1. $\operatorname{proj}\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}\left((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n\right)$

— All Signs Are Wrong! —

I understand Drinfel'd and Alekseev−Torossian, I don't understand Etingof−Kazhdan yet, and I'm clueless about Kontsevich

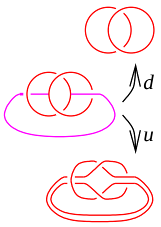
Dror Bar−Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier−1006/

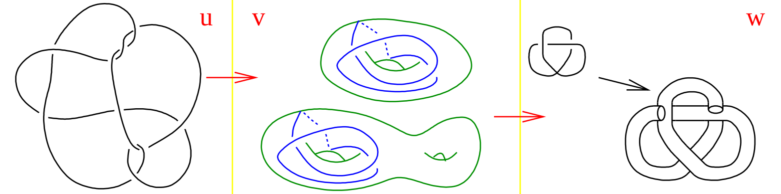
Cans and Can't Yets.

$\left(\begin{array}{c}\text{arbitrary algebraic}\\ \text{structure}\end{array}\right) \xrightarrow[\text{machine}]{\text{projectivization}} \left(\begin{array}{c}\text{a problem in}\\ \text{graded algebra}\end{array}\right)$

- Feed knot-things, get Lie algebra things.
- (u-knots)→(Drinfel'd associators).
- (w-knots)→(K-V-A-E-T).
- Dream: (v-knots)→(Etingof-Kazhdan).
- Clueless: (???)→(Kontsevich)?
- Goals: add to the Knot Atlas, produce a working AKT and touch ribbon 1-knots, rip benefits from *truly* understanding quantum groups.

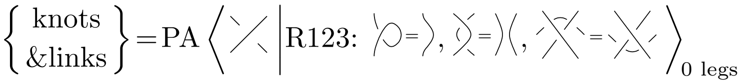
www.katlas.org



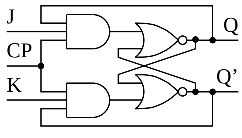


u-Knots

(PA :=Planar Algebra)



Circuit Algebras

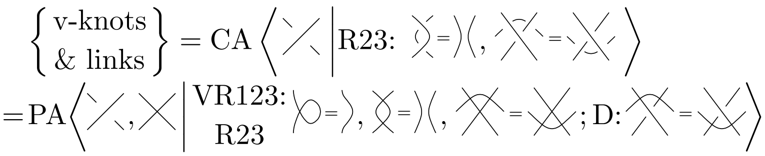


A J-K Flip Flop

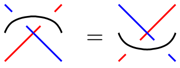
Infineon HYS64T64020HDL-3.7-A 512MB RAM

v-Tangles and w-Tangles

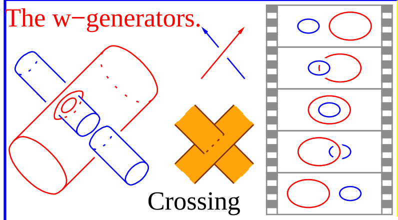
(CA :=Circuit Algebra)

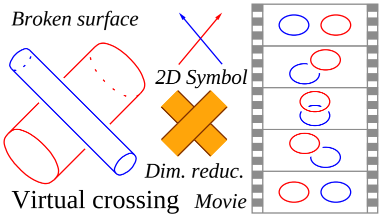


{w-Tangles} = v-Tangles / OC:

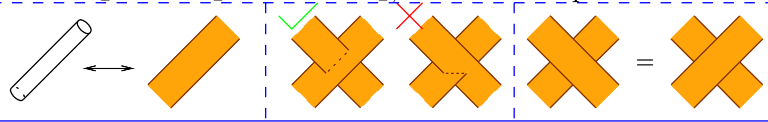


The w−generators.

