The Hardest Math I've Ever Really Used

Dror Bar−Natan, Mathcamp July 2009, http://www.math.toronto.edu/~drorbn/Talks/Mathcamp−0907

Picture credits. Mona: Leonrado; Map 1: en.wikipedia.org/wiki/Greenhouse_gas; Smokestacks: gbuapcd.org/complaint.htm; Penguin: brentpabst.com/bp/2007/12/15/BrentGoesPenguin.aspx; Map 2: flightpedia.org; Segway: co2calculator.wordpress.com/2008/10;

Abstract. What's the hardest math I've ever used in real life? Me, myself, directly — not by using a cellphone or a GPS device that somebody else designed. And in "real life" - not while studying or teaching mathematics?
I use addition and subtraction daily, adding up bills or calculating change. I use percentages often, though mostly it is just "add 15 percents". I seldom use multiplication and division: when I buy in bulk, or when I need to know how many tiles I need to replace my kitchen floor. I've used powers twice in my life, doing calculations related to mortgages. I've used a tiny bit of 2×2 linear algebra for a tiny bit of non-math-related computer graphics I've played with. And for a long time, that was all. In my talk I will tell you how recently a math topic discovered only in the 1800s made a brief and modest appearance in my non-mathematical life. There are many books devoted to that topic and a lot of active research. Yet for all I know, nobody ever needed the actual gory formulas for such a simple reason before.

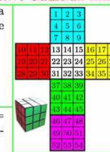
I could be a Mathematician...

Dror Bar-Natan: Talks: Mathcamp-0907:

Non-Commutative Gaussian Elimination and Rubik's Cube

Joint study with Itai Bar-Natan

The Problem. Let $G=\langle g_1,\dots,g_\alpha\rangle$ be a subgroup of S_n, with $n=O(100)$. Before you die, understand G:
1. Compute $|G|$.
2. Given $\sigma\in S_n$, decide if $\sigma\in G$.
3. Write a $\sigma\in G$ in terms of $g_1,\dots,g_\alpha$.
4. Produce *random* elements of G.



The Commutative Analog. Let $V=\operatorname{span}(v_1,\dots,v_\alpha)$ be a subspace of $\mathbb{R}^n$. Before you die, understand V.

Solution: Gaussian Elimination. Prepare an empty table,

1	2	3	4	⋯	$n-1$	n

Space for a vector $u_4\in V$, of the form $u_4=(0,0,0,1,*,\dots,*)$; 1 := "the pivot".

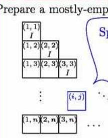
Based on an algorithm by

Otto Schreier Charles Sims

See also *Permutation Group Algorithms* by Ákos Seress.

Feed $v_1,\dots,v_\alpha$ in order. To feed a non-zero v, find its pivotal position i.
1. If box i is empty, put v there.
2. If box i is occupied, find a combination v' of v and u_i that eliminates the pivot, and feed v'.

Non-Commutative Gaussian Elimination
Prepare a mostly-empty table,



Space for a $\sigma_{i,j}\in S_n$ of the form $(1,2,\dots,i-2,i-1,j,*,*,\dots,*)$
So $\sigma_{i,j}$ fixes $1,\dots,i-1$, sends "the pivot" i to j and goes wild afterwards, and $\sigma_{i,j}^{-1}$ "does sticker j".

Feed $g_1,\dots,g_\alpha$ in order. To feed a non-identity σ, find its pivotal position i and let $j:=\sigma(i)$.
1. If box (i,j) is empty, put σ there.
2. If box (i,j) contains $\sigma_{i,j}$, feed $\sigma':=\sigma_{i,j}^{-1}\sigma$.

The Twist. When done, for every occupied (i,j) and (k,l), feed $\sigma_{i,j}\sigma_{k,l}$. Repeat until the table stops changing.

Claim. The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table T.

Claim. Anything fed in T is a monotone product in T:
f was fed $\Rightarrow f\in M_1:=\{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n}:\forall i, j_i\geq i \ \&\ \sigma_{i,j_i}\in T\}$

