{-1}Bg(A,B)+g(A,C)^{-1}Cg(A,C)=0\quad\text{for } A+B+C=0,\tag{5.14}$$

$$g(X^{12},X^{23}+X^{24})g(X^{13}+X^{23},X^{34})=g(X^{23},X^{34})g(X^{12}+X^{13},X^{24}+X^{34})g(X^{12},X^{23}),\tag{5.15}$$

where the X^{ij} satisfy (5.1). $\mathrm{GRT}_1(k)$ is a group with the operation

$$(g_1\circ g_2)(A,B)=g_1(g_2(A,B)Ag_2(A,B)^{-1},B)\cdot g_2(A,B).\tag{5.16}$$

On $\mathrm{GRT}_1(k)$ there is an action of k^*, given by $\tilde{g}(A,B)=g(c^{-1}A,c^{-1}B)$, $c\in k^*$. The semidirect product of k^* and $\mathrm{GRT}_1(k)$ we denote by $\mathrm{GRT}(k)$. The Lie algebra $\mathfrak{grt}_1(k)$ of the group $\mathrm{GRT}_1(k)$ consists of the series $\psi\in\mathfrak{fr}_k(A,B)$ such that

$$\psi(B,A)=-\psi(A,B),\tag{5.17}$$

$$\psi(C,A)+\psi(B,C)+\psi(A,B)=0\quad\text{for } A+B+C=0,\tag{5.18}$$

$$[B,\psi(A,B)]+[C,\psi(A,C)]=0\quad\text{for } A+B+C=0,\tag{5.19}$$

$$\psi(X^{12},X^{23}+X^{24})+\psi(X^{13}+X^{23},X^{34})=\psi(X^{23},X^{34})+\psi(X^{12}+X^{13},X^{24}+X^{34})+\psi(X^{12},X^{23}),\tag{5.20}$$

where the X^{ij} satisfy (5.1). A commutator $\langle\ ,\ \rangle$ in $\mathfrak{grt}_1(k)$ is of the form

$$\langle\psi_1,\psi_2\rangle=[\psi_1,\psi_2]+D_{\psi_2}(\psi_1)-D_{\psi_1}(\psi_2),\tag{5.21}$$

where $[\psi_1,\psi_2]$ is the commutator in $\mathfrak{fr}_k(A,B)$ and D_ψ is the derivation of $\mathfrak{fr}_k(A,B)$ given by $D_\psi(A)=[\psi,A]$, $D_\psi(B)=0$. The algebra $\mathfrak{grt}_1(k)$ is

Proposition 5.1. *The action of* $\mathrm{GT}(k)$ *on* $M(k)$ *is free and transitive.*

Proof. If $(\mu,\varphi)\in M(k)$ and $(\overline{\mu},\overline{\varphi})\in M(k)$, then there is exactly one f such that $\overline{\varphi}(A,B)=f(\varphi(A,B)e^A\varphi(A,B)^{-1},e^B)\cdot\varphi(A,B)$. We need to show that $(\lambda,f)\in\mathrm{GT}(k)$, where $\lambda=\overline{\mu}/\mu$. We prove (4.10). Let G_n be the semidirect product of S_n and $\exp\mathfrak{a}_n^k$. Consider the homomorphism $B_n\to G_n$ that takes σ_i into

$$\varphi(X^{1i}+\cdots+X^{i-1,i},X^{i,i+1})^{-1}\sigma^{i,i+1}e^{\mu X^{i,i+1}/2}\varphi(X^{1i}+\cdots+X^{i-1,i},X^{i,i+1}),$$

where $\sigma^{ij}\in S_n$ transposes i and j. It induces a homomorphism $K_n\to\exp\mathfrak{a}_n^k$, and therefore a homomorphism $\alpha_n\colon K_n(k)\to\exp\mathfrak{a}_n^k$, where $K_n(k)$ is the k-pro-unipotent completion of K_n. It is easily shown that the left- and right-hand sides of (4.10) have the same images in $\exp\mathfrak{a}_4^k$. It remains to prove that α_n is an isomorphism. The algebra Lie $K_n(k)$ is topologically generated by the elements ξ_{ij}, $1\le i<j\le n$, with defining relations obtained from (4.7)–(4.9) by substituting $x_{ij}=\exp\xi_{ij}$. The principal parts of these relations are the same as in (5.1), while $(\alpha_n)_*(\xi_{ij})=\mu X^{ij}+$ {lower terms}, where $(\alpha_n)_*\colon \mathrm{Lie}\,K_n(k)\to\mathfrak{a}_n^k$ is induced by the homomorphism α_n. Therefore α_n is an isomorphism, i.e., (4.10) is proved. (4.3) is obvious. To prove (4.4), we can interpret it in terms of K_3 and argue as in the proof of (4.10), or, what is equivalent, make the substitution

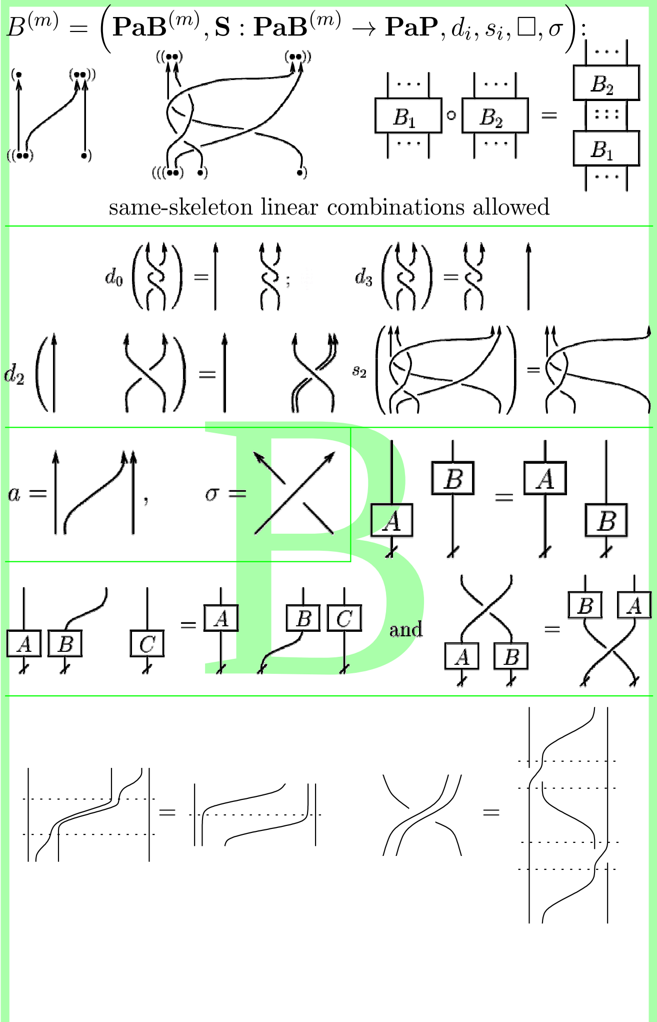
$$X_1=e^A,\quad X_2=e^{-A/2}\varphi(B,A)e^B\varphi(B,A)^{-1}e^{A/2},\quad X_3=\varphi(C,A)e^C\varphi(C,A)^{-1},\tag{5.4}$$

where $A+B+C=0$. ●

From Drinfel'd's *On quasitriangular Quasi-Hopf algebras and a group closely connected with* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991) 829–860.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Newton-1301/

The Main Course



ASSO

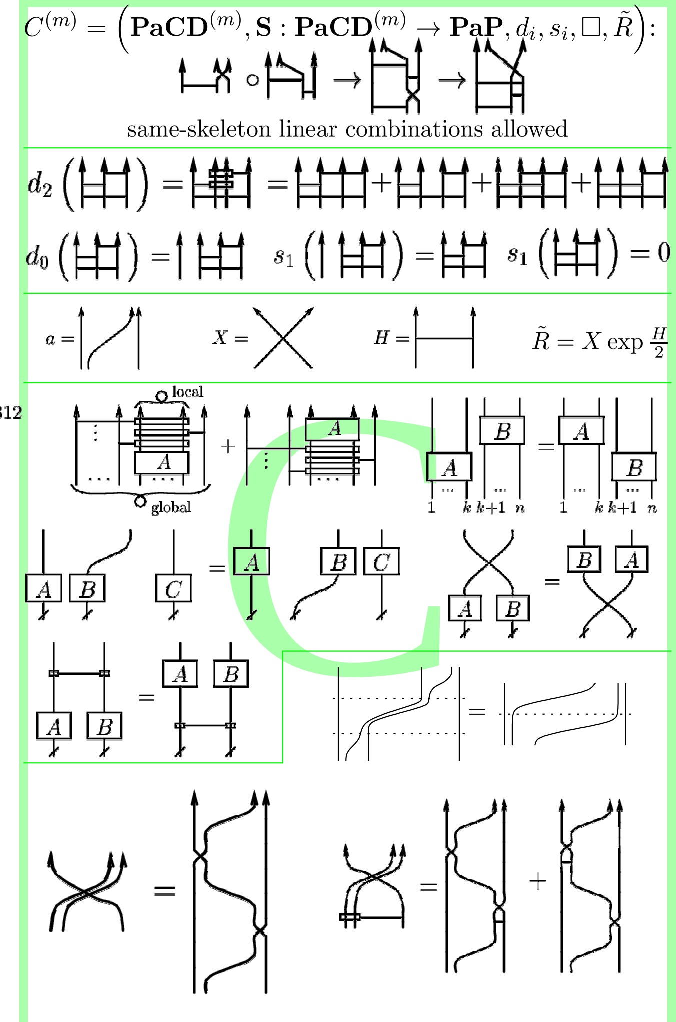
$$d_4\Phi \cdot d_2\Phi \cdot d_0\Phi = d_1\Phi \cdot d_3\Phi$$

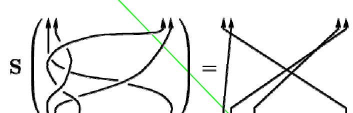
$$d_1 \exp\left(\pm\frac{1}{2}t^{12}\right) =$$

$$\Phi \cdot \exp\left(\pm\frac{1}{2}t^{23}\right) \cdot (\Phi^{-1})^{132} \cdot \exp\left(\pm\frac{1}{2}t^{13}\right) \cdot \Phi^{312}$$

$$s_1\Phi = s_2\Phi = s_3\Phi = 1$$

$$\square\Phi = \Phi \otimes \Phi$$





PaP

GT

GRT

$$d_4\Gamma \cdot d_2\Gamma \cdot d_0\Gamma = d_1\Gamma \cdot d_3\Gamma$$

$$1 = \Gamma \cdot (\Gamma^{-1})^{132} \cdot \Gamma^{312}$$

$$d_1 t^{12} = \Gamma \cdot \left(t^{23} \cdot (\Gamma^{-1})^{132} + (\Gamma^{-1})^{132} \cdot t^{13}\right) \cdot \Gamma^{312}$$

$$e^{\epsilon(t^{13}+t^{23})} = \Gamma \cdot e^{\epsilon t^{23}} \cdot (\Gamma^{-1})^{132} \cdot e^{\epsilon t^{13}} \cdot \Gamma^{312}$$

I have a nifty
Free Lie calculator.
Wanna play?

Braids and the Grothendieck−Teichmuller Group, II

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Newton-1301/

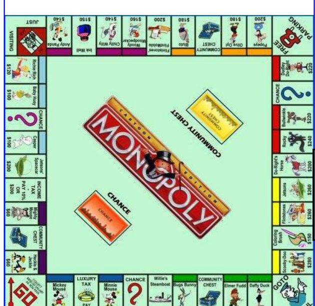
With $\omega\epsilon\beta$:= http://www.math.toronto.edu/drorbn/Talks/Hamburg-1208

A Quick Introduction to Khovanov Homology

Dror Bar–Natan,
Hamburg, August 2012

Abstract. I will tell the Kauffman bracket story of the Jones polynomial as Kauffman told it in 1987, then the Khovanov homology story as Khovanov told it in 1999, and finally the "local Khovanov homology" story as I understood it in 2003. At the end of our 90 minutes we will understand what is a "Jones homology", how to generalize it to tangles and to cobordisms between tangles, and why it is computable relatively efficiently. But we will say nothing about more modern stuff — the Rasmussen invariant, Alexander and HOMFLYPT knot homologies, and the categorification of sl_2 and other Lie algebras.

Why Bother?



$\omega\epsilon\beta$/webcrafterz

$\omega\epsilon\beta$/onemoretap

$\omega\epsilon\beta$/cheat-pcgame

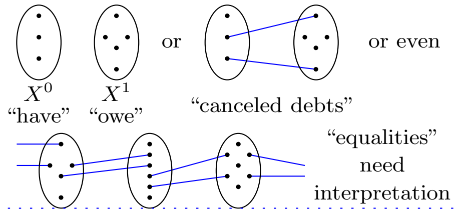
What is Categorification=Concretization=deabstraction? "3" is {cow, cow, cow} and {pig, pig, pig} and many other things...

... categorification is choosing which 3 it is!

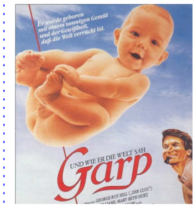
$\mathbb{N}$. Natural numbers $\mapsto$ finite sets, equalities $\mapsto$ bijections, inequalities $\mapsto$ injections and surjections:

$$\binom{2n}{n} = \sum \binom{n}{k}^2 \mapsto \binom{X\times\{1,2\}}{|X|} \leftrightarrow \bigcup \binom{X}{k}\times\binom{X}{k}$$

$\mathbb{Z}$. Negative numbers:



The Philosophy Corner

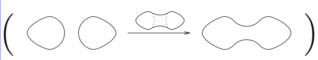


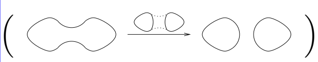
Weaker Categorification. Do the same in the category of vector spaces: "3" becomes V s.t. $\dim V = 3$, or better, $V^\bullet = (\cdots V^{r-1}\to V^r\to V^{r+1}\cdots)$ s.t. $d^2=0$ and $\chi(V^\bullet) := \sum(-1)^r\dim V^r = 3 = \sum(-1)^r\dim H^r$. Equalities become homotopies between complexes.

Khovanov: $K(L)$ is a chain complex of graded $\mathbb{Z}$-modules;
$V=\mathrm{span}\langle v_+, v_-\rangle$; $\deg v_\pm = \pm 1$; $q\dim V = q+q^{-1}$;

$$K(\bigcirc^k) = V^{\otimes k};\quad K(\times) = \text{Flatten}\Big(0\to \underset{\text{height }0}{K(\asymp\!\!\!\!\!)(\,)\{1\}} \to \underset{\text{height }1}{K(\asymp)\{2\}}\to 0\Big);$$

$$K(\times) = \text{Flatten}\Big(0\to \underset{\text{height }-1}{K(\asymp)\{-2\}} \to \underset{\text{height }0}{K()(\,)\{-1\}}\to 0\Big);$$

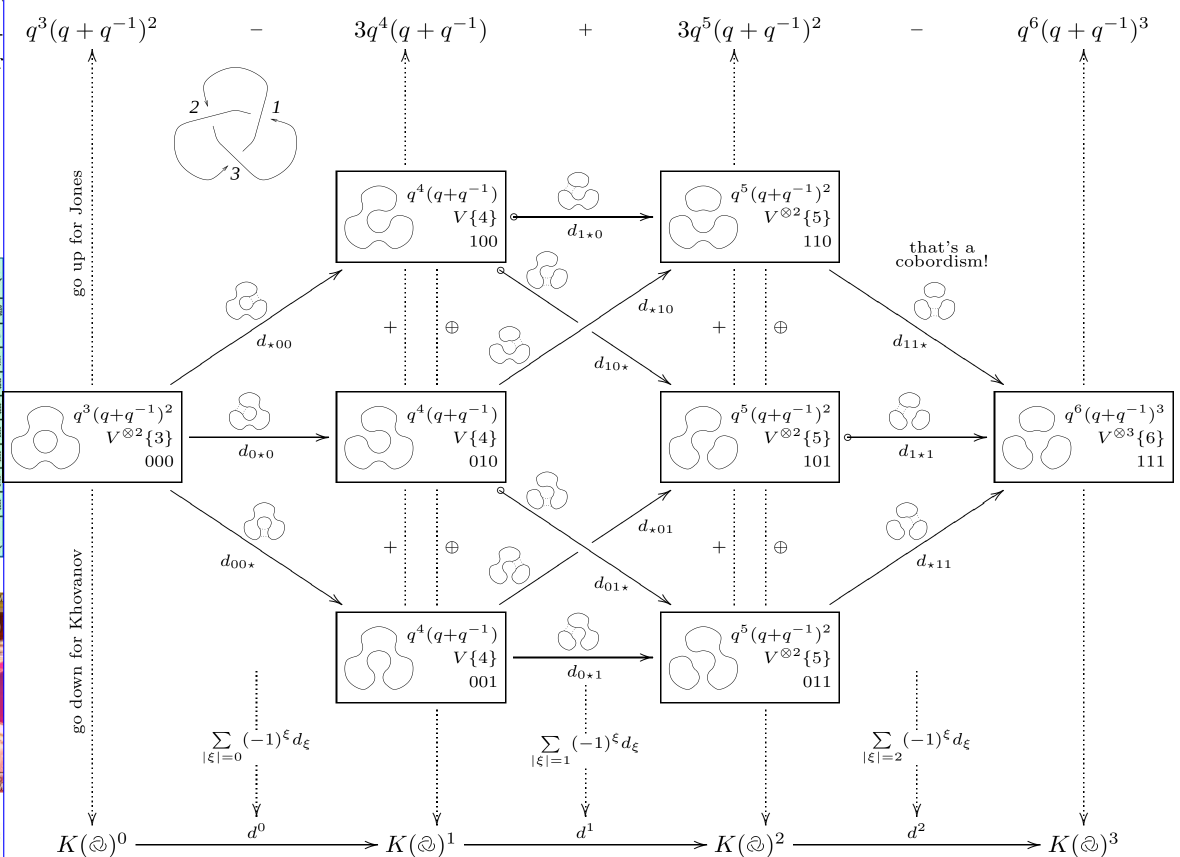
 $\longrightarrow (V\otimes V\xrightarrow{m} V)$ $\quad m: \begin{cases} v_+\otimes v_- \mapsto v_- & v_+\otimes v_+\mapsto v_+\\ v_-\otimes v_+\mapsto v_- & v_-\otimes v_-\mapsto 0\end{cases}$

 $\longrightarrow (V\xrightarrow{\Delta} V\otimes V)$ $\quad \Delta: \begin{cases} v_+\mapsto v_+\otimes v_- + v_-\otimes v_+\\ v_-\mapsto v_-\otimes v_-\end{cases}$

Categorifying $\mathbb{Z}[q^{\pm 1}]$. $f=\sum a_j q^j$ becomes $V=\bigoplus V_j$ s.t. $q\dim V := \sum q^j\dim V_j = f$, or better, $V^\bullet = (\cdots V^{r-1}\to V^r\to V^{r+1}\cdots)$ s.t. $d^2=0$, $\deg d = 0$, and $\chi_q(V^\bullet) := \sum(-1)^r q\dim V^r = f = \sum(-1)^r q\dim H^r$.

Note. Setting $V\{l\}_j := V_{j-l}$, we get $q\dim V\{l\} = q^l\, q\dim V$.

Example: $\quad = \quad q+q^3+q^5-q^9.$



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j=\star$) $\quad = \quad K(\circledast)$.

Theorem 1. The graded Euler characteristic of $K(L)$ is $J(L)$.

Theorem 2. The homology $\mathrm{Kh}(L)$ of $K(L)$ is a link invariant.

Theorem 3. $\mathrm{Kh}(L)$ is strictly stronger than $J(L)$: $J(\bar{5}_1)=J(10_{132})$ yet $\mathrm{Kh}(\bar{5}_1)\neq\mathrm{Kh}(10_{132})$.

References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and my http://www.math.toronto.edu/~drorbn/papers/Categorification/.

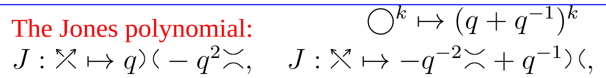
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

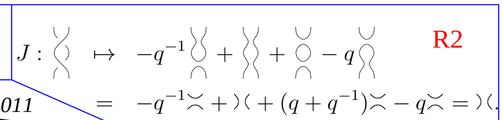
Dror Bar–Natan: Talks: Hamburg–1208

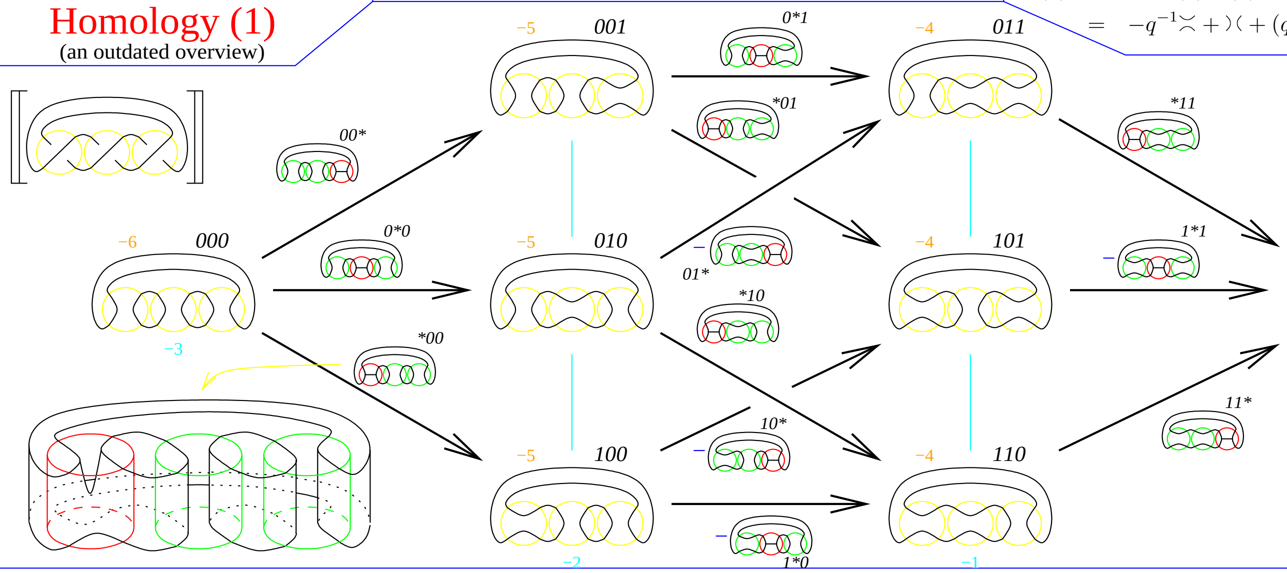
Local Khovanov Homology (1)

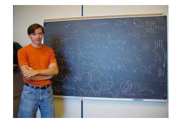
(an outdated overview)

The Jones polynomial:



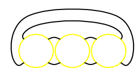
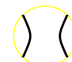
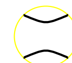


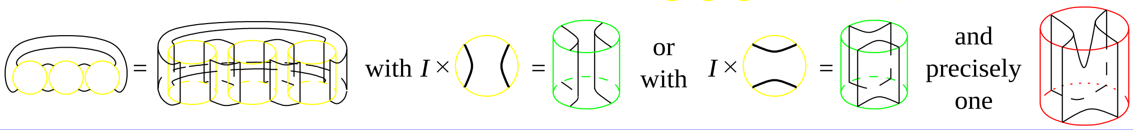




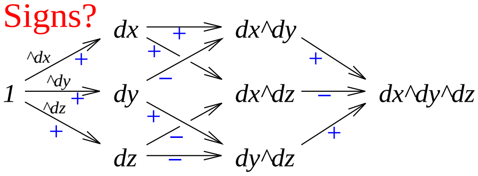
Mikhail Khovanov

What is it?

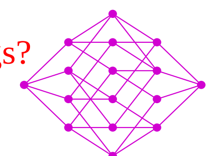
A cube for each knot/link projection; Vertices: All fillings of  with  or with .

Edges: All fillings of $I\times$  with $I\times$)(= … or with $I\times$ ≍ = … and precisely one.

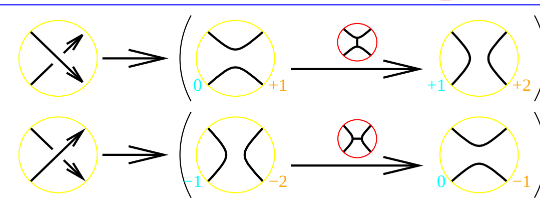
Signs?



More crossings?



General Crossings



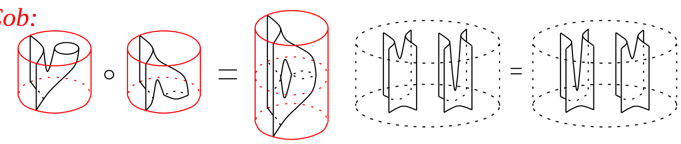
Where does it live?

In *Kom(Mat(<Cob> / {S, T, G, NC})) / homotopy*

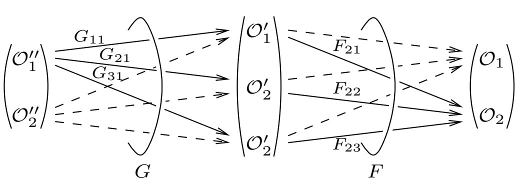
Kom: Complexes Mat: Matrices

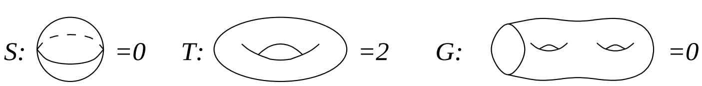
Cob: Cobordisms <...>: Formal lin. comb.

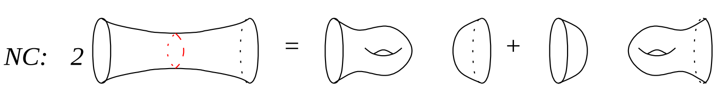
Cob:



Mat(C):







Complexes:

$$\Omega = \left(\Omega^{-n_-} \longrightarrow \Omega^{-n_-+1} \longrightarrow \cdots \qquad \longrightarrow \Omega^{n_+}\right)$$

Morphisms:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F^{r-1}} & & \downarrow{\scriptstyle F^r} & & \downarrow{\scriptstyle F^{r+1}} & & \\
\cdots & \longrightarrow & \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} & \longrightarrow & \cdots
\end{array}$$

Homotopies:

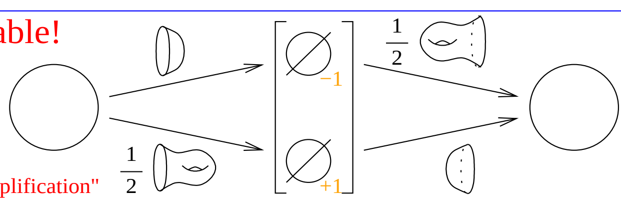
$$\begin{array}{ccccc}
\Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} \\
F^{r-1}\downdownarrows G^{r-1} & \swarrow{\scriptstyle h^r} & F^r\downdownarrows G^r & \swarrow{\scriptstyle h^{r+1}} & F^{r+1}\downdownarrows G^{r+1} \\
\Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1}
\end{array}$$

$$F^r - G^r = h^{r+1}d^r + d^{r-1}h^r$$

All arrows in an arbitrary additive category!

Computable!

via "complex simplification"



The Main Point. "The cube", Kh(L), is an up-to-homotopy invariant of knots and links. It's Euler characteristic is the Jones polynomial, yet it is strictly stronger than the Jones polynomial. It is functorial (in the appropriate sense) and practically computable.

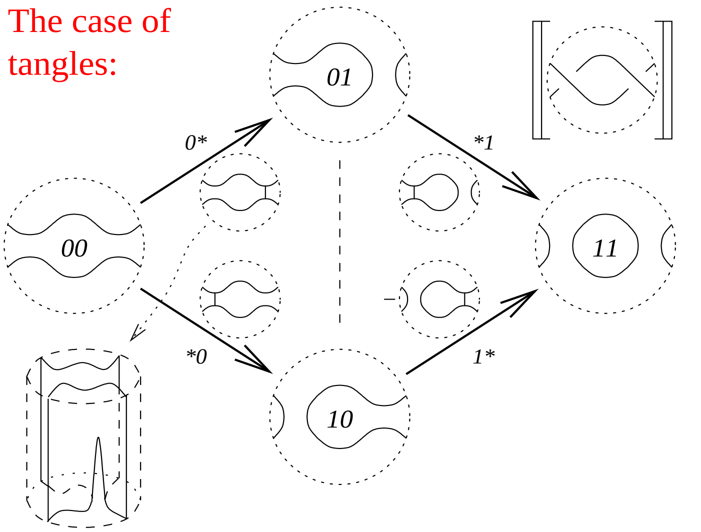
The Categorification Speculative Paradigm. • Every object in math is the Euler characteristic of a complex.
• Every operation lifts to an operation between complexes.
• Every identity remains true, up to homotopy.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

Dror Bar−Natan: Talks: Hamburg−1208

Local Khovanov Homology (2)

The case of tangles:



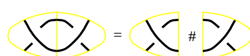
The Reduction Lemma. If ϕ is an isomorphism then the complex

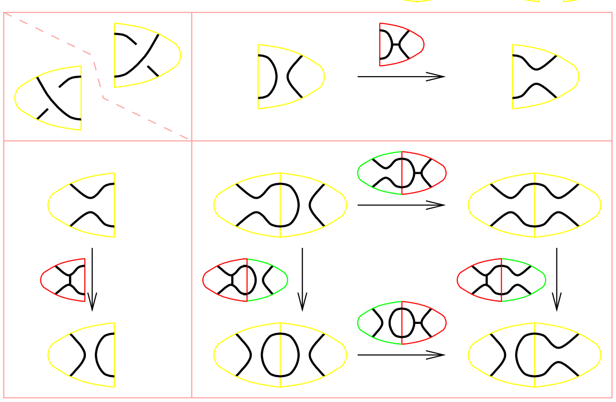
$$[C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]$$

is isomorphic to the (direct sum) complex

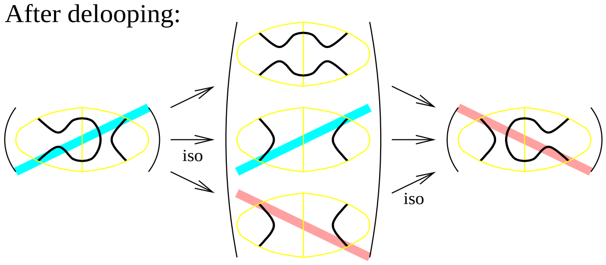
$$[C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]$$

Invariance under R2.



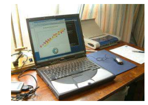
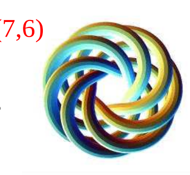


After delooping:



Kurt Reidemeister

I mean business.

In 1 day  says T(7,6) 

Old techniques: ~1,000 years, ~1GGb RAM. (now down to seconds)

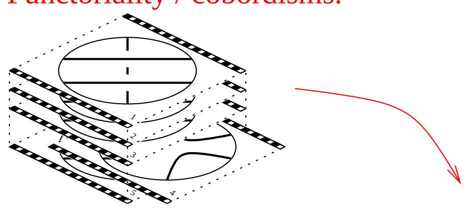
$\dim_j H_r$ is given by:

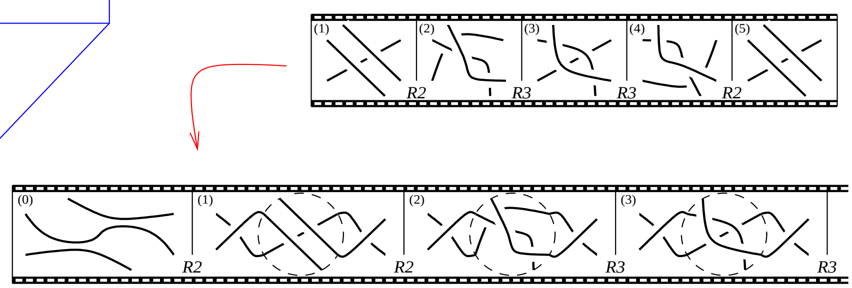
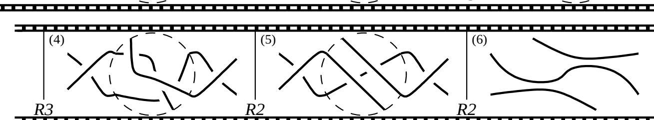
$j \setminus r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
57																	1			1
55																	1	1		
53															1	2		1	1	
51													1	1		2	1			
49														3	1		1			
47												3	1		1					
45										2		1	2							
43								1		1	2									
41						1		1	2		1									
39						1	1		1											
37				1	1		1													
35					1															
33			1																	
31	1																			
29	1																			

the alternating knots zone

Functoriality / cobordisms.

M. Jacobsson

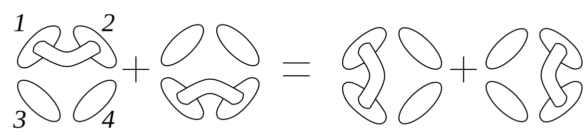


J. Rasmussen: Leads to a no-analysis proof of a conjecture by Milnor.

A more general theory:

Remove G and NC, add

4Tu: 

(minor further revisions are necessary)

"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (modified)

Visit! Edit!

http://katlas.org

http://www.math.toronto.edu/~drorbn/papers/Cobordism/
http://www.math.toronto.edu/~drorbn/papers/FastKh/
http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

The Most Important Missing Infrastructure Project in Knot Theory

January-23-12
10:12 AM

An "infrastructure project" is hard (and sometimes non-glorious) work that's done now and pays off later.

An example, and the most important one within knot theory, is the tabulation of knots up to 10 crossings. I think it precedes Rolfsen, yet the result is often called "the Rolfsen Table of Knots", as it is famously printed as an appendix to the famous book by Rolfsen. There is no doubt the production of the Rolfsen table was hard and non-glorious. Yet its impact was and is tremendous. Every new thought in knot theory is tested against the Rolfsen table, and it is hard to find a paper in knot theory that doesn't refer to the Rolfsen table in one way or another.

A second example is the Hoste-Thistlethwaite tabulation of knots with up to 17 crossings. Perhaps more fun to do as the real hard work was delegated to a machine, yet hard it certainly was: a proof is in the fact that nobody so far had tried to replicate their work, not even to a smaller crossing number. Yet again, it is hard to overestimate the value of that project: in many ways the Rolfsen table is "not yet generic", and many phenomena that appear to be rare when looking at the Rolfsen table become the rule when the view is expanded. Likewise, other phenomena only appear for the first time when looking at higher crossing numbers.

But as I like to say, knots are the wrong object to study in knot theory. Let me quote (with some variation) my own (with Dancso) "WKO" paper:

> Studying knots on their own is the parallel of studying cakes and pastries as they come out of the bakery - we sure want to make them our own, but the theory of desserts is more about the ingredients and how they are put together than about the end products. In algebraic knot theory this reflects through the fact that knots are not finitely generated in any sense (hence they must be made of some more basic ingredients), and through the fact that there are very few operations defined on knots (connected sums and satellite operations being the main exceptions), and thus most interesting properties of knots are transcendental, or non-algebraic, when viewed from within the algebra of knots and operations on knots (see [AKT-CFA]).
>
> The right objects for study in knot theory are thus the ingredients that make up knots and that permit a richer algebraic structure. These are braids (which are already well-studied and tabulated) and even more so tangles and tangled graphs.

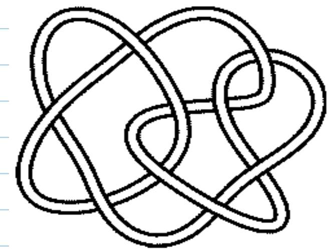
Thus in my mind the most important missing infrastructure project in knot theory is the tabulation of tangles to as high a crossing number as practical. This will enable a great amount of testing and experimentation for which the grounds are now still missing. The existence of such a tabulation will greatly impact the direction of knot theory, as many tangle theories and issues that are now ignored for the lack of scope, will suddenly become alive and relevant. The overall influence of such a tabulation, if done right, will be comparable to the influence of the Rolfsen table.

Aside. What are tangles? Are they embedded in a disk? A ball? Do they have an "up side" and a "down side"? Are the strands oriented? Do we mod out by some symmetries or figure out the action of some symmetries? Shouldn't we also calculate the affect of various tangle operations (strand doubling and deletion, juxtapositions, etc.)? Shouldn't we also enumerate virtual tangles? w-tangles? Tangled graphs?

In my mind it would be better to leave these questions to the tabulator. Anything is better than nothing, yet good tabulators would try to tabulate the more general things from which the more special ones can be sieved relatively easily, and would see that their programs already contain all that would be easy to implement within their frameworks. Counting legs is easy and can be left to the end user. Determining symmetries is better done along with the enumeration itself, and so it should.

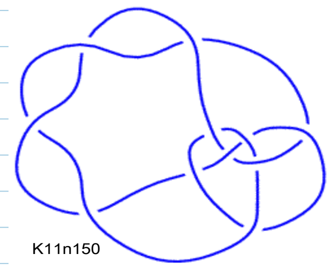
An even better tabulation should come with a modern front-end - a set of programs for basic manipulations of tangles, and a web-based "tangle atlas" for an even easier access.

Overall this would be a major project, well worthy of your time.



(KnotPlot image)

9_42 is Alexander Stoimenow's favourite

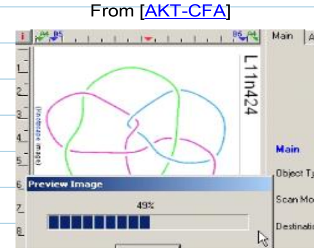


K11n150

(Knotscape image)

The interchange of I-95 and I-695, northeast of Baltimore. (more)

From [AKT-CFA]



From [FastKh]

http://katlas.org/

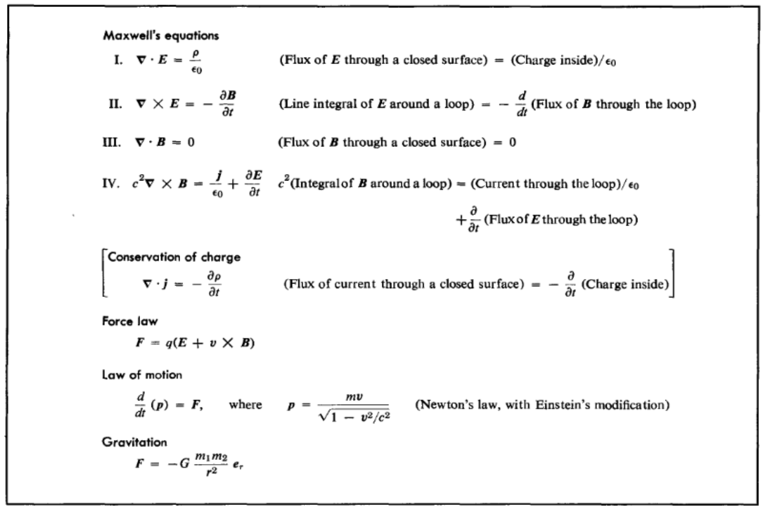
(Source: http://katlas.math.toronto.edu/drorbn/AcademicPensieve/2012-01/)

A Bit on Maxwell's Equations

Prerequisites.

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- Integration by parts: $\int \omega \wedge d\eta = -(-1)^{\deg\omega} \int (d\omega) \wedge \eta$ on domains that have no boundary.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q \mapsto \int_a^b \left(\frac{1}{2}m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

Table 18–1 Classical Physics



Maxwell's equations	
I. $\nabla \cdot E = \frac{\rho}{\epsilon_0}$	(Flux of E through a closed surface) = (Charge inside)/ϵ_0
II. $\nabla \times E = -\frac{\partial B}{\partial t}$	(Line integral of E around a loop) = $-\frac{d}{dt}$(Flux of B through the loop)
III. $\nabla \cdot B = 0$	(Flux of B through a closed surface) = 0
IV. $c^2\nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$	c^2(Integral of B around a loop) = (Current through the loop)/ϵ_0 $+\frac{\partial}{\partial t}$(Flux of E through the loop)
[**Conservation of charge**	
$\nabla \cdot j = -\frac{\partial \rho}{\partial t}$	(Flux of current through a closed surface) = $-\frac{\partial}{\partial t}$(Charge inside)]
Force law	
$F = q(E + v \times B)$	
Law of motion	
$\frac{d}{dt}(p) = F$, where $p = \frac{mv}{\sqrt{1 - v^2/c^2}}$	(Newton's law, with Einstein's modification)
Gravitation	
$F = -G\frac{m_1 m_2}{r^2} e_r$	

The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The *Vector Field* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2}||dA||^2 dtdxdydz + J \wedge A}$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

These are the Maxwell equations! Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$dJ = 0 \Longrightarrow$	$\frac{\partial\rho}{\partial t} + \operatorname{div} j = 0$	"conservation of charge"
$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
	$\operatorname{curl} E = -\frac{\partial B}{\partial t}$	that's how generators work!
$d * F = J \Longrightarrow$	$\operatorname{div} E = -\rho$	"electrostatics"
	$\operatorname{curl} B = -\frac{\partial E}{\partial t} + j$	that's how electromagnets work!

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise. With $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e^2}(ds + eA)$ to derive Feynman's "law of motion" and "force law".

The Pure Virtual Braid Group is Quadratic[1]

Dror Bar-Natan and Peter Lee in Oregon, August 2011
http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
foots & refs on PDF version, page 3

Abstract Generalities

Let K be a unital algebra over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; so $K/I \xrightarrow[\epsilon]{\sim} \mathbb{F}$.

Definition. Say that K is *quadratic* if its associated graded $\operatorname{gr}K=\bigoplus_{p=0}^{\infty} I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=q(K)=\langle V=I/I^2\rangle/\langle R_2=\ker(\bar\mu_2 : V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (q is a lovely functor). Then K is quadratic iff the obvious $\mu : A\to \operatorname{gr}K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

The Overall Strategy. Consider the "singularity tower" of (K,I) (here ":" means $\otimes_K$ and μ is (always) multiplication):

$$\cdots\ I^{:p+1}\xrightarrow{\mu_{p+1}} I^{:p}\xrightarrow{\mu_p} I^{:p-1}\longrightarrow\cdots\longrightarrow K$$

We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$. Hence we ask:

- What's $I^{:p}/\mu(I^{:p+1})$?
- How injective is this tower?

