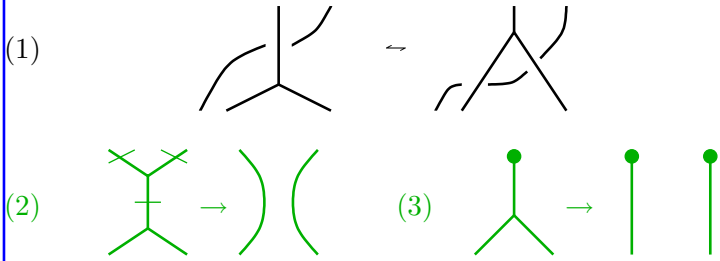
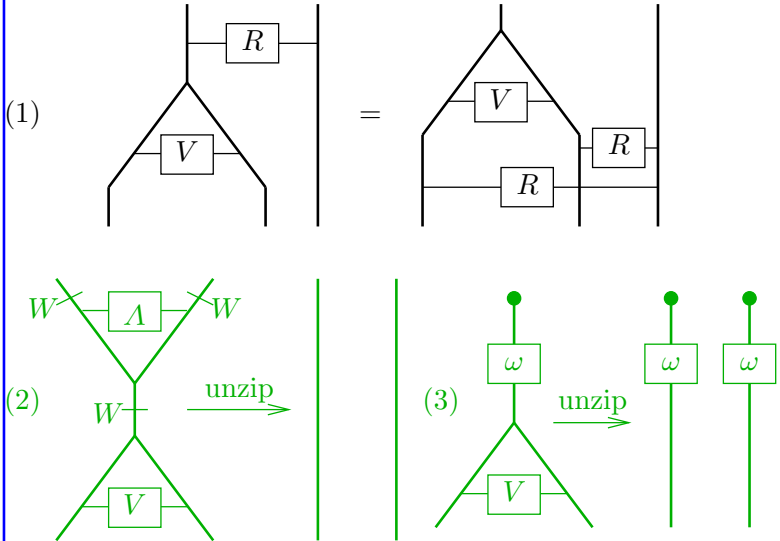


Convolutions on Lie Groups and Lie Algebras and Ribbon 2–Knots, Page 2

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \uparrow \in \mathcal{A}^w(\uparrow \uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow \uparrow)$ so that



Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{U}(I\mathfrak{g}) \rightarrow \hat{U}(I\mathfrak{g})/\hat{U}(\mathfrak{g}) = \hat{S}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{U}(I\mathfrak{g})$, with W the automorphism of $\hat{U}(I\mathfrak{g})$ induced by flipping the sign of \mathfrak{g}^* , with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{U}(I\mathfrak{g}) \otimes \hat{U}(\mathfrak{g})$ there exist $\omega \in \hat{S}(\mathfrak{g}^*)$ and $V \in \hat{U}(I\mathfrak{g})^{\otimes 2}$ so that

- $$\begin{aligned} (1) \quad & V(\Delta \otimes 1)(R) = R^{13}R^{23}V \text{ in } \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g}) \\ (2) \quad & V \cdot SWV = 1 \qquad (3) \quad (c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega \end{aligned}$$

Unitary statement. There exists $\omega \in \text{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\text{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

- (1) $V\widehat{e^{x+y}} = \widehat{e^x e^y} V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$ -valued functions)
 (2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x \omega_y$

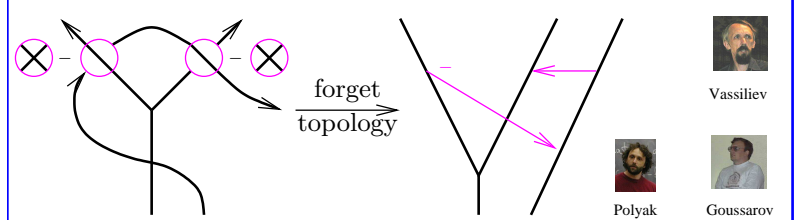
Group-Algebra statement. There exists $\omega^2 \in \text{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \text{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2 = j^{1/2}$)

$$\iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)\omega_{x+y}^2 e^{x+y} = \iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)\omega_x^2 \omega_y^2 e^x e^y. \quad (\text{shhh, this is Duflo})$$