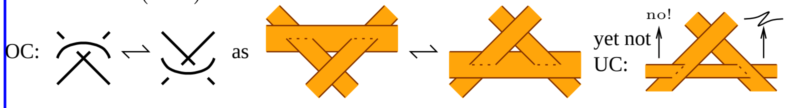


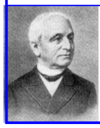


A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:



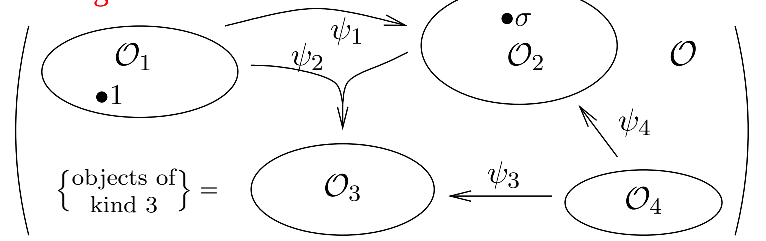


"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

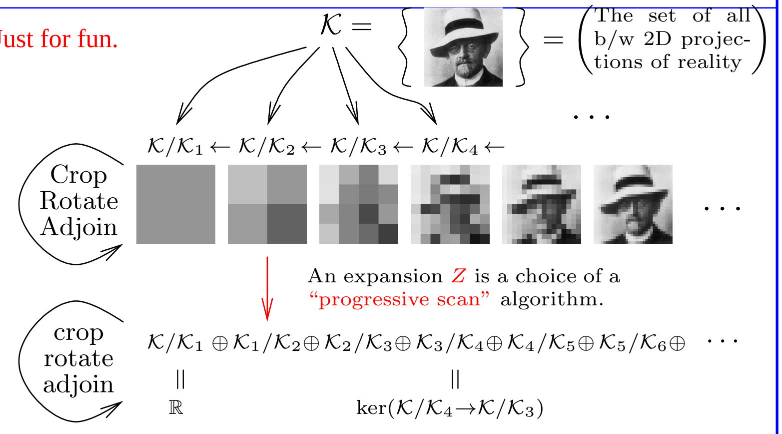
Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\begin{array}{ccccccccc} \text{ops} \circlearrowright \mathcal{K} & = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 \quad \supset \ldots \\ \Downarrow & & & & & & \downarrow Z & & \\ \text{ops} \circlearrowright \operatorname{gr}\mathcal{K} & := & \mathcal{K}_0/\mathcal{K}_1 & \oplus & \mathcal{K}_1/\mathcal{K}_2 & \oplus & \mathcal{K}_2/\mathcal{K}_3 & \oplus & \mathcal{K}_3/\mathcal{K}_4 \oplus \ldots \end{array}$$

An **expansion** is a filtered $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A **homomorphic expansion** is an expansion that respects all relevant "extra" operations.

Reality. $\operatorname{gr}\mathcal{K}$ is often too hard. An $\mathcal{A}$-expansion is a graded "guess" $\mathcal{A}$ with a surjection $\tau : \mathcal{A} \to \operatorname{gr}\mathcal{K}$ and a filtered $Z : \mathcal{K} \to \mathcal{A}$ for which $(\operatorname{gr} Z) \circ \tau = I_{\mathcal{A}}$. An $\mathcal{A}$-expansion confirms $\mathcal{A}$ and yields an ordinary expansion. Same for "homomorphic".

Just for fun.



An expansion Z is a choice of a "progressive scan" algorithm.

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products"). In this case, set $\operatorname{proj}\mathcal{K} := \operatorname{gr}\mathcal{K}$.

Examples. 1. The projectivization of a group is a graded associative algebra.

2. Pure braids — PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr} PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

3. Quandle: a set Q with an op $\wedge$ s.t.

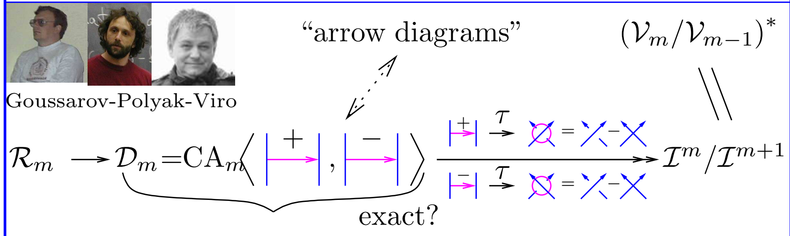
$$1 \wedge x = 1, \quad x \wedge 1 = x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

$\operatorname{proj} Q$ is a graded Leibniz algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