Lemma. $I^{:p}/\mu(I^{:p+1})\simeq (I/I^2)^{\otimes p}=V^{\otimes p}$; set $\pi : I^{:p}\to V^{\otimes p}$.

Flow Chart.

$$\begin{tikzcd}
\boxed{\text{Any }(K,I)} \arrow[r, "\text{Prop 1}"] & \boxed{\text{2-local}} \arrow[r, "\text{Prop 2}"] & \boxed{\text{Quadratic}} \\
\boxed{K=PvB_n} \arrow[r, "\text{Thm S}", "\text{by Peter}"'] & \boxed{\text{Hutchings' Criterion}} \arrow[r] & \boxed{\text{2-injective}} \arrow[u]
\end{tikzcd}$$

Proposition 1. The sequence

$$\mathfrak{R}_p:=\bigoplus_{j=1}^{p-1}\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)\xrightarrow{\partial} I^{:p}\xrightarrow{\mu_p} I^{:p-1}$$

is exact, where $\mathfrak{R}_2:=\ker\mu : I^{:2}\to I$; so (K,I) is "2-local".

The Free Case. If J is an augmentation ideal in $K=F=\langle x_i\rangle$, define $\psi : F\to F$ by $x_i\mapsto x_i+\epsilon(x_i)$. Then $J_0:=\psi(J)$ is $\{w\in F : \deg w>0\}$. For J_0 it is easy to check that $\mathfrak{R}_2=\mathfrak{R}_p=0$, and hence the same is true for every J.

The General Case. If $K=F/\langle M\rangle$ (where M is a vector space of "moves") and $I\subset K$, then $I=J/\langle M\rangle$ where $J\subset F$. Then $I^{:p}=J^{:p}\big/\sum J^{:j-1}:\langle M\rangle:J^{:p-j}$ and we have

$$\begin{tikzcd}
J^{:p} \arrow[rr, "\mu_F", "\text{1-1}"'] \arrow[d, "\pi_p", "\text{onto}"'] & & J^{:p-1} \arrow[d, "\pi_{p-1}"', "\text{onto}"] \\
I^{:p}=J^{:p}\big/\sum J^{:}:\langle M\rangle:J^{:} \arrow[rr, "\mu"] & & I^{:p-1}=J^{:p-1}\big/\sum J^{:}:\langle M\rangle:J^{:}
\end{tikzcd}$$

So[2] $\ker(\mu)=\pi_p\left(\mu_F^{-1}(\ker\pi_{p-1})\right)=\pi_p\left(\sum\mu_F^{-1}\left(J^{:}:\langle M\rangle:J^{:}\right)\right)=\sum\pi_p\left(J^{:}:\mu_F^{-1}\langle M\rangle:J^{:}\right)=\sum I^{:}:\mathfrak{R}_2:I^{:}=:\sum_{j=1}^{p-1}\mathfrak{R}_{p,j}$.

$\mathfrak{R}_2$ is simpler than may seem! It's an "augmentation bimodule" ($I\mathfrak{R}_2=0=\mathfrak{R}_2 I$ thus $xr=\epsilon(x)r=r\epsilon(x)=rx$ for $x\in K$ and $r\in\mathfrak{R}_2$), and hence $\mathfrak{R}_2=\pi_2(\mu_F^{-1}M)$.

$$\begin{tikzcd}
J^{:2} \arrow[r, "\mu_F", "\text{1-1}"'] \arrow[d, "\pi_2"] & J\supset M \arrow[d, "\pi_1"] \\
I^{:2} \arrow[r, "\mu"] & I=J/\langle M\rangle
\end{tikzcd}$$

$\mathfrak{R}_p$ is simpler than may seem! In $\mathfrak{R}_{p,j}=I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}$ the I factors may be replaced by $V=I/I^2$. Hence

$$\mathfrak{R}_p\simeq\bigoplus_{j=1}^{p-1}V^{\oplus j-1}\otimes\pi_2(\mu_F^{-1}M)\otimes V^{\otimes p-j-1}.$$

Claim. $\pi(\mathfrak{R}_{p,j})=R_{p,j}$; namely,

$$\pi\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)=V^{\otimes j-1}\otimes R_2\otimes V^{\otimes p-j-1}.$$

Why Care?

- In abstract generality, $\operatorname{gr}K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z : K\to\hat{A}$, becomes wonderful mathematics:

K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torossian [KV, AT]

2-Injectivity. A (one-sided infinite) sequence

$$\cdots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}} K_p\xrightarrow{\delta_p}\cdots\longrightarrow K_0=K$$

is "injective" if for all $p>0$, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\cdots\longrightarrow\frac{K_{p+1}}{\ker\delta_{p+1}}\xrightarrow{\bar\delta_{p+1}}\frac{K_p}{\ker\delta_p}\xrightarrow{\bar\delta_p}\frac{K_{p-1}}{\ker\delta_{p-1}}\longrightarrow\cdots$$

is injective; i.e. if for all p, $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if its singularity tower is 2-injective.

Proposition 2. If (K,I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get $\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}$. But $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}\big/\sum\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)$. But that's the degree p piece of $q(K)$.

The X Lemma (inspired by [Hut]).

$$\begin{tikzcd}
A_0 \arrow[dr, "\alpha_0"] & & C_0 \\
& B \arrow[ur, "\beta_0"] \arrow[dr, "\beta_1"] & \\
A_1 \arrow[ur, "\alpha_1"] & & C_1
\end{tikzcd}$$

Michael Hutchings

will compute homology for food

If the above diagram is Conway ($\asymp$) exact, then its two diagonals have the same "2-injectivity defect". That is, if $A_0\to B\to C_0$ and $A_1\to B\to C_1$ are exact, then $\ker(\beta_1\circ\alpha_0)/\ker\alpha_0\simeq\ker(\beta_0\circ\alpha_1)/\ker\alpha_1$.

Proof. $\frac{\ker(\beta_1\circ\alpha_0)}{\ker\alpha_0}\xrightarrow[\alpha_0]{\sim}\ker\beta_1\cap\operatorname{im}\alpha_0$

$$=\ker\beta_0\cap\operatorname{im}\alpha_1\xleftarrow[\alpha_1]{\sim}\frac{\ker(\beta_0\circ\alpha_1)}{\ker\alpha_1}.$$

The Hutchings Criterion [Hut]. The singularity tower of (K,I) is 2-injective iff on the right, $\ker(\pi\circ\partial)=\ker(\partial)$. That is, iff every "diagrammatic syzygy" is also a "topological syzygy".

$$\begin{tikzcd}
\mathfrak{R}_p \arrow[dr, "\partial"] & & I^{:p-1} \\
& I^{:p} \arrow[ur, "\mu_p"] \arrow[dr, "\pi"] & \\
I^{:p+1} \arrow[ur, "\mu_{p+1}"] & & V^{\otimes p}
\end{tikzcd}$$

Conclusion. We need to know that (K,I) is "syzygy complete" — that every diagrammatic syzygy is also a topological syzygy, that $\ker(\pi\circ\partial)=\ker(\partial)$.

More at http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/

The Pure Virtual Braid Group is Quadratic, II

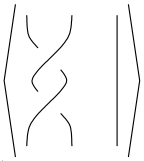
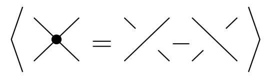
Examples and Interpretations

Dror Bar-Natan and Peter Lee in Oregon, August 2011

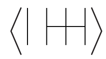
http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/

Example. (goes back to [Koh])

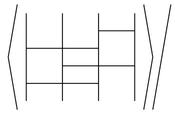
T. Kohno

$K =$  $I =$ 

$(K/I^{p+1})^\star = (\text{invariants of type } p) =: \mathcal{V}_p$

$(I^p/I^{p+1})^\star = \mathcal{V}_p/\mathcal{V}_{p-1} \quad V = \langle t^{ij} | t^{ij} = t^{ji} \rangle =$ 

$\ker \bar\mu_2 = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle 4\text{T relations} \rangle$

$A = q(K) = \left(\text{horizontal chord diagrams mod 4T}\right) =$  $/$ 4T

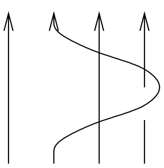
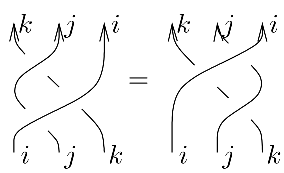
Z: universal finite type invariant, the Kontsevich integral.

PvB_n is the group

$$\langle \sigma_{ij} : 1 \leq i \neq j \leq n \rangle \Big/ \begin{array}{c} \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \end{array}$$

L. Kauffman [Kau, KL]

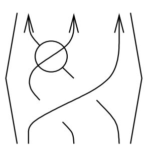
of "pure virtual braids" ("braids when you look", "blunder braids"):

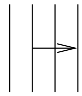
$\sigma_{24} =$  R3: 

The Main Theorem [Lee]. PvB_n is quadratic.

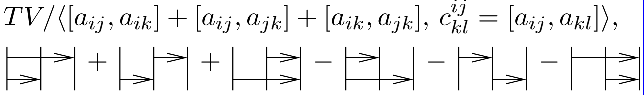
$A_n = q(PvB_n)$. [GPV]

Goussarov-Polyak-Viro

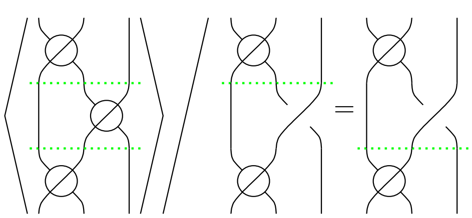
$I =$  with $\tilde\sigma_{ij} = \sigma_{ij} - 1$, the "semi-virtual crossing".

$V = I/I^2 = \left\langle \begin{array}{c}\text{v-braids} \\ \text{with one semi-virtual crossing}\end{array} \right\rangle \Big/ (\text{virtual} = \text{crossing})$
$= \langle a_{ij} \rangle_{1 \leq i \neq j \leq n}$ $\quad a_{24} =$ 

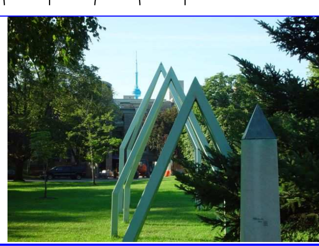
$A_n = TV / \langle [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}],\ c^{ij}_{kl} = [a_{ij}, a_{kl}] \rangle,$

$y_{ijk} =$ 

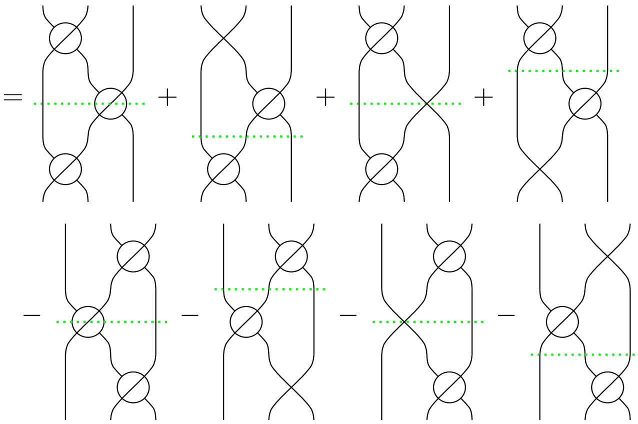
$I^{:p}$.



James Gillespie's Sightline #2 (1984) is a syzygy, and (arguably) Toronto's largest sculpture. Find it next to University of Toronto's Hart House.



$\mathfrak{R}_2(PvB_n)$ is generated as a vector space by C^{ij}_{kl} and

$Y_{ijk} :=$ 

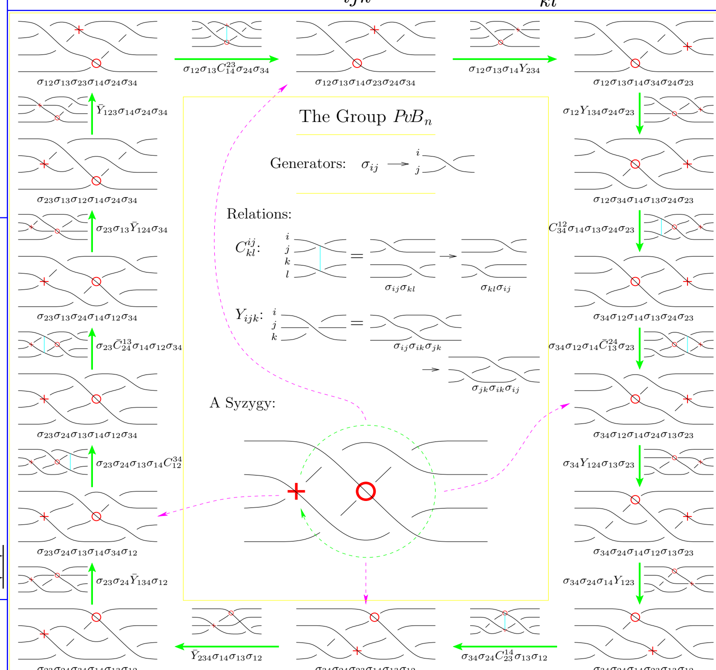
Syzygy Completeness, for PvB_n, means:

$$\mathfrak{R}_p = \bigoplus_{j=1}^{p-1} \mathfrak{R}_{p,j} \xrightarrow{\partial} I^{:p} \xrightarrow{\pi} V^{\otimes p}$$

$\{\tilde\sigma_{12} : \underline{Y_{345}} : \tilde\sigma_{67} : \ldots\} \longrightarrow$

$\{\tilde\sigma_{12} : Y_{345} : \tilde\sigma_{67} : \ldots\} \longrightarrow \{a_{12} y_{345} a_{67} \ldots\}$

Is every relation between the y_{ijk}'s and the c^{ij}_{kl}'s also a relation between the Y_{ijk}'s and the C^{ij}_{kl}'s?



Theorem S. Let D be the free associative algebra generated by symbols a_{ij}, y_{ijk} and c^{ij}_{kl}, where $1 \leq i, j, k, l \leq n$ are distinct integers. Let D_0 be the part of D with only a_{ij} symbols and let D_1 be the span of the monomials in D having only a_{ij} symbols, with exactly one exception that may be either a y_{ijk} or a c^{ij}_{kl}. Let $\partial : D_1 \to D_0$ be the map defined by

$$\begin{aligned} y_{ijk} &\mapsto [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}], \\ c^{ij}_{kl} &\mapsto [a_{ij}, a_{kl}]. \end{aligned}$$

Then $\ker \partial$ is generated by a family of elements readable from the picture above and by a few similar but lesser families.

More at http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/

Footnotes

1. Following a homonymous paper and thesis by Peter Lee [Lee]. All serious work here is his and was extremely patiently explained by him to DBN. Page design by the latter.
2. The proof presented here is broken. Specifically, at the very end of the proof of the "general case" of Proposition 1 the sum that makes up $\ker\pi_{p-1}$ is interchaged with μ_F^{-1}. This is invalid; in general it is not true that $T^{-1}(U+V) = T^{-1}(U) + T^{-1}(V)$, when T is a linear transformation and U and V are subspaces of its target space. We thank Alexander Polishchuk for noting this gap. A handwritten non-detailed fix can be found at `http://katlas.math.toronto.edu/drorbn/AcademicPensieve/Projects/Quadraticity/`, especially under "Oregon Handout Post Mortem". A fuller fix will be made available at a later time.

References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* arXiv:0802.4300.

[BN1] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[BN2] D. Bar-Natan, *Facts and Dreams About v-Knots and Etingof-Kazhdan,* talk presented at the Swiss Knots 2011 conference. Video and more at `http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/`.

[BN3] D. Bar-Natan, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne,* paper and related files at `http://www.math.toronto.edu/~drorbn/papers/WKO/`.

[BND] D. Bar-Natan, and Z. Dancso, *Homomorphic Expansions for Knotted Trivalent Graphs,* arXiv:1103.1896.

[BEER] L. Bartholdi, B. Enriquez, P. Etingof, and E. Rains, *Groups and Lie algebras corresponding to the Yang-Baxter equations,* Journal of Algebra **305-2** (2006) 742-764, arXiv:math.RA/0509661.

[Dri] V. G. Drinfel'd, *Quasi-Hopf Algebras,* Leningrad Math. J. **1** (1990) 1419–1457 and *On Quasitriangular Quasi-Hopf Algebras and a Group Closely Connected with Gal*$(\bar{\mathbb{Q}}/\mathbb{Q})$*,* Leningrad Math. J. **2** (1991) 829–860.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica, New Series **2** (1996) 1–41, arXiv:q-alg/9506005, and *Quantization of Lie Bialgebras, II,* Selecta Mathematica, New Series **4** (1998) 213–231, arXiv:q-alg/9701038.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[Hav] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants,* Hebrew University PhD thesis, September 2002, arXiv:math.QA/0211031.

[Hut] M. Hutchings, *Integration of singular braid invariants and graph cohomology,* Transactions of the AMS **350** (1998) 1791–1809.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions,* Invent. Math. **47** (1978) 249–272.

[Kau] L. H. Kauffman, *Virtual Knot Theory,* European J. Comb. **20** (1999) 663–690, arXiv:math.GT/9811028.

[KL] L. H. Kauffman and S. Lambropoulou, *Virtual Braids,* Fundamenta Mathematicae **184** (2005) 159–186, arXiv:math.GT/0407349.

[Koh] T. Kohno, *Monodromy representations of braid groups and Yang-Baxter equations,* Ann. Inst. Fourier **37** (1987) 139–160.

[Lee] P. Lee, *The Pure Virtual Braid Group is Quadratic,* in preparation. See links at `http://www.math.toronto.edu/drorbn/Talks/Oregon-1108/`.

Lecture 2 Handout

More on Chern-Simons Theory and Feynman Diagrams

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

After $A \mapsto A/\sqrt{k}$, and setting $\hbar = \frac{1}{\sqrt{k}}$:

$$Z(\gamma) = \int_{A \in \Omega^1(\mathbb{R}^3, \mathfrak{g})} \mathcal{D}A \, tr_R \, hol_\gamma(A) \, e^{\frac{i}{4\pi} \underbrace{\int_{\mathbb{R}^3} tr(A \wedge dA + \frac{2\hbar}{3} A \wedge A \wedge A)}_{CS(A)}}$$

where $tr_R hol_\gamma(A) = tr_R \left(1 + \hbar \int ds \, A(\dot\gamma(s)) + \hbar^2 \int_{s_1 < s_2} A(\dot\gamma(s_1)) A(\dot\gamma(s_2)) + \ldots\right)$

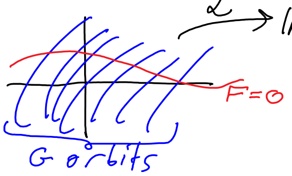
Trouble! "d" is not invertible!

Gauge Invariance: $CS(A)$ is invariant under $A \mapsto A + \delta A$, $\delta A = -(dc + \hbar[A, c])$, $c \in \Omega^0(\mathbb{R}, \mathfrak{g})$

Back to the drawing board....

Suppose $\mathcal{L}(x)$ on $\mathbb{R}^n$ is invariant under a k-dimensional group G w/ Lie algebra $\mathfrak{g} = \langle g_a \rangle$, and suppose $F: \mathbb{R}^n \to \mathbb{R}^k$ is such that $F = 0$ is a section of the G-action:

$$G \to \mathbb{R}^n \xrightarrow{F} \mathbb{R}^k$$



Then

$$\int_{\mathbb{R}^n} dx \, e^{i\mathcal{L}} \sim \int_{\mathbb{R}^n} dx \, e^{i\mathcal{L}} \delta(F(x)) \cdot \det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

$$\sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^k} d\phi \, e^{i(\mathcal{L} + F(x)\cdot\phi)} \det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

Perturbation theory for determinants?

$$\det(J_0 + \hbar J_1(x)) = \det(J_0) \sum_m \hbar^m \, Tr\left(\Lambda^m J_0^{-1}\right) \cdot \left(\Lambda^m J_1(x)\right)$$

Berezin Fermionic Anti-commuting Variables: $\int d^k\bar{c} \, d^k c \, e^{i\bar{c}_a J^a_b c^b} \sim \det(J)$

So

$$Z \sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^k} d\phi \int d^k\bar{c} \int d^k c \, e^{i\mathcal{L}_{tot}} \text{ where}$$

$$\mathcal{L}_{tot} = \underbrace{\mathcal{L}(x)}_{\text{the original}} + \underbrace{F(x)\cdot\phi}_{\text{gauge-fixing}} + \underbrace{\bar{c}_a \left(\frac{\partial F^a}{\partial g_b}\right) c^b}_{\text{"ghosts"}}$$

In Chern-Simons, w/ $F(A) := d^*A = \partial_i A^i$, get

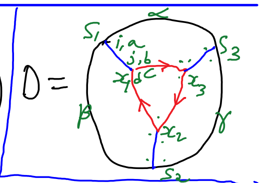
$$\mathcal{L}_{tot} = \frac{k}{4\pi} \int_{\mathbb{R}^3} tr(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A + \phi \, \partial_i A^i + \bar{c} \, \partial_i(\partial^i + ad A^i) c$$

So we have

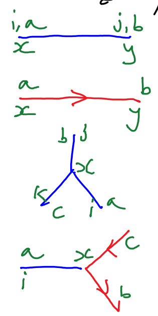
* A bosonic quadratic term involving $\binom{A}{\phi}$.
* A fermionic quadratic term involving $\bar{c}, c$.
* A cubic interaction of 3 A's.
* A cubic $A\bar{c}c$ vertex.
* Funny A and γ "holonomy" vertices along γ.

After much crunching:

$$Z(\gamma) = \sum_{m=0}^{\infty} \hbar^m \sum_{\text{Feynman diags } D} \oint \mathcal{E}(D) \quad D =$$



where $\mathcal{E}(D)$ is constructed as follows:

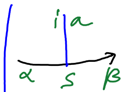


$\longrightarrow \epsilon_{ijk} t^{ab} \frac{i(x-y)^k}{2|x-y|^3}$

$\longrightarrow \frac{t^{ab}}{4\pi|x-y|}$

$\longrightarrow \frac{i}{2\pi} \int_{\mathbb{R}^3} t_{abc} \epsilon^{ijk}$

$\longrightarrow \frac{1}{2\pi} \int_{\mathbb{R}^3} dz \, t_{abc} \, \underbrace{\partial^i_x}_{\text{acting only in } b\text{-direction}}$



$\int_{S^1} ds \, R^{\alpha}_{a\beta} \dot\gamma^i(s)$

$(-)$ sign for each red loop.

By a bit of a miracle, this boils down to a configuration space integral, which in itself, can be reduced to a pre-image count.

... But I run out of steam for tonight...

Banks like knots.

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107/

Lecture 3 Handout

The Basics of Finite-Type Invariants of Knots

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

Definition. A knot invariant is any function whose domain is {knots}. Really, we mean a computable function whose target space is understandable; e.g.

$$C:\{\text{knots}\}/(\text{Reidemeister moves}) \to \mathbb{Z}[z]$$

Example. The Conway polynomial is given by

$$C(\overcrossing) - C(\undercrossing) = zC(\smoothing)$$

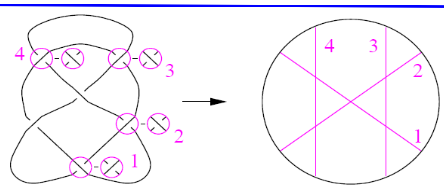
and

$$C(\underbrace{\bigcirc\bigcirc\cdots\bigcirc}_{k}) = \begin{cases} 1 & k=1 \\ 0 & k>1 \end{cases}$$

Exercise. Pick your favourite bank and compute the Conway polynomial of its logo.

INTERLINK

Definition. Any $V:\{\text{knots}\}\to$ Abelian Group A can be extended to "knots w/ double points"



using $V(\times\!\!\!\bullet) = V(\overcrossing) - V(\undercrossing)$. (Think "differentiation")

Definition. V is of type m if always

$$V(\underbrace{\times\!\!\!\bullet\ \times\!\!\!\bullet \cdots\cdots \times\!\!\!\bullet}_{m+1}) = 0 \qquad (\text{think "polynomial"})$$

Conjecture. Finite type invariants separate knots.

Theorem. If $C(K) = \sum_{m=0}^{\infty} V_m(K)z^m$ then V_m is of type m.

Proof. $C(\times\!\!\!\bullet) = C(\overcrossing) - C(\undercrossing) = zC(\smoothing)$ $\square$

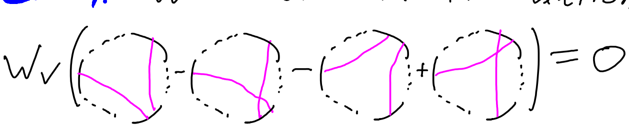
Let V be of type m; then $V^{(m)}$ is constant:

$$V(\underbrace{\times\!\!\!\bullet\cdots\times\!\!\!\bullet}_{m}\overcrossing) = V(\times\!\!\!\bullet\cdots\times\!\!\!\bullet\undercrossing)$$

So $W_V := V^{(m)} = V\big|_{m\text{-singular knots}}$ is really a function on m-chord diagrams: $W_V:\{\text{chord diagrams}\}\to A$

Claim. W_V satisfies the 4T relation:

$$W_V\Big(\ \ \Big) - \ \ - \ \ + \ \ \Big) = 0$$



Proof. $V(\ \ ) = V(\ \ )$ $\quad x\cdot\cdot x\ \ m-2 \qquad x\cdot\cdot x\ \ m-2$ $\square$

Exercise for Lecture 2. Use $\int_{\mathbb{R}} e^{-x^2/2} = \sqrt{2\pi}$, Fubini's theorem, and polar coordinates to compute $\int_{\mathbb{R}^n} e^{-\|x\|^2/2}d^nx$ in two different ways and hence to deduce the volume of S^{n-1}, the $(n-1)$-dimensional sphere.

Exercise. 1. Determine the "weight system" W_{V_m} of the m-th coefficient of the Conway polynomial and verify that is satisfies 4T.
2. Learn somewhere about the Jones polynomial, and do the same for its coefficients.

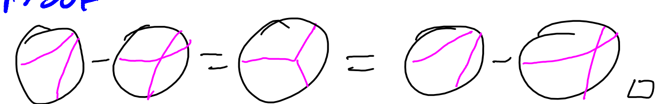
Theorem. (The Fundamental Theorem) Every "weight system", i.e. every linear functional W on $\mathcal{A} := \{\text{chord diagrams}\}/4T$ is the m'th derivative of a type m invariant: $\forall W\ \exists V$ s.t. $W = W_V$

M. Kontsevich

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

Theorem. $\mathcal{A}^{\text{today}} \cong \mathcal{A}^{\text{Monday}}$

Proof



$\square$

Proposition. The fundamental thm holds iff there exists an expansion: $Z:\mathcal{K}\to\hat{\mathcal{A}}$ s.t. if K is m-singular, then

$$Z(K) = D_K + \text{higher degrees.}$$

Proof.

$$\begin{tikzcd} \mathcal{K} \arrow[r, "Z"] \arrow[dr, "V"'] & \mathcal{A} \arrow[d, "W"] \\ & \mathbb{Q} \end{tikzcd} \qquad \square$$

Also see my old paper, "On the Vassiliev knot invariants" (google will find...)

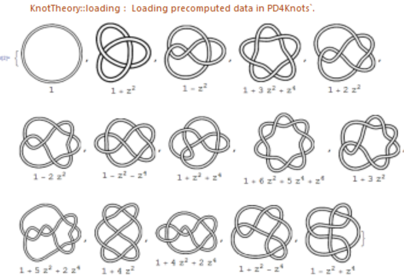
```
In[1]:= << KnotTheory`

Loading KnotTheory` version of August 22, 2010, 13:36:57.55.
Read more at http://katlas.org/wiki/KnotTheory.

In[2]:= Column[{
      Import[
       "C:\\drorbn\\AcademicPensieve\\2011-07\\RolfsenKnots\\"
        <> ToString@#[[1]] <>
        "." <> ToString@#[[2]] <> "_240.gif"],
      Conway[#][z]
     }, Center
    ] & /@ AllKnots[{0, 7}]

KnotTheory::loading : Loading precomputed data in PD4Knots`.
```

J.H. Conway



Low and High Algebra in the "u" Case

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

The big picture, "u" Case.

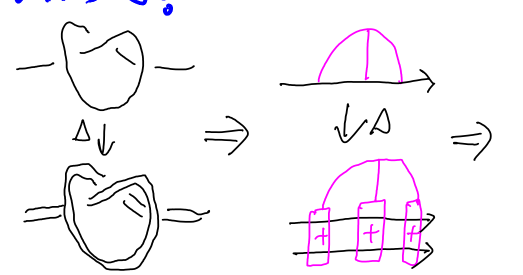
Topology — Combinatorics

$\mathcal{K} \xrightarrow[\text{expansion}]{Z} \mathcal{A} \xrightarrow{\text{choose } \mathfrak{g}} U(\mathfrak{g})$, and $\mathcal{A} \xrightarrow{\text{choose } \mathfrak{g}, R} \mathbb{Q}$

$\mathcal{K} = \{\text{knots etc.}\}$, $\mathcal{A} = \{\text{chord diagrams}\}/IHX, STU, 4T$

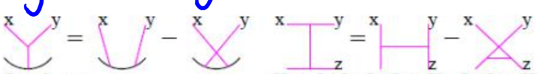
$\underbrace{\qquad}_{\text{high algebra}} \underbrace{\qquad}_{\text{low algebra}}$

What's Δ?



$U(\mathfrak{g}) \xrightarrow{\Delta} U(\mathfrak{g})^{\otimes 2}$

Very low algebra.

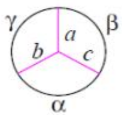


[x,y] = xy − yx   [[x,y],z]=[x,[y,z]]−[y,[x,z]]

More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [a,b], c\rangle \qquad X_a v_\beta = \sum_{\gamma} r^\beta_{a\gamma} v_\gamma$$

and then

$$W_{\mathfrak{g},R}: \quad$$  $$\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

Exercise. Find a fast method to find $W_{\mathfrak{g},R}(D)$ when $\mathfrak{g} = \mathfrak{gl}_n$, $R = \mathbb{R}^n$? Is it related to the Conway polynomial?

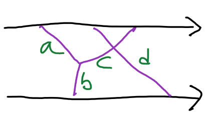
Universal Representation Theory.

Inspired by $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$, set $U(\mathfrak{g}) = \langle \text{words in } \mathfrak{g}\rangle / [x,y] = xy - yx$

* Every rep of $\mathfrak{g}$ extends to $U(\mathfrak{g})$.
* $\exists\, \Delta: U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes 2}$ by "word splitting", as must be for $R_1 \otimes R_2$.

Exercise. With $\mathfrak{g} = \langle x, y\rangle / [x,y] = x$, determine $U(\mathfrak{g})$. Guess a generalization.

Low algebra. $\mathcal{A}(\uparrow\uparrow) \to U(\mathfrak{g})^{\otimes 2}$ via

 $$\longrightarrow \sum_{a\text{-}d} f_{abc} \begin{pmatrix} X_a X_d X_c \\ \otimes \\ X_b X_d \end{pmatrix}$$

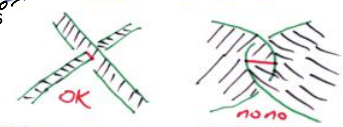
& likewise, $\mathcal{A}(\uparrow_n) \to U(\mathfrak{g})^{\otimes n}$ $\Rightarrow$ $\mathcal{A}(\uparrow_n)$ is "universal universal rep. theory"!

A "Homomorphic Expansion" $Z: \mathcal{K} \to \mathcal{A}$

is an expansion that intertwines all relevant algebraic ops. If $\mathcal{K}$ is finitely presented, finding Z is **High Algebra.**

$\{\text{Ribbon knots}\} \overset{\uparrow\ \text{more or less}}{=} \{u\gamma : \gamma \in \mathcal{K}(O\text{-}O),\ d\gamma = OO\}$

where Ribbon means



and

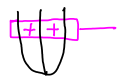
Algebraic knot Theory:

$$\begin{array}{ccccc} & \nearrow^{d} & \mathcal{K}(OO) & \xrightarrow{Z} & \mathcal{A}(OO) \ni OO \\ \mathcal{K}(O\text{-}O) & \xrightarrow{\quad Z \quad} & & & \mathcal{A}(O\text{-}O) \\ & \searrow_{u} & \mathcal{K}(O\text{-}O) & \xrightarrow{Z} & \mathcal{A}(O) \end{array}$$

with $\mathcal{A}(O\text{-}O) \xrightarrow{d} \mathcal{A}(OO)$ and $\mathcal{A}(O\text{-}O) \xrightarrow{u} \mathcal{A}(O)$.

So

$$Z(\{\text{Ribbon knots}\}) \subset \{u\alpha : d\alpha = Z(OO)\} \subset \mathcal{A}(O\text{-}O)$$

VI  $= 0$, follows from  $=$ 

An Associator:

Quantum Algebra's "root object"

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$$\begin{array}{ccc} ((AB)C)D & \longrightarrow & (AB)(CD) \\ & (\Delta 11)\Phi & \\ \downarrow \Phi 1 & & \downarrow (11\Delta)\Phi \\ (A(BC))D & & A(B(CD)) \\ \searrow (1\Delta 1)\Phi & & \nearrow 1\Phi \\ & A((BC)D) & \end{array}$$

$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

The hexagon? Never heard of it.

See Also. B-N & Dancso, arXiv: 1103.1896

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107/

Facts and Dreams About v−Knots and Etingof−Kazhdan, 1

Dror Bar−Natan at Swiss Knots 2011
http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/

This is an overview with too many and not enough details. I apologize.

Foots & refs on PDF version, page 3.

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan [EK] quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.

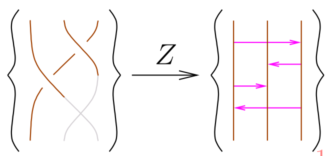
Abstract Generalities. (K, I): an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the "I-adic completion". $\operatorname{gr}_I K := \widehat{\bigoplus} I^m/I^{m+1}$ has a product μ, especially, $\mu_{11}\colon (C = I/I^2)^{\otimes 2} \to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \widehat{FC}/\langle\ker \mu_{11}\rangle$ of K surjects using μ on $\operatorname{gr} K$.

Peter Lee

The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filterred $Z\colon K \to \mathcal{A}$ for which $\operatorname{gr} Z : \operatorname{gr} K \to \mathcal{A}$ inverts μ.

Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way. 6

Example 2. For $K = \mathbb{Q}PvB_n$ = "braids when you look", [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one.



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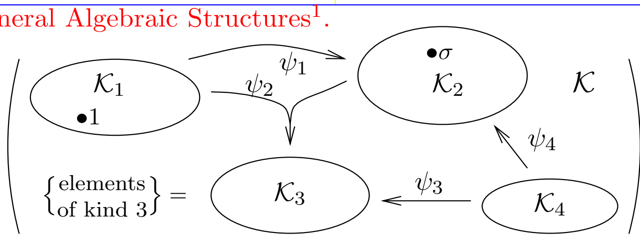
General Algebraic Structures[1].



- Has kinds, elements, operations, and maybe constants.
- Must have "the free structure over some generators".
- We always allow formal linear combinations. 14

All still works!

Example 3. Quandle: a set K with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x = x \wedge x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

$\mathcal{A}(K)$ is a graded Leibniz[2] algebra: Roughly, set $\bar{v} := (v-1)$ (these generate I!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

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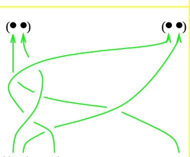
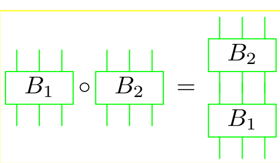
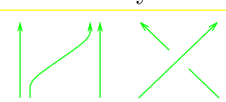
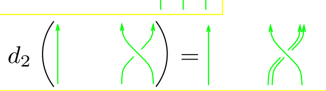
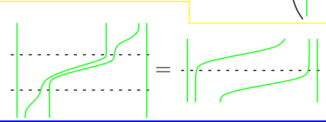
Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an A_n-associator, and the Grothendieck-Teichmüller story[3] arises naturally.

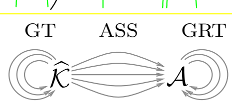












Drinfel'd [Dr1,2]

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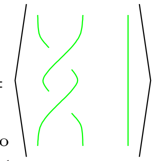
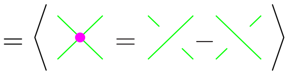
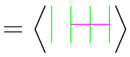
Example 1.

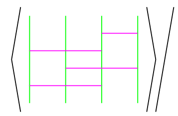
T. Kohno

$K = $  $\quad I = $ 

$$(K/I^{m+1})^\star = (\text{invariants of type } m) =: \mathcal{V}_m$$

$$(I^m/I^{m+1})^\star = \mathcal{V}_m/\mathcal{V}_{m-1} \quad C = \langle t^{ij} | t^{ij} = t^{ji}\rangle = $$ 

$$\ker \mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}]\rangle = \langle 4\text{T relations}\rangle$$

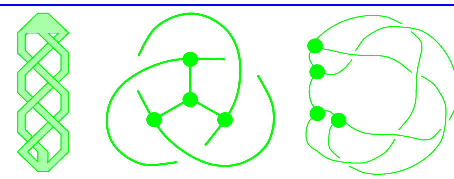
$$\mathcal{A} = A_n = \begin{pmatrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{pmatrix} = $$ 

Z: universal finite type invariant, the Kontsevich integral. 9

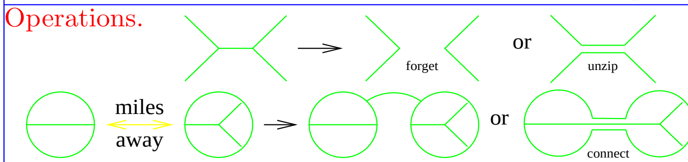
Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients. 11

Example 5 - Knotted Trivalent Graphs.

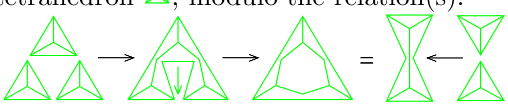
D. Thurston [Th]

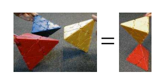


Operations.



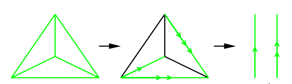
Presentation. KTG is generated by ribbon twists and the tetrahedron, modulo the relation(s):





(+more)

Claim. With $\Phi := Z(\text{tetrahedron})$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

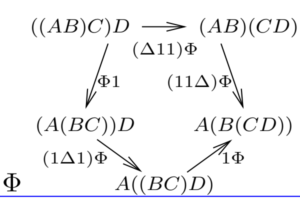


$:= \Phi \in \mathcal{A}(\uparrow_3)$

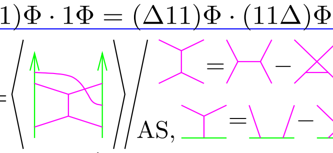
A $\mathcal{U}(\mathfrak{g})$-Associator:

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

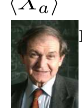


$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

$\mathcal{A}(\uparrow_2) := $ 

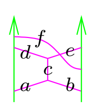
$$\xrightarrow[\text{metrized } \mathfrak{g}\ \ \mathfrak{g} = \langle X_a\rangle]{\text{Given a}} \mathcal{U}(\mathfrak{g})^{\otimes 2}$$

(deg $= \frac{1}{2}\#\{$trivalent vertices$\}$)



Penrose

Cvitanovic



$$\longrightarrow \sum_{a,b,c,d,e,f=1}^{\dim \mathfrak{g}} f_{abc} f_{dce} X_a X_d X_f \otimes X_b X_f X_e$$

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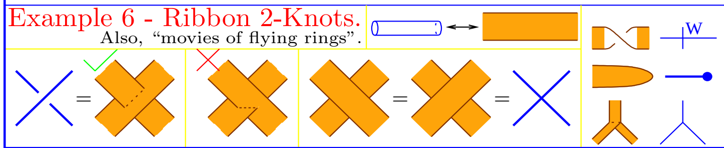
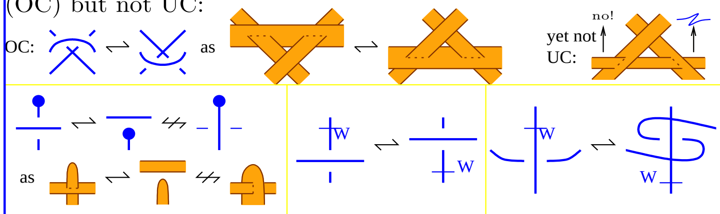
Video and more at http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/

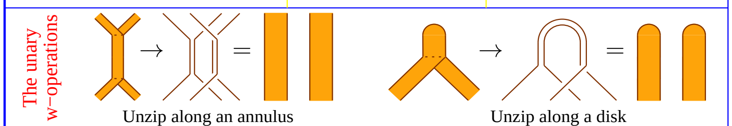
Facts and Dreams About v−Knots and Etingof−Kazhdan, 2

Example 6 - Ribbon 2-Knots. Also, "movies of flying rings".



