THE DUFLO THEOREM IMPLIES THE KASHIWARA-VERGNE CONJECTURE

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Abstract. The purpose of this note is to show that either (a minor variant of) the Duflo theorem on the isomorphism between $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ and $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ implies the Kashiwara-Vergne conjecture [KV] on the local equivalence between the convolution algebra of invariant functions on a Lie algebra and the convolution algebra of invariant functions on its Lie group, or that the author of this note is confused about something simple. The Kashiwara-Vergne conjecture was first proven by much harder means by Alekseev and Meinrenken [AM].

Added May 25, 2010. Project abandoned (back in June 2009) — I misunderstood KV to think that their "convolutions" statement was the main point, whereas the main point for real is the free-Lie statement.

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1. Introduction

It is well understood that the Kashiwara-Vergne conjecture (KV, see 3.3) is strictly stronger and much harder than the Duflo theorem (see 3.2). Indeed nearly thirty years have passed between the stating of KV in [KV] and its proof in [AM] (Duflo's theorem is even older, and its weaker status was asserted already in [KV]).

I think I found a new way of relating Duflo and KV, showing that Duflo implies KV in just a few pages that could have easily been written in 1978. As I am not an expert, it may well be that I am simply misunderstanding something or that I am confused about something simple.

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To allow experts a quick access to my argument, I am reversing the traditional order of mathematical proceedings, putting the prerequisites and lemmas later than the main proof. Hence the main proof appears in a telegraphic form in Section 2, with many parenthetical references to the subsections of Section 3 in which I cover the lemmas and prerequisites in some detail and add a few comments.

I should state outright that the version of Duflo's theorem I am using is slightly stronger than the original and that the version of KV that I am proving is slightly weaker than the original. For me Duflo is about a completed graded version $\mathcal{U}^\hbar(\mathfrak{g})$ of the Lie algebra in question and not just about $\mathcal{U}(\mathfrak{g})$ (precisely, $\mathcal{U}^\hbar(\mathfrak{g}) := \mathcal{T}(\mathfrak{g})[\hbar]/\langle xy - yx = \hbar[x,y]\rangle^\wedge$, see 3.1) and KV is merely about smooth functions supported near the identity, rather than about distributions supported on certain cones. To my understanding these differences are minor, but hey, I've already admitted I'm not an expert. Is this the source of my misunderstanding?

2. The Proof

2.1. **Three Exponentials.** Let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra. One may define three different exponential maps on $\mathfrak{g}$:

- $\exp_\mathcal{S} : \mathfrak{g} \to \hat{\mathcal{S}}(\mathfrak{g})$ is the formal exponential defined by the formula $\exp_\mathcal{S}(x) = e^x := \sum_n \frac{x^n}{n!}$, with values in the completed symmetric algebra $\hat{\mathcal{S}}(\mathfrak{g})$ of $\mathfrak{g}$ (see 3.1). It is a group homomorphism relative to the additive group structure on $\mathfrak{g}$. Namely, $\exp_\mathcal{S}(x+y) = \exp_\mathcal{S}(x)\exp_\mathcal{S}(y)$.
- $\exp_\mathcal{S} : \mathfrak{g} \to \hat{\mathcal{U}}(\mathfrak{g})$ is the formal exponential defined by the same formula, $\exp_\mathcal{U}(x) := e^x = \sum_n \frac{x^n}{n!}$, only with values in the completed graded universal enveloping algebra $\hat{\mathcal{S}}(\mathfrak{g})$ of $\mathfrak{g}$ (see 3.1). Unless $\mathfrak{g}$ is Abelian, it is not a group homomorphism. Namely, in general $\exp_\mathcal{U}(x+y) \neq \exp_\mathcal{U}(x)\exp_\mathcal{U}(y)$.
- $\exp_G : \mathfrak{g} \to G$ is the standard exponential map of a Lie algebra into its Lie group. It is an invertible diffeomorphism on a small neighborhood of $0 \in \mathfrak{g}$ with values in a small neighborhood of the identity $1 \in G$, but unless G is Abelian, it is not a group homomorphism. Namely, in general $\exp_G(x+y) \neq \exp_G(x)\exp_G(y)$.

Let us define $\tau_0 := \exp_\mathcal{S}$, $\tau_1 := \exp_\mathcal{U} \circ \exp_G^{-1}$. Let $\chi : \hat{\mathcal{S}}(\mathfrak{g}) \to \hat{\mathcal{U}}(\mathfrak{g})$ denote the unique graded vector space isomorphism which maps $e^x \in \hat{\mathcal{S}}(\mathfrak{g})$ to $e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. One may verify that χ is the total symmetrization map, which is not an algebra homomorphism. In summary, we have the following commutative diagram, in which the horizontal arrows are group homomorphisms (but the vertical ones in general are not) and in which the left vertical arrow is a diffeomorphism and the right vertical arrow is a graded vector space isomorphism:

$$\begin{array}{ccc} (\mathfrak{g},+) \ni x & \xrightarrow{\tau_0=\exp_\mathcal{S}} & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow{\scriptstyle \exp_G} & \searrow^{\exp_\mathcal{U}} & \downarrow{\scriptstyle \chi} \\ (G,\cdot) \ni e^x & \xrightarrow{\tau_1} & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) \end{array} \tag{1}$$