The Generators

```
In[1]:= gs = {
  purple = P[18,27,36,4,5,6,7,8,9,3,11,12,13,14,15,16,17,
    45,2,20,21,22,23,24,25,26,44,1,29,30,31,32,33,34,35,43,
    37,38,39,40,41,42,10,19,28,52,49,46,53,50,47,54,51,48],
  white = P[1,2,3,4,5,6,16,25,34,10,11,9,15,24,33,39,17,
    18,19,20,8,14,23,32,38,26,27,28,29,7,13,22,31,37,35,36,
    12,21,30,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54],
  green = P[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,
    19,20,21,22,23,24,25,26,27,31,32,33,34,35,36,48,47,46,
    39,42,45,38,41,44,37,40,43,30,29,28,49,50,51,52,53,54],
  blue = P[3,6,9,2,5,8,1,4,7,54,53,52,10,11,12,13,14,15,
    19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,
    37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,18,17,16],
  red = P[13,2,3,22,5,6,31,8,9,12,21,30,37,14,15,16,17,
    18,11,20,29,40,23,24,25,26,27,10,19,28,43,32,33,34,35,
    36,46,38,39,49,41,42,52,44,45,1,47,48,4,50,51,7,53,54],
  yellow = P[1,2,48,4,5,51,7,8,54,10,11,12,13,14,3,18,27,
    36,19,20,21,22,23,6,17,26,35,28,29,30,31,32,9,16,25,34,
    37,38,15,40,41,24,43,44,33,46,47,39,49,50,42,52,53,45]
};  Enter
```

Theorem. $G=M_1$. *G^{-1} is more fun!*

$G=M_1:=\{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n}:\forall i, j_i\geq i \text{ and } \sigma_{i,j_i}\in T\}$.

Proof. The inclusions $M_1\subset G$ and $\{g_1,\dots,g_\alpha\}\subset M_1$ are obvious. The rest follows from the following

Lemma. M_1 is closed under multiplication.

Proof. By backwards induction. Let

$$M_k:=\{\sigma_{k,j_k}\cdots\sigma_{n,j_n}:\forall i\geq k, j_i\geq i \text{ and } \sigma_{i,j_i}\in T\}.$$

Clearly $M_nM_n\subset M_n$. Now assume that $M_5M_5\subset M_5$ and show that $M_4M_4\subset M_4$. Start with $\sigma_{8,j}M_4\subset M_4$:

$$\sigma_{8,j}(\sigma_{4,j_4}M_5)\overset{1}{=}(\sigma_{8,j}\sigma_{4,j_4})M_5\overset{2}{\subset}M_4M_5$$

$$\overset{3}{=}\sigma_{4,j_4}(M_5M_5)\overset{4}{\subset}\sigma_{4,j_4}M_5\subset M_4$$

(1: associativity, 2: thank the twist, 3: associativity and tracing i_4, 4: induction). Now the general case

$$(\sigma_{4,j_4'}\sigma_{5,j_5'}\cdots)(\sigma_{4,j_4}\sigma_{5,j_5}\cdots)$$

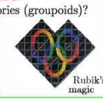
falls like a chain of dominos.

Problem Solved!

Homework Problem 1. Can you do cosets?

Homework Problem 2. Can you do categories (groupoids)?

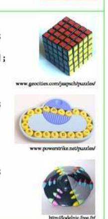
7	9	2	5
1	4	8	3
6	10	11	12
13	14	15	



Rubik's magic

A Demo Program

```
In[2]:= ($RecursionLimit = 2^16;
  n = 54;
  P /: p_P ** P[a___] := p[[{a}]];
  Inv[p_P] := P @@ Ordering[p];
  Feed[P @@ Range[n]] := Null;
  Feed[p_P] := Module[{i, j},
    For[i = 1, p[[i]] == i, ++i];
    j = p[[i]];
    If[Head[s[i, j]] === P,
      Feed[Inv[s[i, j]] ** p],
    (* Else *) s[i, j] = p;
      Do[If[Head[s[k, l]] == P,
        Feed[s[i, j] ** s[k, l]];
        Feed[s[k, l] ** s[i, j]]
      ], {k, n}, {l, n}]
    ]]);  Enter
```

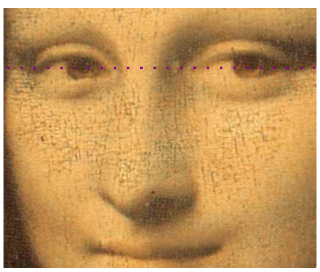


The Results

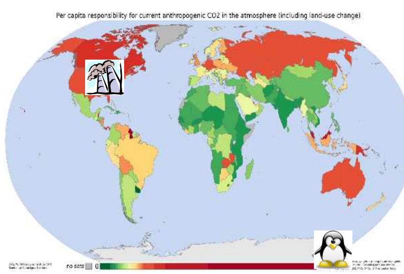
```
In[3]:= (Feed[#]; Product[1 + Length[Select[Range[n], Head[s[i, #]] === P &]], {i, n}]) & /@ gs   Enter
Out[3]= {4, 16, 159993501696000, 21119142223872000, 43252003274489856000, 43252003274489856000}
```

http://www.math.toronto.edu/~drorbn/Talks/Mathcamp-0907/ and http://www.math.toronto.edu/~drorbn/Misc/SchreierSimsRubik/

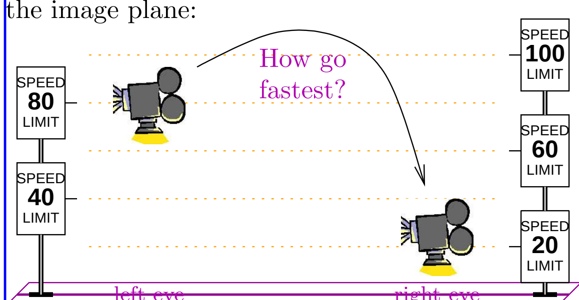
...or an Art Historian...