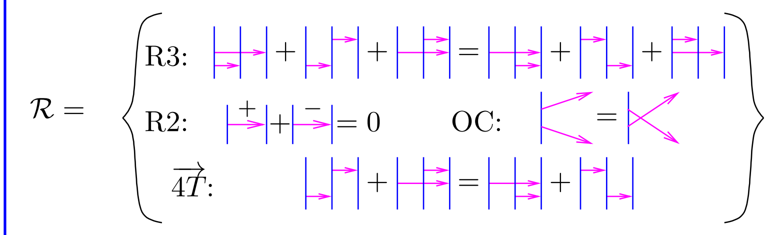
$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

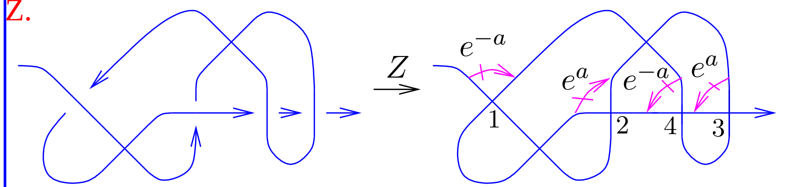
Kashiwara, Vergne, Alekseev, Enriquez, Torossian.

1. $\operatorname{proj}\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n)$, continued.

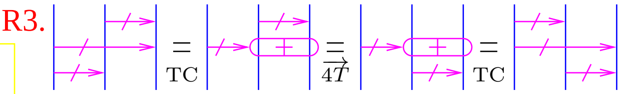


Imperfect Thumb-Rule. Take R3 (say), substitute ⤧ → ⤧ + ⤧, keep the lowest degree terms that don't immediately die:

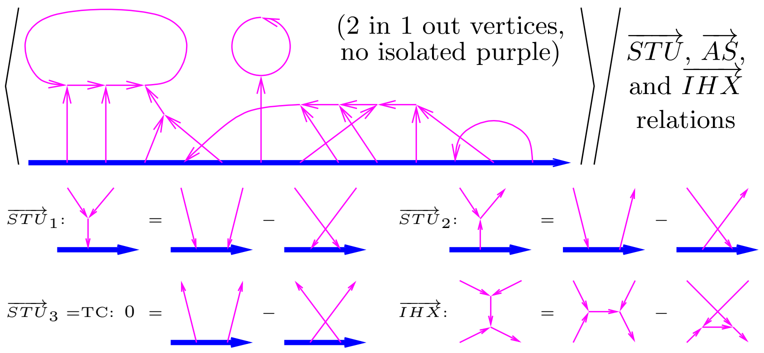




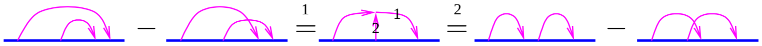
Thm. $Z \succ A$, maybe $\succ\!\succ$.



The Bracket-Rise Theorem. $\mathcal{A}^w(\uparrow_1)$ is isomorphic to

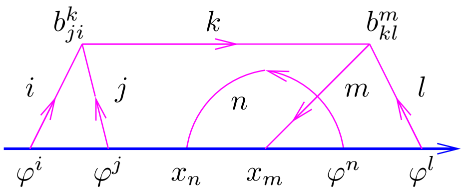


Proof.



Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

To Lie Algebras. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij} x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via

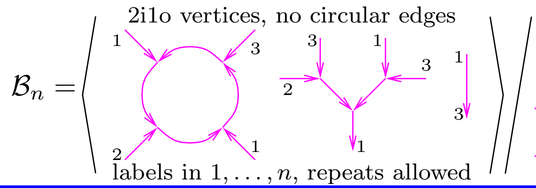



Penrose

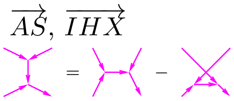

Cvitanovic

$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij} b^m_{kl} \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$

Theorem (PBW, "$\mathcal{U}(I\mathfrak{g})^{\otimes n} \cong \mathcal{S}(I\mathfrak{g})^{\otimes n}$"). As vector spaces, $\mathcal{A}^w(\uparrow_n) \cong \mathcal{B}_n$, where