The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:

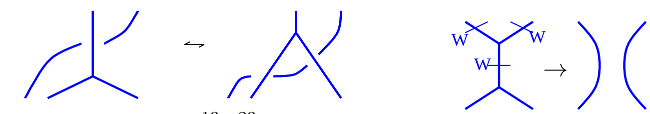




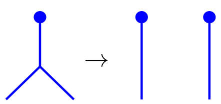
Trivalent w-Tangles. 30

$$\text{wTT} = \text{PA}\left\langle \begin{array}{c}\text{w-}\\ \text{generators}\end{array} \middle| \begin{array}{c}\text{w-}\\ \text{relations}\end{array} \middle| \begin{array}{c}\text{unary w-}\\ \text{operations}\end{array} \right\rangle = \text{CA}\left\langle \begin{array}{c}\text{same}\\ \text{w/o} \times\end{array}\right\rangle$$

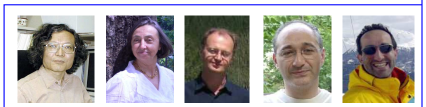
Theorem. There exists a homomorphic expansion Z for wTT. In particular, Z respects $R4$ and intertwines annulus and disk unzips:



$V\cdot(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow_3)$ $\qquad$ $VV^* = I$ in $\mathcal{A}^w(\uparrow_2)$



$V\cdot\Delta(\omega) = \omega\otimes\omega$ in $\mathcal{A}^w(\uparrow_2)$



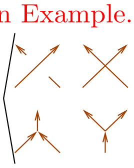
Kashiwara-Vergne-Alekseev-Enriquez-Torrosian

Alekseev-Torossian [AT] (equivalent to Kashiwara-Vergne [KV]). There are elements $F\in \mathrm{TAut}_2$ and $a\in\mathfrak{tr}_1$ such that

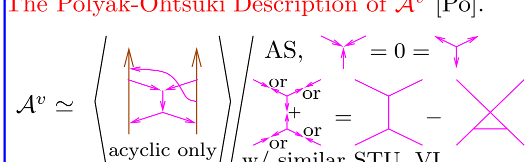
$$F(x+y) = \log e^x e^y \quad\text{and}\quad jF = a(x)+a(y)-a(\log e^x e^y).$$ 33

Theorem. That's equivalent to a homomorphic expansion for wTT.

The Main Example.

vTT= PA⟨  | R234, VR234, D, yet not UC, OC | unzips ⟩ $= \widetilde{\text{CA}}\left\langle \begin{array}{c}\text{same}\\ \text{w/o} \times\end{array}\right\rangle$

The Polyak-Ohtsuki Description of $\mathcal{A}^v$ [Po].



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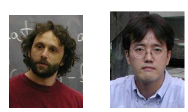
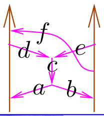


Polyak $\quad$ Ohtsuki

$\mathcal{A}^v$ pairs with Lie bialgebras. Let $\mathfrak{g}_+$ be a Lie bialgebra with basis X_a, bracket $[\cdot,\cdot]$, cobracket δ, dual $\mathfrak{g}_- = \mathfrak{g}_+^\star$, dual basis X^a for $\mathfrak{g}_-$, double $\mathfrak{g} = \mathfrak{g}_+\oplus\mathfrak{g}_-$, structure constants $[X_a, X_b] = \sum b_{ab}^c X_c$ and co-structure constants $\delta(X_a) = \sum c_a^{bc} X_b\otimes X_c$. Then



$$\longrightarrow \sum_{a,b,c,d,e,f=1}^{\dim\mathfrak{g}} b_{de}^c c_c^{ba} X_a X^d X_f \otimes X_b X^f X^e \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$$ 39

Forbidden Theorem [EK, Ha, ?]. There exists a homomorphic expansion Z for vTT.

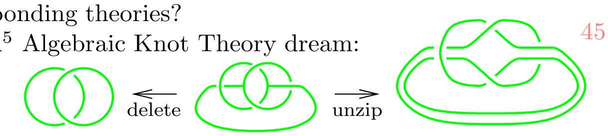
Why Forbidden (to me)?

- Minor statement details may be off.
- No fully written proof.
- I don't understand the proof.
- There isn't yet a knot-theoretic view of the proof, like there is in the w-case. 42

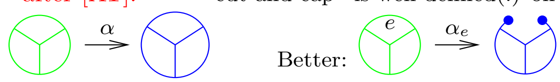
Haviv $\quad$ Etingof $\quad$ Kazhdan

Why Should We Care?

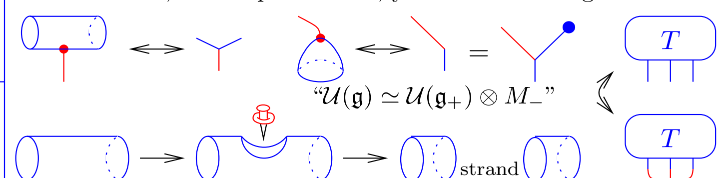
- A gateway into the forbidden territory of "quantum groups".
- Abstractly more pleasing: We study the things, and not just their representations.
- $\mathcal{A}^v$ is sometimes easier than $\mathcal{A}^u$: Alexander, say, arises easily from the 2D Lie algebra[4].
- Potentially, $\mathcal{A}^v$ has many more "internal quotients" than there are Lie bialgebras. What are they and what are the corresponding theories?
- My old[5] Algebraic Knot Theory dream: 45

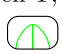
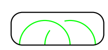


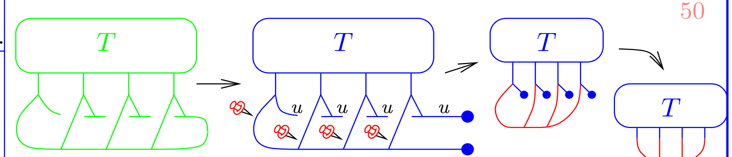
$V\to\Phi^{1\text{-loop}}$ after [AT]. "cut and cap" is well-defined(!) on $\mathcal{K}^u$

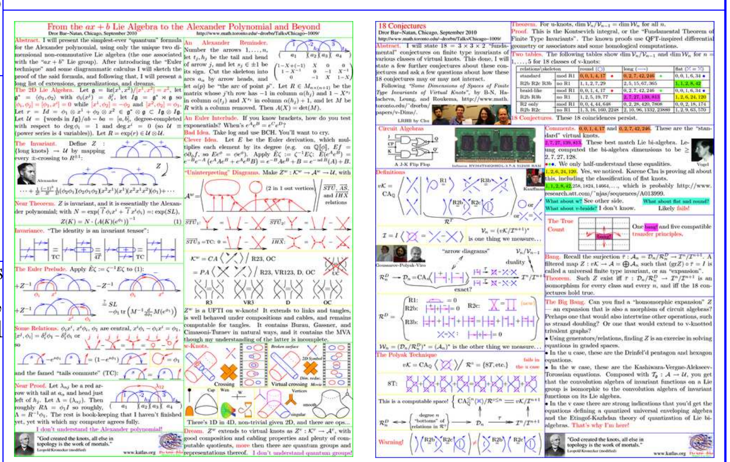


$\Phi\to V$ after [AET]. In $\mathcal{K}^{\overline{w}}$ allow tubes and strands and tube-strand vertices, allow "punctures", yet allow no "tangles".

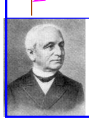


The generators of $\mathcal{K}^{\overline{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V). With T any classical tangle, esp.  or , consider the "sled" 50



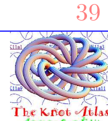


Alexander is easy! $\qquad$ In Chicago, [BN4] $\qquad$ Many kinds of virtuals!



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org



Help Needed!

Video and more at http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/

Footnotes

1. I probably mean "a functor from some fixed "structure multi-category" to the multi-category of sets, extended to formal linear combinations".
2. A Leibniz algbera is a Lie algebra minus the anti-symmetry of the bracket; I have previously erroneously asserted that here $\mathcal{A}(K)$ is Lie; however see the comment by Conant attached to this talk's video page.
3. See my paper [BN1] and my talk/handout/video [BN3].
4. See [BN5] and my talk/handout/video [BN4].
5. Not so old and not quite written up. Yet see [BN2].

References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld associators, Braid groups and explicit solutions of the KashiwaraVergne equations,* Pub. Math. de L'IHES **112-1** (2010) 143–189, arXiv:arXiv:0903.4067.

[BEER] L. Bartholdi, B. Enriquez, P. Etingof, and E. Rains, *Groups and Lie algebras corresponding to the YangBaxter equations,* Jornal of Algebra **305-2** (2006) 742-764, arXiv:math.RA/0509661.

[BN1] D. Bar-Natan, *On Associators and the Grothendieck-Teichmüller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[BN2] D. Bar-Natan, *Algebraic Knot Theory — A Call for Action,* web document, 2006, `http://www.math.toronto.edu/~drorbn/papers/AKT-CFA.html`.

[BN3] D. Bar-Natan, *Braids and the Grothendieck-Teichmüller Group,* talk given in Toronto on January 10, 2011, `http://www.math.toronto.edu/~drorbn/Talks/Toronto-110110/`.

[BN4] D. Bar-Natan, *From the $ax+b$ Lie Algebra to the Alexander Polynomial and Beyond,* talk given in Chicago on September 11, 2010, `http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/`.

[BN5] D. Bar-Natan, *Finite Type Invariants of w-Knotted Objects: From Alexander to Kashiwara and Vergne,* in preparation, online at `http://www.math.toronto.edu/~drorbn/papers/WKO/`.

[Dr1,2] V. G. Drinfel'd, *Quasi-Hopf Algebras,* Leningrad Math. J. **1** (1990) 1419–1457 and *On Quasitriangular Quasi-Hopf Algebras and a Group Closely Connected with $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$,* Leningrad Math. J. **2** (1991) 829–860.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica, New Series **2** (1996) 1–41, arXiv:q-alg/9506005.

[Ha] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants,* Hebrew University PhD thesis, September 2002, arXiv:math.QA/0211031.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions,* Invent. Math. **47** (1978) 249–272.

[Lee] P. Lee, *The Pure Virtual Braid Group is Quadratic,* in preparation.

[Po] M. Polyak, *On the Algebra of Arrow Diagrams,* Let. Math. Phys. **51** (2000) 275–291.

[Th] D. P. Thurston, *The Algebra of Knotted Trivalent Graphs and Turaev's Shadow World,* Geometry & Topology Monographs **4** (2002) 337-362, arXiv:math.GT/0311458.

Plan

1. (8 minutes) The Peter Lee setup for (K, I), "all interesting graded equations arise in this way".
2. (3 minutes) Example: the pure braid group (mention PvB, too).
3. (3 minutes) Generalized algebraic structures.
4. (1 minute) Example: quandles.
5. (4 minutes) Example: parenthesized braids and horizontal associators.
6. (6 minutes) Example: KTGs and non-horizontal associators. ("Bracket rise" arises here).
7. (8 minutes) Example: wKO's and the Kashiwara-Vergne equations.
8. (12 minutes) vKO's, bi-algebras, E-K, what would it mean to find an expansion, why I care (stronger invariant, more interesting quotients).
9. (5 minutes) wKO's, uKO's, and Alekseev-Enriquez-Torossian.

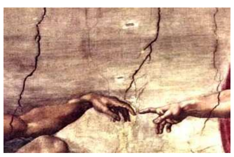
Video and more at `http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/`

Cosmic Coincidences and Several Other Stories, 1

Dror Bar−Natan at the University of Tennessee
March 4, 2011, http://www.math.toronto.edu/~drorbn/Talks/Tennessee−1103/

Abstract. In the first half of my talk I will tell a cute and simple story — how given a knot in $\mathbb{R}^3$ one may count all possible "cosmic coincidences" associated with that knot, and how this count, appropriately packaged, becomes an invariant Z with values in some space $\mathcal{A}$ of linear combinations of certain trivalent graphs.
In the second half of my talk I will describe (rather sketchily, I'm afraid) a part of the story surrounding Z and $\mathcal{A}$: How the same Z also comes from quantum field theory, Feynman diagrams, and configuration space integrals. How $\mathcal{A}$ is a space of universal formulas which make sense in every metrized Lie algebra and how specific choices for that Lie algebra correspond to various famed knot invariants. How Z solves a universal topological problem, and how solving for Z is solving some universal Lie-algebraic problem. All together, this is the u-story.
In the remaining time I will mention several other Z's and $\mathcal{A}$'s and the parallel (yet sometimes interwoven) stories surrounding them — the v-story, and w-story, and perhaps also the p-story. Each of these stories is clearly still missing some chapters.

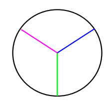
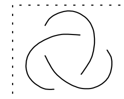
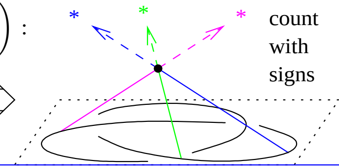
Creation of Adam



Michelangelo

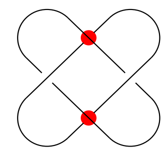
Disclaimer
We'll concentrate on the beauty and ignore the cracks.

$\langle D, K\rangle_{\text{⛩}} :=$ (The signed Stonehenge pairing of D and K) :

$D=$  $K=$  $\Longrightarrow$ 

The Gaussian linking number

$lk(\text{link}) = \sum_{\text{vertical chopsticks}} (\text{signs}) = \langle \text{O—O}, \text{link} \rangle_{\text{⛩}}$



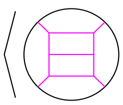
C.F. Gauss

The generating function of all cosmic coincidences:

D. Thurston

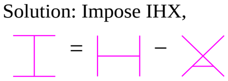
$$Z(K) := \lim_{N\to\infty} \sum_{3\text{-valent } D} \frac{\langle D, K\rangle_{\text{⛩}} D}{2^c c! \binom{N}{e}} \cdot \left(\begin{array}{c}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{array}\right) \in \mathcal{A}(\circlearrowleft)$$

N := # of stars
c := # of chopsticks
e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span $\left\langle\right.$  $\left.\right\rangle$ / (oriented vertices, AS: $Y + Y = 0$, & more relations)

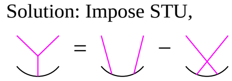
When deforming, catastrophes occur when:

A plane moves over an intersection point − Solution: Impose IHX,



(see below)

An intersection line cuts through the knot − Solution: Impose STU,



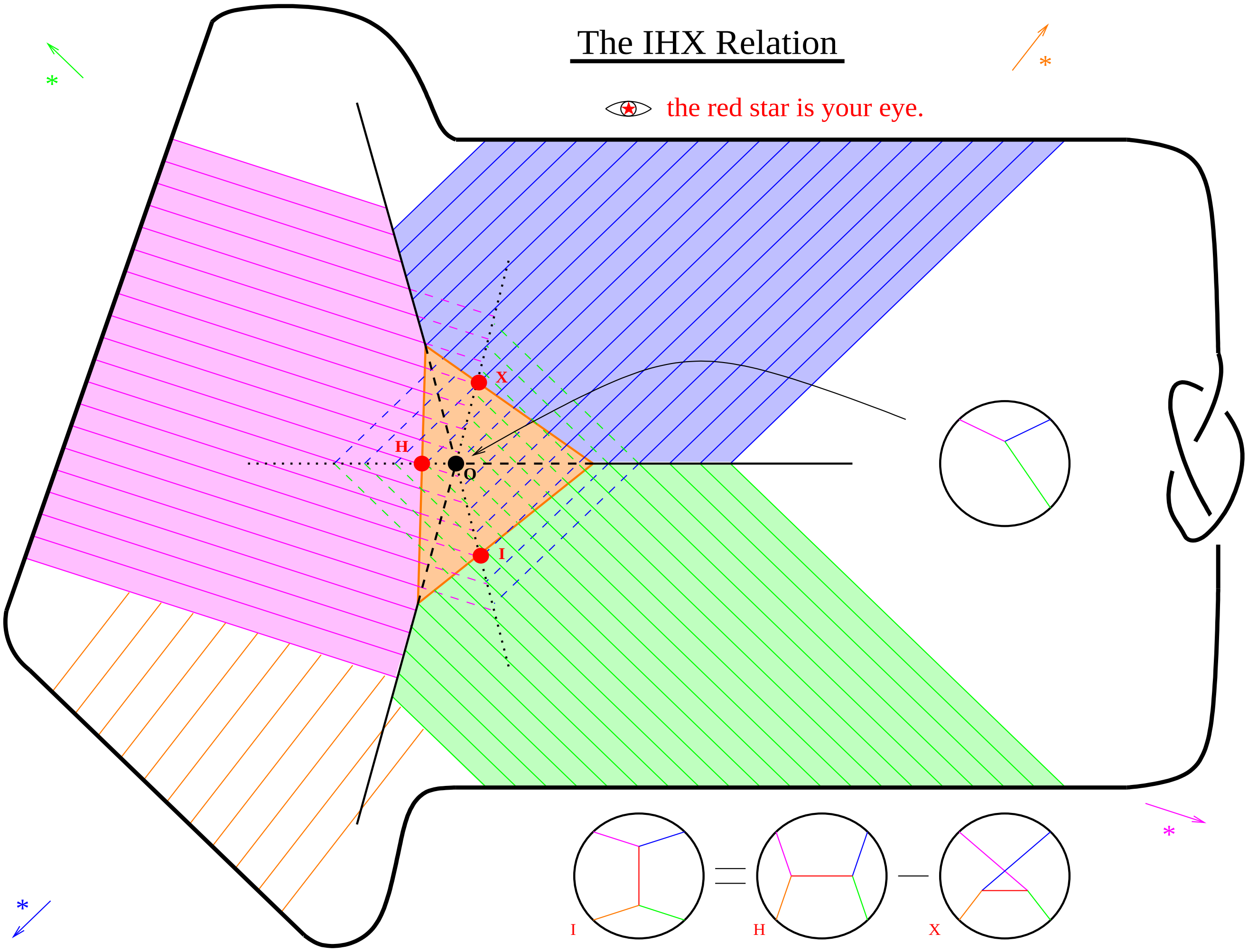
(similar argument)

The Gauss curve slides over a star − Solution: Multiply by a framing−dependent counter−term.

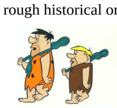
(not shown here)

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

The IHX Relation



The Cast
in rough historical order



The Neolithic People

Carl Friedrich Gauss
Edward Witten
Victor Vassiliev
Mikhail Goussarov

Maxim Kontsevich

Raoul Bott

Clifford Taubes

Thang Le

Jun Murakami

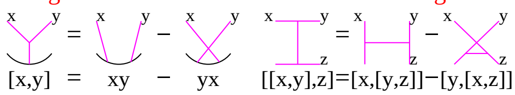
Tomotada Ohtsuki

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Tennessee-1103/ and at http://www.math.toronto.edu/~drorbn/Talks/Caen-1206/#Colloquium

Cosmic Coincidences and Several Other Stories, 2

Dror Bar−Natan at the University of Tennessee
March 4, 2011, http://www.math.toronto.edu/~drorbn/Talks/Tennessee−1103/

"Low Algebra" and universal formulae in Lie algebras.



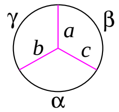

Sophus Lie

More precisely, let $\mathfrak g = \langle X_a\rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha\rangle$ be a representation. Set

$$f_{abc} := \langle [X_a, X_c], X_c\rangle \qquad X_a v_\beta = \sum_\gamma r^\beta_{a\gamma} v_\gamma$$

and then

$$W_{\mathfrak g,R}:$$



$$\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

$W_{\mathfrak g,R}\circ Z$ is often interesting:

$\mathfrak g = sl(2) \longrightarrow$  The Jones polynomial

$\mathfrak g = sl(N) \longrightarrow$  Przytycki The HOMFLYPT polynomial

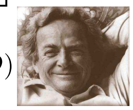
$\mathfrak g = so(N) \longrightarrow$  The Kauffman polynomial

Chern-Simons-Witten theory and Feynman diagrams.

$$\int_{\mathfrak g\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb R^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]$$

$$\longrightarrow \sum_{\substack{D:\text{ Feynman}\\ \text{diagram}}} W_{\mathfrak g}(D) \not\sum \mathcal{E}(D) \longrightarrow \sum_{\substack{D:\text{ Feynman}\\ \text{diagram}}} D \not\sum \mathcal{E}(D)$$


Witten


Feynman

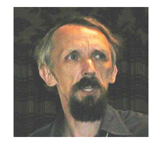
Definition. V is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

Theorem. All knot polynomials (Conway, Jones, etc.) are of finite type.

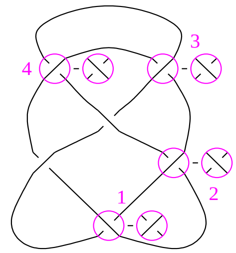
Conjecture. (Taylor's theorem) Finite type invariants separate knots.

Theorem. *Z(K)* is a universal finite type invariant!

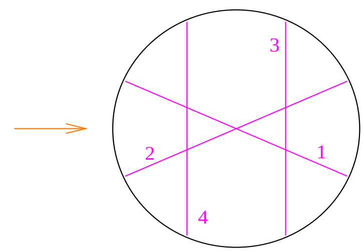
(sketch: to dance in many parties, you need many feet).


Vassiliev


Goussarov



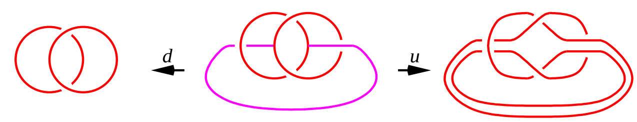
The Miller Institute knot



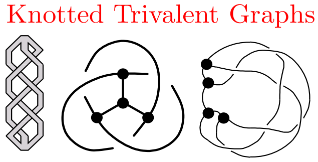
Knots are the wrong objects to study in knot theory!

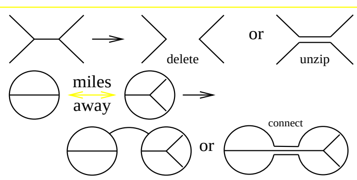
They are not finitely generated and they carry no interesting operations.

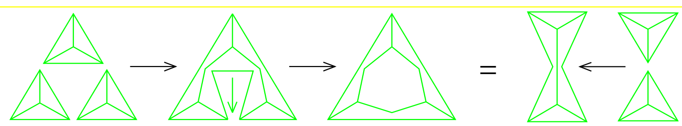
Algebraic Knot Theory



Knotted Trivalent Graphs







Theorem (∼, "High Algebra"). A homomorphic Z is the same as a "Drinfel'd Associator".


Drinfel'd

The u→v→w & p Stories

explained | sketched | could explain | could explain, gaps remain | more gaps then explains | mystery

	Topology	Combinatorics	Low Algebra	High Algebra	Counting Coincidences Conf. Space Integrals	Quantum Field Theory	Graph Homology
u-Knots	The usual Knotted Objects (KOs) in 3D — braids, knots, links, tangles, knotted graphs, etc.	Chord diagrams and Jacobi diagrams, modulo $4T$, STU, IHX, etc.	Finite dimensional metrized Lie algebras, representations, and associated spaces.	The Drinfel'd theory of associators.	Today's work. Not beautifully written, and some detour-forcing cracks remain.	Perturbative Chern-Simons-Witten theory.	The "original" graph homology.
v-Knots	Virtual KOs — "algebraic", "not embedded"; KOs drawn on a surface, mod stabilization.	Arrow diagrams and v-Jacobi diagrams, modulo $6T$ and various "directed" STUs and IHXs, etc.	Finite dimensional Lie bi-algebras, representations, and associated spaces.	Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras.	No clue.	No clue.	No clue.
w-Knots	Ribbon 2D KOs in 4D; "flying rings". Like v, but also with "overcrossings commute".	Like v, but also with "tails commute". Only "two in one out" internal vertices.	Finite dimensional co-commutative Lie bi-algebras ($\mathfrak g \ltimes \mathfrak g^*$), representations, and associated spaces.	The Kashiwara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups / algebras.	No clue.	Probably related to 4D BF theory.	Studied.
p-Objects	No clue.	"Acrobat towers" with 2-in many-out vertices.	Poisson structures.	Deformation quantization of poisson manifolds.	Configuration space integrals are key, but they don't reduce to counting.	 Work of Cattaneo.	Studied. Hyperbolic geometry ?

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Tennessee-1103/ and at http://www.math.toronto.edu/~drorbn/Talks/Caen-1206/#Colloquium

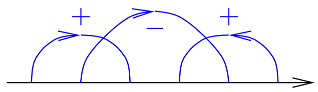
From the $ax+b$ Lie Algebra to the Alexander Polynomial and Beyond

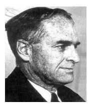
Dror Bar−Natan, Chicago, September 2010 http://www.math.toronto.edu/~drorbn/Talks/Chicago−1009/

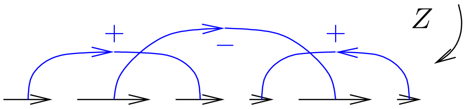
Abstract. I will present the simplest-ever "quantum" formula for the Alexander polynomial, using only the unique two dimensional non-commutative Lie algebra (the one associated with the "$ax+b$" Lie group). After introducing the "Euler technique" and some diagrammatic calculus I will sketch the proof of the said formula, and following that, I will present a long list of extensions, generalizations, and dreams.

The 2D Lie Algebra. Let $\mathfrak{g} = \mathfrak{lie}(x^1,x^2)/[x^1,x^2]=x^2$, let $\mathfrak{g}^* = \langle \phi_1,\phi_2\rangle$ with $\phi_i(x^j)=\delta_i^j$, let $I\mathfrak{g}=\mathfrak{g}^*\rtimes\mathfrak{g}$ so $[\phi_i,\phi_j]=[\phi_1,x^i]=0$ while $[x^1,\phi_2]=-\phi_2$ and $[x^2,\phi_2]=\phi_1$. Let $r = Id = \phi_1\otimes x^1+\phi_2\otimes x^2 \in \mathfrak{g}^*\otimes\mathfrak{g}\subset I\mathfrak{g}\otimes I\mathfrak{g}$. Let $\mathcal{U}=\{\text{words in } I\mathfrak{g}\}/ab-ba=[a,b]$, degree-completed with respect to $\deg\phi_i=1$ and $\deg x^i=0$ (so $\mathcal{U}\equiv$ (power series is 4 variables)). Let $R=\exp(r)\in\mathcal{U}\otimes\mathcal{U}$.

The Invariant. Define Z : {long knots} $\to\mathcal{U}$ by mapping every $\pm$-crossing to $R^{\pm1}$:




Alexander

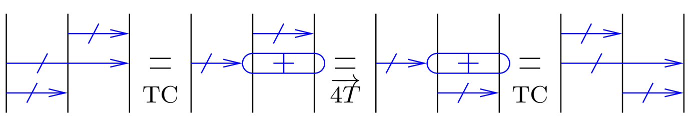


$$\cdots+\frac{1}{2!}\frac{(-1)^3}{3!}\frac{1}{1!}(\phi_2\phi_1)(\phi_2\phi_1\phi_2)(x^2x^1)(x^1)(x^2x^1x^2)(\phi_1)+\cdots$$

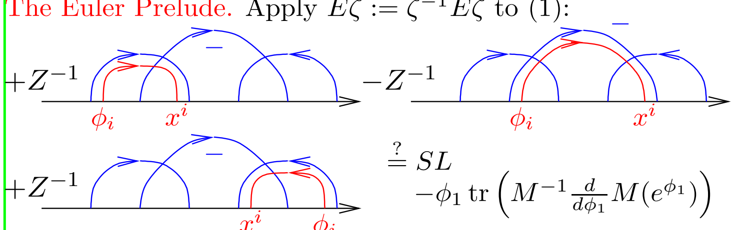
Near Theorem. Z is invariant, and it is essentially the Alexander polynomial; with $N=\exp(\overrightarrow{l}\,\phi_ix^i+\overleftarrow{l}\,x^i\phi_i)=:\exp(SL)$,

$$Z(K)=N\cdot\left(A(K)(e^{\phi_1})\right)^{-1} \qquad (1)$$

Invariance. "The identity is an invariant tensor":

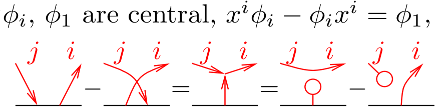


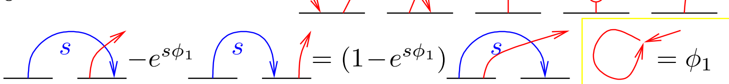
The Euler Prelude. Apply $\tilde{E}\zeta := \zeta^{-1}E\zeta$ to (1):



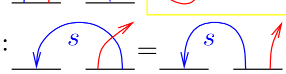
$$\stackrel{?}{=} SL - \phi_1\operatorname{tr}\left(M^{-1}\frac{d}{d\phi_1}M(e^{\phi_1})\right)$$

Some Relations. ϕ_ix^i, $x^i\phi_i$, ϕ_1 are central, $x^i\phi_i-\phi_ix^i=\phi_1$, $[x^j,\phi_i]=\delta_i^j\phi_1-\delta_1^j\phi_i$ or so

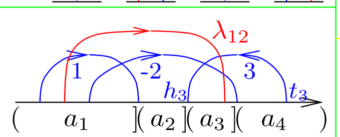




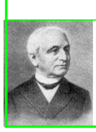
and the famed "tails commute" (TC):



Near Proof. Let $\lambda_{\alpha j}$ be a red arrow with tail at a_α and head just left of h_j. Let $\Lambda=(\lambda_{\alpha j})$. Then roughly $R\Lambda=\phi_1 I$ so roughly, $\Lambda=R^{-1}\phi_1$. The rest is book-keeping that I haven't finished yet, yet with which my computer agrees fully.



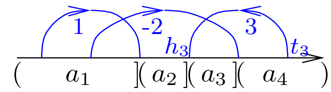
I don't understand the Alexander polynomial!



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

An Alexander Reminder. Number the arrows $1,\dots,n$, let t_j,h_j be the tail and head of arrow j, and let $s_j\in\pm1$ be its sign. Cut the skeleton into arcs a_α by arrow heads, and let $\alpha(p)$ be "the arc of point p". Let $R\in M_{n\times(n+1)}$ be the matrix whose j'th row has -1 in column $\alpha(h_j)$ and $1-X^{s_j}$ in column $\alpha(t_j)$ and X^{s_j} in column $\alpha(h_j)+1$, and let M be R with a column removed. Then $A(X)=\det(M)$.



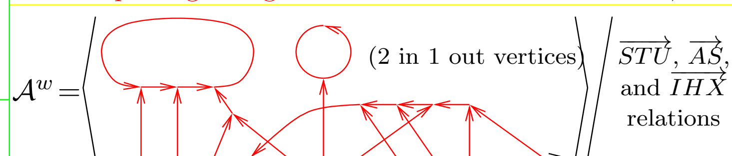
$$\begin{pmatrix} 1-X+(-1) & X & 0 & 0\\ 1-X^{-1} & 0 & -1 & X^{-1}\\ 0 & -1 & X & 1-X\end{pmatrix}$$

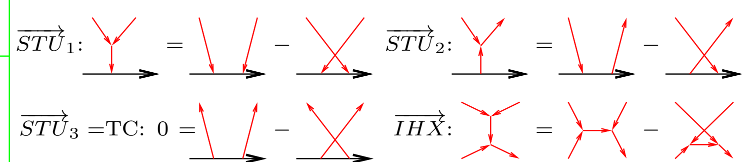
An Euler Interlude. If you know brackets, how do you test exponentials? When's $e^Ae^B=e^Ce^D$?

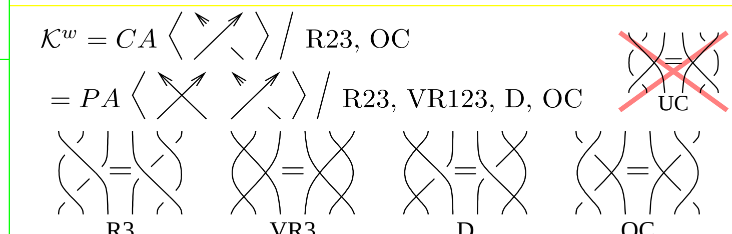
Bad Idea. Take log and use BCH. You'll want to cry.

Clever Idea. Let E be the Euler derivation, which multiplies each element by its degree (e.g. on $\mathbb{Q}[\![\phi]\!]$, $Ef=\phi\partial_\phi f$, so $Ee^\phi=\phi e^\phi$). Apply $\tilde{E}\zeta:=\zeta^{-1}E\zeta$: $\tilde{E}(e^Ae^B)=e^{-B}e^{-A}\left(e^AAe^B+e^Ae^BB\right)=e^{-B}Ae^B+B=e^{-\operatorname{ad}B}(A)+B$.

"Uninterpreting" Diagrams. Make $Z^w:\mathcal{K}^w\to\mathcal{A}^w\to\mathcal{U}$, with

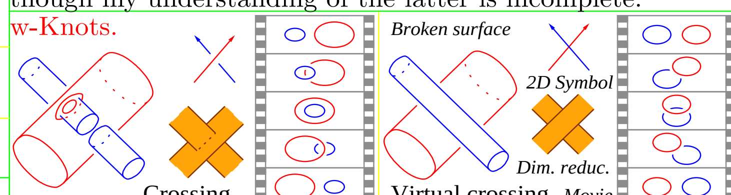


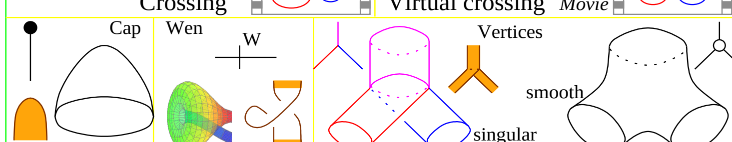




Z^w is a UFTI on w-knots! It extends to links and tangles, is well behaved under compositions and cables, and remains computable for tangles. It contains Burau, Gassner, and Cimasoni-Turaev in natural ways, and it contains the MVA though my understanding of the latter is incomplete.

w-Knots.





There's 1D in 4D, non-trivial given 2D, and there are ops...

Dream. Z^w extends to virtual knots as $Z^v:\mathcal{K}^v\to\mathcal{A}^v$, with good composition and cabling properties and plenty of computable quotients, more then there are quantum groups and representations thereof. I don't understand quantum groups!

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

18 Conjectures

Dror Bar-Natan, Chicago, September 2010
http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

Abstract. I will state $18 = 3 \times 3 \times 2$ "fundamental" conjectures on finite type invariants of various classes of virtual knots. This done, I will state a few further conjectures about these conjectures and ask a few questions about how these 18 conjectures may or may not interact.

Following *"Some Dimensions of Spaces of Finite Type Invariants of Virtual Knots"*, by B-N, Halacheva, Leung, and Roukema, http://www.math.toronto.edu/~drorbn/papers/v-Dims/.


LRHB by Chu

Theorem. For u-knots, $\dim \mathcal{V}_n/\mathcal{V}_{n-1} = \dim \mathcal{W}_n$ for all n.

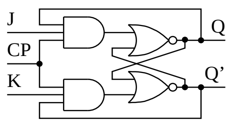
Proof. This is the Kontsevich integral, or the "Fundamental Theorem of Finite Type Invariants". The known proofs use QFT-inspired differential geometry or associators and some homological computations.

Two tables. The following tables show $\dim \mathcal{V}_n/\mathcal{V}_{n-1}$ and $\dim \mathcal{W}_n$ for $n = 1, \dots, 5$ for 18 classes of v-knots:

relations\skeleton		round (◯)	long (⟶)	flat (⤫ = ⤫)
standard	mod R1	0, 0, 1, 4, 17 •	0, 2, 7, 42, 246 •	0, 0, 1, 6, 34 •
R2b R2c R3b	no R1	1, 1, 2, 7, 29	2, 5, 15, 67, 365	1, 1, 2, 8, 42
braid-like	mod R1	0, 0, 1, 4, 17 •	0, 2, 7, 42, 246 •	0, 0, 1, 6, 34 •
R2b R3b	no R1	1, 2, 5, 19, 77	2, 7, 27, 139, 813	1, 2, 6, 24, 120
R2 only	mod R1	0, 0, 4, 44, 648	0, 2, 28, 420, 7808	0, 0, 2, 18, 174
R2b R2c	no R1	1, 3, 16, 160, 2248	2, 10, 96, 1332, 23880	1, 2, 9, 63, 570

18 Conjectures. These 18 coincidences persist.

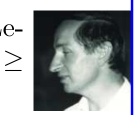
Circuit Algebras


A J-K Flip Flop


Infineon HYS64T64020HDL-3.7-A 512MB RAM

Comments. 0, 0, 1, 4, 17 and 0, 2, 7, 42, 246. These are the "standard" virtual knots.

2, 7, 27, 139, 813. These best match Lie bi-algebra. Leung computed the bi-algebra dimensions to be $\geq 2, 7, 27, 128$.


Vogel

•••. We only half-understand these equalities.

1, 2, 6, 24, 120. Yes, we noticed. Karene Chu is proving all about this, including the classification of flat knots.

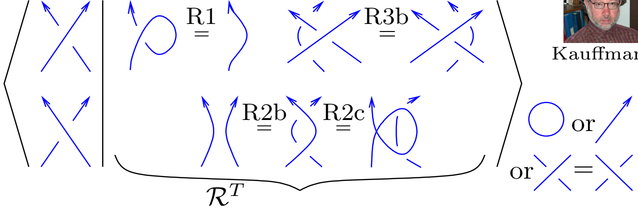
1, 1, 2, 8, 42, 258, 1824, 14664, ..., which is probably http://www.research.att.com/~njas/sequences/A013999.

What about w? See other side. What about flat and round? Likely **fails!**

What about v-braids? I don't know.

Definitions

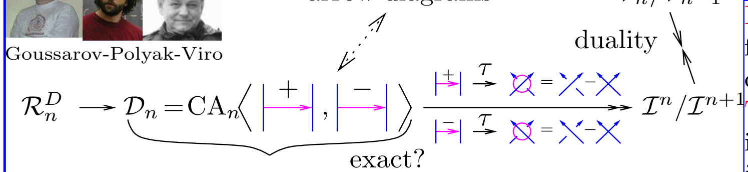

Kauffman

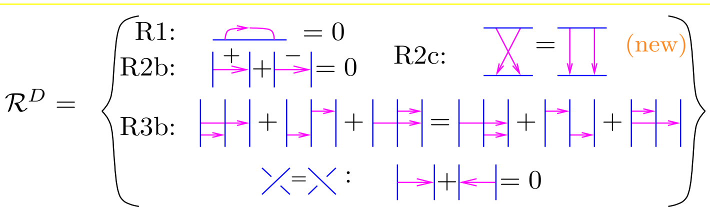

$v\mathcal{K} = \mathrm{CA}_{\mathbb{Q}}\langle \dots \rangle$, R1, R3b, R2b, R2c, or, or, $\mathcal{R}^T$

$\mathcal{I} = I\left\langle \dots \right\rangle$

$\mathcal{V}_n = (v\mathcal{K}/\mathcal{I}^{n+1})^*$ is one thing we measure...


Goussarov-Polyak-Viro

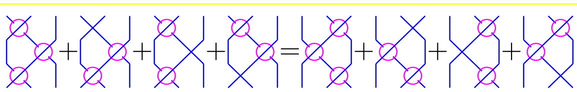

"arrow diagrams", $\mathcal{V}_n/\mathcal{V}_{n-1}$, duality, $\mathcal{R}^D_n \longrightarrow \mathcal{D}_n = \mathrm{CA}_n\langle \dots \rangle \xrightarrow{\tau} \mathcal{I}^n/\mathcal{I}^{n+1}$, exact?


$\mathcal{R}^D =$ R1: $= 0$; R2b: $= 0$; R2c: (new); R3b; $= 0$

$\mathcal{W}_n = (\mathcal{D}_n/\mathcal{R}^D_n)^* = (\mathcal{A}_n)^*$ is the other thing we measure...

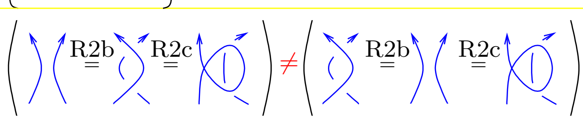
The Polyak Technique

$v\mathcal{K} = \mathrm{CA}_{\mathbb{Q}}\langle \dots \rangle / \mathcal{R}^\circ = \{8T, \text{etc.}\}$ — fails in the u case

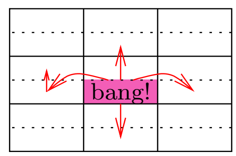

8T:

This is a computable space! $\left\{ \mathrm{CA}^{\leq n}_{\mathbb{Q}}\langle \dots \rangle / \mathcal{R}^{\circ \leq n} = v\mathcal{K}/\mathcal{I}^{n+1} \right.$

$\mathcal{R}^D_n \longleftrightarrow \left\{ \begin{array}{c}\text{degree } n \\ \text{"bottoms" of} \\ \text{relations in } \mathcal{R}^\circ\end{array} \right\} \longrightarrow \mathcal{D}_n \xrightarrow{\tau} \mathcal{I}^n/\mathcal{I}^{n+1}$


Warning! R2b, R2c $\neq$ R2b, R2c

The True Count


One **bang!** and five compatible **transfer principles**.

Bang. Recall the surjection $\bar{\tau} : \mathcal{A}_n = \mathcal{D}_n/\mathcal{R}^D_n \to \mathcal{I}^n/\mathcal{I}^{n+1}$. A filtered map $Z : v\mathcal{K} \to \mathcal{A} = \bigoplus \mathcal{A}_n$ such that $(\mathrm{gr} Z) \circ \bar{\tau} = I$ is called a universal finite type invariant, or an "expansion".

Theorem. Such Z exist iff $\bar{\tau} : \mathcal{D}_n/\mathcal{R}^D_n \to \mathcal{I}^n/\mathcal{I}^{n+1}$ is an isomorphism for every class and every n, and iff the 18 conjectures hold true.

The Big Bang. Can you find a "homomorphic expansion" Z — an expansion that is also a morphism of circuit algebras? Perhaps one that would also intertwine other operations, such as strand doubling? Or one that would extend to v-knotted trivalent graphs?

- Using generators/relations, finding Z is an exercise in solving equations in graded spaces.
- In the u case, these are the Drinfel'd pentagon and hexagon equations.
- In the w case, these are the Kashiwara-Vergne-Alekseev-Torossian equations. Composed with $\mathcal{T}_{\mathfrak{g}} : \mathcal{A} \to \mathcal{U}$, you get that the convolution algebra of invariant functions on a Lie group is isomorphic to the convolution algebra of invariant functions on its Lie algebra.
- In the v case there are strong indications that you'd get the equations defining a quantized universal enveloping algebra and the Etingof-Kazhdan theory of quantization of Lie bi-algebras. **That's why I'm here!**


"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

1. $\operatorname{proj}\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}\left((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n\right)$

— All Signs Are Wrong! —

I understand Drinfel'd and Alekseev−Torossian, I don't understand Etingof−Kazhdan yet, and I'm clueless about Kontsevich

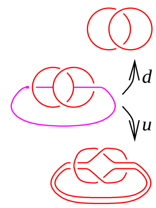
Dror Bar−Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier−1006/

Cans and Can't Yets.