...or an Environmentalist.

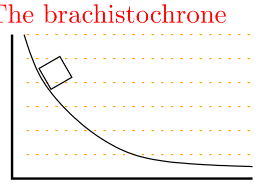


Goal. Find the least-blur path to go from Mona's left eye to Mona's right eye in fixed time. Alternatively, fix your blur-tolerance, and find the fastest path to do the same. For fixed blur, our camera moves at a speed proportional to its distance from the image plane:



The Mona Plane

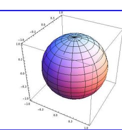
The brachistochrone



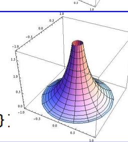
$$mgh = \frac{1}{2}mv^2$$

Bernoulli on Newton. "I recognize the lion by his paw".

```
ParametricPlot3D[{
  Sin[u] Cos[v],
  Sin[u] Sin[v],
  Cos[u]
 }, {u, 0, π}, {v, 0, 2 π}]
```



```
ParametricPlot3D[{
  Sech[u] Cos[v],
  Sech[u] Sin[v],
  u - Tanh[u]
 }, {u, 0, ⅇ}, {v, 0, 2 π}]
```

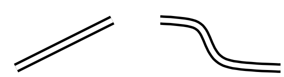


Flatlanders airline route map

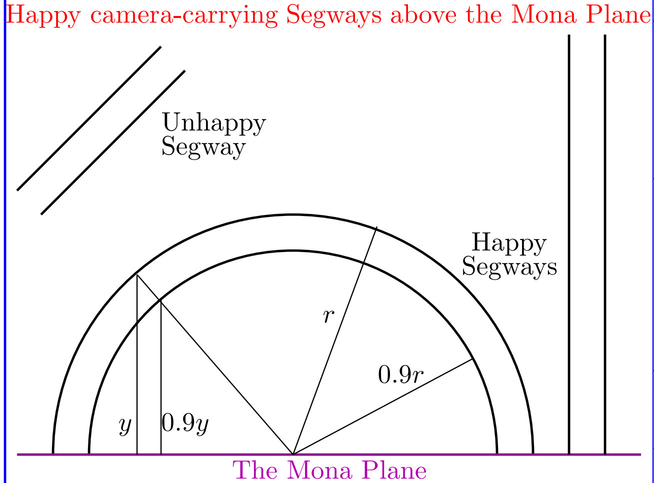
576
252
167
131
112
103
100
103
112

The Happy Segway Principle

A Segway is happy iff both its wheels are

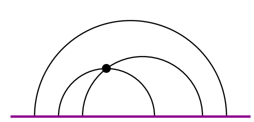


Happy camera-carrying Segways above the Mona Plane



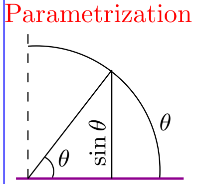
The Mona Plane

The Lobachevsky Space



Two parallels through one point

Parametrization

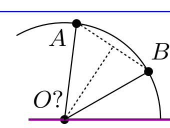


$$\theta'(t)=\sin\theta(t)$$

$$\Downarrow$$

$$\theta=2\arctan e^t$$

Rotations, translations and some basic geometry also occur



The Actual Code

```
x = p1.x-p2.x; y = p1.y-p2.y;
d1 = p1.d; d2 = p2.d;
norm = sqrt(x*x + y*y);
a = x/norm; b = y/norm;
x1p = a*x + b*y;
x0 = (x1p + (d1*d1-d2*d2)/x1p)/2;
r = sqrt((x1p-x0)*(x1p-x0)+d1*d1);
x1pp = (x1p-x0)/r; x2pp = -x0/r;
theta1 = acos(x1pp);
theta2 = acos(x2pp);
t1 = log(tan(theta1/2));
t2 = log(tan(theta2/2));
t3 = t1 + s*(t2-t1);
theta3 = 2*atan(exp(t3));
x3pp = cos(theta3);
d3pp = sin(theta3);
x3p = x0 + r*x3pp;
p3.d = r*d3pp;
p3.x = p2.x + a*x3p;
p3.y = p2.y + b*x3p;
```

I despise the real numbers !!!

Ops used. $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, cos, sin, tan, arccos, arctan, log, exp.