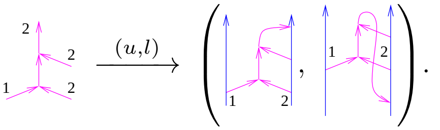

Kontsevich



Wheels and Trees. With $\mathcal{P}$ for $\mathcal{P}$rimitives,

$$\begin{tikzcd} 0 \arrow[r] & \langle\text{wheels}\rangle \arrow[r, "\iota"] & \mathcal{PA}^w(\uparrow_n) \arrow[r, "\pi"] & \langle\text{trees}\rangle \arrow[l, bend right, "u"'] \arrow[l, bend left, "l"] \arrow[r] & 0, \end{tikzcd}$$

with



So $\operatorname{proj}\mathcal{K}^w(\uparrow_n) \cong \mathcal{U}(\langle\text{trees}\rangle \ltimes \langle\text{wheels}\rangle)$.


trees atop a wheel, and a little prince.

Some A-T Notions. $\mathfrak{a}_n$ is the vector space with basis $x_1, \ldots, x_n$, $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ is the free Lie algebra, $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ is the free associative algera "of words", $\mathrm{tr} : \mathrm{Ass}_n^+ \to \mathfrak{tr}_n = \mathrm{Ass}_n^+/(x_{i_1}x_{i_2}\cdots x_{i_m} = x_{i_2}\cdots x_{i_m}x_{i_1})$ is the "trace" into "cyclic words", $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ are all the derivations, and

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

are "tangential derivations", so $D \leftrightarrow (a_1, \ldots, a_n)$ is a vector space isomorphism $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_n \mathfrak{lie}_n$. Finally, $\operatorname{div} : \mathfrak{tder}_n \to \mathfrak{tr}_n$ is $(a_1, \ldots, a_n) \mapsto \sum_k \mathrm{tr}(x_k(\partial_k a_k))$, where for $a \in \mathrm{Ass}_n^+$, $\partial_k a \in \mathrm{Ass}_n$ is determined by $a = \sum_k (\partial_k a) x_k$, and $j : \mathrm{TAut}_n = \exp(\mathfrak{tder}_n) \to \mathfrak{tr}_n$ is $j(e^D) = \frac{e^D - 1}{D} \cdot \operatorname{div} D$.

Theorem. Everything matches. $\langle\text{trees}\rangle$ is $\mathfrak{a}_n \oplus \mathfrak{tder}_n$ as Lie algebras, $\langle\text{wheels}\rangle$ is $\mathfrak{tr}_n$ as $\langle\text{trees}\rangle$ / $\mathfrak{tder}_n$-modules, $\operatorname{div} D = \iota^{-1}(u - l)(D)$, and $e^{uD}e^{-lD} = e^{jD}$.

Differential Operators. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\operatorname{ad} x$: $(x\varphi)(y) := \varphi([x, y])$.

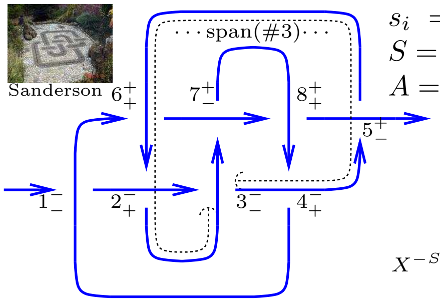
Trees become vector fields and $uD \mapsto lD$ is $D \mapsto D^*$. So $\operatorname{div} D$ is $D - D^*$ and $jD = \log(e^D(e^D)^*) = \int_0^1 dt\, e^{tD} \operatorname{div} D$.

Special Derivations. Let $\mathfrak{sder}_n = \{D \in \mathfrak{tder}_n : D(\sum x_i) = 0\}$.

Theorem. $\mathfrak{sder}_n = \pi\alpha(\operatorname{proj}\text{u-tangles})$, where α is the obvious map $\operatorname{proj}\text{u-tangles} \to \operatorname{proj}\text{w-tangles}$.

Proof. After decoding, this becomes Lemma 6.1 of Drinfel'd's amazing $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ paper.

The Alexander Theorem.



$T_{ij} = |\mathrm{low}(\#j) \in \mathrm{span}(\#i)|$,
$s_i = \mathrm{sign}(\#i)$, $d_i = \mathrm{dir}(\#i)$,
$S = \mathrm{diag}(s_i d_i)$,
$A = \det\left(I + T(I - X^{-S})\right)$.