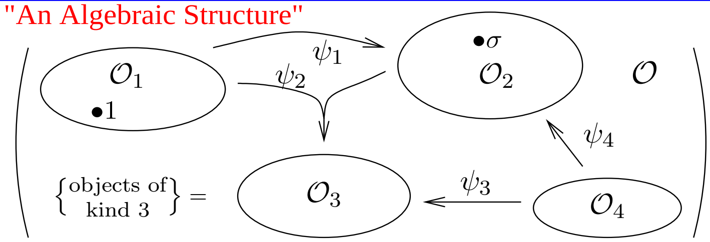
$\left(\text{arbitrary algebraic structure}\right) \xrightarrow[\text{machine}]{\text{projectivization}} \left(\text{a problem in graded algebra}\right)$

- Feed knot-things, get Lie algebra things.
- (u-knots)→(Drinfel'd associators).
- (w-knots)→(K-V-A-E-T).
- Dream: (v-knots)→(Etingof-Kazhdan).
- Clueless: (???)→(Kontsevich)?
- Goals: add to the Knot Atlas, produce a working AKT and touch ribbon 1-knots, rip benefits from *truly* understanding quantum groups.

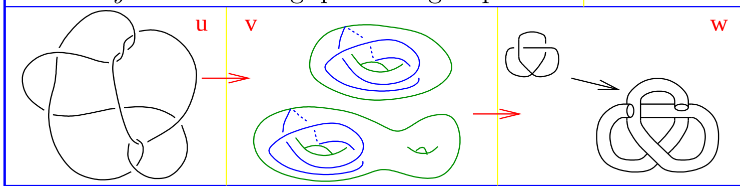
www.katlas.org



"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

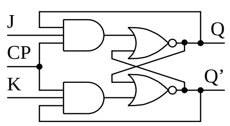


u-Knots

(PA :=Planar Algebra)

$\left\{\begin{array}{c}\text{knots}\\ \text{\&links}\end{array}\right\} = \text{PA}\langle \ \text{R123:} \ \rangle_{0\text{ legs}}$

Circuit Algebras



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v-Tangles and w-Tangles

(CA :=Circuit Algebra)

$\left\{\begin{array}{c}\text{v-knots}\\ \text{\& links}\end{array}\right\} = \text{CA}\langle \ \text{R23:} \ \rangle$

$= \text{PA}\langle \ \text{VR123: R23} \ ; \text{D:} \ \rangle$

{w-Tangles} = v-Tangles / OC:

Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

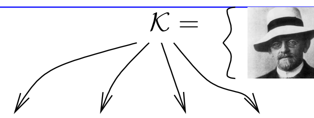
$$\begin{array}{ccccccccc} \text{ops} \circlearrowright \mathcal{K} = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 & \supset \dots \\ \Downarrow & & & & & \downarrow Z & & & \\ \text{ops} \circlearrowright \operatorname{gr}\mathcal{K} := & \mathcal{K}_0/\mathcal{K}_1 & \oplus & \mathcal{K}_1/\mathcal{K}_2 & \oplus & \mathcal{K}_2/\mathcal{K}_3 & \oplus & \mathcal{K}_3/\mathcal{K}_4 & \oplus \dots \end{array}$$

An expansion is a filtered $Z:\mathcal{K}\to\operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

Reality. $\operatorname{gr}\mathcal{K}$ is often too hard. An $\mathcal{A}$-expansion is a graded "guess" $\mathcal{A}$ with a surjection $\tau:\mathcal{A}\to\operatorname{gr}\mathcal{K}$ and a filtered $Z:\mathcal{K}\to\mathcal{A}$ for which $(\operatorname{gr}Z)\circ\tau = I_{\mathcal{A}}$. An $\mathcal{A}$-expansion confirms $\mathcal{A}$ and yields an ordinary expansion. Same for "homomorphic".

Just for fun.

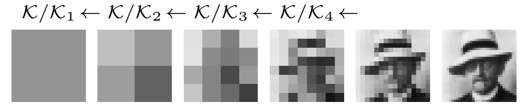
$\mathcal{K} = \left\{ \ \right\} = \left(\text{The set of all b/w 2D projections of reality}\right)$



$\dots$

$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow$

Crop Rotate Adjoin



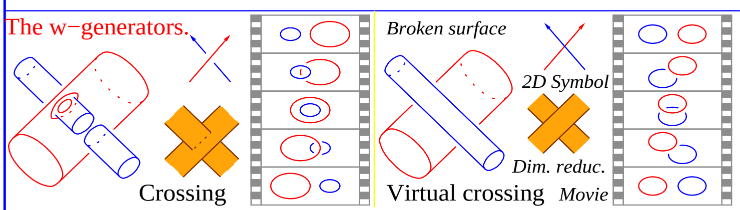
$\dots$

An expansion Z is a choice of a "progressive scan" algorithm.

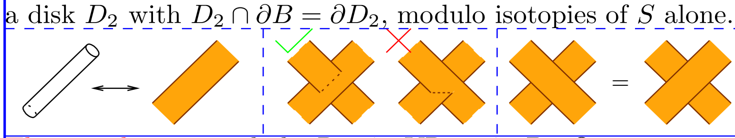
crop rotate adjoin

$\underset{\mathbb{R}}{\underset{\|}{\mathcal{K}/\mathcal{K}_1}} \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \underset{\ker(\mathcal{K}/\mathcal{K}_4\to\mathcal{K}/\mathcal{K}_3)}{\underset{\|}{\mathcal{K}_3/\mathcal{K}_4}} \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \ \cdots$

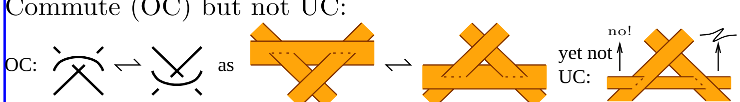
The w−generators.



A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2\cap\partial B = \partial D_2$, modulo isotopies of S alone.



The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:



Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle(\mathcal{K}_1)^m\rangle$ (using all available "products"). In this case, set $\operatorname{proj}\mathcal{K} := \operatorname{gr}\mathcal{K}$.

Examples. 1. The projectivization of a group is a graded associative algebra.

2. Pure braids — PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr}PB_n$ is generated by $t_{ij} := x_{ij}-1$, modulo the $4T$ relations $[t_{ij}, t_{ik}+t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

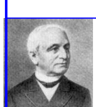
3. Quandle: a set Q with an op $\wedge$ s.t.

$$1\wedge x = 1, \quad x\wedge 1 = x, \qquad \text{(appetizers)}$$

$$(x\wedge y)\wedge z = (x\wedge z)\wedge(y\wedge z). \qquad \text{(main)}$$

$\operatorname{proj}Q$ is a graded Leibniz algebra: Roughly, set $\bar{v} := (v-1)$ (these generate $I!$), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x}\wedge\bar{y})\wedge\bar{z} = (\bar{x}\wedge\bar{z})\wedge\bar{y} + \bar{x}\wedge(\bar{y}\wedge\bar{z}).$$



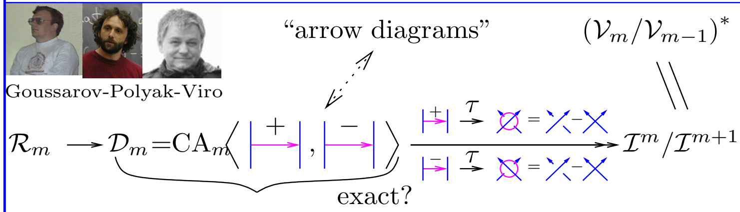
"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

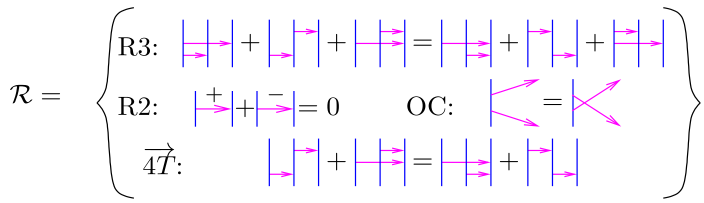
Kashiwara, Vergne, Alekseev, Enriquez, Torossian.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/

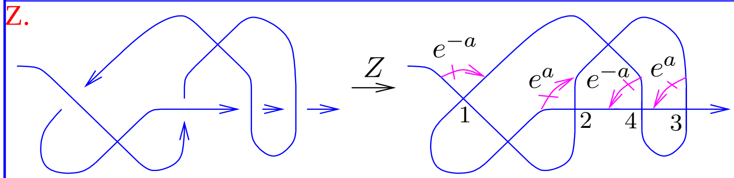
1. proj $\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}\left((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n\right)$, continued.

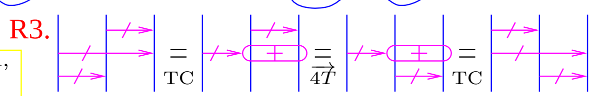


Imperfect Thumb-Rule. Take R3 (say), substitute $\to$ + , keep the lowest degree terms that don't immediately die:



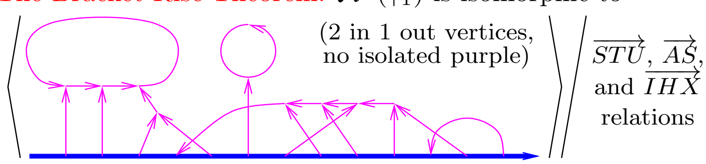
Z.

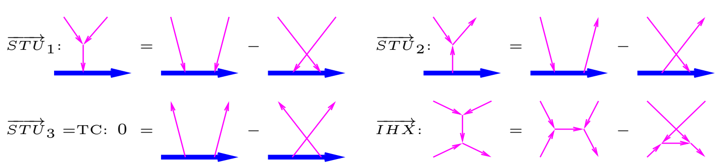




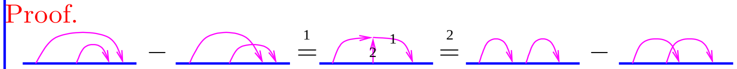
Thm. $Z \succ A$, maybe $\succ\!\succ$.

The Bracket-Rise Theorem. $\mathcal{A}^w(\uparrow_1)$ is isomorphic to



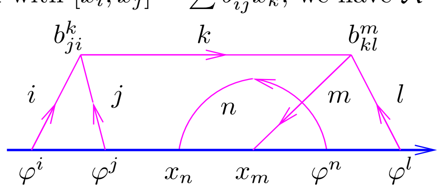


Proof.



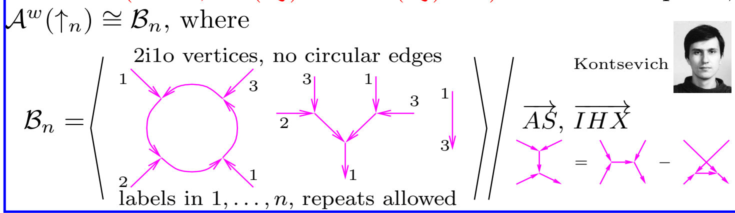
Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

To Lie Algebras. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim \mathfrak{g}} b_{ij}^k b_{kl}^m \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$

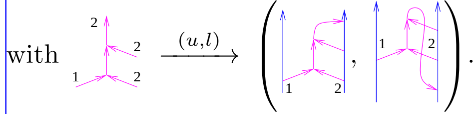
Theorem (PBW, "$\mathcal{U}(I\mathfrak{g})^{\otimes n} \cong \mathcal{S}(I\mathfrak{g})^{\otimes n}$"). As vector spaces, $\mathcal{A}^w(\uparrow_n) \cong \mathcal{B}_n$, where



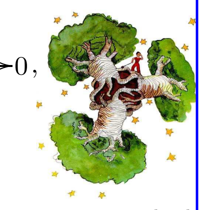
Wheels and Trees. With $\mathcal{P}$ for $\mathcal{P}$rimitives,

$$0 \longrightarrow \langle\text{wheels}\rangle \xrightarrow{\iota} \mathcal{PA}^w(\uparrow_n) \underset{l}{\overset{u}{\underset{\pi}{\rightleftarrows}}} \langle\text{trees}\rangle \longrightarrow 0,$$

with



So proj $\mathcal{K}^w(\uparrow_n) \cong \mathcal{U}(\langle\text{trees}\rangle \ltimes \langle\text{wheels}\rangle)$.



trees atop a wheel, and a little prince.

Some A-T Notions. $\mathfrak{a}_n$ is the vector space with basis $x_1, \dots, x_n$, $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ is the free Lie algebra, $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ is the free associative algera "of words", $\mathrm{tr} : \mathrm{Ass}_n^+ \to \mathfrak{tr}_n = \mathrm{Ass}_n^+ / (x_{i_1} x_{i_2} \cdots x_{i_m} = x_{i_2} \cdots x_{i_m} x_{i_1})$ is the "trace" into "cyclic words", $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ are all the derivations, and

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

are "tangential derivations", so $D \leftrightarrow (a_1, \dots, a_n)$ is a vector space isomorphism $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_n \mathfrak{lie}_n$. Finally, $\mathrm{div} : \mathfrak{tder}_n \to \mathfrak{tr}_n$ is $(a_1, \dots, a_n) \mapsto \sum_k \mathrm{tr}(x_k(\partial_k a_k))$, where for $a \in \mathrm{Ass}_n^+$, $\partial_k a \in \mathrm{Ass}_n$ is determined by $a = \sum_k (\partial_k a) x_k$, and $j : \mathrm{TAut}_n = \exp(\mathfrak{tder}_n) \to \mathfrak{tr}_n$ is $j(e^D) = \frac{e^D - 1}{D} \cdot \mathrm{div}\, D$.

Theorem. Everything matches. $\langle\text{trees}\rangle$ is $\mathfrak{a}_n \oplus \mathfrak{tder}_n$ as Lie algebras, $\langle\text{wheels}\rangle$ is $\mathfrak{tr}_n$ as $\langle\text{trees}\rangle$ / $\mathfrak{tder}_n$-modules, $\mathrm{div}\, D = \iota^{-1}(u - l)(D)$, and $e^{uD} e^{-lD} = e^{jD}$.

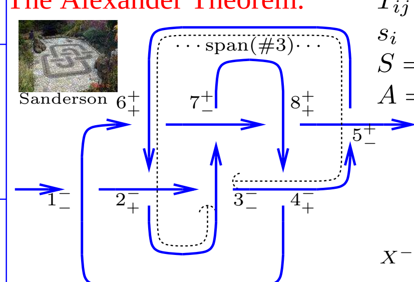
Differential Operators. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\, x$: $(x\varphi)(y) := \varphi([x, y])$.

Trees become vector fields and $uD \mapsto lD$ is $D \mapsto D^*$. So $\mathrm{div}\, D$ is $D - D^*$ and $jD = \log(e^D (e^D)^*) = \int_0^1 dt e^{tD}\, \mathrm{div}\, D$.

Special Derivations. Let $\mathfrak{sder}_n = \{D \in \mathfrak{tder}_n : D(\sum x_i) = 0\}$.
Theorem. $\mathfrak{sder}_n = \pi\alpha(\mathrm{proj\,u\text{-}tangles})$, where α is the obvious map proj u-tangles $\to$ proj w-tangles.
Proof. After decoding, this becomes Lemma 6.1 of Drinfel'd's amazing $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ paper.

The Alexander Theorem.



$T_{ij} = |\mathrm{low}(\#j) \in \mathrm{span}(\#i)|$,
$s_i = \mathrm{sign}(\#i)$, $d_i = \mathrm{dir}(\#i)$,
$S = \mathrm{diag}(s_i d_i)$,
$A = \det\left(I + T(I - X^{-S})\right)$.

$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0 \\ 0&0&1&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1&1 \\ 0&1&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$X^{-S} = \mathrm{diag}(\frac{1}{X}, X, \frac{1}{X}, X, X, \frac{1}{X}, X, \frac{1}{X})$.

Conjecture. For u-knots, A is the Alexander polynomial.
Theorem. With $w : x^k \mapsto w_k =$ (the k-wheel),

$$Z = N \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(e^x)\right)\right)$$

mod $w_k w_l = w_{k+l}$, $Z = N \cdot A^{-1}(e^x)$

This is the ultimate Alexander invariant! computable in polynomial time, local, composes well, behaves under cabling. Seems to significantly generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers. But it's ugly, and much work remains.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/

2. w−Knots, Alekseev−Torossian, and baby Etingof−Kazhdan

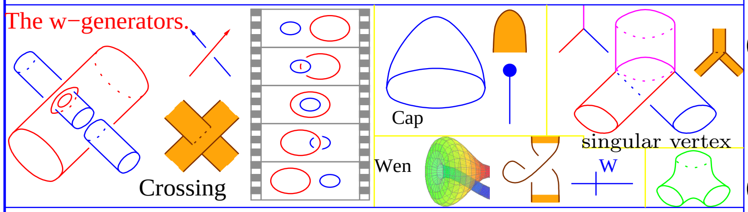
I understand Drinfel'd and Alekseev−Torossian, I don't understand Etingof−Kazhdan yet, and I'm clueless about Kontsevich

Dror Bar−Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier−1006/

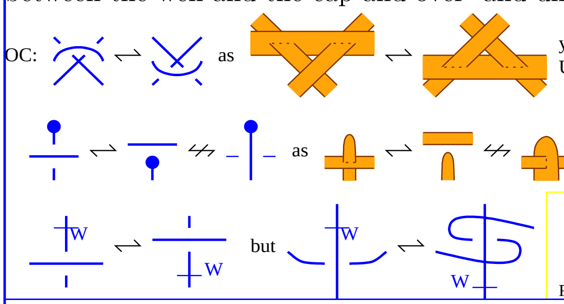
Trivalent w-Tangles.

$$\text{wTT} = \text{CA}\left\langle \begin{array}{c}\text{w-}\\ \text{generators}\end{array} \middle| \begin{array}{c}\text{w-}\\ \text{relations}\end{array} \middle| \begin{array}{c}\text{unary w-}\\ \text{operations}\end{array} \right\rangle$$

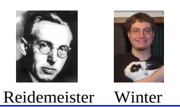
The w−generators.



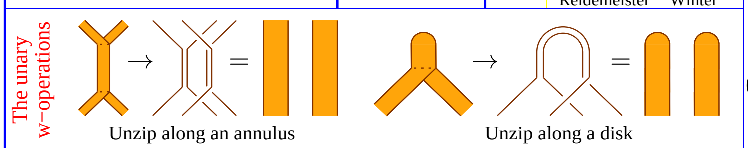
The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC, $W^2=1$, and funny interactions between the wen and the cap and over- and under-crossings:



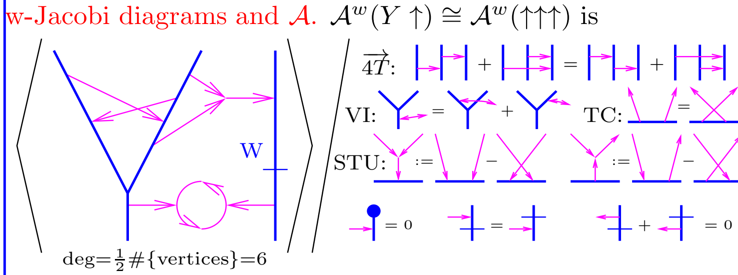
Challenge. Do the Reidemeister!



The unary w−operations



w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



An Associator:

$$(AB)C \xrightarrow{\Phi\in\mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

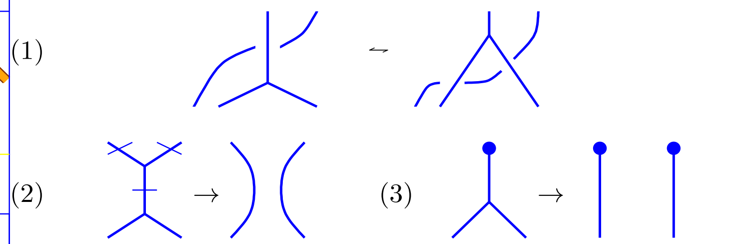
satisfying the "pentagon",

$$\begin{tikzcd}
((AB)C)D \arrow[rr, "(\Delta 11)\Phi"] \arrow[d, "\Phi 1"'] & & (AB)(CD) \arrow[d, "(11\Delta)\Phi"] \\
(A(BC))D \arrow[dr, "(1\Delta 1)\Phi"'] & & A(B(CD)) \\
& A((BC)D) \arrow[ur, "1\Phi"'] &
\end{tikzcd}$$

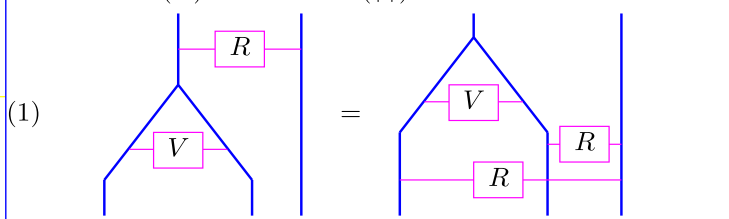
$$\Phi 1\cdot(1\Delta 1)\Phi\cdot 1\Phi = (\Delta 11)\Phi\cdot(11\Delta)\Phi$$

The hexagon? Never heard of it.

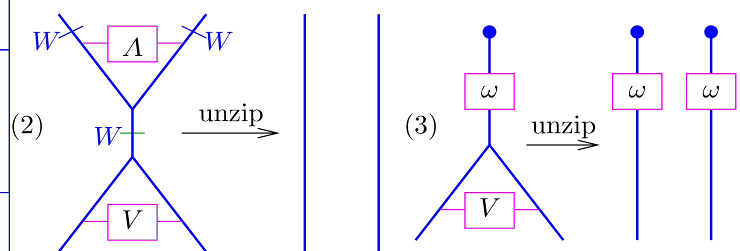
Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp$ $\in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega\in\mathcal{A}^w(\uparrow)$ and $V\in\mathcal{A}^w(\uparrow\uparrow)$ so that



(1) $V\cdot(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow\uparrow\uparrow)$



(2) $VV^*=I$ in $\mathcal{A}^w(\uparrow\uparrow)$ (3) $V\cdot\Delta(\omega)=\omega\otimes\omega$ in $\mathcal{A}^w(\uparrow\uparrow)$

Alekseev-Torossian statement. There are elements $F\in\text{TAut}_2$ and $a\in\mathfrak{tr}_1$ such that

$$F(x+y) = \log e^x e^y \quad\text{and}\quad jF = a(x)+a(y)-a(\log e^x e^y).$$

Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

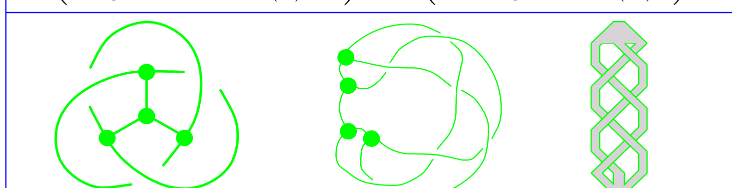
Proof. Write $V=e^c e^{uD}$ with $c\in\mathfrak{tr}_2$, $D\in\mathfrak{tder}_2$, and $\omega=e^b$ with $b\in\mathfrak{tr}_1$. Then (1) $\Leftrightarrow e^{uD}(x+y)e^{-uD}=\log e^x e^y$, (2) $\Leftrightarrow I = e^c e^{uD}(e^{uD})^* e^c = e^{2c}e^{jD}$, and (3) $\Leftrightarrow e^c e^{uD} e^{b(x+y)} = e^{b(x)+b(y)} \Leftrightarrow e^c e^{b(\log e^x e^y)} = e^{b(x)+b(y)}$ $\Leftrightarrow c = b(x)+b(y)-b(\log e^x e^y)$.

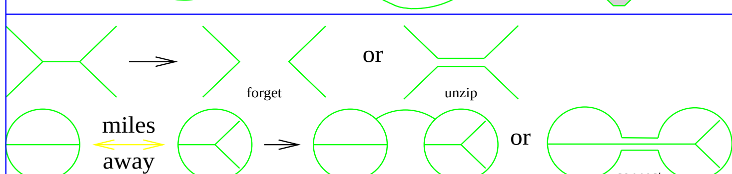
The Alekseev-Torossian Correspondence.

$$\{\text{Drinfel'd Associators}\} \leftrightarrows \{\text{Solutions of KV}\}.$$

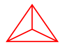
We need an even bigger algebraic structure!

$$\left(\begin{array}{c}\text{green knotted trivalent}\\ \text{graphs in } \mathbb{R}^3\ (u)\end{array}\right) \xrightarrow{\alpha_e} \left(\begin{array}{c}\text{blue tubes and red}\\ \text{strings in } \mathbb{R}^4\ (\overline{w})\end{array}\right)$$

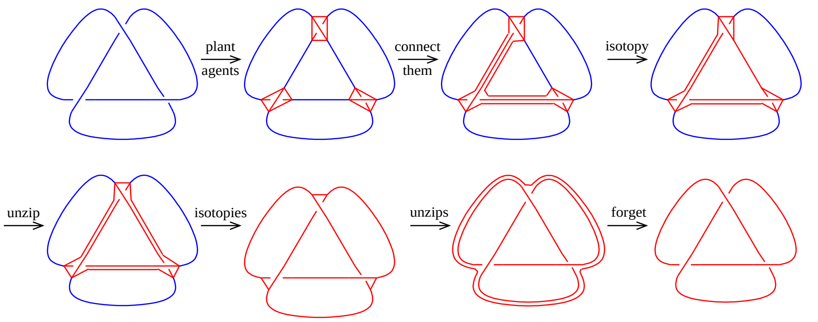


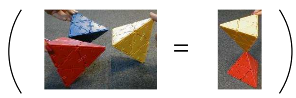


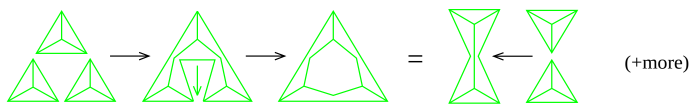
2. w−Knots, Alekseev−Torossian, and baby Etingof−Kazhdan, continued.

Using moves, KTG is generated by ribbon twists and the tetrahedron  :

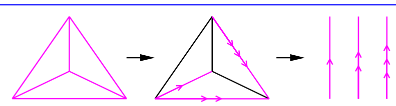
All strands here are green

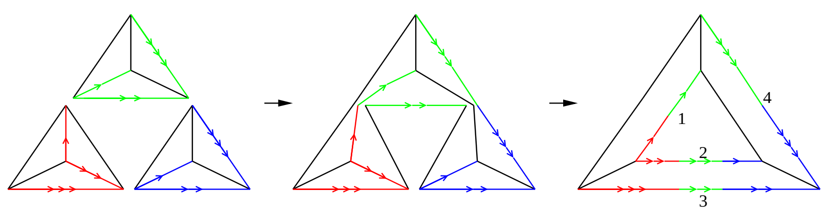


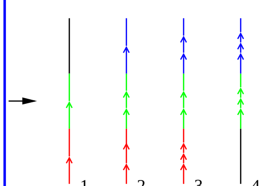
Modulo the relation(s): 



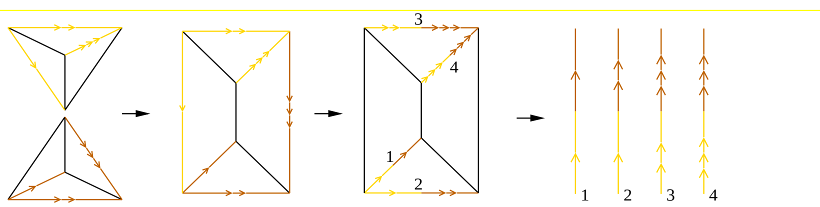
Claim. With $\Phi := Z(\text{tetrahedron})$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

Proof.  $:= \Phi \in \mathcal{A}(\uparrow_3)$



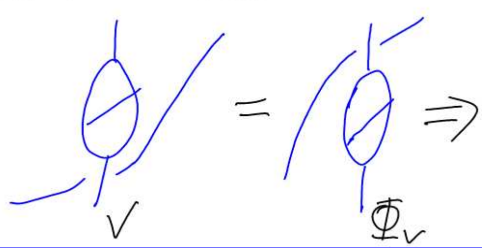


$= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$



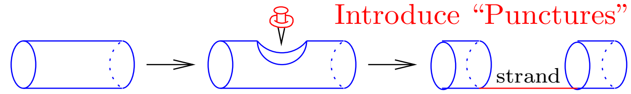
$= (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$

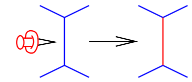
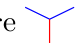
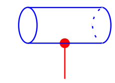
{SolKV} → {"Associators"}: Trivial – a tetrahedron has 4 vertices.

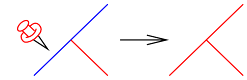


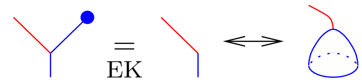
$\Phi_V \in \mathfrak{sder}_3$, so it is a sum of unrooted trees.

Introduce "Punctures".



So  where  means  .

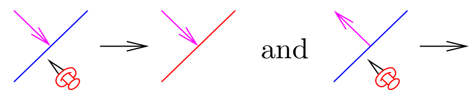
Punctures expand to the nearest Y-vertex: 

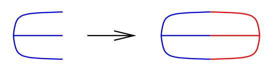
Note. 

$\mathcal{K}^{\overline{w}}$. Allow tubes and strands and tube-strand vertices as above, yet allow only "compact" knots — nothing runs to ∞.

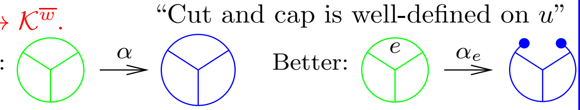
$\mathcal{K}^w \leftrightarrow \mathcal{K}^{\overline{w}}$ equivalence. $\mathcal{K}^w$ has a homomorphic expansion iff $\mathcal{K}^{\overline{w}}$ has a homomorphic expansion.

$\Longrightarrow$ Puncture $\mathcal{A}$ and Z:



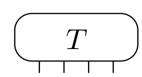
$\Longleftarrow$  (makes sense, and CA ops can be emulated)

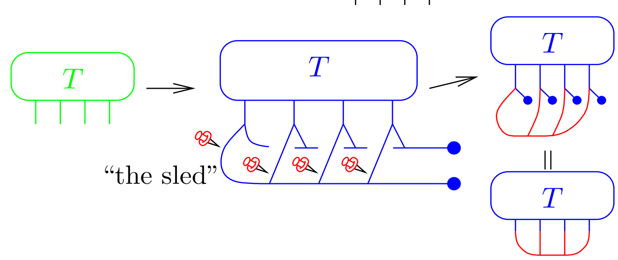
$\mathcal{K}^u \to \mathcal{K}^{\overline{w}}$. "Cut and cap is well-defined on u"

Light: 

Theorem. The generators of $\mathcal{K}^{\overline{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V.

Sketch.

so enough to write any  . Here go:

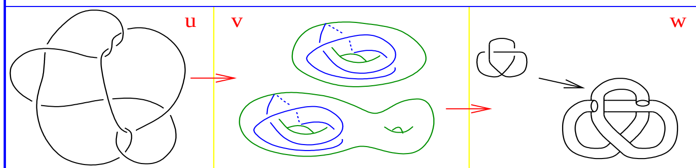


Dror Bar–Natan, Goettingen, April 2010

Plans and Dreams

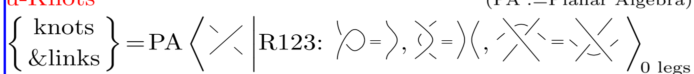
$$\begin{pmatrix}\text{arbitrary algebraic}\\ \text{structure}\end{pmatrix} \xrightarrow[\text{machine}]{\text{projectivization}} \begin{pmatrix}\text{a problem in}\\ \text{graded algebra}\end{pmatrix}$$

- Feed knot-things, get Lie algebra things.
- Feed u-knots, get Drinfel'd associators.
- Feed w-knots, get Kashiware-Vergne-Alekseev-Torossian.
- Dream: Feed v-knots, get Etingof-Kazhdan.
- Dream: Knowing the question whose answer is 42, or E-K, will be useful to algebra and topology.

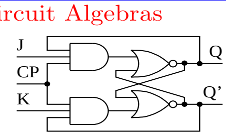


u-Knots

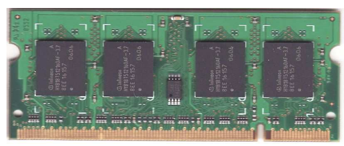
(PA :=Planar Algebra)



Circuit Algebras



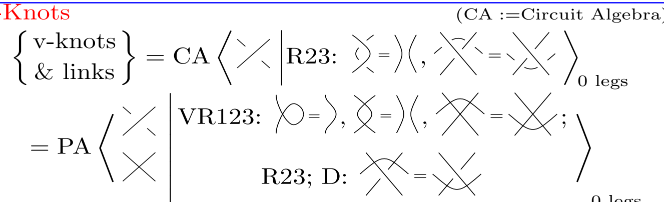
A J-K Flip Flop



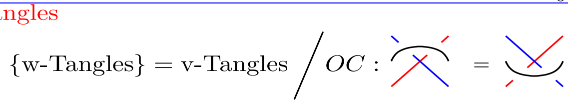
Infineon HYS64T64020HDL-3.7-A 512MB RAM

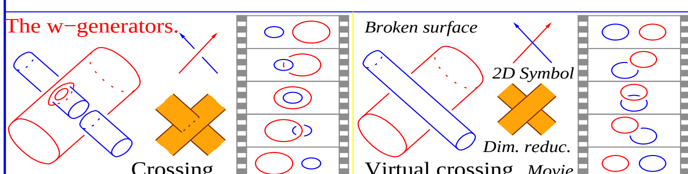
v-Knots

(CA :=Circuit Algebra)

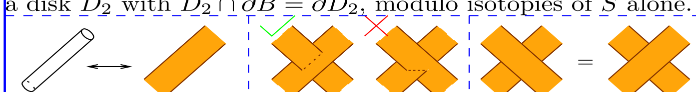


w-Tangles

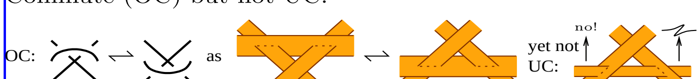




A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

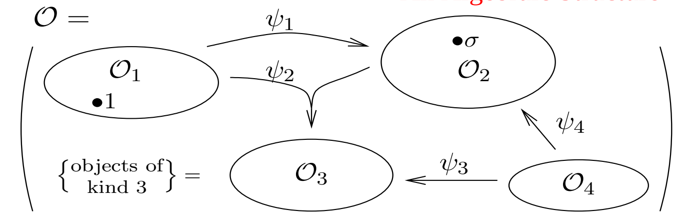


The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC:



Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

"An Algebraic Structure"



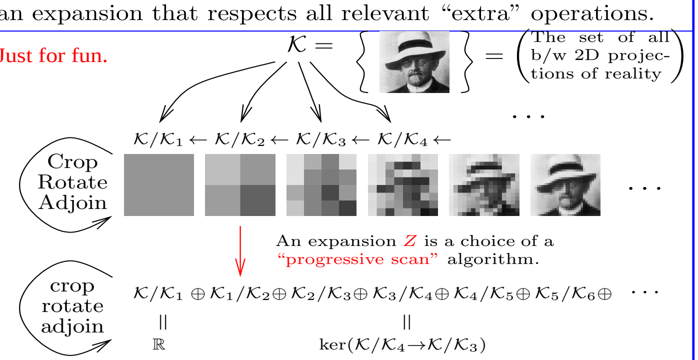
- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\begin{array}{ccccccccccc} \text{ops} \circlearrowleft \mathcal{K} & = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 & \supset & \dots \\ \Downarrow & & & & & & \downarrow Z & & & & \\ \text{ops} \circlearrowleft \operatorname{gr}\mathcal{K} & := & \mathcal{K}_0/\mathcal{K}_1 & \oplus & \mathcal{K}_1/\mathcal{K}_2 & \oplus & \mathcal{K}_2/\mathcal{K}_3 & \oplus & \mathcal{K}_3/\mathcal{K}_4 & \oplus & \dots \end{array}$$

An **expansion** is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A **homomorphic expansion** is an expansion that respects all relevant "extra" operations.

Just for fun.



Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products").

Examples. 1. The projectivization of a group is a graded associative algebra. 2. Quandle: a set Q with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

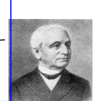
$\operatorname{proj} Q$ is a graded Leibniz algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

Our case(s).

$$\mathcal{K} \xrightarrow[\substack{\text{solving finitely many}\\ \text{equations in finitely}\\ \text{many unknowns}}]{Z:\ \text{high algebra}} \mathcal{A} := \operatorname{proj}\mathcal{K} \xrightarrow[\substack{\text{low algebra: pic-}\\ \text{tures represent}\\ \text{formulas}}]{\substack{\text{given a "Lie"}\\ \text{algebra } \mathfrak{g}}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

$\mathcal{K}$ is knot theory or topology; $\operatorname{proj}\mathcal{K} = \bigoplus \mathcal{I}^m/\mathcal{I}^{m+1}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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Video and more at http://www.math.toronto.edu/~drorbn/Talks/Goettingen-1004/

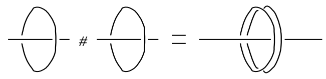
Day 2 − u, v, w: combinatorics, low and high algebra

Dror Bar−Natan, Goettingen, April 2010
http://www.math.toronto.edu/~drorbn/Talks/Goettingen−1004/

The Scheme. Topology → Combinatorics → Lie Theory via

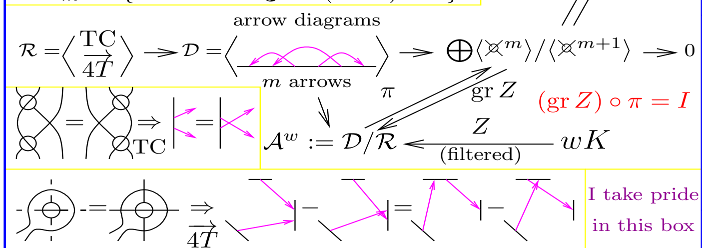
$$\mathcal{K}\xrightarrow[\text{equations, unknowns}]{Z:\ \text{high algebra}}\mathcal{A}=\operatorname{proj}\mathcal{K}=\bigoplus\mathcal{I}^m/\mathcal{I}^{m+1}\xrightarrow[\text{pictures}\to\text{formulas}]{\mathcal{T}_{\mathfrak{g}}:\ \text{low algebra}}\text{“}\mathcal{U}(\mathfrak{g})\text{”}$$

$1+1=2$, on an abacus, implies Duflo's $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}}\cong\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ (with T. Le and D. Thurston).

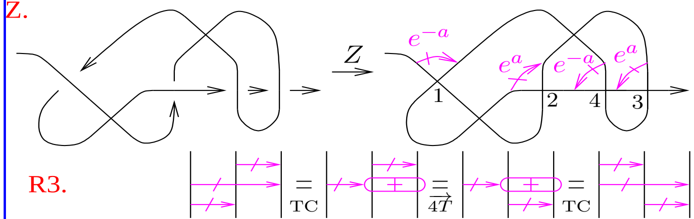


The Finite Type Story. With $\asymp := \times - \times$ set $\mathcal{V}_m := \{V : wK\to\mathbb{Q} : V(\asymp^{>m})=0\}$.

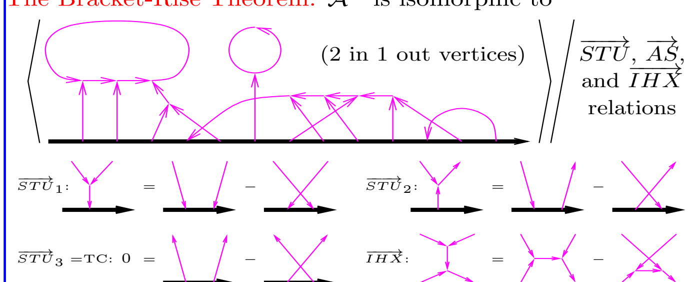
$$\bigoplus(\mathcal{V}_m/\mathcal{V}_{m-1})^*$$



Z.

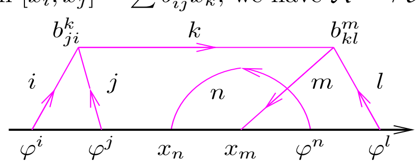


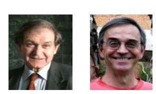
The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to



Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

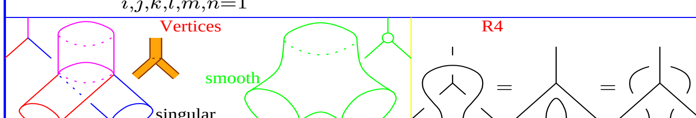
Low Algebra. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i,x_j]=\sum b^k_{ij}x_k$, we have $\mathcal{A}^w\to\mathcal{U}$ via





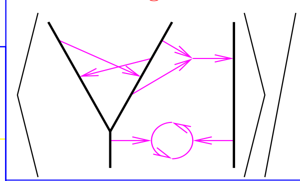
Penrose Cvitanovic

$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij}b^m_{kl}\varphi^i\varphi^j x_n x_m\varphi^l\in\mathcal{U}(I\mathfrak{g}:=\mathfrak{g}^*\rtimes\mathfrak{g})$$

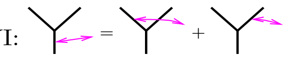


Kashiwara Vergne Alekseev Torossian

w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow)\cong\mathcal{A}^w(\uparrow\uparrow\uparrow)$ is

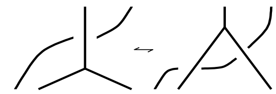


same relations, plus

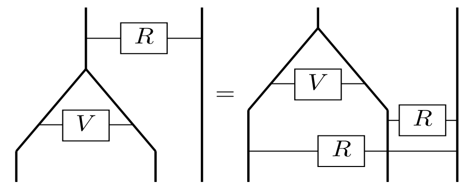


$\deg=\frac12\#\{\text{vertices}\}=6$

Knot-Theoretic statement (simplified). There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$.