2.2. **Formal Fourier-Laplace Transforms.** In general, if $\tau := \Gamma \to A$ is a product-respecting map of a finite group Γ into an algebra A, then $Tf := \sum_{\gamma\in\Gamma} f(\gamma)\tau(\gamma)$ maps $\mathrm{Fun}(\Gamma)$, the linear space of all scalar-valued functions on Γ to A, *carrying the convolution product* $\star$ *on* $\mathrm{Fun}(\Gamma)$ *to the algebras product of* A. We wish to imitate this observation in the continuous case, using the fact that τ_0 and τ_1 are group homomorphisms. To achieve this, fix a translation invariant measure dx on $\mathfrak{g}$ (it is anyway unique up to a constant factor) and let dg denote the the left-invariant Haar measure on G normalized to that at $1 \in G$ it agrees with the measure dx at $0 \in \mathfrak{g}$ using $\exp_G$ to identify an infinitesimal neighborhood of $0 \in \mathfrak{g}$ with an infinitesimal neighborhood of $1 \in G$.

Let $\mathrm{Fun}_0(\mathfrak{g})$ denote the linear space of smooth functions on $\mathfrak{g}$ supported in a small neighborhood of $0 \in \mathfrak{g}$ and let $L_0 : \mathrm{Fun}_0(\mathfrak{g}) \to \hat{\mathcal{S}}(\mathfrak{g})$ be defined by "linear extension" of τ_0, that is, by $L_0\psi := \int_{\mathfrak{g}} dx\psi(x)\tau_0(x) = \int_{\mathfrak{g}} dx\psi(x)e^x$ for $\psi \in \mathrm{Fun}_0(\mathfrak{g})$. Our integrand (and hence also the resulting integral) is valued in the vector space $\hat{\mathcal{S}}(\mathfrak{g})$.

It is worthwhile to note that L_0 is a formal analogue of the Laplace transform for functions on $\mathfrak{g}$ (here $\mathfrak{g}$ can be replaced by an arbitrary vector space; also, if we insert a $\sqrt{-1}$ inside the exponential defining τ_0, we get a similar formal analogue of the Fourier transform). Indeed, L_0 has the following easily verified properties:

- L_0 maps delta functions to exponentials.
- L_0 converts translations to multiplication by exponentials.
- L_0 converts convolutions to products.
- L_0 converts "multiplication by a polynomial $\varphi \in \mathcal{S}(\mathfrak{g}^*)$" to "applying the differential operator D_φ" (recall that $\mathcal{S}(\mathfrak{g}^*)$ acts on $\hat{\mathcal{S}}(\mathfrak{g})$ as "differential operators").

Likewise let $\mathrm{Fun}_1(G)$ denote the linear space of smooth functions on G supported in a small neighborhood of the identity $1 \in G$) and let $L_1 : \mathrm{Fun}_1(G) \to \hat{\mathcal{U}}(\mathfrak{g})$ be defined by "linear extension" of τ_1, that is, by $L_1\phi := \int_G dg\phi(x)\tau_1(g)$ for $\phi \in \mathrm{Fun}_1(G)$. Our integrand (and hence also the resulting integral) is valued in the vector space $\hat{\mathcal{U}}(\mathfrak{g})$.

Let $j : \mathfrak{g} \to \mathbb{R}$ the Jacobian of $\exp_G$; more precisely, it is the density of dg relative to the pushforward of dx via $\exp_G$. By our choice of normalization, $j(0) = 1$. Let $\Phi' : \mathrm{Fun}_1(G) \to \mathrm{Fun}_0(\mathfrak{g})$ be given by $(\Phi' f)(x) := j(x)f(\exp_G x)$ for $x \in \mathfrak{g}$. Consider the following "linear extension" of the diagram in (1):

$$\begin{array}{ccc} (\mathrm{Fun}_0(\mathfrak{g}), \star) & \xrightarrow{L_0} & \hat{\mathcal{S}}(\mathfrak{g}) \\ \uparrow{\scriptstyle \Phi'} & & \downarrow{\scriptstyle \chi} \\ (\mathrm{Fun}_1(G), \star) & \xrightarrow{L_1} & \hat{\mathcal{U}}(\mathfrak{g}) \end{array} \tag{2}$$

By a change of variables formula, this diagram is commutative and by a simple calculation the horizontal arrows in it are algebra homomorphisms. The vertical arrows in (2) are vector space isomorphisms but not homomorphisms. Finally L_0 and hence also L_1 is injective. Indeed if $\varphi \in \mathcal{S}(\mathfrak{g}^*)$ is a polynomial on $\mathfrak{g}$, then for any $\psi \in \mathrm{Fun}_0(\mathfrak{g})$ we have that $\int dx\varphi(s)\psi(x) = \langle \varphi, L_0(\psi) \rangle$ using the obvious pairing $\langle \cdot, \cdot \rangle : \mathcal{S}(\mathfrak{g}^*) \otimes \hat{\mathcal{S}}(\mathfrak{g}) \to \mathbb{R}$. Hence all the moments of ψ are readable from $L_0(\psi)$, so if the latter is 0 the former must be 0 too.

MORE.

3. Prerequisites, Lemmas, Comments

3.1. **A graded completion of $\mathcal{U}(\mathfrak{g})$.**

3.2. **The Duflo Theorem.**

3.3. **The Kashiwara-Vergne Conjecture (KV).**

References

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