$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0\\ 0&0&1&0&1&0&0&0\\ 0&1&0&0&1&0&0&0\\ 0&1&0&0&1&0&1&0\\ 0&1&0&1&0&1&1&1\\ 0&1&0&1&0&0&1&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$$X^{-S} = \mathrm{diag}(\tfrac{1}{X}, X, \tfrac{1}{X}, X, X, \tfrac{1}{X}, X, \tfrac{1}{X}).$$

Conjecture. For u-knots, A is the Alexander polynomial.

Theorem. With $w : x^k \mapsto w_k =$ (the k-wheel),

$$Z = N \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(e^x)\right)\right)$$

mod $w_k w_l = w_{k+l}$,
$Z = N \cdot A^{-1}(e^x)$

This is the ultimate Alexander invariant! computable in polynomial time, local, composes well, behaves under cabling. Seems to significantly generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers. But it's ugly, and much work remains.

2. w−Knots, Alekseev−Torossian, and baby Etingof−Kazhdan

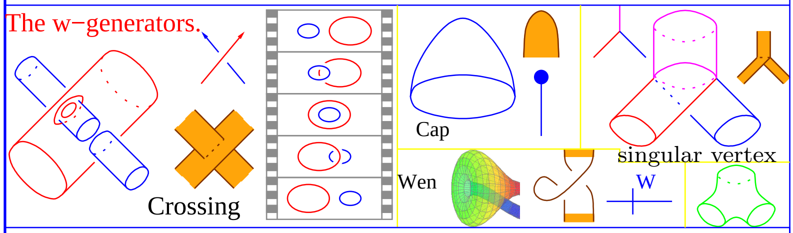
I understand Drinfel'd and Alekseev−Torossian, I don't understand Etingof−Kazhdan yet, and I'm clueless about Kontsevich

Dror Bar−Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier−1006/

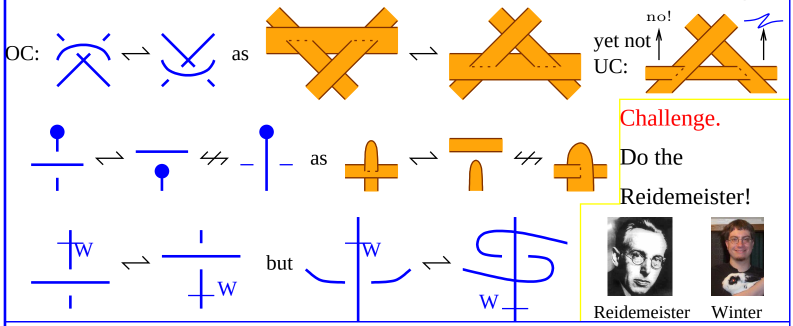
Trivalent w-Tangles.

$$\text{wTT} = \text{CA}\left\langle \begin{matrix}\text{w-}\\ \text{generators}\end{matrix} \,\middle|\, \begin{matrix}\text{w-}\\ \text{relations}\end{matrix} \,\middle|\, \begin{matrix}\text{unary w-}\\ \text{operations}\end{matrix}\right\rangle$$

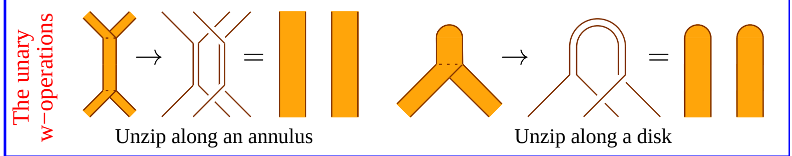
The w-generators.