Diagrammatic statement (simplified). Let $R = \exp \leftrightarrow \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $V\in\mathcal{A}^w(\uparrow\uparrow)$ so that



Algebraic statement (simplified). With $r\in\mathfrak{g}^*\otimes\mathfrak{g}$ the identity element and with $R=e^r\in\hat{\mathcal{U}}(I\mathfrak{g})\otimes\hat{\mathcal{U}}(\mathfrak{g})$ there exist $V\in\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that $V(\Delta\otimes 1)(R)=R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}\otimes\hat{\mathcal{U}}(\mathfrak{g})$

Unitary statement (simplified). There exists a unitary tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x\times\mathfrak{g}_y)$ so that $V\widehat{e^{x+y}}=\widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

Unitary $\Longleftrightarrow$ Algebraic. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$: $\varphi\in\mathfrak{g}^*$ becomes a multiplication operator, and $x\in\mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\operatorname{ad}x$: $(x\varphi)(y):=\varphi([x,y])$.

Group-Algebra statement (simplified). For every $\phi,\psi\in\mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$:

$$\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)e^{x+y}=\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)e^xe^y.$$

(shhh, this is Duflo)

Unitary $\Longrightarrow$ Group-Algebra. $\iint e^{x+y}\phi(x)\psi(y)=\langle 1,e^{x+y}\phi(x)\psi(y)\rangle=\langle V1,Ve^{x+y}\phi(x)\psi(y)\rangle=\langle 1,e^xe^yV\phi(x)\psi(y)\rangle=\langle 1,e^xe^y\phi(x)\psi(y)\rangle=\iint e^xe^y\phi(x)\psi(y)$.

Convolutions statement (Kashiwara-Vergne, simplified). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, and let $\Phi:\mathrm{Fun}(G)\to\mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x):=f(\exp x)$. Then if $f,g\in\mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f)\star\Phi(g)=\Phi(f\star g)$.

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau:G\to A$ is multiplicative then $(\mathrm{Fun}(G),\star)\to(A,\cdot)$ via $L:f\mapsto\sum f(a)\tau(a)$. For Lie $(G,\mathfrak{g})$,

$$\begin{tikzcd}
(\mathfrak{g},+)\ni x \arrow[r, "\tau_0=\exp_{\mathcal{S}}"] \arrow[d, "\exp_G"'] \arrow[dr, "\exp_{\mathcal{U}}"] & e^x\in\hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] \\
(G,\cdot)\ni e^x \arrow[r, "\tau_1"] & e^x\in\hat{\mathcal{U}}(\mathfrak{g})
\end{tikzcd}
\quad\text{so}\quad
\begin{tikzcd}
\mathrm{Fun}(\mathfrak{g}) \arrow[r, "L_0"] \arrow[d, "\Phi^{-1}"'] & \hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] \\
\mathrm{Fun}(G) \arrow[r, "L_1"] & \hat{\mathcal{U}}(\mathfrak{g})
\end{tikzcd}$$

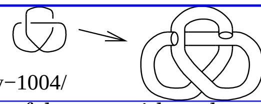
with $L_0\psi=\int\psi(x)e^xdx\in\hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi=\int\psi(x)e^x\in\hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i\in\mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1)\star\Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1\star\psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are “Laplace transforms”)

$$\star\text{ in }G:\iint\psi_1(x)\psi_2(y)e^xe^y\qquad\star\text{ in }\mathfrak{g}:\iint\psi_1(x)\psi_2(y)e^{x+y}$$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Goettingen-1004/

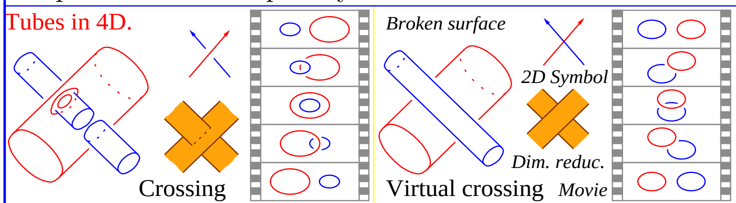
w−Knots from Z to A

Dror Bar−Natan, Luminy, April 2010
http://www.math.toronto.edu/~drorbn/Talks/Luminy−1004/

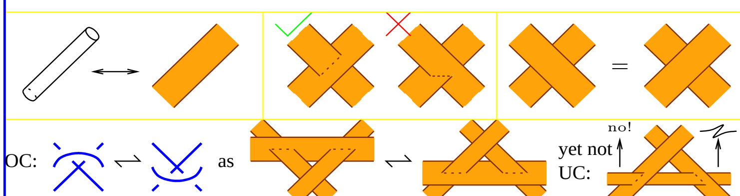


Abstract I will define w-knots, a class of knots wider than ordinary knots but weaker than virtual knots, and show that it is quite easy to construct a universal finite invariant Z of w-knots. In order to study Z we will introduce the "Euler Operator" and the "Infinitesimal Alexander Module", at the end finding a simple determinant formula for Z. With no doubt that formula computes the Alexander polynomial A, except I don't have a proof yet.

Tubes in 4D.

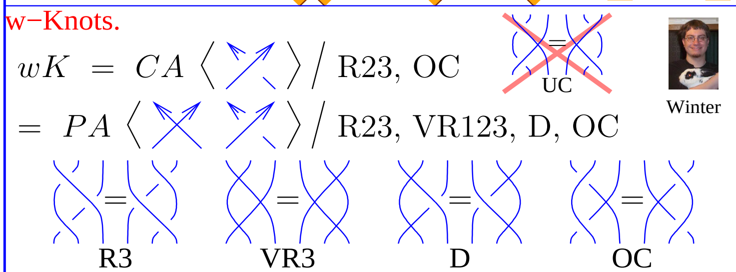


A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

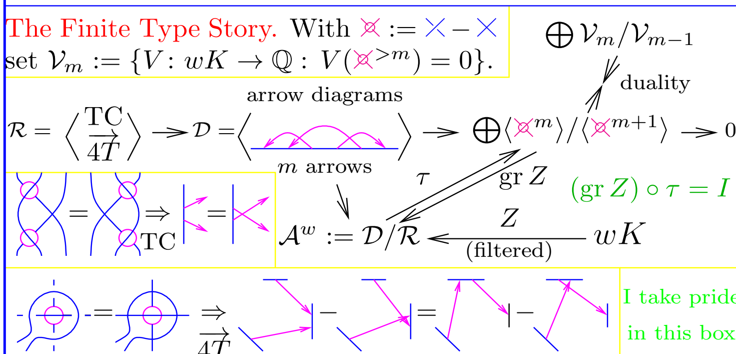


w−Knots.

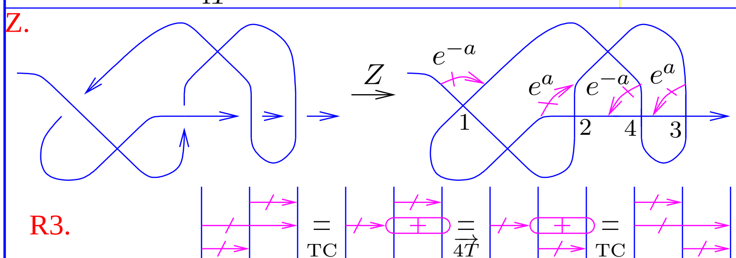
$$wK = CA\left\langle \; \right\rangle / \text{R23, OC}$$
$$= PA\left\langle \; \right\rangle / \text{R23, VR123, D, OC}$$

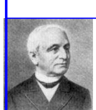


The Finite Type Story. With $\asymp := \times - \times$ set $\mathcal{V}_m := \{V : wK \to \mathbb{Q} : V(\asymp^{>m}) = 0\}$.

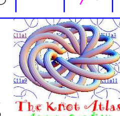


Z.





"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)



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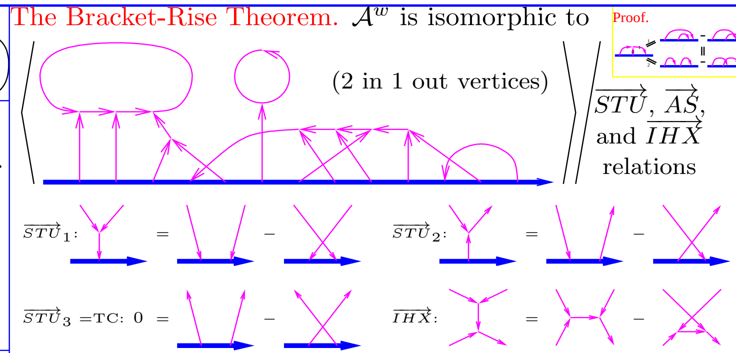
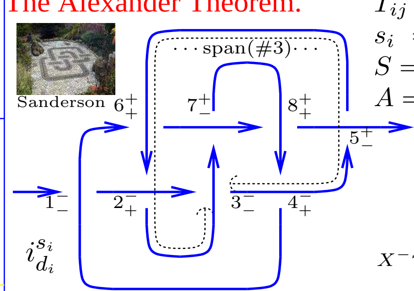
The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to



Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

Habiro - can you do better?

The Alexander Theorem.



$T_{ij} = |\text{low}(\#j) \in \text{span}(\#i)|$, $s_i = \text{sign}(\#i)$, $d_i = \text{dir}(\#i)$, $S = \text{diag}(s_i d_i)$, $A = \det\left(I + T(I - X^{-S})\right)$.

$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0\\ 0&0&1&0&1&0&0&0\\ 0&1&0&0&1&0&0&0\\ 0&1&0&0&1&0&1&0\\ 0&1&0&1&0&1&1&1\\ 0&1&0&1&0&0&1&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$X^{-S} = \text{diag}(\frac{1}{X}, X, \frac{1}{X}, X, X, \frac{1}{X}, X, \frac{1}{X})$.

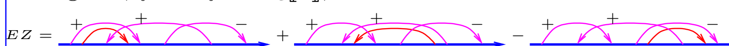
Conjecture. For u-knots, A is the Alexander polynomial.

Theorem. With $w : x^k \mapsto w_k =$ (the k-wheel),

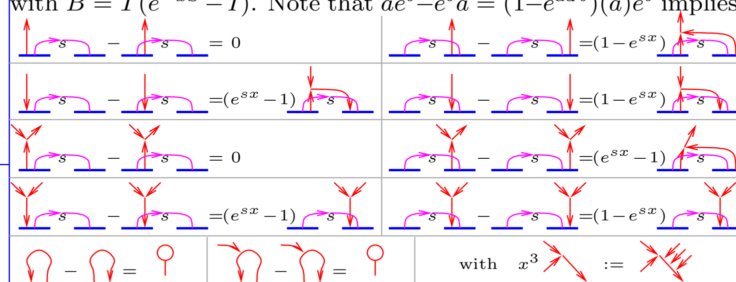
$$Z = N \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(e^x)\right)\right)$$

mod $w_k w_l = w_{k+l}$, $Z = N \cdot A^{-1}(e^x)$

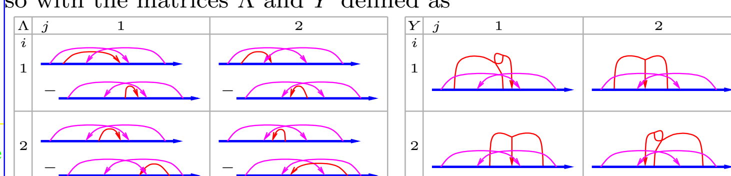
Proof Sketch. Let E be the Euler operator, "multiply anything by its degree", $f \mapsto xf'$ in $\mathbb{Q}[\![x]\!]$, so $Ee^x = xe^x$ and



We need to show that $Z^{-1}EZ = N' - \text{tr}\left((I-B)^{-1}TSe^{-xS}\right)w_1$, with $B = T(e^{-xS} - I)$. Note that $ae^b - e^b a = (1 - e^{\text{ad}\, b})(a)e^b$ implies



so with the matrices Λ and Y defined as



we have $EZ - N'' = \text{tr}(S\Lambda)$, $\Lambda = -BY - Te^{-xS}w_1$, and $Y = BY + Te^{-xS}w_1$. The theorem follows.

So What? • Habiro-Shima did this already, but not quite. (HS: *Finite Type Invariants of Ribbon 2-Knots, II*, Top. and its Appl. **111** (2001).)
• New (?) formula for Alexander, new (?) "Infinitesimal Alexander Module". Related to Lescop's arXiv:1001.4474?
• An "ultimate Alexander invariant": local, composes well, behaves under cabling. Ought to also generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers.
• Tip of the Alekseev-Torossian-Kashiwara-Vergne iceberg (AT: *The Kashiwara-Vergne conjecture and Drinfeld's associators*, arXiv:0802.4300).
• Tip of the v-knots iceberg. May lead to other polynomial-time polynomial invariants. "A polynomial's worth a thousand exponentials".
Also see http://www.math.toronto.edu/˜drorbn/papers/WKO/

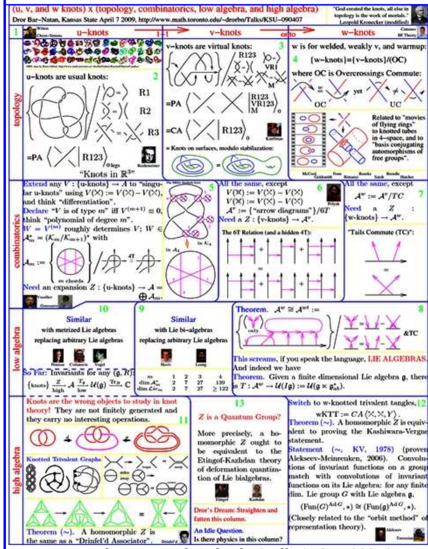
Video and more at http://www.math.toronto.edu/~drorbn/Talks/Luminy-1004/

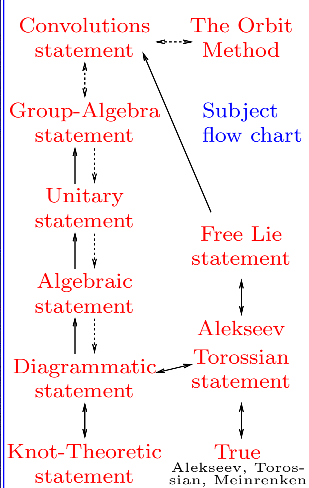
Convolutions on Lie Groups and Lie Algebras and Ribbon 2−Knots

Dror Bar−Natan, Bonn August 2009, http://www.math.toronto.edu/~drorbn/Talks/Bonn−0908

Disclaimer: Rough edges remain!

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)





www.math.toronto.edu/~drorbn/Talks/KSU−090407

Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\text{ops} \circlearrowright \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

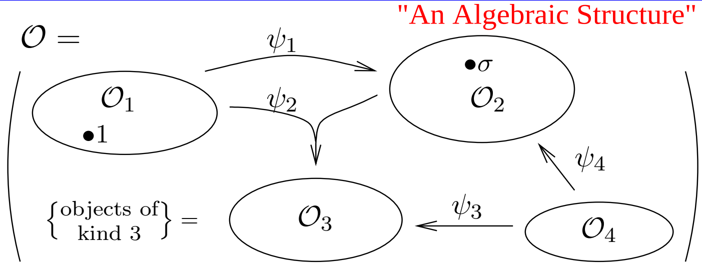
$$\Downarrow \qquad \downarrow Z$$

$$\text{ops} \circlearrowright \operatorname{gr}\mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An **expansion** is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A **homomorphic expansion** is an expansion that respects all relevant "extra" operations.

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products").

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Example: Pure Braids. PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr} PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

Our case(s).

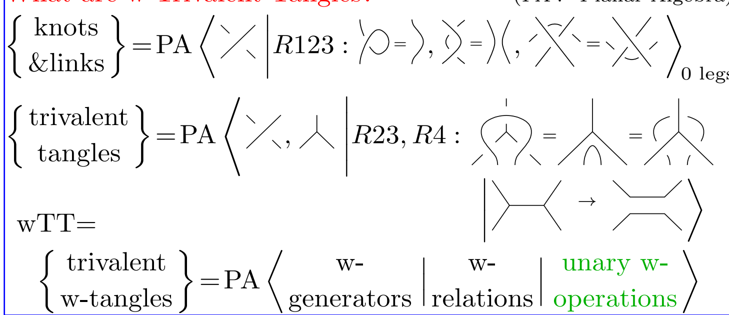
$$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z:\ \text{high algebra}} \mathcal{A} := \operatorname{gr}\mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a "Lie" algebra } \mathfrak{g}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

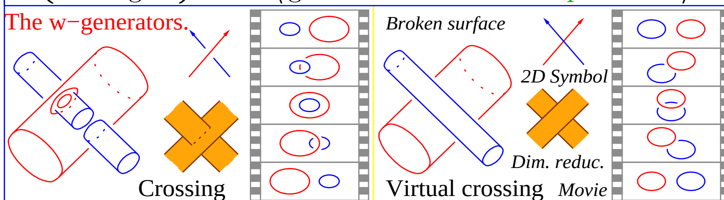
$\mathcal{K}$ is knot theory or **topology**; $\operatorname{gr}\mathcal{K}$ is finite **combinatorics**: bounded-complexity diagrams modulo simple relations.

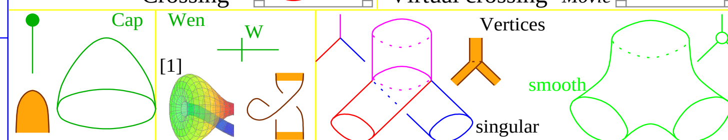
[1] http://qlink.queensu.ca/~4lb11/interesting.html
Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

29/5/10, 8:42am

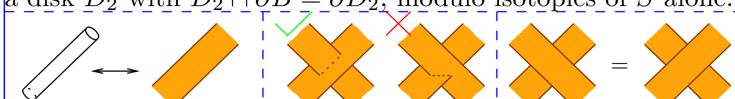
What are w-Trivalent Tangles? (PA :=Planar Algebra)



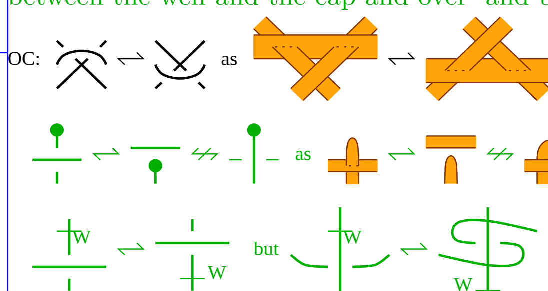




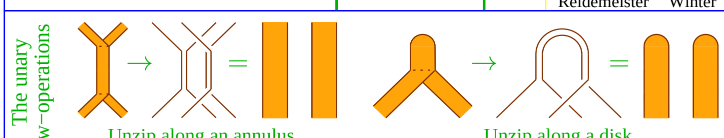
A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



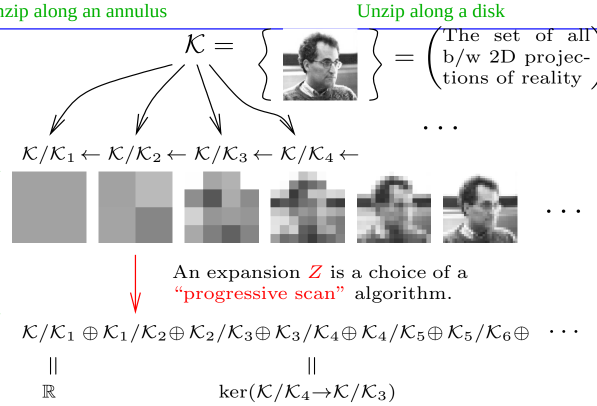
The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:



Challenge. Do the Reidemeister!

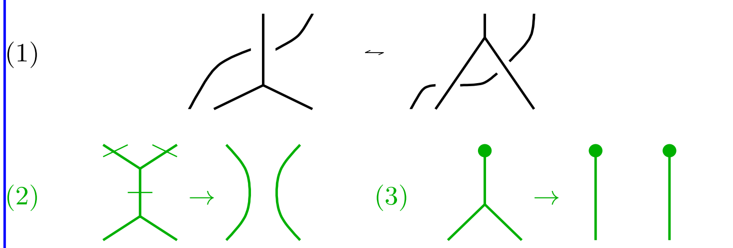


Just for fun.

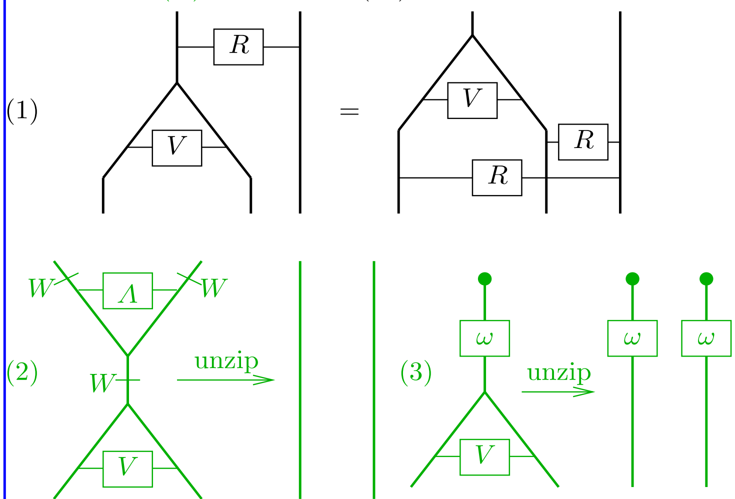


Video and more at http://www.math.toronto.edu/~drorbn/Talks/Bonn-0908/

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \overrightarrow{\text{H}} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega \in \hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

(2) $V \cdot SWV = 1$ (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \text{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\text{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

(2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x\omega_y$

Group-Algebra statement. There exists $\omega^2 \in \text{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \text{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2 = j^{1/2}$)

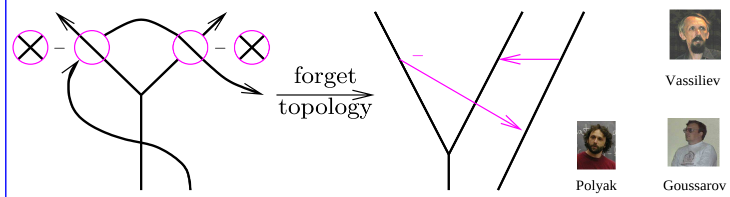
$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_{x+y}e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_x\omega^2_y e^x e^y.$$

(shhh, this is Duflo)

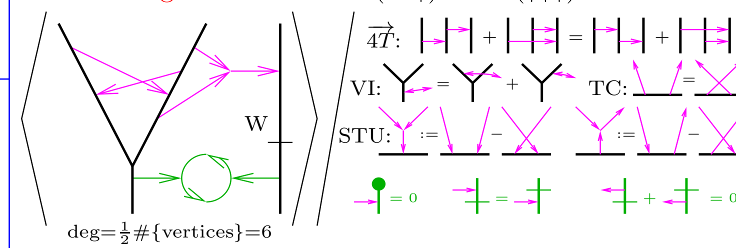
Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j : \mathfrak{g} \to \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$, and let $\Phi : \text{Fun}(G) \to \text{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

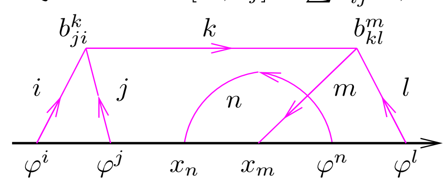
From wTT to $\mathcal{A}^w$. $\text{gr}_m\,\text{wTT} := \{m\text{−cubes}\}/\{(m+1)\text{−cubes}\}$:



w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij}x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij}b^m_{kl}\varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g})$$

Unitary $\Longleftrightarrow$ Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\text{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\text{ad}\,x$: $(x\varphi)(y) := \varphi([x, y])$.
- $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ is "the constant term".

Unitary $\Longrightarrow$ Group-Algebra. $\iint \omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

$$= \langle \omega_{x+y}, \omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle = \langle V\omega_{x+y}, Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$$
$$= \langle \omega_x\omega_y, e^x e^y V\phi(x)\psi(y)\omega_{x+y}\rangle = \langle \omega_x\omega_y, e^x e^y \phi(x)\psi(y)\omega_x\omega_y\rangle$$
$$= \iint \omega^2_x\omega^2_y e^x e^y \phi(x)\psi(y).$$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\text{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

$$\begin{tikzcd}
(\mathfrak{g},+) \ni x \arrow[r, "\tau_0=\exp_{\mathcal{S}}"] \arrow[d, "\exp_G"] \arrow[dr, "\exp_{\mathcal{U}}"] & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] & \text{so} & \text{Fun}(\mathfrak{g}) \arrow[r, "L_0"] \arrow[d, "\Phi^{-1}"] & \hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] \\
(G,\cdot) \ni e^x \arrow[r, "\tau_1"] & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) & & \text{Fun}(G) \arrow[r, "L_1"] & \hat{\mathcal{U}}(\mathfrak{g})
\end{tikzcd}$$

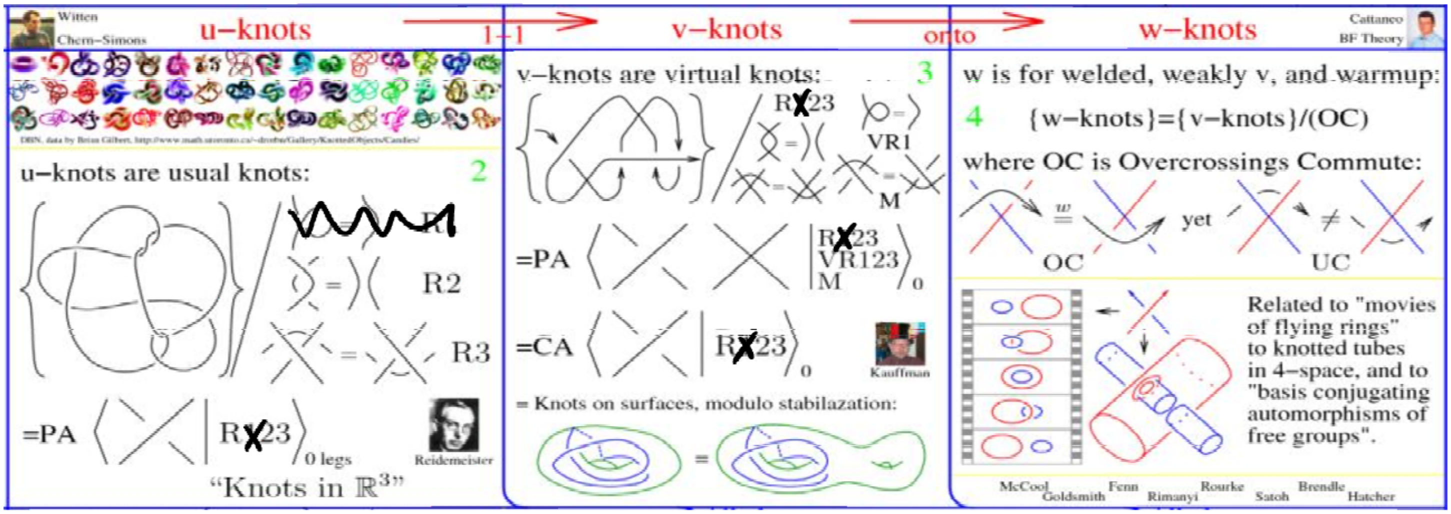
with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \text{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

$\star$ in G : $\iint \psi_1(x)\psi_2(y)e^x e^y$ $\star$ in $\mathfrak{g}$: $\iint \psi_1(x)\psi_2(y)e^{x+y}$

We skipped...

- The Alexander polynomial and Milnor numbers.
- u-Knots, Alekseev-Torossian, and Drinfel'd associators.
- v-Knots, quantum groups and Etingof-Kazhdan.
- BF theory and the successful religion of path integrals.
- The simplest problem hyperbolic geometry solves.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Bonn-0908/



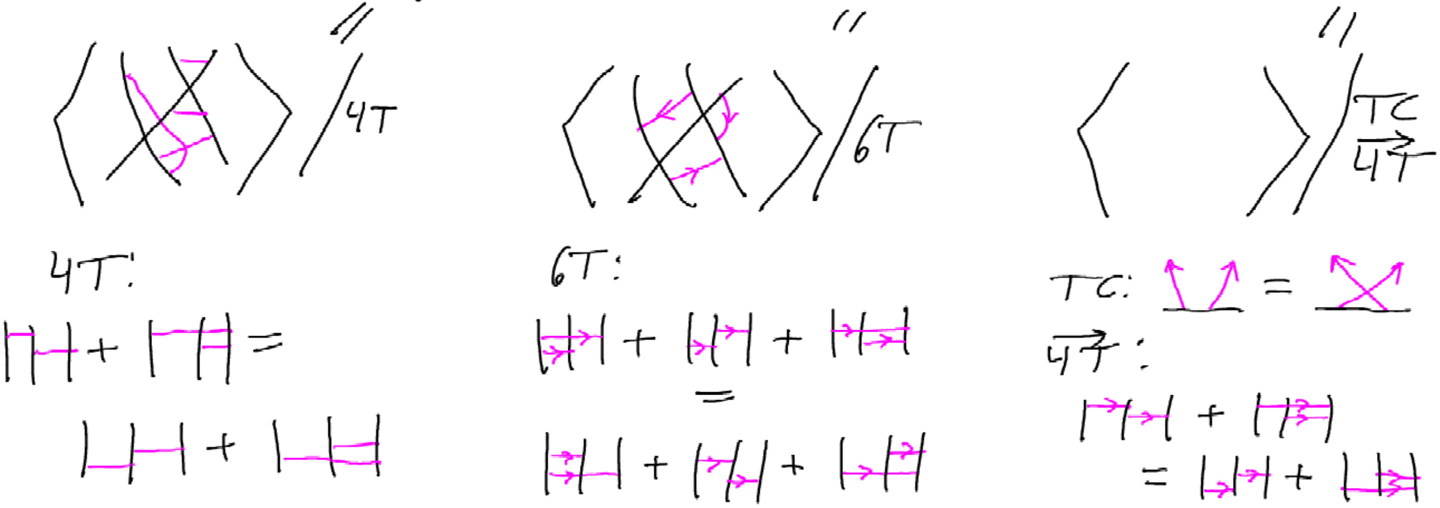
$$\begin{array}{ccccc} \mathcal{K}^u & \longrightarrow & \mathcal{K}^v & \longrightarrow & \mathcal{K}^w \\ \downarrow Z^u & & \downarrow Z^v & & \downarrow Z^w \\ \mathcal{A}^u & \longrightarrow & \mathcal{A}^v & \longrightarrow & \mathcal{A}^w \end{array}$$

Expansion exists, Eg., using the Kontsevich integral. No homomorphic expansion!

wide open

Homomorphic Z^v exists?

$\sigma_{ij} \to e^{a_{ij}}$



$$\begin{array}{ccc} \downarrow \tau^u & \downarrow \tau^v & \downarrow \tau^w \quad \text{Today} \\ U(\mathfrak{g})^{\otimes C} & U(\mathfrak{g}_+ \oplus \mathfrak{g}_-)^{\otimes C} & U(I\mathfrak{g})^{\otimes C} \end{array}$$

For any metrized f.d. Lie algebra $\mathfrak{g}$

For any f.d. Lie bialgebra $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$

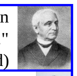
For any f.d. Lie algebra $\mathfrak{g}$

More at http://www.math.toronto.edu/~drorbn/Talks/Trieste-0905/

(u, v, and w knots) x (topology, combinatorics, low algebra, and high algebra)

Dror Bar−Natan, Kansas State April 7 2009, http://www.math.toronto.edu/~drorbn/Talks/KSU−090407

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)



1 Witten Chern−Simons

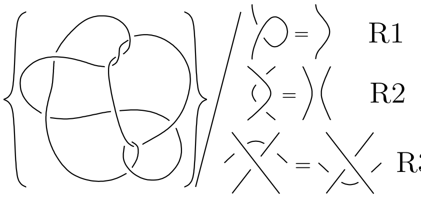
u−knots $\xrightarrow{1-1}$ v−knots $\xrightarrow{\text{onto}}$ w−knots

Cattaneo BF Theory

topology

DBN, data by Brian Gilbert, http://www.math.utoronto.ca/~drorbn/Gallery/KnottedObjects/Candies/

u−knots are usual knots: 2



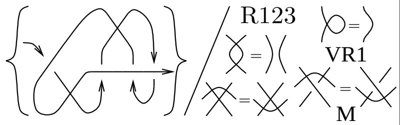
R1
R2
R3

$=\text{PA}\,\langle\ \ \mid \text{R123}\rangle_{0\text{ legs}}$

Reidemeister

"Knots in $\mathbb{R}^3$"

v−knots are virtual knots: 3



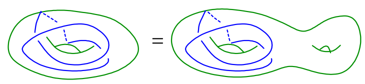
R123
VR1
M

$=\text{PA}\,\langle\ \ \mid \text{R123, VR123, M}\rangle_0$

$=\text{CA}\,\langle\ \ \mid \text{R123}\rangle_0$

Kauffman

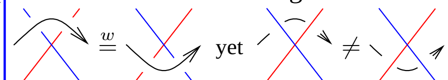
= Knots on surfaces, modulo stabilazation:



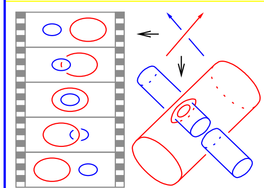
w is for welded, weakly v, and warmup:

4 {w−knots}={v−knots}/(OC)

where OC is Overcrossings Commute:



OC yet UC



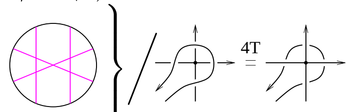
Related to "movies of flying rings" to knotted tubes in 4−space, and to "basis conjugating automorphisms of free groups".

McCool Goldsmith Fenn Rimanyi Rourke Satoh Brendle Hatcher

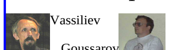
combinatorics

Extend any V : {u-knots} $\to A$ to "singular u-knots" using $V(\text{⤫}) := V(\text{⤫}) - V(\text{⤫})$, and think "differentiation".
Declare "V is of type m" iff $V^{(m+1)} \equiv 0$, think "polynomial of degree m".
$W = V^{(m)}$ roughly determines V; $W \in \mathcal{A}_m^\star = (\mathcal{K}_m/\mathcal{K}_{m+1})^\star$ with

$$\mathcal{A}_m := \left\{ \quad \right\}_{m\text{ chords}} \Big/ \quad \overset{4T}{=}$$

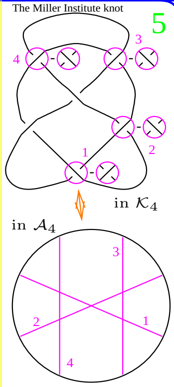


Need an expansion Z : {u-knots} $\to \mathcal{A} = \bigoplus \mathcal{A}_m$.



Vassiliev Goussarov

The Miller Institute knot 5



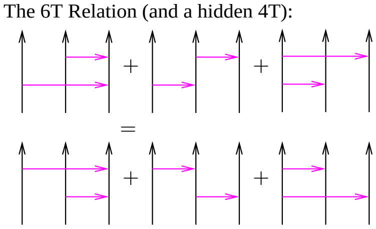
in $\mathcal{K}_4$

in $\mathcal{A}_4$

All the same, except 6
$V(\text{⤫}) := V(\text{⤫}) - V(\text{⤫})$
$V(\text{⤫}) := V(\text{⤫}) - V(\text{⤫})$
$\mathcal{A}^v :=$ {"arrow diagrams"}$/6T$
Need a Z : {v-knots} $\to \mathcal{A}^v$.

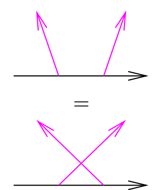
Polyak

The 6T Relation (and a hidden 4T):



All the same, except 7
$\mathcal{A}^w := \mathcal{A}^v/TC$
Need a Z : {w-knots} $\to \mathcal{A}^w$.

"Tails Commute (TC)":



low algebra

10

Similar

with metrized Lie algebras replacing arbitrary Lie algebras

Penrose Cvitanovic Vogel

So Far: Invariants for any $(\mathfrak{g}, R)$:

$$\{\text{knots}\} \xrightarrow[\text{high}]{Z} \mathcal{A} \xrightarrow[\text{low}]{\mathcal{T}_\mathfrak{g}} \mathcal{U}(\mathfrak{g}) \xrightarrow{\text{Tr}_R} \mathbb{C}$$

9

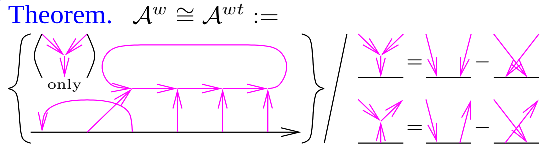
Similar

with Lie bi−algebras replacing arbitrary Lie algebras

Haviv Leung

m	1	2	3	4
$\dim \mathcal{A}_m^v$	2	7	27	139
$\dim \mathcal{L}ie_m$	2	7	27	≥ 122

Theorem. $\mathcal{A}^w \cong \mathcal{A}^{wt} :=$ 8

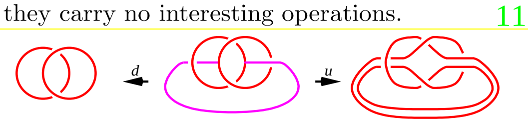


only &TC

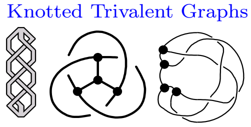
This screams, if you speak the language, LIE ALGEBRAS. And indeed we have
Theorem. Given a finite dimensional Lie algebra $\mathfrak{g}$, there is $T : \mathcal{A}^w \to \mathcal{U}(I\mathfrak{g}) := \mathcal{U}(\mathfrak{g} \ltimes \mathfrak{g}^\star_{ab})$.

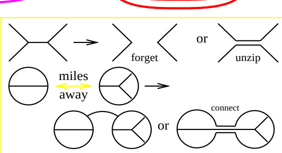
high algebra

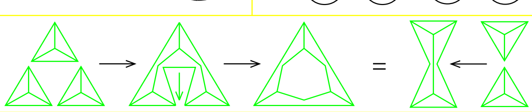
Knots are the wrong objects to study in knot theory! They are not finitely generated and they carry no interesting operations. 11



Knotted Trivalent Graphs







Theorem (∼). A homomorphic Z is the same as a "Drinfel'd Associator".

Drinfel'd

13

Z is a Quantum Group?

More precisely, a homomorphic Z ought to be equivalent to the Etingof-Kazhdan theory of deformation quantization of Lie bialgebras.

Etingof Kazhdan

Dror's Dream: Straighten and fatten this column.

An Idle Question.
Is there physics in this column?

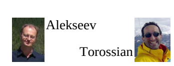
Switch to w-knotted trivalent tangles, 12

$$\text{wKTT} := CA\,\langle \text{⤫}, \text{⤫}, Y \rangle.$$

Theorem (∼). A homomorphic Z is equivalent to proving the Kashiwara-Vergne statement.
Statement (∼, KV, 1978) (proven Alekseev-Meinrenken, 2006). Convolutions of invariant functions on a group match with convolutions of invariant functions on its Lie algebra: for any finite dim. Lie group G with Lie algebra $\mathfrak{g}$,

$$(\text{Fun}(G)^{\text{Ad}\,G}, \star) \cong (\text{Fun}(\mathfrak{g})^{\text{Ad}\,G}, \star).$$

(Closely related to the "orbit method" of representation theory).



Alekseev Torossian

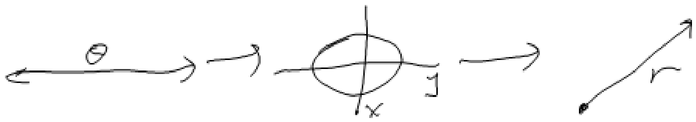
More at http://www.math.toronto.edu/~drorbn/Talks/KSU-090407/

10 Minutes on Homology

Dror Bar-Natan, Bogota, February 23 2009
http://www.math.toronto.edu/~drorbn/Talks/Bogota-0902/
and http://katlas.math.toronto.edu/drorbn/AcademicPensieve/2009-02/

Motivation

Homology measures our failure to construct all solutions of a given equation:



$$\mathbb{R}^1_\theta \xrightarrow{d^1} \mathbb{R}^2_{x,y} \xrightarrow{d^2} \mathbb{R}^1_r$$

$$\theta \mapsto \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \qquad \begin{pmatrix}x\\ y\end{pmatrix} \mapsto \sqrt{x^2+y^2}$$

$$d^2 \circ d^1 = \text{const}$$

$$V \xrightarrow{d_1} W \xrightarrow{d_2} Z$$

$$\operatorname{im} d_1 \subset \ker d_2 \Leftrightarrow d_2 \circ d_1 = 0$$

$$H(W) := \ker d_2 / \operatorname{im} d_1$$

Definition

A "complex" is a long chain of "paramatrization problems":

$$\Omega = \left(\dots \to \Omega^{r-1} \xrightarrow{d^{r-1}} \Omega^r \xrightarrow{d^r} \Omega^{r+1} \to \dots\right)$$

s.t. $d^2 = 0$ or $\operatorname{im}(d) \subset \ker(d)$

Homology:

$$H^r(\Omega) := \ker d^r / \operatorname{im} d^{r-1}$$

The "parametrization failure" at step r.

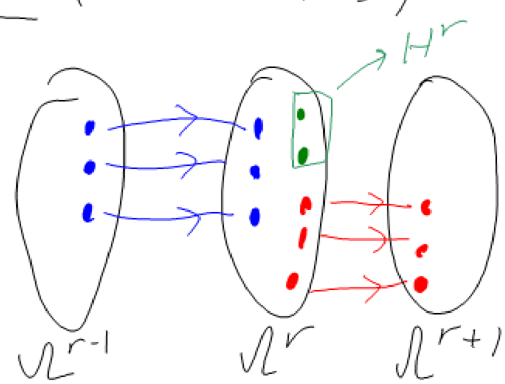
[I don't understand why "long" complexes are so common ☹]

Euler Characteristic

Theorem If everything is finite, then

$$\sum (-1)^r \dim \Omega^r = \sum (-1)^r \dim H^r =: \chi(\Omega)$$

Proof (more or less)



Morphisms and Homotopy

Morphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega_0^{r-1} \arrow[r, "d^{r-1}"] \arrow[d, "F^{r-1}"'] & \Omega_0^r \arrow[r, "d^r"] \arrow[d, "F^r"'] & \Omega_0^{r+1} \arrow[r] \arrow[d, "F^{r+1}"'] & \cdots \\
\cdots \arrow[r] & \Omega_1^{r-1} \arrow[r, "d^{r-1}"] & \Omega_1^r \arrow[r, "d^r"] & \Omega_1^{r+1} \arrow[r] & \cdots
\end{tikzcd}$$

Homotopies:

$$\begin{tikzcd}
\Omega_0^{r-1} \arrow[r, "d^{r-1}"] \arrow[d, shift right, "F^{r-1}"'] \arrow[d, shift left, "G^{r-1}"] & \Omega_0^r \arrow[r, "d^r"] \arrow[d, shift right, "F^r"'] \arrow[d, shift left, "G^r"] \arrow[dl, "h^r"'] & \Omega_0^{r+1} \arrow[d, shift right, "F^{r+1}"'] \arrow[d, shift left, "G^{r+1}"] \arrow[dl, "h^{r+1}"'] \\
\Omega_1^{r-1} \arrow[r, "d^{r-1}"] & \Omega_1^r \arrow[r, "d^r"] & \Omega_1^{r+1}
\end{tikzcd}$$

$$F^r - G^r = h^{r+1}d^r + d^{r-1}h^r$$

If there are $\Omega_0 \underset{g}{\overset{f}{\rightleftarrows}} \Omega_1$ s.t. $f \circ g \sim I_{\Omega_1}$ and $g \circ f \sim I_{\Omega_2}$, then "$\Omega_0$ & Ω_1 are homotopy equivalent" [and they have equal homology]

© | << < ? > >> | Dror Bar-Natan: Talks:

Algebraic Knot Theory

Colloquium in Mathematics

University of Copenhagen, October 9, 2008

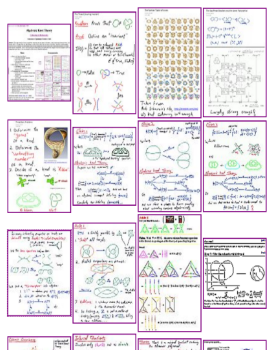
Abstract. The right objects of study in algebraic topology are not spaces, but rather, "spaces and maps between them". In a similar spirit I will argue that the right things to study in knot theory are not knots, but rather, "knotted trivalent graphs", as in the world of knotted trivalent graphs (and the basic operations between them) many interesting properties of honest knots become "definable". Thus I find myself again studying the good old Kontsevich integral - the best example I know of an algebraic knot theory - but my perspective this time is completely different.

Menu

- Some very basic knot theory.
  - The three-colouring invariant.
  - The Rolfsen table.
  - The Kauffman bracket and the Jones polynomial.
- Three things we'd like to understand:
  - The genus of a knot.
  - The crossing number of a knot.
  - Ribbon knots.
- TG-algebras and TG-morphisms.
  - Aside 1: KTG is finitely generated (and presented).
  - Aside 2: And it is related to Drinfel'd associators.
- A word about "definable" sets.
- A very strong TG-morphism exists! (But it is too hard...)
- The envelope of the Alexander polynomial. (Must be doable, but I don't know how).
- Internal quotients. (Likely more than just Lie algebra; may have "moduli" rather than just discrete points).
- Open questions and propaganda.

Also see my talk in Aarhus, June 2007. Much progress was made since, but the introductory talk (this one) remains more or less the same.

Transparencies



The Three-Colouring Invariant

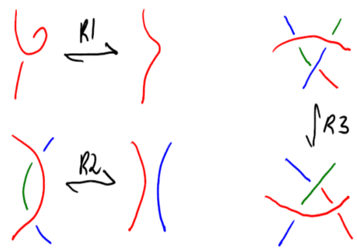
October-08-08
11:30 AM

Problem Prove that ○ ≠ (trefoil).

Proof Define an "invariant"

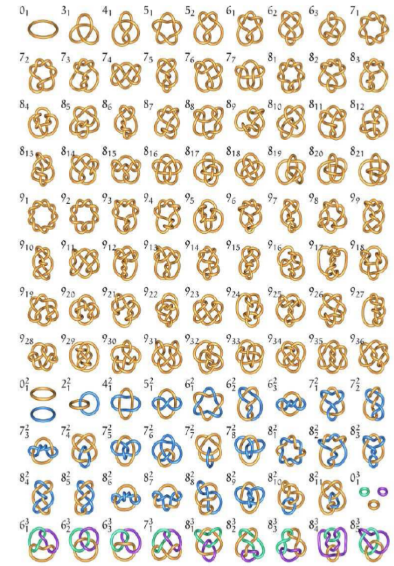
$I(D) := \left|\begin{array}{l} D \text{ can be coloured RGB} \\ \text{so that all colours are} \\ \text{used, and every crossing} \\ \text{is either mono- or tri-chromatic} \end{array}\right| \in \{\text{True}, \text{False}\}$

○ → False   (trefoil) → True



The Rolfsen Table of Knots

October-08-08
11:31 AM



Taken from
Rob Scharein's site, http://knotplot.com/zoo/
⇒ Knot Colouring isn't enough.

The Kauffman Bracket and the Jones Polynomial

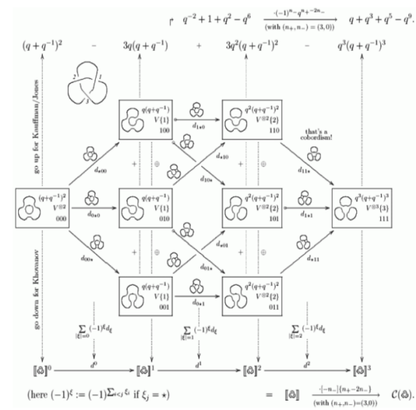
October-08-08
11:32 AM

$\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$ (0-smoothing, 1-smoothing)

$\langle \bigcirc^k \rangle = (q+q^{-1})^k$

$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$

(n_+, n_-) count $(\nwarrow, \nearrow)$

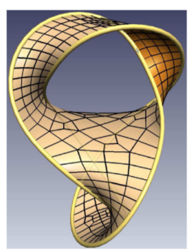


Largely strong enough?

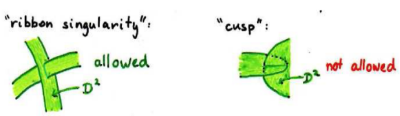
Three Basic Problems

October-08-08
12:49 PM

1. Determine the "genus" of a knot.
2. Determine the "unknotting number" of a knot.
3. Decide if a knot is "Ribbon".



Drawn using SeifertView, http://www.win.tue.nl/~vanwijk/seifertview/



(image by Zsuzsanna Dancso)

Claim 1

$\mathcal{K}(\bigcirc) \supset \left\{\begin{array}{c}\text{Knots bounding} \\ \text{a surface of} \\ \text{genus 1}\end{array}\right\} = \{\partial_\tau \gamma : \gamma \in \mathcal{K}(\text{band-graph})\}$

knottings of a band-graph

where

∂_τ: is the "topological boundary" operator.

Algebraic Knot Theory: Suppose we had invariants Z:

$$\begin{array}{ccc} \mathcal{K}(\text{band-graph}) & \xrightarrow{Z} & \mathcal{A}(\text{band-graph}) \\ \downarrow \partial_\tau & & \downarrow \partial_A \\ \left\{\begin{array}{c}\text{genus 1} \\ \text{knots}\end{array}\right\} = \operatorname{im} \partial_\tau \subset \mathcal{K}(\bigcirc) & \xrightarrow{Z} & \mathcal{A}(\bigcirc) \end{array}$$

Then $Z\left(\left\{\begin{array}{c}\text{genus 1} \\ \text{knots}\end{array}\right\}\right) \subset \operatorname{im} \partial_A$ and we have an algebraic invariant detecting $\{\text{genus} \geq 1\}$.

Similarly for detecting $\{\text{genus} \geq 7\}$...

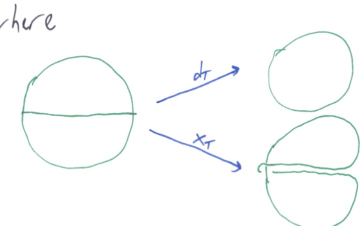
Claim 2

$\mathcal{K}(\bigcirc) \supset \left\{\begin{array}{c}\text{Knots of unknotting} \\ \text{number 1}\end{array}\right\} = \left\{ \chi\gamma : \begin{array}{c} \gamma \in \mathcal{K}(\Theta) \\ d\gamma = \bigcirc \end{array} \right\}$

knottings of Θ

the unknot.

where



Algebraic Knot Theory:

$$\begin{array}{ccc} & \mathcal{K}(\bigcirc) \xrightarrow{Z} \mathcal{A}(\bigcirc) & \\ \mathcal{K}(\Theta) & \xrightarrow{Z} & \mathcal{A}(\Theta) \\ & \mathcal{K}(\bigcirc) \xrightarrow{Z} \mathcal{A}(\bigcirc) \ni \bigcirc & \end{array}$$

(χ_τ, d_τ, χ_A, d_A)

So $Z\left(\left\{\begin{array}{c}\text{unknotting} \\ \text{number 1}\end{array}\right\}\right) \subset \left\{ \chi_A \alpha : \begin{array}{c} \alpha \in \mathcal{A}(\Theta) \\ d_A\alpha = Z(\bigcirc) \end{array} \right\}$

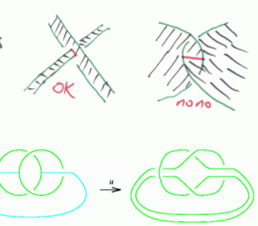
and we stand a chance to learn something about unknotting numbers algebraically.

Claim 3

$\{\text{Ribbon knots}\} \overset{\text{more or less}}{=} \left\{ u\gamma : \begin{array}{c} \gamma \in \mathcal{K}(\bigcirc\!\!-\!\!\bigcirc) \\ d\gamma = \bigcirc\bigcirc \end{array} \right\}$

Where:

Ribbon means



Algebraic Knot Theory:

$$\begin{array}{ccc} & \mathcal{K}(\bigcirc\bigcirc) \xrightarrow{Z} \mathcal{A}(\bigcirc\bigcirc) \ni \bigcirc\bigcirc & \\ \mathcal{K}(\bigcirc\!\!-\!\!\bigcirc) & \xrightarrow{Z} & \mathcal{A}(\bigcirc\!\!-\!\!\bigcirc) \\ & \mathcal{K}(\bigcirc) \xrightarrow{Z} \mathcal{A}(\bigcirc) & \end{array}$$

(d, u)

So $Z\left(\left\{\begin{array}{c}\text{Ribbon} \\ \text{knots}\end{array}\right\}\right) \subset \{u\alpha : d\alpha = Z(\bigcirc\bigcirc)\} \subset \mathcal{A}(\bigcirc\!\!-\!\!\bigcirc)$

And we stand a chance to find a counterexample to $\{\text{Ribbon}\} = \{\text{Slice}\}$!

More at http://www.math.toronto.edu/~drorbn/Talks/Copenhagen-081009/

So many interesting properties of knots are definable using Knotted Trivalent Graphs (KTGs) (Fully labeled, oriented, Framed:)

and the basic operations between them:

"delete" "unzip"

We seek a "TG-morphism" into algebra:

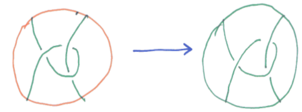
1. $\forall\Gamma$ an algebraic space $\mathcal{A}(\Gamma)$, $Z_\Gamma : \mathcal{K}(\Gamma) \to \mathcal{A}(\Gamma)$.
2. $d, u, \#$ defined on the $\mathcal{A}(\Gamma)$'s.
3.
$$\begin{array}{ccc} \mathcal{K}(\Gamma) & \xrightarrow{Z} & \mathcal{A}(\Gamma) \\ \downarrow u_e & & \downarrow u_e \\ \mathcal{K}(u_e\Gamma) & \xrightarrow{Z} & \mathcal{A}(u_e\Gamma) \end{array}$$ etc.

Aside 1

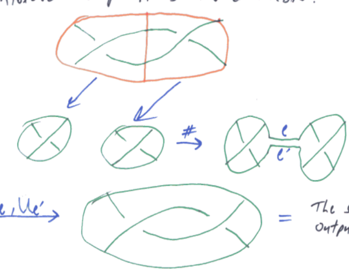
KTG is Finitely generated, by and :

The shielded generators of tangled graphs.

1. "Shield" all tangles:



2. Shielded compositions are definable:



The shielded output.

3. Relations: 1. whatever makes this well-defined. 2. The Reidemeister moves.

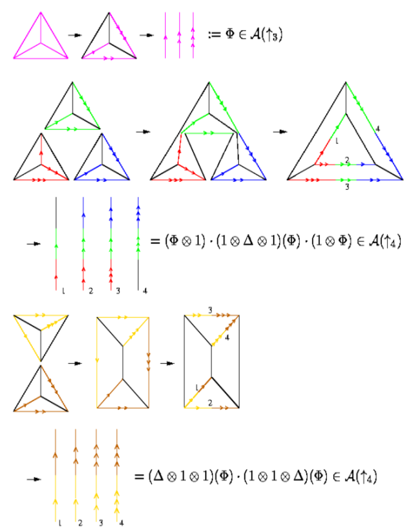
4. So finding a Z is just a matter of finding/guessing $Z(\triangle)$ & $Z(\boxtimes)$, solving a few relations....

Aside 2

Lest you think it is easy...

Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi Hopf algebras.

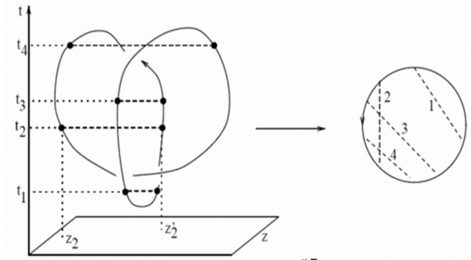
Proof.



Abstract [edit]

We construct a (very) well-behaved invariant of knotted trivalent graphs using only the Kontsevich integral, in three steps.

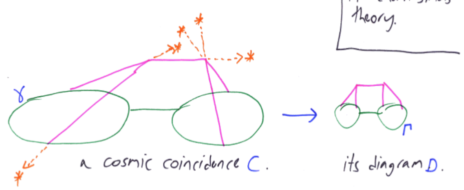
Step 1 - The Naive Kontsevich Integral [edit]



$$Z_0(K) = \sum_{m;\ t_1<\ldots<t_m;\ P=\{(z_i,z_i')\}} \frac{(-1)^{\#P_\downarrow}}{(2\pi i)^m} D_P \bigwedge_{i=1}^m \frac{dz_i - dz_i'}{z_i - z_i'}$$

We define the "naive Kontsevich integral" Z_1 of a knotted trivalent graph or a slice thereof as in the "standard" picture above, except generalized to graphs in the obvious manner.

Cosmic Coincidences:



a cosmic coincidence C. its diagram D.

witten would call it "Chern-Simons" theory.

Definition (Dylan)

$Z(\gamma) = \sum_C D \in \mathcal{A}(\Gamma) :=$ AS: IHX: STU: Vinv:

(lots of work is hidden here, and some unknowns)

Behavior under / → // is predictable.

Internal Quotients

Involve only chords and no strands.

Examples:

1. $=0$ "The Lie algebra is 3-dimensional" (i.e., it is $sl(2)$...)

2. $=0$ "Diagrams are (very) shallow" (related to $gl(1|1)$, Alexander)

3. $=0$ "pentagons stay out"

4,5,.... use your imagination.

classification?

Theorem There is a minimal quotient containing the Alexander polynomial.

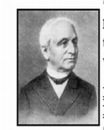
Proof Use the "internal kernel" of the Alexander weight system:

1. 2. 3. $\pm$ $\pm$ cyc perms $=0$.

What remains is a polynomial amount of information!

Conjecture This explains everything that we know about the Alexander polynomial.

Some propaganda...



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

Visit! Edit!

I don't understand... (more)

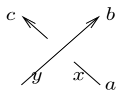
- Who needs "trivalent" in "knotted trivalent graphs"? (more)
- The notion of "framing". (more)
- Configuration space integrals / perturbative Chern-Simons theory for knotted graphs. (more)
- **The Alexander polynomial.** (more)
- The Lieberum "$gl(1|1)$" associator. (more)
- Must associators be so hard? Why? (more)
- The relationship between "genus" and "finite type". (more)
- The relationship between "gropes" and "finite type". (more)
- ***TG*-ideals / internal quotients.**
- What exactly are *TG*-algebras? What are the syzygies among the relations between their generators?
- Virtual knots.
- Quantum groups(!) (more)
- The work of Etingof and Kazhdan.
- The polynomiality of knot polynomials.
- Functional equations.
- The Eilenberg-Zilber theorem.
- Which other interesting classes of knots/links are "TG-definable"?

More at http://www.math.toronto.edu/~drorbn/Talks/Copenhagen-081009/

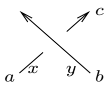
Dror Bar−Natan: Talks: Sandbjerg−0810:

The Penultimate Alexander Invariant

A Definition of the MVA (From [Ar])



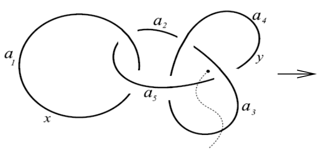
$$\to \begin{array}{c|ccc} & a & b & c \\ \hline c & -1 & 1-x & y \end{array}$$



$$\to \begin{array}{c|ccc} & a & b & c \\ \hline c & -y & x-1 & 1 \end{array}$$

Joint with
Jana Archibald

$$A = \frac{(-)^{i+j}\det(M_i^j)}{w_i(t_i-1)}\prod_k t_k^{\frac{\mathrm{rot}(k)-\mu(k)}{2}}$$

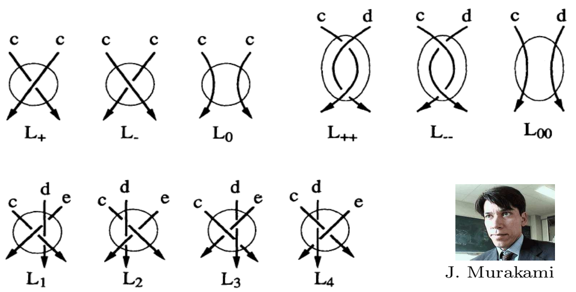


$\to$

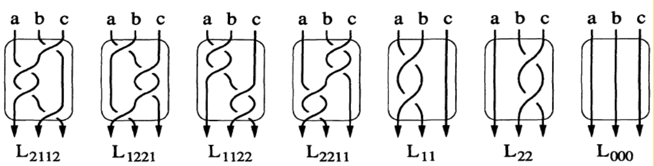
	a_1	a_2	a_3	a_4	a_5
a_1	$y-1$	0	0	0	$1-x$
a_2	$1-y$	x	0	0	-1
a_3	$-y$	0	1	$y-1$	0
a_4	0	0	$-y$	1	$y-1$
a_5	0	0	$y-1$	$-y$	1

$\to \quad xy(1-y+y^2)$

Relations by J. Murakami (From [MJ])



J. Murakami



The Naik−Stanford Double Delta Relation (From [NS])

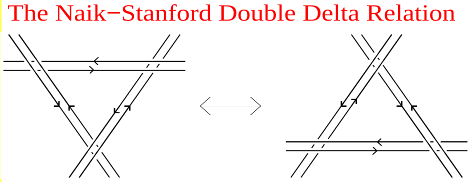


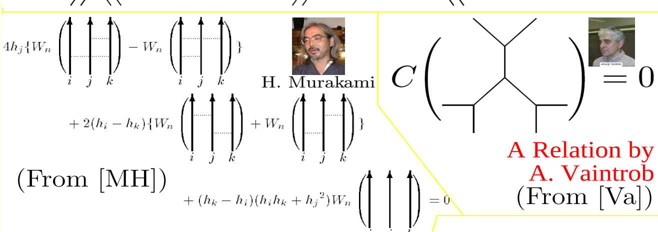
image
not
found

S. Naik T. Stanford

H. Murakami

(From [MH])

A Relation by H. Murakami



A Relation by
A. Vaintrob
(From [Va])

There's Lots More!

"God created the knots,
all else in topology
is the work of mortals"
Leopold Kronecker (paraphrased)

Visit!

Edit!

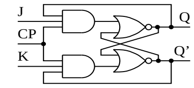
http://katlas.org

This handout and further links are at
http://www.math.toronto.edu/~drorbn/Talks/Sandbjerg−0810/

Our Goal. Prove all these relations uniformly, at maximal confidence and minimal brain utilization.
$\Rightarrow$ We need an "Alexander Invariant" for arbitrary tangles, easy to define and compute and well-behaved under tangle compositions; better, "virtual tangles".

Circuit Algebras

* Have "circuits" with "ends",
* Can be wired arbitrarily.
* May have "relations" − de−Morgan, etc.



A J−K Flip Flop

Example $\mathcal{VT} = CA\langle\times,\times\rangle/R23 = PA\langle\times,\times,\times\rangle/R23, VR123, MR3$

Reminders from linear algebra. If X is a (finite) set,

$$\Lambda^k(X) := \langle k\text{-tuples in } X, \text{ modulo anti-symmetry}\rangle$$

$$\Lambda^{\mathrm{top}}(X) := \langle |X|\text{-tuples in } X, \text{ modulo anti-symmetry}\rangle$$

$$\Lambda^{1/2}(X) := \langle (|X|/2)\text{-tuples in } X, \text{ modulo anti-symmetry}\rangle.$$

If $Y \subset X^m$, the "interior multiplication" $i_Y : \Lambda^k(X) \to \Lambda^{k-m}(X)$ is anti-symmetric in Y.

Definition. An "Alexander half density with input strands X^{in} and output strands X^{out}" is an element of

$$\mathrm{AHD}(X^{\mathrm{in}}, X^{\mathrm{out}}) := \Lambda^{\mathrm{top}}(X^{\mathrm{out}}) \otimes \Lambda^{1/2}(X^{\mathrm{in}} \cup X^{\mathrm{out}}).$$

Often we extend the coefficients to some polynomial ring without warning.

Definition. If $\alpha_i \otimes p_i \in \mathrm{AHD}(X_i^{\mathrm{in}}, X_i^{\mathrm{out}}$ (for $i = 1, 2$), and $G = (X_1^{\mathrm{in}} \cup X_2^{\mathrm{in}}) \cap (X_1^{\mathrm{out}} \cup X_2^{\mathrm{out}})$ is the set of "gluable legs", the "gluing" in $\mathrm{AHD}(X_1^{\mathrm{in}} \cup X_2^{\mathrm{in}} - G, X_1^{\mathrm{out}} \cup X_2^{\mathrm{out}} - G)$ is

$$i_G(\alpha_1 \wedge \alpha_2) \otimes i_G(p_1 \wedge p_2).$$

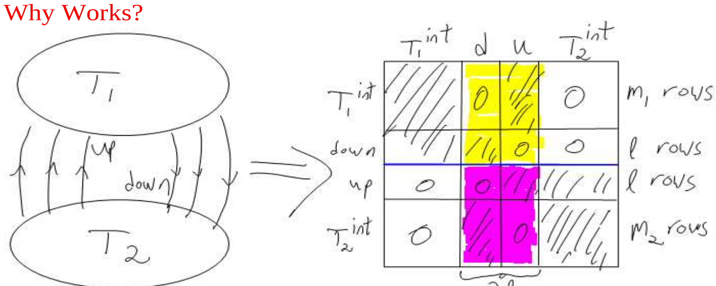
Claim. This makes AHD a circuit algebra.

Definition. The "Penultimate Alexander Invariant" is defined using

$$pA : \overset{k\ \ j}{\underset{l\ \ i}{\times}} \mapsto (j \wedge k) \otimes \begin{pmatrix} l \wedge i + (t_i - 1) l \wedge j - t_l l \wedge k \\ +i \wedge j + t_l j \wedge k \end{pmatrix}$$

$$pA : \overset{l\ \ k}{\underset{i\ \ j}{\times}} \mapsto (k \wedge l) \otimes \begin{pmatrix} t_j i \wedge j - t_j i \wedge l + j \wedge k \\ +(t_i - 1) j \wedge l + k \wedge l \end{pmatrix}$$

Why Works?



Every "rook arrangement" in the above picture must have exactly l rooks in the yellow zone and l rooks in the purple zone. So for T_1 we only care about the minors in which exactly l of the $2l$ middle columns are dropped, and the rest is signs...

Weaknesses. Exponential, no understanding of cablings, no obvious "meaning". The ultimate Alexander invariant should address all that...

Challenge. Can you categorify this?

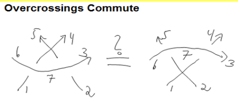
More at http://www.math.toronto.edu/~drorbn/Talks/Sandbjerg-0810/

We Mean Business

```
1          (* WP: Wedge Product *)
2  WSort[expr_] := Expand[expr /. w_W :> Signature[w]*Sort[w]];
3  WP[0, _] = WP[_, 0] = 0;
4  WP[a_, b_] := WSort[Distribute[a ** b] /.
5      (c1_. * w1_W) ** (c2_. * w2_W) :> c1 c2 Join[w1, w2]];
6
7          (* IM: Interior Multiplication *)
8  IM[{}, expr_] := expr;
9  IM[i_, w_W] := If[FreeQ[w, i], 0,
10     -(-1)^Position[w, i][[1,1]]*DeleteCases[w, i] ];
11 IM[{is___, i_}, w_W] := IM[{is}, IM[i, w]];
12 IM[is_List, expr_] := expr /. w_W :> IM[is, w]
13
14         (* pA on Crossings *)
15 pA[Xp[i_,j_,k_,l_]] := AHD[(t[i]==t[k])(t[j]==t[l]), {i,l}, W[j,k],
16     W[l,i] + (t[i]-1)W[l,j] - t[l]W[l,k] + W[i,j] + t[l]W[j,k] ];
17 pA[Xm[i_,j_,k_,l_]] := AHD[(t[i]==t[k])(t[j]==t[l]), {i,j}, W[k,l],
18     t[j]W[i,j] - t[j]W[i,l] + W[j,k] + (t[i]-1)W[j,l] + W[k,l] ]
19
20         (* Variable Equivalences *)
21 ReductionRules[Times[]] = {};
22 ReductionRules[Equal[a_, b__]] := (# -> a)& /@ {b};
23 ReductionRules[eqs_Times] := Join @@ (ReductionRules /@ List@@eqs)
24
25         (* AHD: Alexander Half Densities *)
26 AHD[eqs_, is_, -os_, p_] := AHD[eqs, is, os, Expand[-p]];
27 AHD /: Reduce[AHD[eqs_, is_, os_, p_]] :=
28   AHD[eqs, Sort[is], WSort[os], WSort[p /. ReductionRules[eqs]]];
29 AHD /: AHD[eqs1_,is1_,os1_,p1_] AHD[eqs2_,is2_,os2_,p2_] := Module[
30   {glued = Intersection[Union[is1, is2], List@@Union[os1, os2]]},
31   Reduce[AHD[
32     eqs1*eqs2 //. eq1_Equal*eq2_Equal /;
33       Intersection[List@@eq1, List@@eq2] =!= {} :> Union[eq1, eq2],
34     Complement[Union[is1, is2], glued],
35     IM[glued, WP[os1, os2]],
36     IM[glued, WP[p1, p2]]
37 ]] ]
38
39         (* pA on Circuit Diagrams *)
40 pA[cd_CircuitDiagram, eqs___] := pA[cd, {}, AHD[Times[eqs], {}, W[], W[]]];
41 pA[cd_CircuitDiagram, done_, ahd_AHD] := Module[
42   {pos = First[Ordering[Length[Complement[List @@ #, done]] & /@ cd]]},
43   pA[Delete[cd, pos], Union[done, List @@ cd[[pos]]], ahd*pA[cd[[pos]]]]
44 ];
45 pA[CircuitDiagram[], _, ahd_AHD] := ahd
```

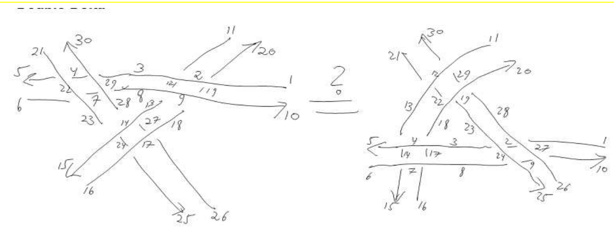
Comments online **2.** `W[i1,i2,...]` represents $i_1 \wedge i_2 \wedge \ldots$. To sort it we `Sort` its arguments and multiply by the `Signature` of the permutation used. **3.** The wedge product of 0 with anything is 0. **4-5.** The wedge product of two things involves applying the `Distribute`eive law, `Join`ing all pairs of `W`'s, and `WSort`ing the result. **8.** Inner multiplying by an empty list of indices does nothing. **9-10.** Inner multiplying a single index yields 0 if that index is not pressent, otherwise it's a sign and the index is deleted. **11-12.** Aftwrwards it's simple recursion. **15-18.** For the crossings `Xp` and `Xm` it is straightforward to determine the incoming strands, the outgoing ones, and the variable equivalences. The associated half-densities are just as in the formulas. **21-23.** The technicalities of imposing variable equivalences are annoying. **26.** That's all we need from the definition of a tensor product. **27-28.** Straightforward simplifications. **29.** The (circuit algebra) product of two Alexander Half Densities: **30.** The `glued` strands are the intersection of the ins and the outs. **32-33.** Merging the variable equivalences is tricky but natural. **34-35.** Removing the `glued` strands from the ins and outs. **36 The Key Point.** The wedge product of the half-densities, inner with the `glued` strands. **40-45.** A quick implementation of a "thin scanning" algorithm for multiple products. The key line is **42**, where we select the next crossing we multiply in to be the crossing with the fewest "loose strands".

▪ Overcrossings Commute



Hence "w-knots"

```
In[15]:= Equal[
           pA[CircuitDiagram[Xp[1, 7, 4, 6], Xp[2, 3, 5, 7]]],
           pA[CircuitDiagram[Xp[2, 7, 5, 6], Xp[1, 3, 4, 7]]] ]
Out[15]= True
```



```
In[10]:= Timing[res4 = pA[#, (t[1] == t[6]) (t[11] == t[16]) (t[21] == t[26])] & /@ {
            CircuitDiagram[
             Xm[19, 1, 20, 2], Xp[11, 3, 12, 2], Xp[3, 30, 4, 29], Xm[4, 21, 5, 22],
             Xp[6, 23, 7, 22], Xm[7, 28, 8, 29], Xm[12, 8, 13, 9], Xp[18, 10, 19, 9],
             Xm[27, 13, 28, 14], Xp[23, 15, 24, 14], Xm[24, 16, 25, 17], Xp[26, 18, 27, 17]
            ],
            CircuitDiagram[
             Xp[1, 28, 2, 27], Xm[2, 23, 3, 24], Xm[17, 3, 18, 4], Xp[13, 5, 14, 4],
             Xm[14, 6, 15, 7], Xp[16, 8, 17, 7], Xp[8, 25, 9, 24], Xm[9, 26, 10, 27],
             Xm[29, 11, 30, 12], Xp[21, 13, 22, 12], Xm[22, 18, 23, 19], Xp[28, 20, 29, 19]
            ]
           }]

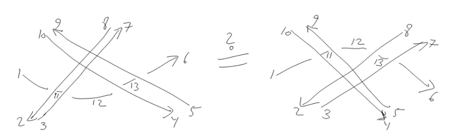
         A very large output was generated. Here is a sample of it:
Out[10]= {9.86, {AHD[(t[1] == t[2] == t[3] == t[4] == t[5] == t[6] == t[7] == t[8] == t[9] == t[10])
              (t[11] == t[12] == t[13] == t[14] == t[15] == t[16] == t[17] == t[18] == t[19] == t[20])
              (t[21] == t[22] == t[23] == t[24] == t[25] == t[26] == t[27] == t[28] == t[29] == t[30]),
            {1, 6, 11, 16, 21, 26}, <<1>>,
            -t[1]^2 t[11]^2 t[21]^2 W[1, 5, 6, 11, 15, 21] + <<2574>>], <<1>>}}
         Show Less | Show More | Show Full Output | Set Size Limit...

In[11]:= Equal @@ (Last /@ res4)
Out[11]= True
```

(The program also prints "False" when appropriate, and computes Alexander polynomials)

More at http://www.math.toronto.edu/~drorbn/Sandbjerg−0810/pA.nb

▪ Commutators Commute



```
In[5]:= Equal[
          pA[CircuitDiagram[Xp[1, 2, 11, 8], Xm[11, 3, 12, 7],
            Xp[12, 4, 13, 10], Xm[13, 5, 6, 9]], t[2] == t[3], t[4] == t[5]],
          pA[CircuitDiagram[Xp[1, 4, 11, 10], Xm[11, 5, 12, 9],
            Xp[12, 2, 13, 8], Xm[13, 3, 6, 7]], t[2] == t[3], t[4] == t[5]] ]
Out[5]= True
```

Question.

Does this specify the Alexander polynomial?

References

[Ar] J. Archibald, *The Weight System of the Multivariable Alexander Polynomial,* arXiv:0710.4885.

[MH] H. Murakami, *A Weight System Derived from the Multivariable Conway Potential Function,* Jour. of the London Math. Soc. **59** (1999) 698–714, arXiv:math/9903108.

[MJ] J. Murakami, *A State Model for the Multi-Variable Alexander Polynoomial,* Pac. Jour. of Math. **157-1** (1993) 109–135.

[NS] S. Naik and T. Stanford, *A Move on Diagrams that Generates S-Equivalence of Knots,* Jour. of Knot Theory and its Ramifications **12-5** (2003) 717–724, arXiv:math/9911005.

[Va] A. Vaintrob, *Melvin-Morton Conjecture and Primitive Feynman Diagrams,* Inter. J. Math. **8** (1997) 537–553, arXiv:q-alg/9605028.

More at http://www.math.toronto.edu/~drorbn/Talks/Sandbjerg-0810/

Dror Bar-Natan: Talks: MSRI-0808:

Projectivization, w-Knots, Kashiwara-Vergne and Alekseev-Torossian

The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

The Projectivization Tentative Speculative Paradigm.

Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

Graded Equations Examples

- $e(x+y)=e(x)e(y)$ in $\mathbb{Q}[[x,y]]$.
- The pentagon and hexagons in $\mathcal{A}(\uparrow_{3,4})$.
- The equations defining a QUEA, the work of Etingof and Kazhdan.
- The Alekseev-Torossian equations in $\mathcal{U}(\mathrm{sder}_n)$ and $\mathcal{U}(\mathrm{tder}_n)$.

sder $\leftrightarrow$ tree-level $\mathcal{A}$
tder $\leftrightarrow$ more

$$F\in\mathcal{U}(\mathrm{tder}_2);\quad F^{-1}e(x+y)F=e(x)e(y)\quad\Longleftrightarrow\quad F\in\mathrm{Sol}_0$$

$$\Phi=\Phi_F:=(F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23}\in\mathcal{U}(\mathrm{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4}=\Phi^{12,3,4}\Phi^{1,2,34}\qquad\text{"the pentagon"}$$

$$t=\frac{1}{2}(y,x)\in\mathrm{sder}_2\text{ satisfies }4T\quad\text{and}\quad r=(y,0)\in\mathrm{tder}_2\text{ satisfies }6T$$

$$R:=e(r)\text{ satisfies Yang-Baxter: }R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$$

$$\text{also }R^{12,3}=R^{13}R^{23}\text{ and }F^{23}R^{1,23}(F^{23})^{-1}=R^{12}R^{13}$$

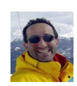
$$\tau(F):=RF^{21}e(-t)\text{ is an involution},\Phi_{\tau(F)}=(\Phi_F^{321})^{-1}$$

$$\mathrm{Sol}_0^\tau:=\{F\colon\tau(F)=F\}\text{ is non-empty; for }F\in\mathrm{Sol}_0^\tau,$$

$$e(t^{13}+t^{23})=\Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

$$\text{and}\quad e(t^{12}+t^{13})=(\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$


Alekseev

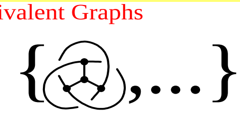

Torossian

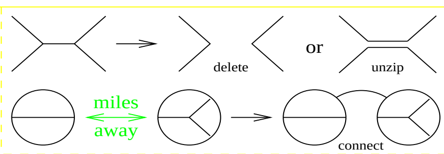
This is just a part of the Alekseev-Torossian work!

So What?

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

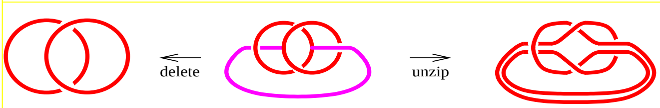
Knotted Trivalent Graphs

$\mathcal{O}(\triangle)=$ 



Theorem. KTG is generated by the unknotted $\triangle$ and the Möbius band, with identifiable relations between them.

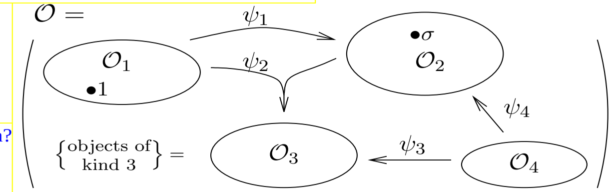
Theorem. $Z(\triangle)$ is equivalent to an associator Φ.



Algebraic Knot Theory

Theorem. $\{\text{ribbon knots}\}\sim\{u\gamma\colon\gamma\in\mathcal{O}(\circ\!\!\circ),\ d\gamma=\bigcirc\!\bigcirc\}$.
Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Defining proj $\mathcal{O}$. The augmentation "ideal":

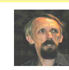
$$I=I_{\mathcal{O}}:=\left\{\begin{matrix}\text{formal differences of ob-}\\\text{jects "of the same kind"}\end{matrix}\right\}.$$

$$\text{Then }I^n:=\left\{\begin{matrix}\text{all outputs of algebraic}\\\text{expressions at least }n\text{ of}\\\text{whose inputs are in }I\end{matrix}\right\},\text{ and}$$

$$\operatorname{proj}\mathcal{O}:=\bigoplus_{n\geq 0}I^n/I^{n+1}\quad\left(\begin{matrix}\text{has same kinds and opera-}\\\text{tions, but different objects}\\\text{and axioms}\end{matrix}\right).$$

Knot Theory Anchors.

- $(\mathcal{O}/I^{n+1})^\star$ is "type n invariants".
- $(I^n/I^{n+1})^\star$ is "weight systems".
- proj $\mathcal{O}$ is $\mathcal{A}$, "chord diagrams".


Vassiliev


Goussarov

Warmup Examples.

- The projectivization of a group is a graded associative algebra.
- A quandle: a set Q with a binary op $\wedge$ s.t.

$$1\wedge x=1,\quad x\wedge 1=x\wedge x=x,\qquad\text{(appetizers)}$$
$$(x\wedge y)\wedge z=(x\wedge z)\wedge(y\wedge z).\qquad\text{(main)}$$

proj Q is a graded Lie algebra: set $\bar v:=(v-1)$ (these generate I!), feed $1+\bar x$, $1+\bar y$, $1+\bar z$ in (main), collect the surviving terms of lowest degree:

$$(\bar x\wedge\bar y)\wedge\bar z=(\bar x\wedge\bar z)\wedge\bar y+\bar x\wedge(\bar y\wedge\bar z).$$

An Expansion is $Z\colon\mathcal{O}\to\operatorname{proj}\mathcal{O}$ s.t. $Z(I^n)\subset(\operatorname{proj}\mathcal{O})_{\geq n}$ and $Z_{I^n/I^{n+1}}=Id_{I^n/I^{n+1}}$ (A "universal finite type invariant"). In practice, it is hard to determine proj $\mathcal{O}$, but easy to guess a surjection $\rho:\mathcal{A}\to\operatorname{proj}\mathcal{O}$. So find $Z'\colon\mathcal{O}\to\mathcal{A}$ with $Z'(I^n)\subset\mathcal{A}_{\geq n}$ and $Z'_{I^n/I^{n+1}}\circ\rho_n=Id_{\mathcal{A}_n}$:

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "Z'"] \arrow[dr, "Z"'] & \mathcal{A} \arrow[d, "\rho"] \arrow[r, leftrightarrow] & \mathcal{A}_n \\
 & \operatorname{proj}\mathcal{O} \arrow[r, leftrightarrow] & (\operatorname{proj}\mathcal{O})_n \arrow[u, "Z'_{I^n/I^{n+1}}"']
\end{tikzcd}$$

Can you make this diagram less confusing?


X-S. Lin

Homomorphic Expansions are expansions that intertwine the algebraic structure on $\mathcal{O}$ and proj $\mathcal{O}$. They provide finite / combinatorial handles on global problems.

The Key Point. If $\mathcal{O}$ is finitely presented, finding a homomorphic expansion is solving finitely many equations with finitely many unknowns, in some graded spaces.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MSRI-0808/

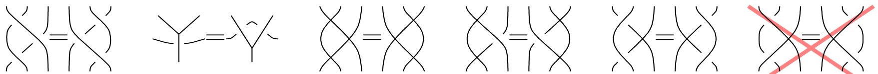
We Mean Business

Trivalent (framed) w-tangles: further operations: delete, unzip.

$wTT = CA$  / R123, R4 (for vertices), F, OC.

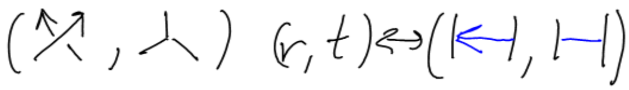
$= PA$  / R1234, F, VR1234, D, OC.

(=tangles in thick surfaces, modulo stabilization)



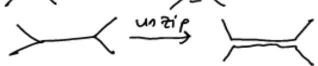
Partial Dictionary.

$(R, F) \longleftrightarrow$ (,) $(r, t) \longleftrightarrow$ (,)



$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \longleftrightarrow$

$FF^! = I \longleftrightarrow$



$F^{-1}e(x+y)F = e(x)e(y)$

$\Updownarrow$

$F^{23}R^{1,23} = R^{12}R^{13}F^{23} \longleftrightarrow$

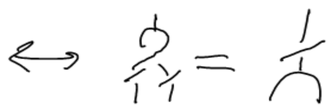
$R^{12,3} = R^{13}R^{23}$

$\Updownarrow$

$F^{12,3}R^{12,3} = R^{13}R^{23}F^{12,3} \longleftrightarrow$

(unforbidding FI makes this automatic)

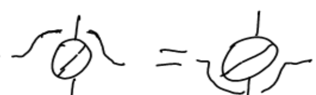
$RF^{21}e(-t) = F \longleftrightarrow$



$\Phi = (F^{12,3})^{-1}(F^{1,2})^{-1}F^{2,3}F^{1,23} \longleftrightarrow$

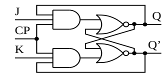
= "The tetrahedron"

$\Phi \in$ sder $\longleftrightarrow$



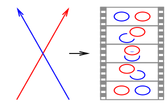
The pentagon and the hexagons follow, with a minor twist, from the fact that we have an unzip-behaved invariant of KTG's.

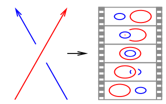
Circuit Algebras

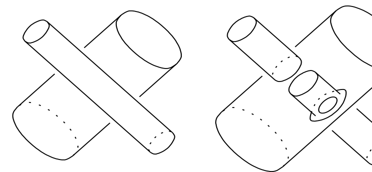


* Have "circuits" with "ends"
* Can be wired arbitrarily.
* May have "relations" - de-Morgan, etc.

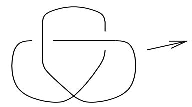
w-braids describe flying rings:

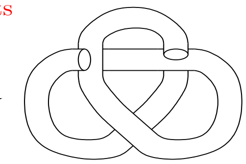






w-knotted objects describe ribbon surfaces in $\mathbb{R}^4$.

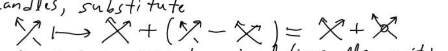


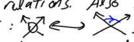


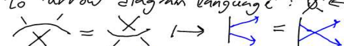
(and π_1 is preserved)

For the Experienced
(and sharp-eyed)

The "Chord Diagrams" — $\mathcal{A}_n^{wt}$. As we did for quandles, substitute

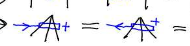


into the various moves, to get relations. Also switch to "arrow diagram language":  Get:

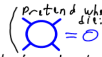


(tails commute)

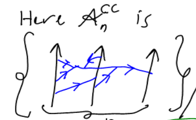
R3 $\longmapsto$  (4T, really 6T)

R4 $\longmapsto$  $= 0$ (vertex invariance)

The "Jacobi Diagrams" — $\mathcal{A}_n^{cc}$.
Theorem. $\mathcal{A}_n^{wt}$ is $\mathcal{A}_n^{cc}$ is $\mathcal{U}(\mathrm{tder}_n)$.

(Pretend wheels die:  $= 0$)

Here $\mathcal{A}_n^{cc}$ is

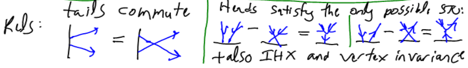


/ rels

trivalent directed trees with only 2-in 1-out vertices

In tensorland, this is "Co-commutative Lie-bialgs"

Rels: tails commute | Heads satisfy the only possible STU:



+ also IHX and vertex invariance

The Map $\alpha\colon \mathcal{A}_n^{tree} \to \mathcal{A}_n^{cc}$:

Theorem. α is an injection on $\mathcal{A}_n^{tree} \cong \mathcal{U}(\mathrm{sder}_n)$. Furthermore, there is a simple characterization of im α, so we can tell "an arrowless element" when we see it.

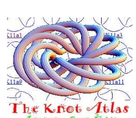
The Main Theorem. (approximate, false as stated) F's in Sol_0^τ are in a bijective correspondance with tree-level associators for ordinary paranthesized tangles (or ordinary knotted trivalent graphs) / with homomorphic expansions for trivalent w-tangles / with solutions of the Kashiwara-Vergne problem.

Extra. Restricted to knots, we get precisely the Alexander polynomial.

Disclaimer. Orientations, rotation numbers, framings, the vertical direction and the cyclic symmetry of the vertex may still make everything uglier. I hope not.

"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)



Visit!

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http://katlas.org

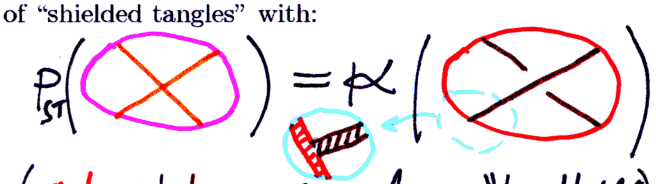
This handout and further links are at http://www.math.toronto.edu/~drorbn/Talks/MSRI-0808/

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MSRI-0808/

A Very Non-Planar Very Planar Algebra

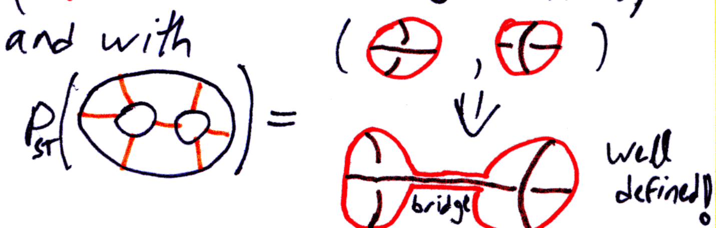
(old work with Dylan Thurston)

Theorem. There exists a skeletal (very) planar algebra of "shielded tangles" with:

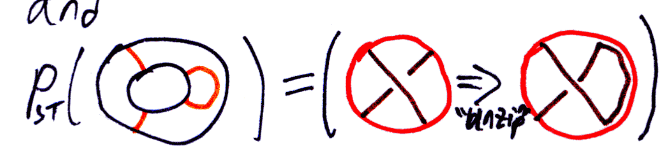


(red and brown are always "knottable")

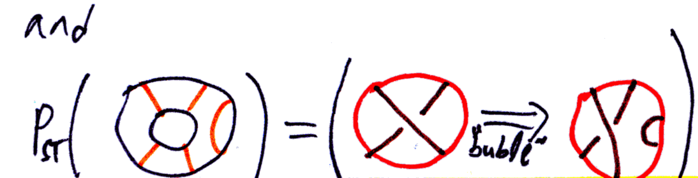
and with



and

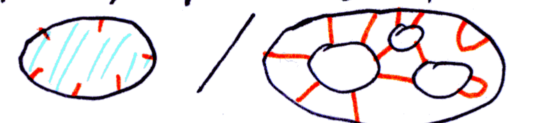


and



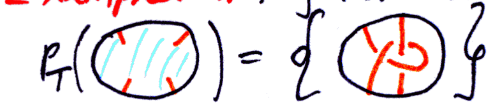
Light entertainment

Definition. A planar algebra has spaces / operations indexed by

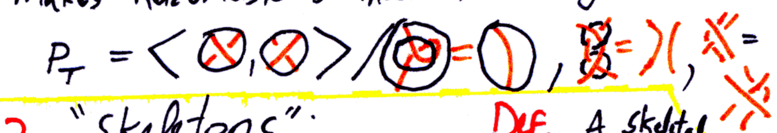


(with obvious compatibilities between ops.)

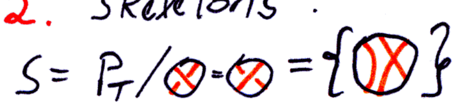
Examples. 1. My favourite — tangles:



makes Reidemeister's Theorem into gens/rels:



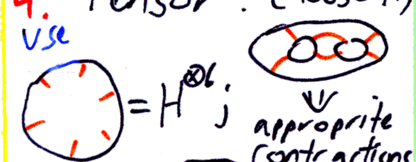
2\. "Skeletons":



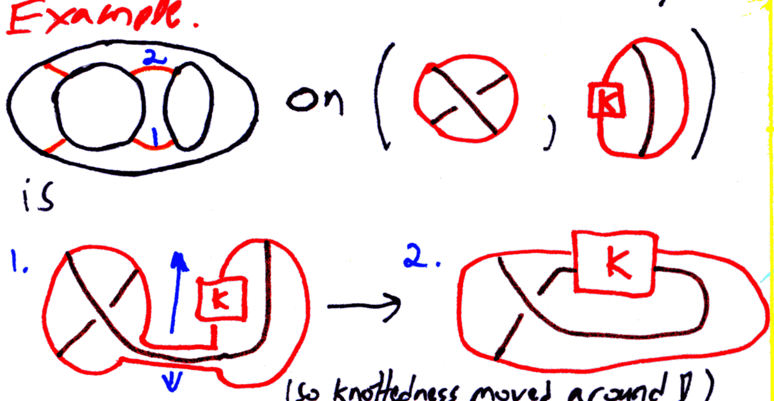
Def. A skeletal planar algebra is "fibred" over S

3\. TL:

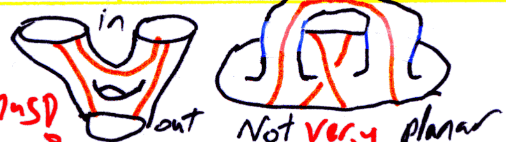
4\. Tensor: choose H, use



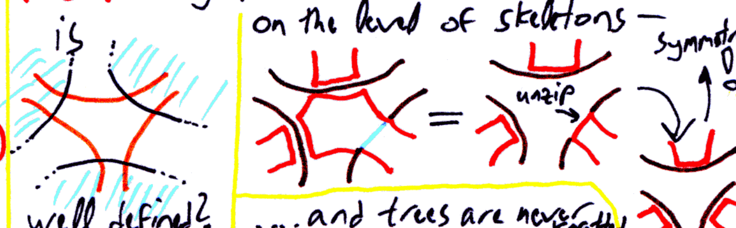
Example.



All make sense in higher genus!



Proof. key point:



The deeper connections

Facts. 1. There is no planar-algebra-structure-respecting universal finite type invariant

2\. But there is one for shielded tangles!

3\. This Z provides a Reidemeister context for the Kontsevich integral!

4\. A cousin of Z is equivalent to the Drinfel'd theory of associators.

Some propaganda

1\. Slides/beamer/powerpoint are evil!

* Can you always sync with the speaker?

* Don't you want to look back at pictures long gone?

2\. Handouts are cool!

Everything's always in front of you, even when you go home.

The Knot Atlas

Visit!

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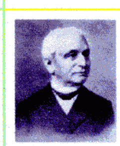
The dreams

Dream — A similar story will be told for "virtual knots", and will provide a topological interpretation of a "universal quantum group". See .../Talks/Hanoi-0708



"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (modified)

http://www.math.toronto.edu/~drorbn/Talks/Fields-0709/

More at http://www.math.toronto.edu/~drorbn/Talks/Fields-0709/

Dror Bar-Natan: Talks: Hanoi-0708: Following Lin:

Expansions for Groups

Riverside, April 2000

Kyoto, September 2001

Vaughan's Hierarchy
(generalized, unauthorized)

- ☺ Computation
- ☺ Formula
- ☺ Proof
- ☹ Theory
- ☹ Dream

See Lin's "Power Series Expansions and Invariants of Links", 1993 Georgia International Topology Conference, AMS/IP Studies in Adv. Math. **2** (1997) 184–202.

The Magnus and Exponential Expansions

$$Z_{1,2}: G_n = \begin{pmatrix}\text{free group}\\ \text{on}\\ X_1,\dots,X_n\end{pmatrix} \to \hat{A}_n = \begin{pmatrix}\text{completed free}\\ \text{associative}\\ \text{algebra on}\\ x_1,\dots,x_n\end{pmatrix}$$

by $X_i \mapsto 1+x_i$ or e^{x_i}

$X_i^{-1} \mapsto 1 - x_i + x_i^2 - \ldots$ or e^{-x_i}.

What's "An Expansion"?

A filtration-preserving *isomorphism* $Z: C(G) \to \mathcal{A}(G)$ where

$$I := \{\textstyle\sum a_i g_i : \sum a_i = 0\} \subset \mathbb{C}G$$

$$\mathbb{C}G = I^0 \supset I^1 \supset I^2 \supset I^3 \supset \cdots$$

$$C(G) := \varprojlim_k \mathbb{C}G/I^k \to \cdots \to \mathbb{C}G/I^2 \to \mathbb{C}G/I \to 0$$

is filtered by $F_m C(G) := \varprojlim_{k>m} I^m/I^k$ and

$$\mathcal{A}(G) := \operatorname{gr} C(G) = \hat{\oplus} I^m/I^{m+1}.$$

So all expansions are "equivalent"

Think duals! $C(G)^\star$ are "finite type invariants". $\mathcal{A}(G)^\star$ are "weight systems". Z is a "universal finite type invariant".

$Z_{1,2}$ are Expansions.

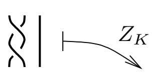
With $Z^0 = Z_1$ or $Z^0 = Z_2$:

$$\begin{tikzcd} G_n \arrow[r, "Z^0"] \arrow[d, "\iota"'] & \hat{A}_n \arrow[d, "\rho"', "x_i \mapsto X_i - 1"] \\ C(G_n) \arrow[ur, "Z^1"] \arrow[r, "Z^2"] & \mathcal{A}(G_n) \end{tikzcd}$$

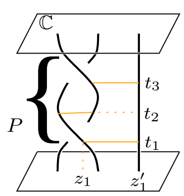
1. ι is automatic.
2. ρ is well-defined.
3. $Z^0|_{I^m} \subset F_m A_n$.
4. Z^0 descends to Z^1.
5. Define Z^2.
6. ρ is surjective.
7. $\operatorname{gr} Z^2$ is the identity.
8. Z^2 is an isomorphism.
9. ρ is an isomorphism.

Everything generalizes, step 2 sometimes becomes tricky.

The Kontsevich Integral for Braids

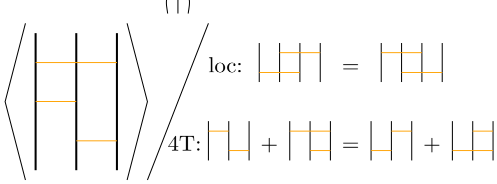


$$\sum_{\substack{m,\ t_1<\ldots<t_m\\ P=\{(z_i,z_i')\}}} \frac{D_P}{(2\pi i)^m} \bigwedge_{i=1}^m \frac{dz_i - dz_i'}{z_i - z_i'}$$



$\Cap$

$\mathcal{A} :=$



M. Kontsevich

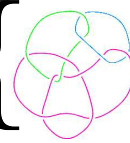
Which other groups / groupoids / categories have expansions?

Dror's Dream / Obsession:

The bigger quest: Understand quantum groups (I don't).

"Unify" quantum groups − find one object that contains all.

Example: One invariant to rule them all:



$\{$ knots $\}$ —The Kontsevich Integral (heavy)→ $\left\{\begin{matrix}\text{chord}\\ \text{diagrams}\\ \text{mod}\\ \text{relations}\end{matrix}\right\}$ —weight systems (light)→ $\left\{\begin{matrix}\text{Alexander,}\\ \text{Jones,}\\ \text{HOMFLY,}\\ \text{Kauffman}\\ \text{and All}\end{matrix}\right\}$

Easy! Universal! A Morphism! Unique! An Isomorphism!

What is a "Quantum Group"?

For now, a "deformation of the trivial" solution in $\mathcal{U}(\mathfrak{g}))^{\otimes *}[[\hbar]]$ of the major equations:

$$(\Delta\otimes 1)\Delta = (1\otimes\Delta)\Delta \qquad R^{-1}\Delta R = \Delta^{op}$$

$$(\Delta\otimes 1)R = R^{23}R^{13} \qquad (1\otimes\Delta)R = R^{12}R^{13}$$

(as well as a few minor equations).

Why care?

Quantum groups computable invariants make!

Visit! katlas.org **Edit!**

Dror's Guess: A unified object exists; we'll need:

1. **Expansions as in Lin / universal finite type invariants.**
2. **Naturality / functoriality.**
3. Knotted graphs, especially trivalent.
4. Associators following Drinfel'd.
5. The work of Etingof and Kazhdan on bialgebras.
6. **Virtual braids / knots / knotted graphs.**
7. Polyak (LMP 54) & Haviv (arXiv:math/0211031) on arrow diagrams.

(and when construction ends, we'll dump the scaffolding)

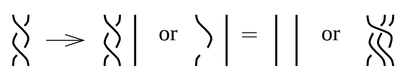
(Quasi?) Natural Expansions

$G \mapsto C(G)$ and $G \mapsto \mathcal{A}(G)$ are functors. Can you choose a ((quasi?) natural) Z satisfying

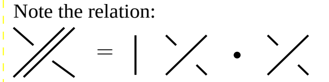
$$\begin{tikzcd} C(G_1) \arrow[r, "C(\Delta)"] \arrow[d, "Z(G_1)"'] & C(G_2) \arrow[d, "Z(G_2)"] \\ \mathcal{A}(G_1) \arrow[r, "\mathcal{A}(\Delta)"] & \mathcal{A}(G_2) \end{tikzcd} \qquad G_1 \xrightarrow{\Delta} G_2$$

$$\begin{pmatrix}\text{quasi: } \mathcal{A}'(\Delta) = \\ J_\Delta^{-1}\mathcal{A}(\Delta)J_\Delta\ ? \\ D_\Delta\mathcal{A}(\Delta)\ ?\end{pmatrix}$$

Perhaps just on a subcategory of **Groups**? Perhaps **Braids** with strands addition, deletion and doubling:



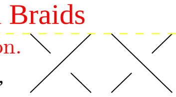
Note the relation:



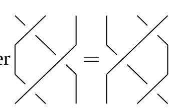
Virtual Braids

crossings are real, strands go virtual

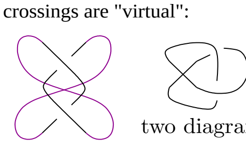
Definition. Crossings,



modulo Reidemeister moves,



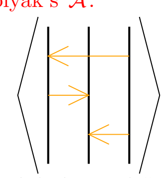
but the linkages between crossings are "virtual":



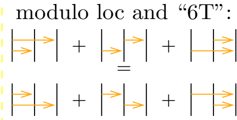
two diagrams of $v2_1$

L. Kauffman

Polyak's $\vec{\mathcal{A}}$.



modulo loc and "6T":



M. Polyak

T. Ohtsuki

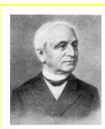
Lie bialgebras. The $\mathfrak{g}$ in a sum $\mathfrak{g}\oplus\mathfrak{g}^\star$ which in itself is a Lie algebra with subalgebras $\mathfrak{g}$ and $\mathfrak{g}^\star$, and in which the tautological metric is invariant.

Why bother? Their deformations are quantum groups, and their diagrammatic universalization is $\vec{\mathcal{A}}$.

Drinfel'd Etingof Kazhdan Haviv

Question Can you interpret quantum groups as (quasi?)-natural expansions on virtual braids?

Dror's Guess: No, but the effort will be worthwhile.



"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (modified)

http://www.math.toronto.edu/~drorbn/Talks/Hanoi-0708/; thanks to Jana Comstock, Peter Lee, Scott Morrison, Dylan Thurston.

More at http://www.math.toronto.edu/~drorbn/Talks/Hanoi-0708/

Dror's Dream Map of Quantum Groups $\subset$ Knot Theory

Drinfel'd, Etingof–Kazhdan Prehistoric

Hallucinations

about Khovanov

C A T E G O R I F I C A T I O N

Some harsh reality.

EK mix tangles and braids and algebras and Verma modules

Night Time Dreams

about Etingof – Kazhdan

$\mathcal{K}/$ translucent vertices $\to$ $\hat{\mathcal{K}}/tv$ $\xrightarrow{Z/tv}$ $\mathrm{gr}\,\hat{\mathcal{K}}/tv \;||\; \mathcal{A}/tv$

Day Time Dreams

about Reshetikhin – Turaev & quasi triangular quasi Hopf algebras

Gens and rels? $\to$ $\mathcal{K}_V$ = K(T)VG Knotted (trivalent?) virtual knots $\to$ $\hat{\mathcal{K}}_V$ $\to$ $\mathrm{gr}\,\hat{\mathcal{K}}_V \;||\; \vec{\mathcal{A}}$ $\xrightarrow{W}$ specific quantum groups

Reality

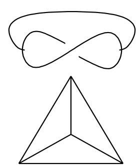
of associators, FT invariants and co–commutative quasi–Hopf algebras

Generators $\to$ $\mathcal{K}$ = KTG Knotted Trivalent Graphs with operations: $\to$ $\hat{\mathcal{K}}$ $\xrightarrow[Z]{\sim}$ $\mathrm{gr}\,\hat{\mathcal{K}} \;//\; \mathcal{A}$ $\xrightarrow{W_{sl(2)}}$ $U(sl(2))$, $\xrightarrow{W_{sl(n)}}$ $U(sl(n))$ etc.

GT, GRT

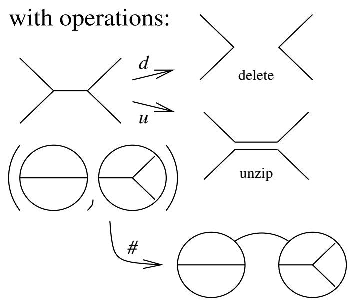
The unipotent completion

This is the Drinfel'd theory of Associators!



Relations

A 5 term relation,
a 6 term relation,
and lesser ones.



Definable subsets:
Genus g knots, ribbon knots, boundary links and more...

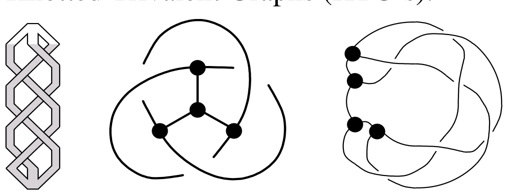
Why should we care?

1. Usefulness!
2. Beauty!
3. Guidance!
4. Confidence!
5. Grothendieck–Teichmuller!

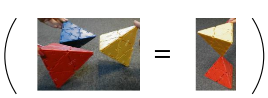
Fushimi–Inari

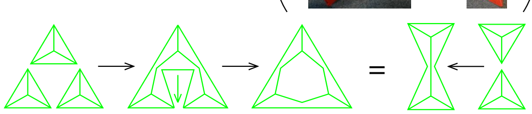
A La Carte Drawings

Knotted Trivalent Graphs (KTG's):



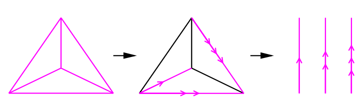
The 5–term relation:



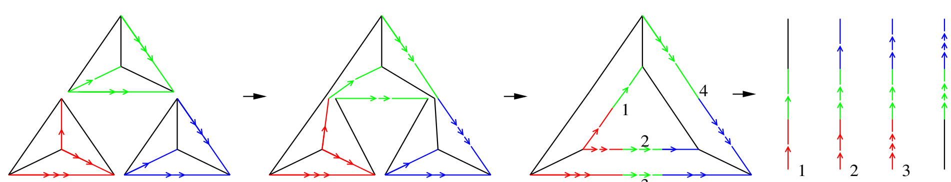


Claim. With $\Phi := Z(\triangle)$, the above relation is equivalent to the Drinfel'd's pentagon equation.

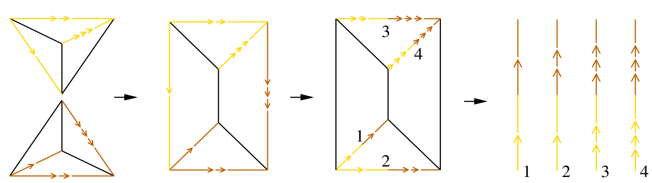
Proof.



$:= \Phi \in \mathcal{A}(\uparrow_3)$



$= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$



$= (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$

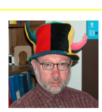

Drinfel'd


Etingof

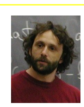

Kazhdan


J. Murakami

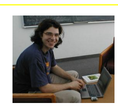

Ohtsuki


Kauffman


Goussarov


Polyak


Viro


D. Thurston


Haviv

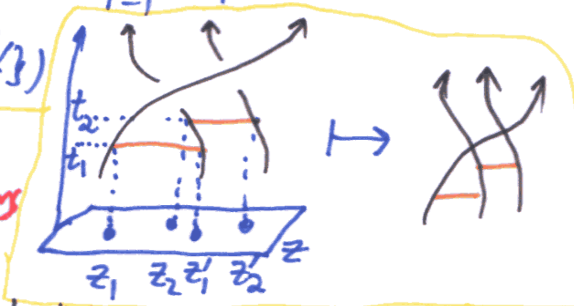
More at http://www.math.toronto.edu/~drorbn/Talks/Kyoto-0705/

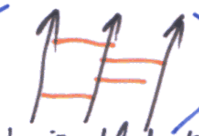
Math 1352 Algebraic Knot Theory – The Knizhnik-Zamolodchikov Connection

Theorem 1. The following is an invariant of braids in $\mathbb{R}_t\times\mathbb{C}_z$ (Fixed endpoints)

$$Z(B)=\oint_{t_1\le\dots\le t_m}\frac{D_P}{(2\pi i)^m}\bigwedge_{i=1}^m\frac{dz_i-dz_i'}{z_i-z_i'}\quad\text{in}\quad \mathcal{A}(\uparrow_n):=\left\langle t^{ij}:1\le i\neq j\le n\right\rangle\Big/\begin{matrix}t^{ij}=t^{ji},\\ [t^{ij},t^{kl}]=0\\ [t^{ij},t^{ik}+t^{jk}]=0\end{matrix}$$

$P=(\{z_i,z_i'\})$



$\mapsto$ $= \left\langle\right.$  $\left.\right\rangle\Big/4T$

horizontal chords.

= "Chord diagrams for braids".

Formal Connections & Curvature.

Let $\Omega\in\Omega^1(M,A)$ (graded unital) with $\deg\Omega=1$.

$\gamma:[0,1]=I\to M$ induces

$\phi:\Delta^m=\{0\le t_1\le\dots\le t_m\le 1\}\to M^m$.

Set $\mathrm{hol}_\gamma(\Omega)=P\exp\int_\gamma\Omega=\oint_{\Delta^m}\phi^*\Omega^m$

where $\Omega^m:=\pi_1^*\Omega\wedge\dots\wedge\pi_m^*\Omega$

Theorem 2. If $F_\Omega:=d\Omega+\Omega\wedge\Omega=0$, then $\mathrm{hol}_\gamma(\Omega)$ is invariant under end-point preserving homotopies of γ.

Proof 2. Let $\Gamma:I_s\times I_t\to M$, $\Phi:I_s\times\Delta^m\to M^m$; By Stokes',

$$\int_{\Delta^m}\Phi_1^*\Omega^m-\int_{\Delta^m}\Phi_0^*\Omega^m=\int_{I\times\Delta^m}d\Phi^*\Omega^m-\int_{I\times\partial\Delta^m}\Phi^*\Omega^m=:A_m-B_m$$

Now

$$A_m=\sum_{k=1}^m(-1)^{k+1}\int_{I\times\Delta^m}\pi_1^*\Omega\wedge\dots\wedge\pi_k^*d\Omega\wedge\dots\wedge\pi_m^*\Omega$$

and

$$B_m=\underbrace{\int_{I\times[t_1=0]}\Phi^*\Omega^m}_{0\ \leftarrow\text{ Fixed endpoints}}\pm\underbrace{\int_{I\times[t_m=1]}\Phi^*\Omega^m}_{0}+\sum_{k=1}^{m-1}(-1)^k\int_{I\times[t_k=t_{k+1}]}\Phi^*\Omega^m$$

$$=\sum_{k=1}^{m-1}(-1)^k\int_{I\times\Delta^{m-1}}\pi_1^*\Omega\wedge\dots\wedge\pi_k^*(\Omega\wedge\Omega)\wedge\dots\wedge\pi_{m-1}^*\Omega$$

and now $\sum A_m=\sum B_m$ by telescopic summation & $F_\Omega=0$.

The KZ connection.

$M=\mathbb{C}^n\setminus\{\text{diagonals}\}$, $A=\mathcal{A}(\uparrow_n)$,

and $\Omega=\sum_{i<j}t^{ij}w_{ij}$ where $w_{ij}:=\dfrac{dz_i-dz_j}{z_i-z_j}\underset{\text{locally}}{=}d\log(z_i-z_j)$

Compute $F_\Omega=d\Omega+\Omega\wedge\Omega$: $dw_{ij}=0$ so $d\Omega=0$.

$$\Omega\wedge\Omega=\sum_{\substack{i<j\\k<l}}t^{ij}t^{kl}w_{ij}\wedge w_{kl}=\underbrace{A}_{|\{ijkl\}|=2}+\underbrace{B}_{=3}+\underbrace{C}_{=4}\quad\text{where}$$

$A=C=0$ as $[t^{ij},t^{kl}]=0$ if $|\{ijkl\}|=2$ or 4 and

$$B=\sum_{\alpha<\beta<\gamma}[t^{\alpha\beta},t^{\beta\gamma}]w_{\alpha\beta}\wedge w_{\beta\gamma}+\text{cyclic perms}$$

Note: by 4T, $[t^{\alpha\beta},t^{\beta\gamma}]=$  $=y^{\alpha\beta\gamma}$

$$=\sum_{\alpha<\beta<\gamma}y^{\alpha\beta\gamma}\underbrace{(w_{\alpha\beta}\wedge w_{\beta\gamma}+\text{cyc perms})}_{=0\text{ by "Arnold's identity"}}=0$$

Proof of 1

Simply take in theorem 2, $\gamma=$ the braid and $\Omega=$ the KZ connection.

Dror Bar-Natan, Feb 13, 2007

More at http://drorbn.net/index.php?title=07-1352/Class_Notes_for_February_13 and at http://www.math.toronto.edu/~drorbn/Talks/Aarhus-1305/ (day 2)

M. Khovanov

Local Differentials and Matrix Factorizations

L. Rozansky

Dror Bar–Natan at UIUC, March 11, 2004, http://www.math.toronto.edu/~drorbn/Talks/UIUC–050311/

Quantum algebra:

Claim. If $ba=qab$ then

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}_q a^k b^{n-k}$$

where

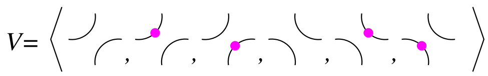
$$(n)_q := 1+q+\ldots+q^{n-1},$$
$$(n)!_q := (1)_q(2)_q\cdots(n)_q,$$
$$\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q(n-k)!_q}.$$

Conjecture: (I. Frenkel, though he may disown this version)

1. Every object in mathematics is the Euler characteristic of a complex.
2. Every operation in mathematics lifts to an operation between complexes.
3. Every identity in mathematics is true up to homotopy at complex–level.

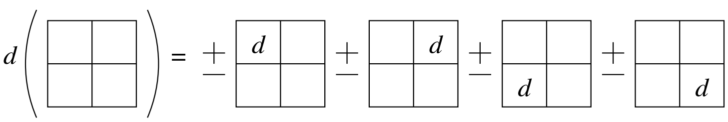
I. Frenkel

Local state spaces:

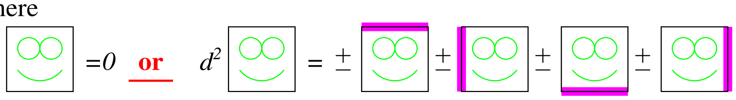


$V=$; $V^{\otimes(4\times5)}=$; ...

Local differentials:

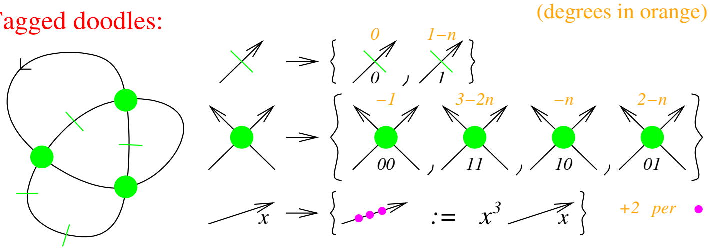


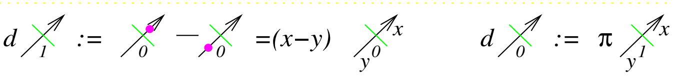
where



d^2 $=0$ or d^2 $=\pm$ $\pm$ $\pm$ $\pm$

Tagged doodles: (degrees in orange)






```
In[1]:= n = 2; π_{i_,j_} := Cancel[(x_i^{n+1} - x_j^{n+1})/(x_i - x_j)]; π_{1,2}
Out[1]= x_1^2 + x_1 x_2 + x_2^2
In[2]:= L = ( 0       x1 - x2
              π_{1,2} 0       );
        Expand[L.L] // MatrixForm
Out[3]//MatrixForm=
   ( x1^3 - x2^3    0
     0              x1^3 - x2^3 )
```

} Set $L=d$ (deg $d = n+1$)

Matrix factorizations:

$$D=\begin{pmatrix}0 & A\\ B & 0\end{pmatrix}$$

$$AB=BA=\omega I$$

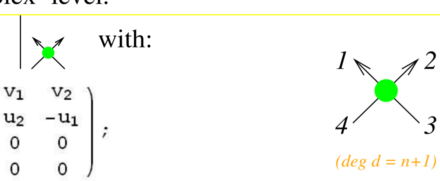
$$\begin{tikzcd}
M^0 \arrow[r, "A"] \arrow[d, shift right, "U^0"'] \arrow[d, shift left, "V^0"] & M^1 \arrow[r, "B"] \arrow[d, shift right, "U^1"'] \arrow[d, shift left, "V^1"] \arrow[dl, "h^1"'] & M^0 \arrow[d, shift right, "U^0"'] \arrow[d, shift left, "V^0"] \arrow[dl, "h^0"'] \\
N^0 \arrow[r, "A'"] & N^1 \arrow[r, "B'"] & N^0
\end{tikzcd}$$

A category, with "complexes", morphisms, homotopies, direct sums and tensor products.

D. Eisenbud

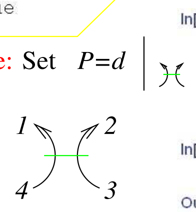
See Khovanov and Rozansky, arXiv:math.QA/0401268

Likewise, set $Q=d$ with:



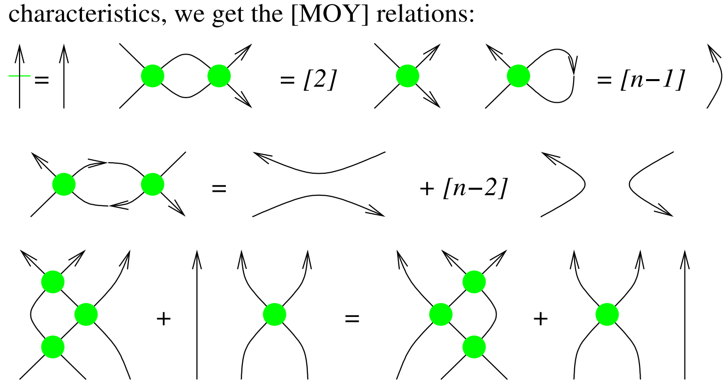

```
In[4]:= Q := ( 0    0    v1   v2
               0    0    u2   -u1
               u1   v2   0    0
               u2   -v1  0    0 );
        {v1, v2} = {x1 + x2 - x3 - x4, x1 x2 - x3 x4};
In[6]:= g[s_, p_] :=
          s^(n+1) + (n + 1) Sum[(-1)^i / i Binomial[n - i, i - 1] s^(n+1-2 i) p^i, {i, 1, (n+1)/2}];
        g[x + y, x y] // Expand
Out[6]= x^3 + y^3
In[7]:= {u1, u2} =
          Cancel[{(g[x1 + x2, x1 x2] - g[x3 + x4, x1 x2]) / v1,
                  (g[x3 + x4, x1 x2] - g[x3 + x4, x3 x4]) / v2}]
Out[7]= {x1^2 - x1 x2 + x2^2 + x1 x3 + x2 x3 + x3^2 + x1 x4 + x2 x4 + 2 x3 x4 + x4^2,
         -3 (x3 + x4)}
In[8]:= ω = u1 v1 + u2 v2 // Expand
Out[8]= x1^3 + x2^3 - x3^3 - x4^3
In[9]:= Simplify[Q.Q == ω IdentityMatrix[4]]
Out[9]= True
```

Example: Set $P=d$




```
In[10]:=
P = ( 0        0        x1 - x4   x2 - x3
      0        0        π_{2,3}   -π_{1,4}
      π_{1,4}  x2 - x3  0         0
      π_{2,3}  x4 - x1  0         0 );
In[11]:=
Simplify[P.P == ω IdentityMatrix[4]]
Out[11]=
True
```

Theorem: (Kh–Ro) Taking homology and then the graded Euler characteristics, we get the [MOY] relations:



[MOY] := Murakami, Ohtsuki, Yamada, Enseignement Math. 44 (1998)

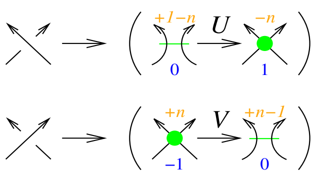
$$[k] := \frac{q^k-q^{-k}}{q-q^{-1}}$$

More at http://www.math.toronto.edu/~drorbn/Talks/UIUC-050311/

The Khovanov–Rozansky Complex

Dror Bar–Natan at UIUC, March 11, 2004, http://www.math.toronto.edu/~drorbn/Talks/UIUC–050311/

Crossings. (height in blue)



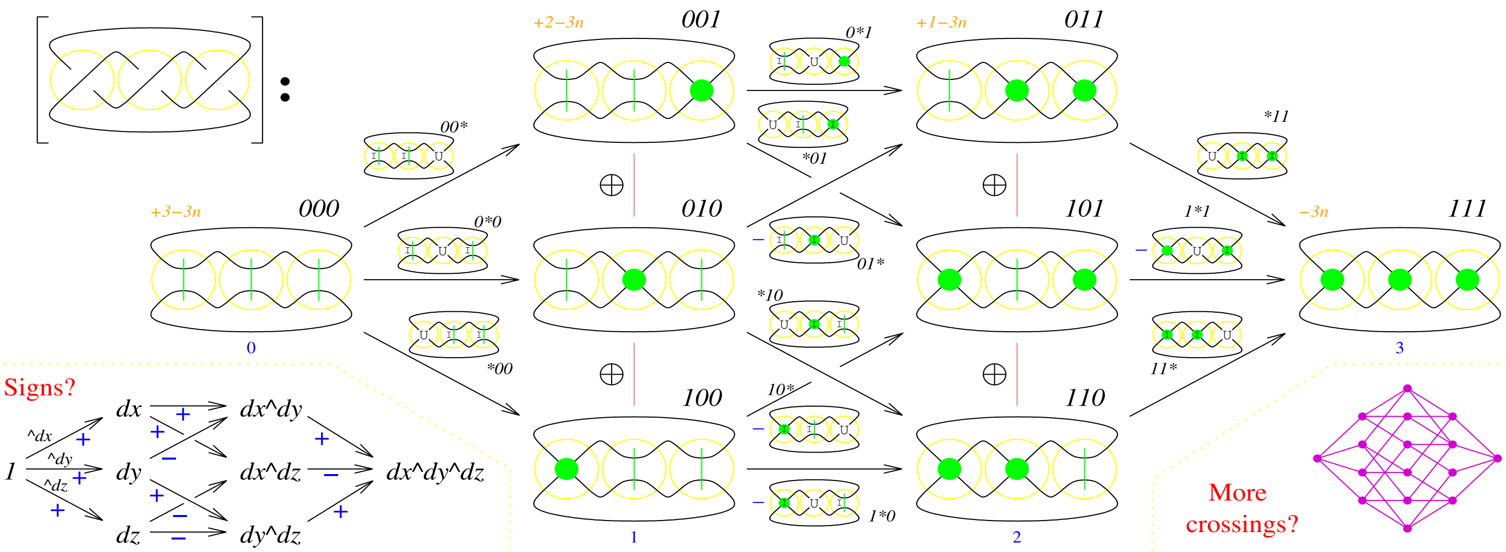
In[12]:=

$$U=\begin{pmatrix} x_4-x_2 & 0 & 0 & 0 \\ \frac{u_1+x_4\,u_2-\pi_{2,3}}{x_1-x_4} & 1 & 0 & 0 \\ 0 & 0 & x_4 & -x_2 \\ 0 & 0 & -1 & 1 \end{pmatrix};\quad V=\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{u_1+x_1\,u_2-\pi_{2,3}}{x_4-x_1} & x_1-x_3 & 0 & 0 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 1 & x_1 \end{pmatrix};$$

```
Simplify[{U.P == Q.U, V.Q == P.V}]
```

Out[12]=

```
{True, True}
```



Signs?

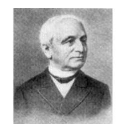
More crossings?

Why am I happy?

1. The ugly formulas for L, Q, U, V; from where they come?
2. Where is the relationship with gl(n), representations and intertwiners?
3. Can you take the Euler characteristic before taking homology?
4. Is this computable?

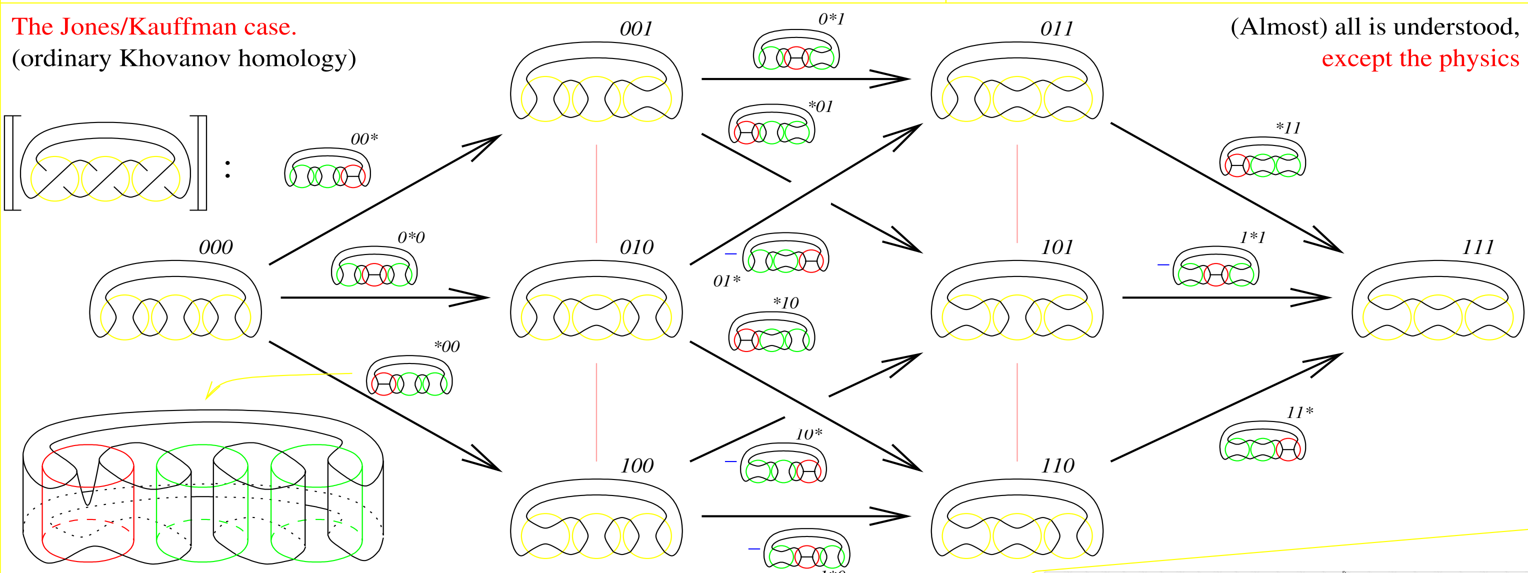
"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)



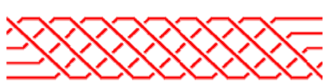
The Jones/Kauffman case.
(ordinary Khovanov homology)

(Almost) all is understood,
except the physics



See my paper "Khovanov homology for tangles and cobordisms", http://www.math.toronto.edu/~drorbn/papers/Cobordism/

A computation example:



t^rq^j	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8	r = 9	r = 10	r = 11	r = 12	r = 13	r = 14	r = 15	r = 16	r = 17	r = 18	r = 19	r = 20	r = 21
j = 63																					1	1
j = 61																			1	1		
j = 59																	1			2	1	
j = 57																	1	3	1			
j = 55															1	3			1			
j = 53													1	2		2	2					
j = 51														3	2							
j = 49												3	2		1							
j = 47										2			2									
j = 45								1		1	2											
j = 43						1		1	2													
j = 41						1	1		1													
j = 39				1	1		1															
j = 37					1																	
j = 35			1																			
j = 33	1																					
j = 31	1																					

More at http://www.math.toronto.edu/~drorbn/Talks/UIUC-050311/

From Stonehenge to Witten Skipping all the Details

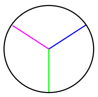
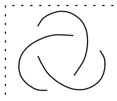
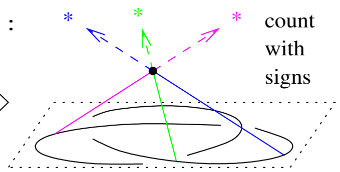
Oporto Meeting on Geometry, Topology and Physics, July 2004

Dror Bar–Natan, University of Toronto

It is well known that when the Sun rises on midsummer's morning over the "Heel Stone" at Stonehenge, its first rays shine right through the open arms of the horseshoe arrangement. Thus astrological lineups, one of the pillars of modern thought, are much older than the famed Gaussian linking number of two knots.

Recall that the latter is itself an astrological construct: one of the standard ways to compute the Gaussian linking number is to place the two knots in space and then count (with signs) the number of shade points cast on one of the knots by the other knot, with the only lighting coming from some fixed distant star.

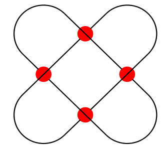
$$\langle D, K\rangle_{\text{π}} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}:$$

$D=$  $K=$  $\Longrightarrow$  count with signs

The Gaussian linking number

$lk($  $) = \frac{1}{2}\sum_{\substack{\text{vertical}\\ \text{chopsticks}}} (\text{signs})$

Carl Friedrich Gauss



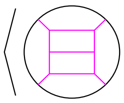
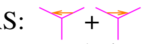
$lk=2$

Thus we consider the generating function of all stellar coincidences:

Dylan Thurston

$$Z(K) := \lim_{N\to\infty} \sum_{\text{3-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\text{π}} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

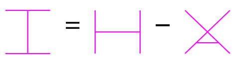
N := # of stars
c := # of chopsticks
e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span $\left\langle\right.$  $\left.\right\rangle$ / oriented vertices, AS:  =0 & more relations

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,



(see below)

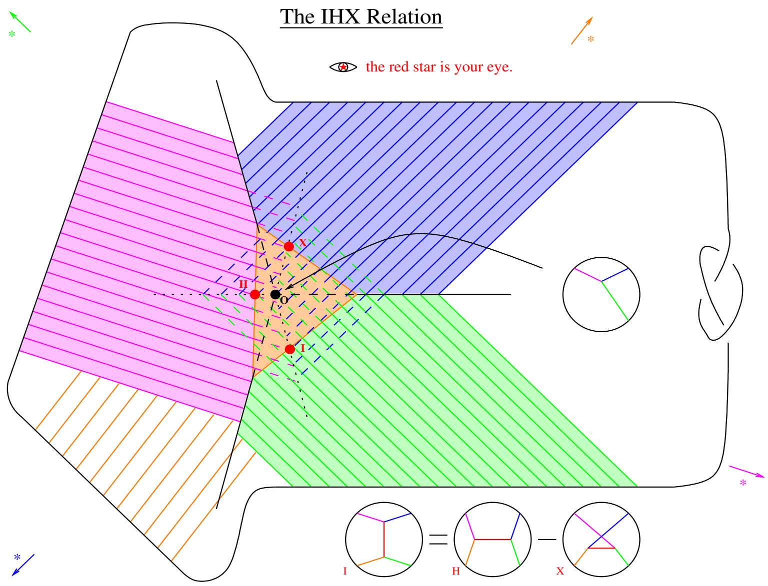
An intersection line cuts through the knot – Solution: Impose STU,

(similar argument)

The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

(not shown here)

The IHX Relation



V: vector space
dV: Lebesgue's measure on V.
Q: A quadratic form on V;
$Q(V)=\langle LV,V\rangle$ where
$L: V\to V^*$ is linear

Compute $I=\int_V dv\, e^{\frac{1}{2}Q+P}$

$$\text{"="} \sum_{m=0}^{\infty}\frac{1}{m!}\int_V dv\, P^m e^{Q/2}$$

$$\sim \sum_{m=0}^{\infty}\frac{1}{m!}P^m(\partial_\psi) e^{-\frac{1}{2}Q^{-1}(\psi)}\Big|_{\psi=0}$$

$\pi=2=i=1$

$$=\sum_{m,n=0}^{\infty}\frac{(-1)^n}{2^n m!\, n!}P^m(\partial)(Q^{-1})^n\Big|_{\psi=0}$$

The Fourier Transform:

$(F: V\to\mathbb{C}) \Rightarrow (\tilde F: V^*\to\mathbb{C})$

via $\tilde F(\psi)=\int_V F(v)e^{-i\langle\psi,v\rangle}dv$.

Simple Facts:

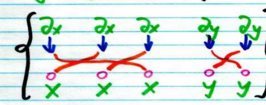
1. $\tilde F(0)=\int_V F(v)dv$.
2. $\frac{\partial}{\partial\psi_i}\tilde F \sim \widetilde{v^i F}$.
3. $\widetilde{(e^{Q/2})} \sim e^{-Q^{-1}/2}$

where $Q^{-1}(\psi):=\langle\psi, L^{-1}\psi\rangle$

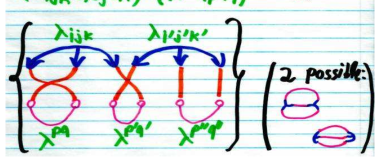
(that's the heart of the Fourier Inversion Formula).

Differentiation and Pairings:

$\partial_x^3\partial_y^2 x^3y^2=3!\cdot 2!$; indeed,

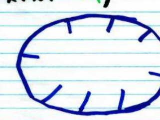


$(\lambda_{ijk}\partial_i\partial_j\partial_k)^2(\lambda^{pq}\psi_p\psi_q)^3$ is



In our case,

* Q is d, so Q^{-1} is an integral operator.
* P is $\frac{2}{3}A\wedge A\wedge A$
* H is the holonomy, itself a sum of integrals along the knot K,

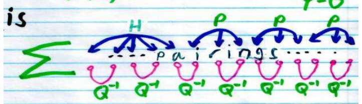


& when the dust settles, we get $Z(K)$!

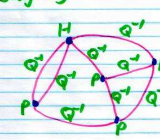
So $\int_V H(v)e^{\frac{1}{2}Q+P}dv$

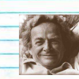
$\sim H(\partial)e^{P(\partial)}e^{-Q^{-1}(\psi)/2}\Big|_{\psi=0}$

is

$\sum$ 

$$=\sum_{\text{Diagrams}} c(D)\begin{pmatrix}\text{Products of}\\ Q^{-1}\text{'s}, P\text{'s}\\ \text{and one } H\end{pmatrix}$$





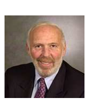
Richard Feynman

It all is perturbative Chern–Simons–Witten theory:

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A)\exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3}\operatorname{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\right]$$

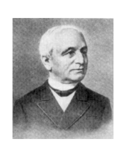
$$\to \sum_{\substack{D:\ \text{Feynman}\\ \text{diagram}}} W_{\mathfrak{g}}(D)\sum\!\!\!\!\!\!/\ \mathcal{E}(D) \to \sum_{\substack{D:\ \text{Feynman}\\ \text{diagram}}} D\sum\!\!\!\!\!\!/\ \mathcal{E}(D)$$

Shiing–shen Chern



James H Simons

"God created the knots, all else in topology is the work of man."



Leopold Kronecker

(modified)

This handout is at http://www.math.toronto.edu/~drorbn/Talks/Oporto–0407

More at `http://www.math.toronto.edu/~drorbn/Talks/Oporto-0407/`

Stonehenge

From Stonehenge to Witten – Some Further Details

Oporto Meeting on Geometry, Topology and Physics, July 2004

Dror Bar–Natan, University of Toronto

Witten

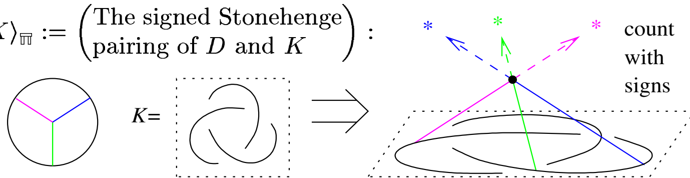
We the generating function of all stellar coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{3\text{-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\Pi} D \cdot \left(\begin{array}{c}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{array}\right) \in \mathcal{A}(\circlearrowleft)$$

$$\langle D, K\rangle_{\Pi} := \left(\begin{array}{l}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{array}\right):$$

count with signs

$D=$ $K=$ $\Longrightarrow$



Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

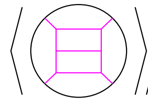
Dylan Thurston

N := # of stars

c := # of chopsticks

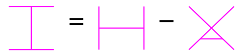
e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span $\left\langle \right\rangle$ / oriented vertices, AS: $+ = 0$, & more relations

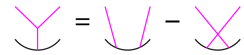


When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,

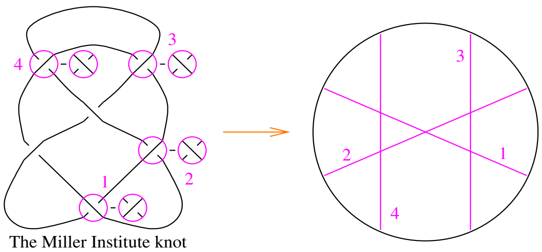


An intersection line cuts through the knot – Solution: Impose STU,



The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right] \longrightarrow \sum_{\substack{D:\text{ Feynman}\\ \text{diagram}}} W_{\mathfrak{g}}(D) \sum\!\!\!\!\!\!/\;\mathcal{E}(D) \longrightarrow \sum_{\substack{D:\text{ Feynman}\\ \text{diagram}}} D \sum\!\!\!\!\!\!/\;\mathcal{E}(D)$$



The Miller Institute knot

Definition. V is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

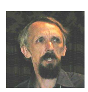
Theorem. All knot polynomials (Conway, Jones, etc.) are of finite type.

Goussarov

Conjecture. (Taylor's theorem) Finite type invariants separate knots.

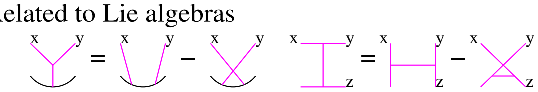
Theorem. $Z(K)$ is a universal finite type invariant!

(sketch: to dance in many parties, you need many feet).



Vassiliev

Related to Lie algebras



[x,y] = xy − yx [[x,y],z]=[x,[y,z]]−[y,[x,z]]

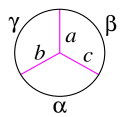
Sophus Lie

More precisely, let $\mathfrak{g} = \langle X_a\rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha\rangle$ be a representation. Set

$$f_{abc} := \langle [a,b], c\rangle \qquad X_a v_\beta = \sum_\beta r^\beta_{a\gamma} v_\gamma$$

and then

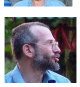
$W_{\mathfrak{g},R}$:



$$\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

$W_{\mathfrak{g},R} \circ Z$ is often interesting:

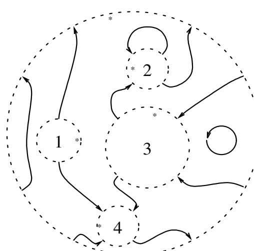
$\mathfrak{g} = sl(2)$ $\longrightarrow$  The Jones polynomial

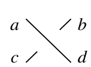
$\mathfrak{g} = sl(N)$ $\longrightarrow$  The HOMFLYPT polynomial

Przytycki

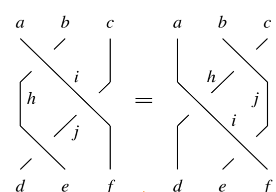
$\mathfrak{g} = so(N)$ $\longrightarrow$  The Kauffman polynomial

Planar algebra and the Yang–Baxter equation



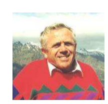


R^{ab}_{cd}

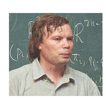


$$R^{ab}_{hi} R^{ic}_{jf} R^{hj}_{de} = R^{ah}_{di} R^{bc}_{hj} R^{ij}_{ef}$$

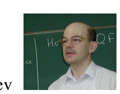
Yang



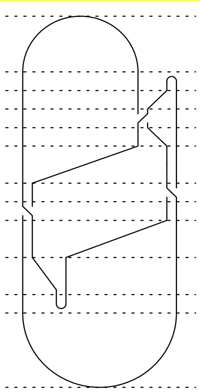
Baxter



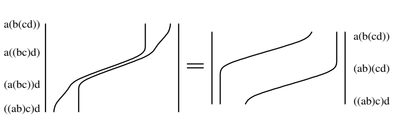
Reshetikhin

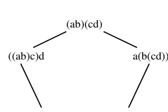


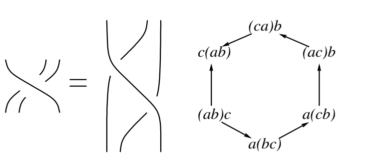
Turaev



Parenthesized tangles, the pentagon and hexagon







Drinfel'd

Kauffman's bracket and the Jones polynomial

$\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$ 0-smoothing 1-smoothing

$\langle \bigcirc^k \rangle = (q+q^{-1})^k$

$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$

(n_+, n_-) count $(\nwarrow\!\!\!\nearrow, \nearrow\!\!\!\nwarrow)$

Claim $\hat{J}(\ ) = \hat{J}(\,)(\,)$

Indeed,

$\langle\ \rangle = \langle\ \rangle - q\langle\ \rangle$

$-q\langle\ \rangle + q^2\langle\ \rangle$

$(q+q^{-1})$

$= -q\langle\,)(\,\rangle$

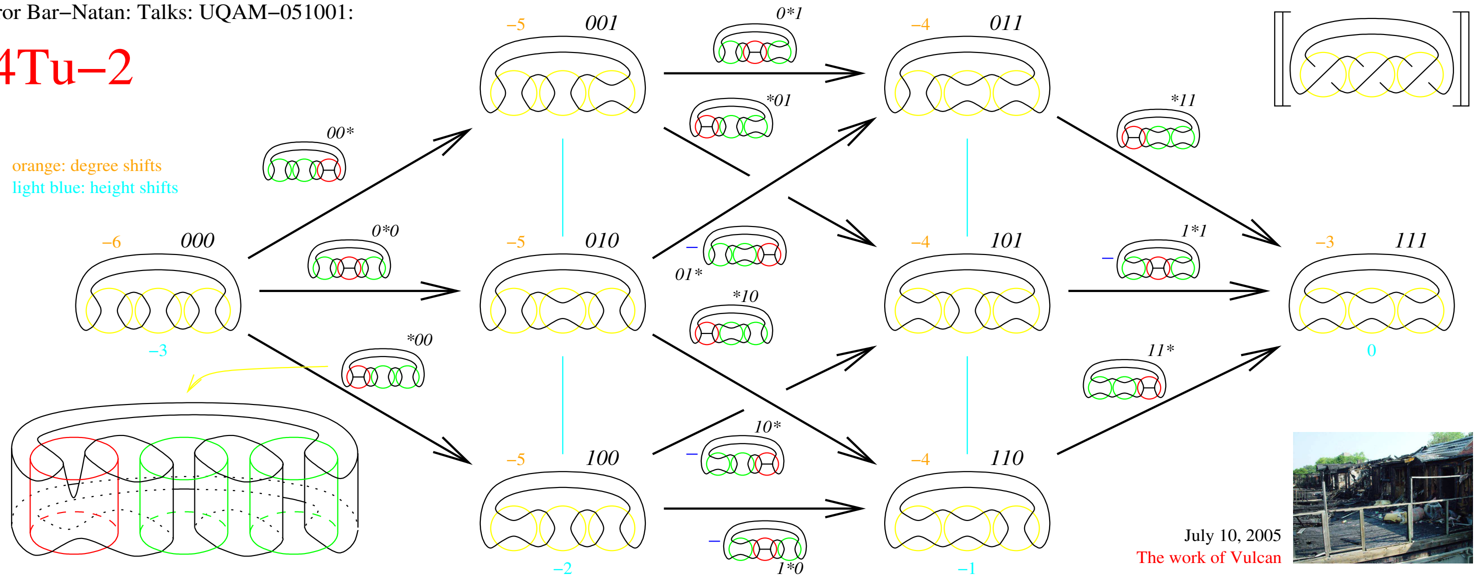
"God created the knots, all else in topology is the work of man." This handout is at http://www.math.toronto.edu/~drorbn/Talks/Oporto–0407

More at http://www.math.toronto.edu/~drorbn/Talks/Oporto-0407/

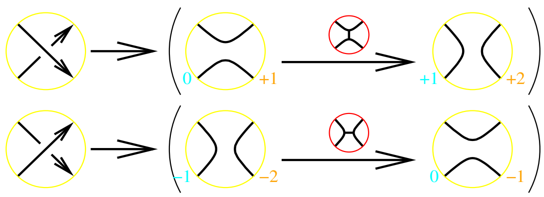
Dror Bar–Natan: Talks: UQAM–051001:

4Tu–2

orange: degree shifts
light blue: height shifts



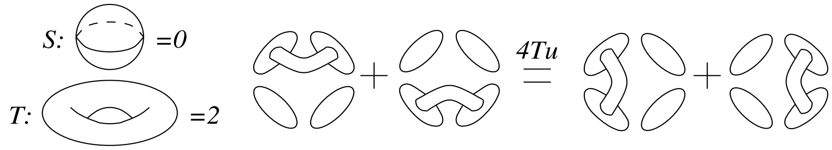
General Crossings



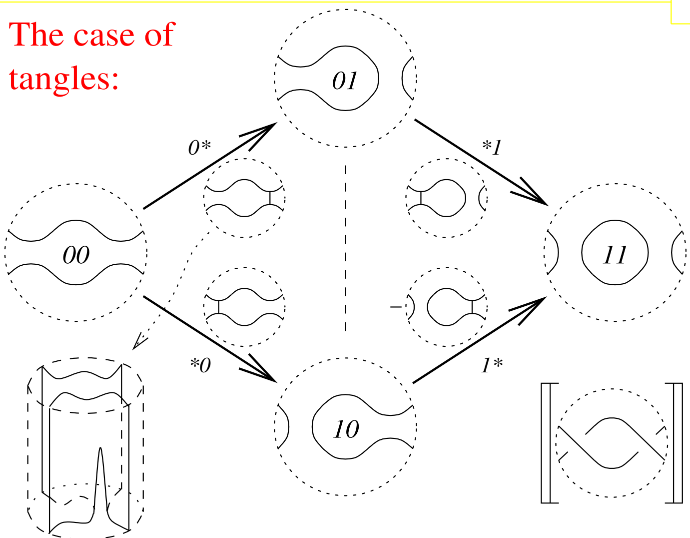
Where does it live?

In *Kom(Mat(<Cob> / {S, T, 4Tu})) / homotopy*

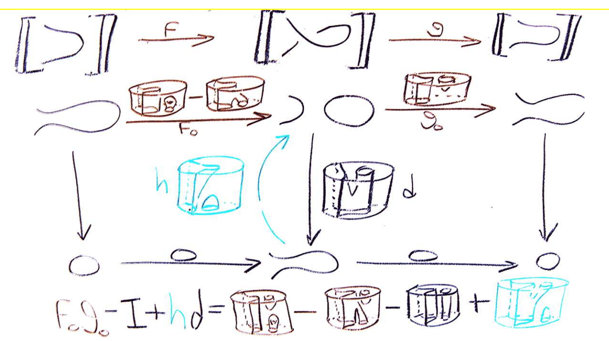
Kom: Complexes Mat: Matrices Cob: Cobordisms <...>: Formal lin. comb.



The case of tangles:



Invariant!



Kurt Reidemeister

The Reduction Lemma. If ϕ is an isomorphism then the complex

$$[C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]$$

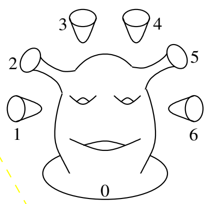
is isomorphic to the (direct sum) complex

$$[C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]$$

The work of Naot.

<surfaces>/4Tu is freely generated by Shrek surfaces

A Shrek surface with 7 boundaries (one distinguished), 3 handles and 2 tubes

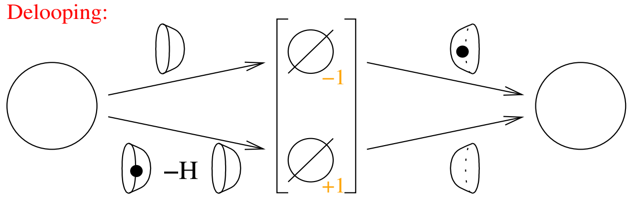


Gad Naot

שרעק

Let ● denote a tube to the distinguished component (the curtain), and let H denote a handle on the curtain. Then

Delooping:

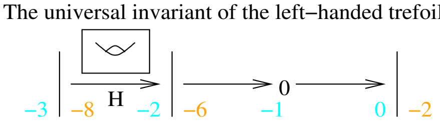


... so the invariant is valued in complexes over a category with just one object and morphisms in Z[H]; all is graded and degH=–2.

The work of Green.

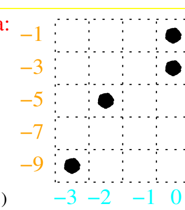
Jeremy Green

The universal invariant of the left–handed trefoil is

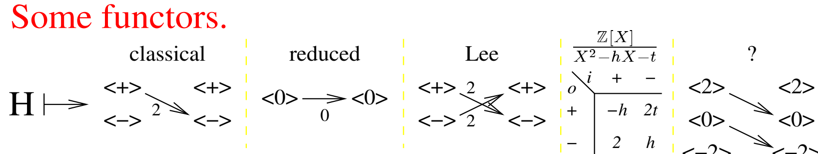


(and the invariant of the 48 crossing T(8,7) is computable in minutes...)

standard data:



Some functors.



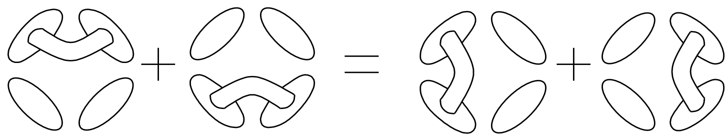
(Lee's spectral sequence and Rasmussen's invariant also recoverable)

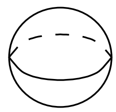
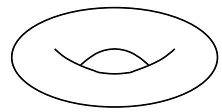
http://www.math.toronto.edu/~drorbn/Talks/UQAM–051001/

More at http://www.math.toronto.edu/~drorbn/Talks/UQAM-051001/

$4Tu$ AND THE "TRUE" SKEIN KHOVANOV HOMOLOGY

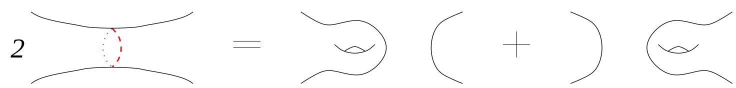
DROR BAR-NATAN

$4Tu$: 

S :  T : 

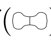
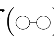
What is it good for?

(1) Cutting necks:



(2) Recovers the good old Khovanov theory,

$\mathcal{F}($$) = \epsilon : \begin{cases} 1 \mapsto v_+ \end{cases}$ $\mathcal{F}($$) = \eta : \begin{cases} v_+ \mapsto 0 \\ v_- \mapsto 1 \end{cases}$

$\mathcal{F}($$) = \Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$ $\mathcal{F}($$) = m : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0. \end{cases}$

(3) Trivially extends to tangles.
(4) Well suited to prove invariance for cobordisms.
(5) Recovers Lee's theory,

$$\Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- + v_+ \otimes v_+ \end{cases} \qquad m : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto v_+. \end{cases}$$

(6) Leads to a new theory (over $\mathbb{Z}/2$ and with $\deg h = -2$),

$$\Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ + hv_+ \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases} \qquad m : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto hv_-. \end{cases}$$

(7) Trivially extends to knots on surfaces.
(8) Non-trivially recovers Khovanov's c,

$$\epsilon : \begin{cases} 1 \mapsto v_+ \end{cases} \qquad \eta : \begin{cases} v_+ \mapsto 0 \\ v_- \mapsto -c \end{cases}$$

$$\Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ + cv_- \otimes v_- \\ v_- \mapsto v_- \otimes v_- \end{cases} \qquad m : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0. \end{cases}$$

(Added June 29, 2004: what appeared to work didn't quite. The recovery of Khovanov's c remains open).

"God created the knots, all else in topology is the work of man."

Leopold Kronecker (modified)

URL: `http://www.math.toronto.edu/~drorbn/papers/Cobordism (and see the ''GWU'' handout)`

Date: May 30, 2004.

More at http://www.math.toronto.edu/~drorbn/Talks/GWU-050213/

A Quick Reference Guide to Khovanov's Categorification of the Jones Polynomial

Dror Bar-Natan, June 12, 2002

The Kauffman Bracket: $\langle\emptyset\rangle = 1$; $\langle \bigcirc L\rangle = (q+q^{-1})\langle L\rangle$; $\langle \times \rangle = \langle \underset{0-\text{smoothing}}{\asymp} \rangle - q\langle \underset{1-\text{smoothing}}{)(} \rangle$.

The Jones Polynomial: $\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-}\langle L\rangle$, where (n_+, n_-) count $(\nwarrow\!\!\!\nearrow, \nearrow\!\!\!\nwarrow)$ crossings.

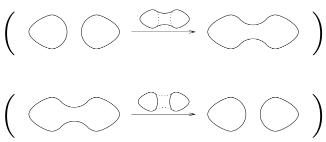
Khovanov's construction: $[\![L]\!]$ — a chain complex of graded $\mathbb{Z}$-modules;

$$[\![\emptyset]\!] = 0 \to \underset{\text{height } 0}{\mathbb{Z}} \to 0; \qquad [\![\bigcirc L]\!] = V \otimes [\![L]\!]; \qquad [\![\times]\!] = \text{Flatten}\left(0 \to \underset{\text{height } 0}{[\![\asymp]\!]} \to \underset{\text{height } 1}{[\![)(]\!]\{1\}} \to 0\right);$$

$$\mathcal{H}(L) = \mathcal{H}\left(\mathcal{C}(L) = [\![L]\!][-n_-]\{n_+ - 2n_-\}\right)$$

$$V = \operatorname{span}\langle v_+, v_-\rangle; \qquad \deg v_\pm = \pm 1; \qquad q\dim V = q + q^{-1} \quad \text{with} \quad q\dim \mathcal{O} := \sum_m q^m \dim \mathcal{O}_m;$$

$$\mathcal{O}\{l\}_m := \mathcal{O}_{m-l} \quad \text{so} \quad q\dim \mathcal{O}\{l\} = q^l\, q\dim \mathcal{O}; \qquad \cdot[s]: \quad \text{height shift by } s;$$

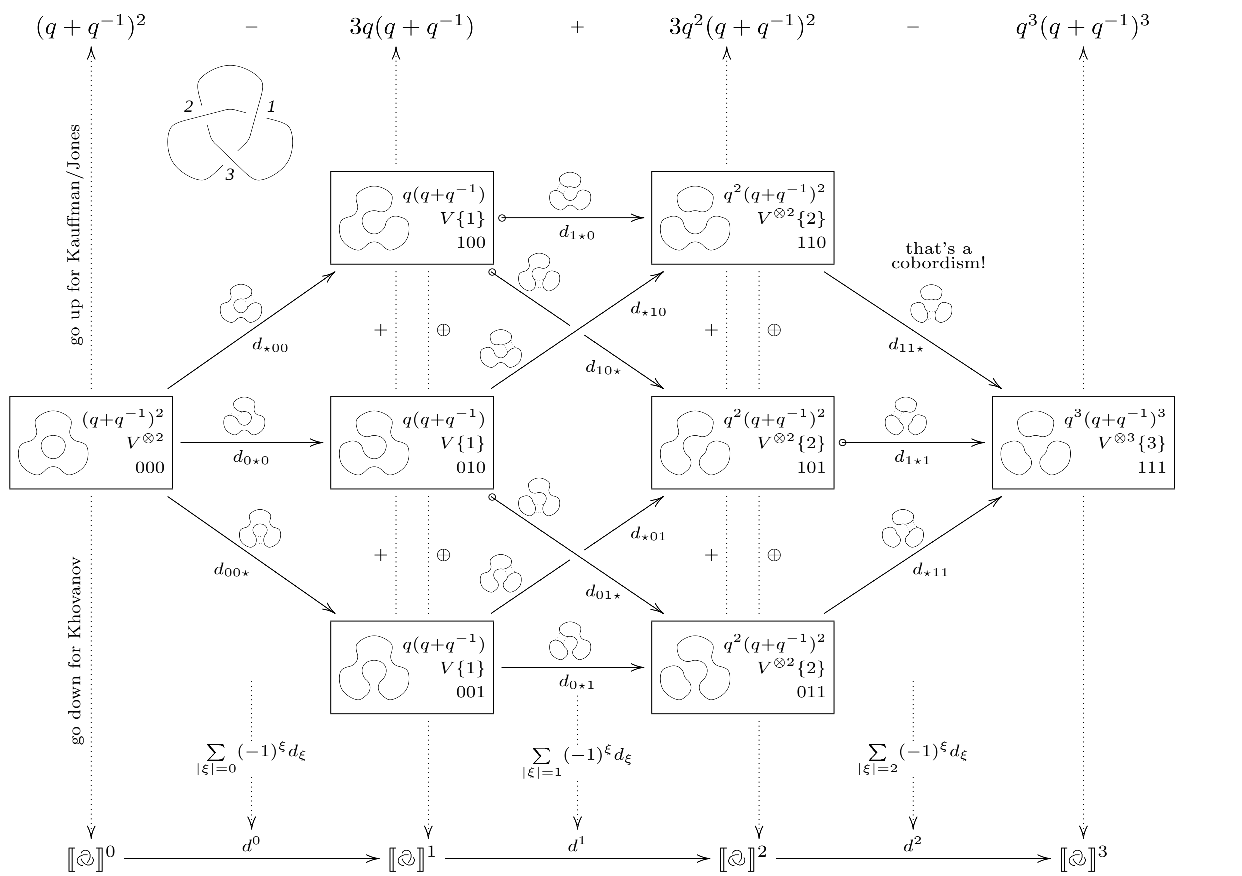


$$\longrightarrow (V \otimes V \xrightarrow{m} V) \qquad m: \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0 \end{cases}$$

$$\longrightarrow (V \xrightarrow{\Delta} V \otimes V) \qquad \Delta: \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

That's a Frobenius Algebra! And a (1 + 1)-dimensional TQFT!

Example: $\quad q^{-2} + 1 + q^2 - q^6 \xrightarrow[\text{(with } (n_+, n_-) = (3,0))]{\cdot(-1)^{n_-} q^{n_+ - 2n_-}} q + q^3 + q^5 - q^9.$



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j = \star$) $\qquad = [\![\text{trefoil}]\!] \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot[-n_-]\{n_+-2n_-\}} \mathcal{C}(\text{trefoil}).$

Theorem 1. The graded Euler characteristic of $\mathcal{C}(L)$ is $\hat{J}(L)$.

Theorem 2. The homology $\mathcal{H}(L)$ is a link invariant and thus so is $Kh_{\mathbb{F}}(L) := \sum_r t^r\, q\dim \mathcal{H}^r_{\mathbb{F}}(\mathcal{C}(L))$ over any field $\mathbb{F}$.

Theorem 3. $\mathcal{H}(\mathcal{C}(L))$ is strictly stronger than $\hat{J}(L)$: $\mathcal{H}(\mathcal{C}(\bar{5}_1)) \neq \mathcal{H}(\mathcal{C}(10_{132}))$ whereas $\hat{J}(\bar{5}_1) = \hat{J}(10_{132})$.

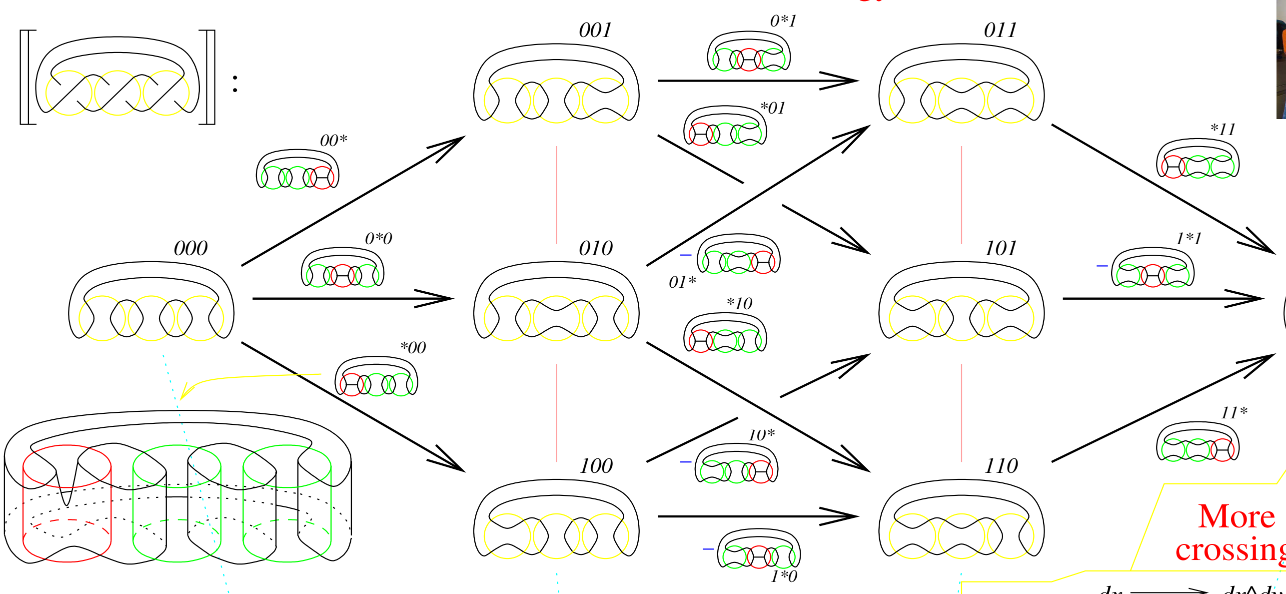
Conjecture 1. $Kh_{\mathbb{Q}}(L) = q^{s-1}\left(1 + q^2 + (1 + tq^4)Kh'\right)$ and $Kh_{\mathbb{F}_2}(L) = q^{s-1}(1+q^2)\left(1 + (1+tq^2)Kh'\right)$ for even $s = s(L)$ and non-negative-coefficients laurent polynomial $Kh' = Kh'(L)$.

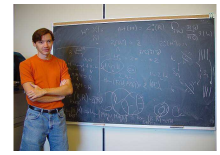
Conjecture 2. For alternating knots s is the signature and Kh' depends only on tq^2.

References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and DBN's `http://www.ma.huji.ac.il/~drorbn/papers/Categorification/`.

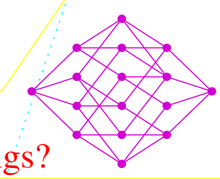
More at http://www.math.toronto.edu/~drorbn/Talks/UWO-040213/

Khovanov Homology



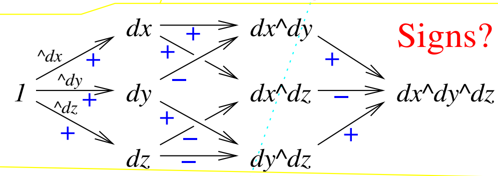


Mikhail Khovanov



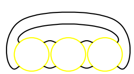
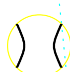
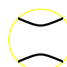
More crossings?

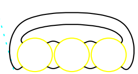
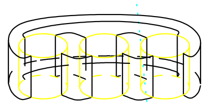
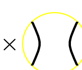
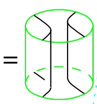
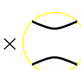
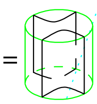
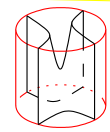
Signs?



What is it?

A cube for each knot/link projection;

Vertices: All fillings of  with  or with  .

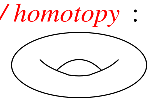
Edges: All fillings of $I\times$  =  with $I\times$  =  or with $I\times$  =  and precisely one  .

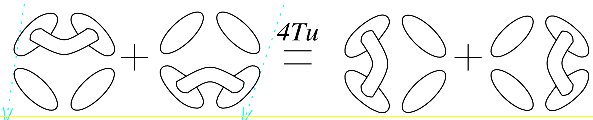
Where does it live?

In *Kom(Mat(<Cob> / {S, T, 4Tu})) / homotopy* :

Kom: Complexes *Cob: Cobordisms*

<...>: Formal lin. comb. *Mat: Matrices*

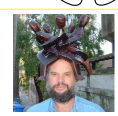
S:  =0 *T:*  =2



Jones/Kauffman?

$$V^{\otimes 3} \longrightarrow (V^{\otimes 2}\oplus V^{\otimes 2}\oplus V^{\otimes 2})\{1\} \longrightarrow (V\oplus V\oplus V)\{2\} \longrightarrow V^{\otimes 2}\{3\}$$

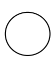
$$(q+q^{-1})^3 \qquad 3q(q+q^{-1})^2 \qquad 3q^2(q+q^{-1}) \qquad q^3(q+q^{-1})^2$$



V. Jones

L. Kauffman

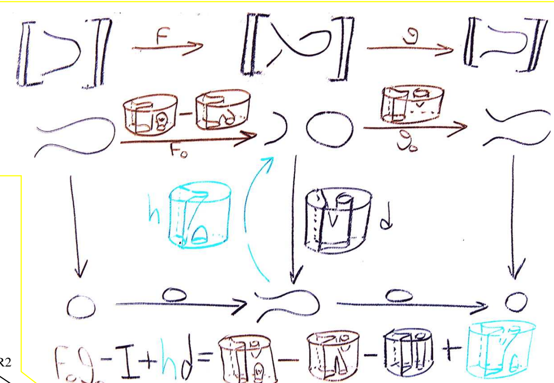
A TQFT takes it to a complex whose graded Euler characteristic is the Jones polynomial.

The key point:  $\longrightarrow V=\langle v_+, v_-\rangle, \quad \deg v_\pm = \pm 1$

$q\text{-}\dim V = q+q^{-1}$

But is it invariant?

(With similar proofs for R–II and R–III)

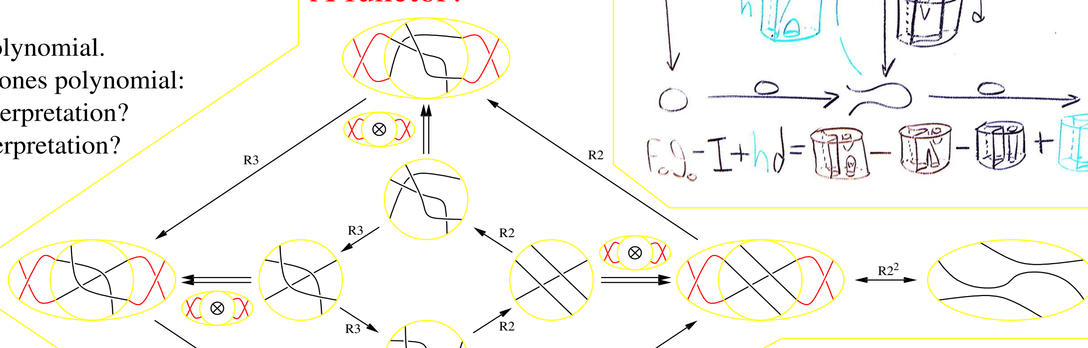


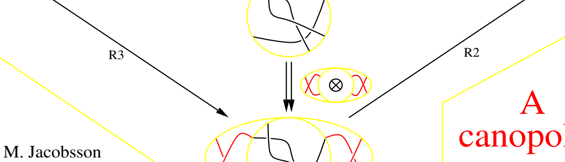
Why is it interesting?

1. It is stronger than the Jones polynomial.
2. It is less understood than the Jones polynomial:
   - a. Does it have a topological interpretation?
   - b. Does it have a "physical" interpretation?
   - c. Does it also work for other quantum invariants?
   - d. Does it work for manifolds and for knots in manifolds?
   - e. Is there a relation with finite–type invariants?
   - f. Does it work for "virtual knots"?
3. Jacobsson, Khovanov: It is a functor!!! (from knots and cobordisms to complexes and morphisms)

M. Jacobsson

A functor?





A canopoly?

See http://www.math.toronto.edu/~drorbn/papers/Cobordism

Dror Bar–Natan, Warszawa, July 2003.

More at http://www.math.toronto.edu/~drorbn/Talks/UWO-040213/