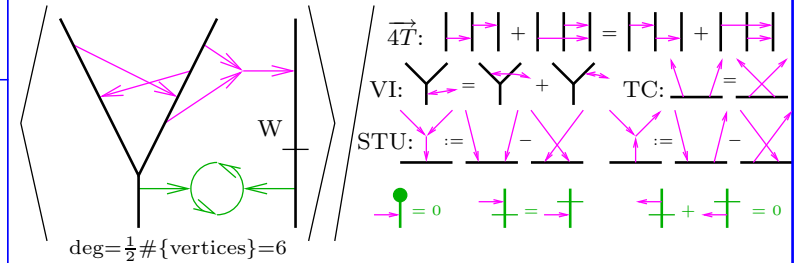
Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let \mathfrak{g} be its Lie algebra, let $j : \mathfrak{g} \rightarrow \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \rightarrow G$, and let $\Phi : \text{Fun}(G) \rightarrow \text{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

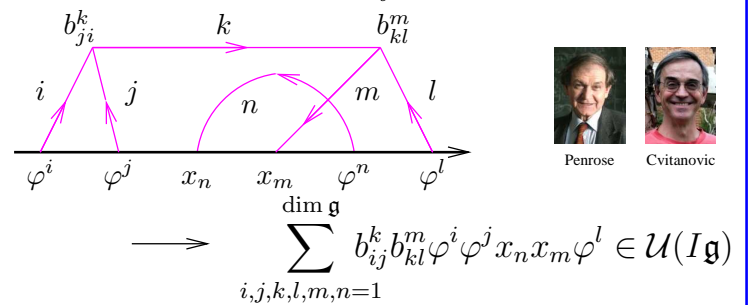
From wTT to \mathcal{A}^w . $\text{gr}_m \text{wTT} := \{m\text{-cubes}\} / \{(m+1)\text{-cubes}\}:$



w-Jacobi diagrams and \mathcal{A} . $\mathcal{A}^w(Y \uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of \mathfrak{g} and \mathfrak{g}^* and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \rightarrow \mathcal{U}$ via

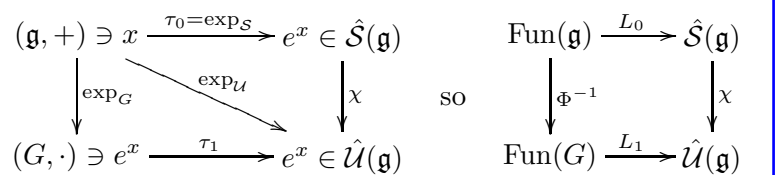


Unitary \iff Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\text{ad } x$: $(x\varphi)(y) := \varphi([x, y])$.
- $c : \dot{\mathcal{U}}(I\mathfrak{g}) \rightarrow \dot{\mathcal{U}}(I\mathfrak{g})/\dot{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ is “the constant term”.

$$\begin{aligned} \text{Unitary} &\implies \text{Group-Algebra.} \quad \iint \omega_{x+y}^2 e^{x+y} \phi(x) \psi(y) \\ &= \langle \omega_{x+y}, \omega_{x+y} e^{x+y} \phi(x) \psi(y) \rangle = \langle V \omega_{x+y}, V e^{x+y} \phi(x) \psi(y) \omega_{x+y} \rangle \\ &= \langle \omega_x \omega_y, e^x e^y V \phi(x) \psi(y) \omega_{x+y} \rangle = \langle \omega_x \omega_y, e^x e^y \phi(x) \psi(y) \omega_x \omega_y \rangle \\ &= \iint \omega_x^2 \omega_y^2 e^x e^y \phi(x) \psi(y). \end{aligned}$$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \rightarrow A$ is multiplicative then $(\text{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie (G, \mathfrak{g}) ,



with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \text{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are “Laplace transforms”)

$$\star \text{ in } G: \iint \psi_1(x)\psi_2(y)e^xe^y \qquad \star \text{ in } \mathfrak{g}: \iint \psi_1(x)\psi_2(y)e^{x+y}$$

We skipped...

- The Alexander polynomial and Milnor numbers.
- u-Knots, Alekseev-Torossian, and Drinfel'd associators.
- The simplest problem hyperbolic geometry solves.
- v-Knots, quantum groups and Etingof-Kazhdan.
- BF theory and the successful religion of path integrals.