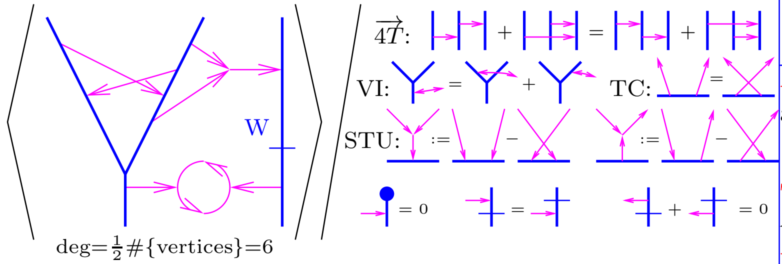
The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:



Challenge. Do the Reidemeister!



w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



An Associator:

$$(AB)C \xrightarrow{\Phi\in\mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

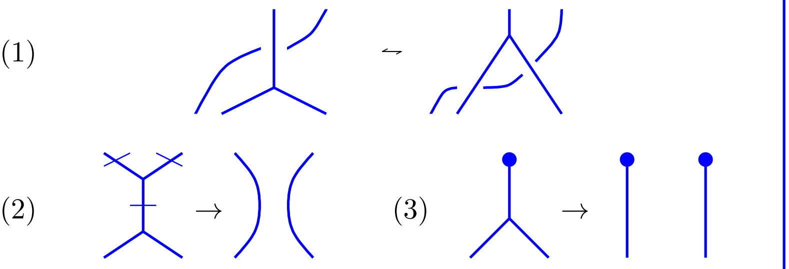
satisfying the "pentagon",

$$\begin{tikzcd}
((AB)C)D \arrow[rr, "(\Delta 11)\Phi"'] \arrow[d, "\Phi 1"] & & (AB)(CD) \arrow[d, "(11\Delta)\Phi"'] \\
(A(BC))D \arrow[r, "(1\Delta 1)\Phi"'] & A((BC)D) \arrow[r, "1\Phi"'] & A(B(CD))
\end{tikzcd}$$

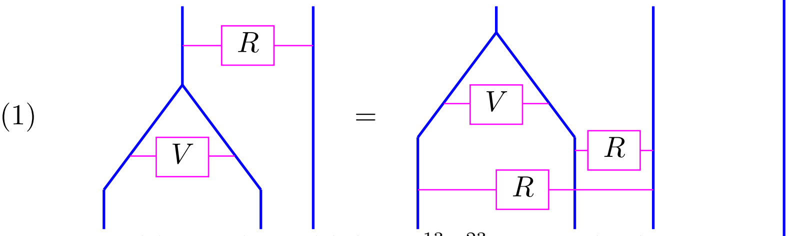
$$\Phi 1\cdot(1\Delta 1)\Phi\cdot 1\Phi = (\Delta 11)\Phi\cdot(11\Delta)\Phi$$

The hexagon? Never heard of it.

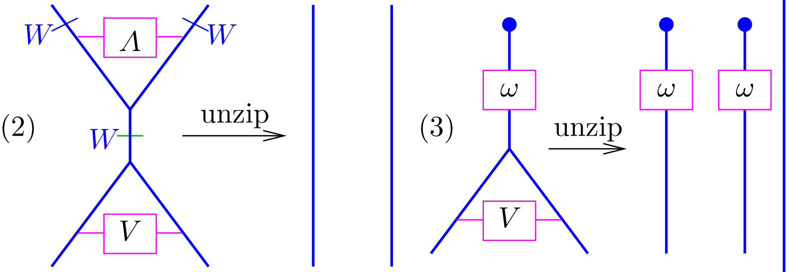
Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \mapsto\!\!| \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



(1) $V\cdot(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow\uparrow\uparrow)$



(2) $VV^* = I$ in $\mathcal{A}^w(\uparrow\uparrow)$ (3) $V\cdot\Delta(\omega) = \omega\otimes\omega$ in $\mathcal{A}^w(\uparrow\uparrow)$

Alekseev-Torossian statement. There are elements $F \in \text{TAut}_2$ and $a \in \mathfrak{tr}_1$ such that

$$F(x+y) = \log e^x e^y \quad \text{and} \quad jF = a(x) + a(y) - a(\log e^x e^y).$$

Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

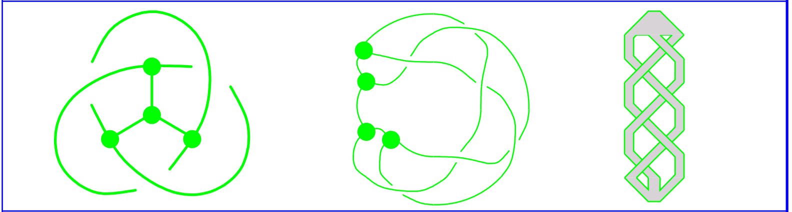
Proof. Write $V = e^c e^{uD}$ with $c \in \mathfrak{tr}_2$, $D \in \mathfrak{tder}_2$, and $\omega = e^b$ with $b \in \mathfrak{tr}_1$. Then (1) $\Leftrightarrow e^{uD}(x+y)e^{-uD} = \log e^x e^y$,
(2) $\Leftrightarrow I = e^c e^{uD}(e^{uD})^* e^c = e^{2c}e^{jD}$, and
(3) $\Leftrightarrow e^c e^{uD} e^{b(x+y)} = e^{b(x)+b(y)} \Leftrightarrow e^c e^{b(\log e^x e^y)} = e^{b(x)+b(y)}$
$\Leftrightarrow c = b(x) + b(y) - b(\log e^x e^y)$.

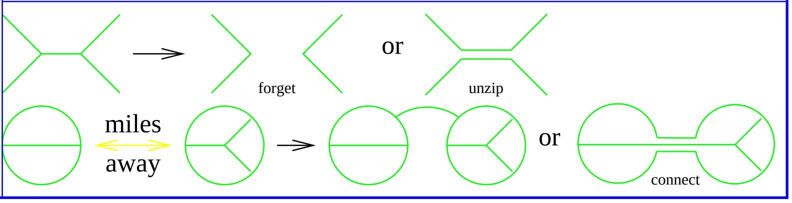
The Alekseev-Torossian Correspondence.

$$\{\text{Drinfel'd Associators}\} \leftrightarrows \{\text{Solutions of KV}\}.$$

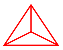
We need an even bigger algebraic structure!

$$\left(\begin{matrix}\text{green knotted trivalent}\\ \text{graphs in } \mathbb{R}^3\ (u)\end{matrix}\right) \xrightarrow{\alpha_e} \left(\begin{matrix}\text{blue tubes and red}\\ \text{strings in } \mathbb{R}^4\ (\overline{w})\end{matrix}\right)$$

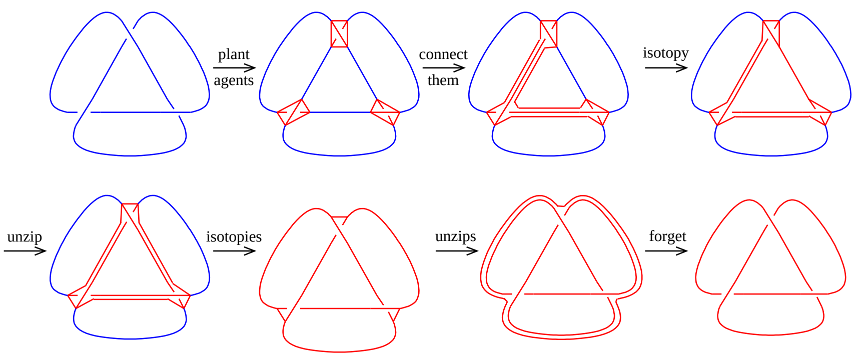


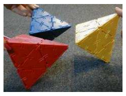


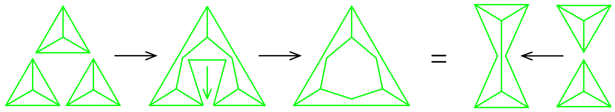
2. w−Knots, Alekseev−Torossian, and baby Etingof−Kazhdan, continued.

Using moves, KTG is generated by ribbon twists and the tetrahedron  :

All strands here are green

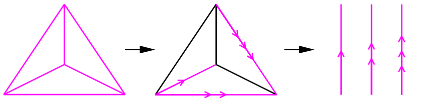


Modulo the relation(s): ( = )

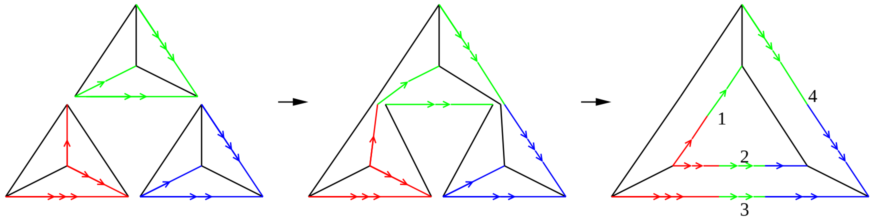


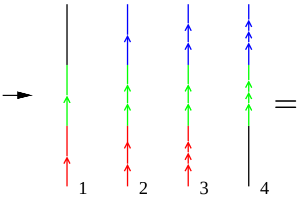
Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

Proof.

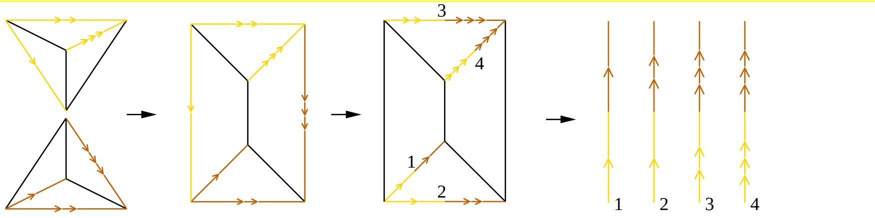


$:= \Phi \in \mathcal{A}(\uparrow_3)$



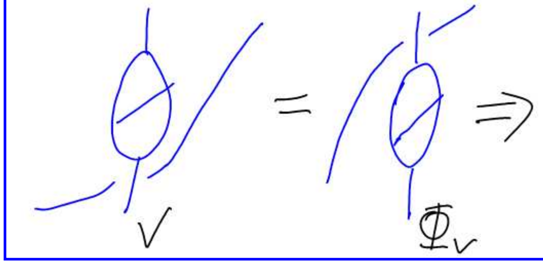


$$= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$$



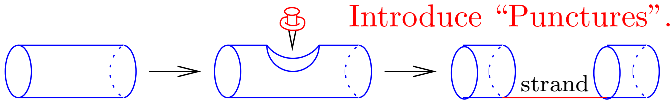
$$= (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$$

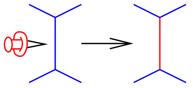
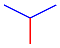
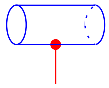
{SolKV}→{"Associators"}: Trivial - a tetrahedron has 4 vertices.

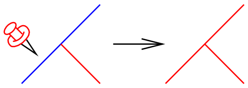


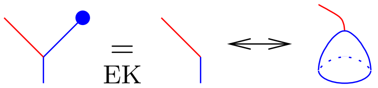
$\Phi_V \in \mathfrak{sder}_3$, so it is a sum of unrooted trees.

Introduce "Punctures".



So  where  means  .

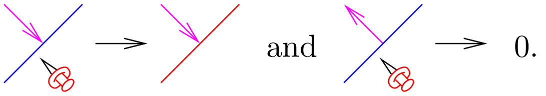
Punctures expand to the nearest Y-vertex: 

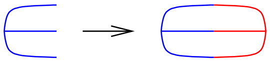
Note. 

$\mathcal{K}^{\overline{w}}$. Allow tubes and strands and tube-strand vertices as above, yet allow only "compact" knots — nothing runs to ∞.

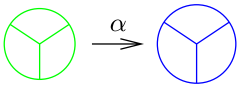
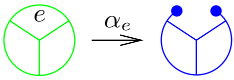
$\mathcal{K}^w \leftrightarrow \mathcal{K}^{\overline{w}}$ equivalence. $\mathcal{K}^w$ has a homomorphic expansion iff $\mathcal{K}^{\overline{w}}$ has a homomorphic expansion.

$\Longrightarrow$ Puncture $\mathcal{A}$ and Z:



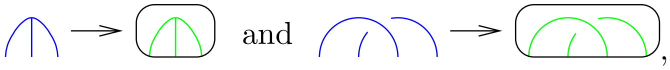
$\Longleftarrow$  (makes sense, and CA ops can be emulated)

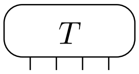
$\mathcal{K}^u \to \mathcal{K}^{\overline{w}}$. "Cut and cap is well-defined on u"

Light:  Better: 

Theorem. The generators of $\mathcal{K}^{\overline{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V.

Sketch.



so enough to write any  . Here go: