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On the Nonrepresentability of Functions of Several Variables in Quadratures*

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To I. M. Gelfand on the occasion of his 90th birthday

Abstract. The paper completes the construction of a multidimensional topological version of differential Galois theory. We construct a rich class of germs of functions of several variables which is closed under superpositions and other natural operations. The main theorem describes the behavior of the monodromy groups of such germs under the natural operations. As a result, we obtain topological obstructions to the representability of functions in quadratures, which give the strongest known statements about unsolvability of equations in closed form.

Key words: multivalued function, monodromy group, differential Galois theory, representability in quadratures.

The present paper is the last in the series of papers [4–6] dealing with topological obstructions to the representability of a function of several variables in quadratures. Similar results for functions of one variable were obtained in my PhD thesis [1–3] written under Arnold's supervision and defended in 1973.

Arnold [9–16] found topological proofs of unsolvability of several problems, including the problem of solving algebraic equations in radicals (cf. [7, 8]), although he ascribed the results concerning the latter problem to Abel, see [7].

In [1–3], a rich class of functions of one variable with infinitely many values has been constructed for which the monodromy group is well defined. Is there a sufficiently rich class of germs of functions of several variables with infinitely many values (including germs of functions representable in generalized quadratures and germs of entire functions of several variables and closed with respect to natural operations such as superposition) possessing the same property? For a long time, I have thought that the answer is "no." In the present paper, I introduce the class of $\mathcal{SC}$-germs, which answers the question in the positive. The proof makes use of the results of [4] concerning the continuability of multivalued analytic functions along their ramification sets.

The main theorem (see Sec. 5) describes the changes occurring in the monodromy groups of $\mathcal{SC}$-germs if one applies natural operations to these germs. This theorem is very close to the main one-dimensional theorem in [2, 3] but uses also new results both of analytic [4] and group-theoretic [6] nature. As a consequence, we obtain topological statements about unsolvability of equations in closed form, which are stronger than their classical counterparts. The reader can find a rather detailed bibliography of classical papers including the fundamental work due to by Liouville, Picard, Vessiot, and Kolchin in [17].

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1. Functions Representable in Quadratures, k-Quadratures, and Generalized Quadratures

One can define a class of germs of analytic functions as follows: take a set of *basic germs* and a list of *admissible operations* and define the *class of germs as the set of all germs obtained from*

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the basic germs by applying the admissible operations. This is exactly the way in which the classes of germs of functions representable in quadratures, k-quadratures, and generalized quadratures are defined. Let us recall these definitions. We take the standard space $\mathbb{C}^n$ with coordinates $x_1, \dots, x_n$.

Definition 1. *A function germ φ at a point $a \in \mathbb{C}^n$ is expressed via function germs* (*at* a)

1) f_1 *and* f_2 *by arithmetic operations* if one of the identities $\varphi = f_1 + f_2$, $\varphi = f_1 - f_2$, $\varphi = f_1 f_2$, or $\varphi = f_1/f_2$ holds (in the last case, we assume that the germ f_2 does not vanish identically);

2) $f_0, \dots, f_k$ *by the solution of an algebraic equation* if the germ f_0 is not identically zero and the identity
$$f_0\varphi^k + f_1\varphi^{k-1} + \dots + f_k = 0$$
holds;

3) $f_1, \dots, f_n$ *by integration* if the identity $d\varphi = \alpha$, where $\alpha = f_1\,dx_1 + \dots + f_n\,dx_n$, holds (for given function germs $f_1, \dots, f_n$, there exists a function germ φ with this property if and only if the 1-form α is closed; in this case, φ is uniquely determined up to an additive constant);

4) $f_1, \dots, f_n$ *by exponentiation of the integral* if the identity $d\varphi = \alpha\varphi$, where $\alpha = f_1\,dx_1 + \dots + f_n\,dx_n$, holds (for given $f_1, \dots, f_n$, the germ φ exists if and only if the 1-form α is closed; in this case, φ is uniquely determined up to a multiplicative constant);

5) $f_1, \dots, f_m$ *and a function germ g at a point $b \in \mathbb{C}^m$ by superposition* if the identity $\varphi = g(f_1, \dots, f_m)$ holds.

Definition 2. The *class of function germs in $\mathbb{C}^n$ representable in quadratures* is defined by the following data. The basic germs are the germs of constant functions (at an arbitrary point of $\mathbb{C}^n$). The admissible operations are the arithmetic operations, integration, and exponentiation of the integral. The *classes of function germs in $\mathbb{C}^n$ representable in k-quadratures and generalized quadratures* are defined in the same way. We only supplement the list of admissible operations with the operation of solving algebraic equations of degree $\leqslant k$ and arbitrary degree, respectively.

Remark. In what follows, we do not consider the operation of exponentiation of the integral: it can be replaced by integration followed by superposition with the exponential. However, in the above definitions, this operation is important: these definitions do not use the absolutely nonalgebralizable operation of superposition. They can be restated almost word for word in the case of abstract differential fields equipped with n pairwise commuting differentiations $\partial/\partial x_1, \dots, \partial/\partial x_n$ and play an important role in differential Galois theory. These generalized definitions of quadratures and generalized quadratures are due to Liouville for $n = 1$ and Kolchin for $n > 1$.

We consider the classes of function germs representable in quadratures, k-quadratures, and generalized quadratures in the spaces $\mathbb{C}^n$ of all dimensions $n \geqslant 1$ simultaneously. One can readily show by repeating the argument due to Liouville in the one-dimensional case that *these classes of function germs are closed with respect to superposition and contain all germs of rational functions of several variables and the germs of all main elementary functions.* (The main elementary functions are those studied at school and often included in calculator's keyboards. Here is the list of these functions: the constants, the independent variable x, the roots $\sqrt[n]{x}$, the exponential $\exp x$, the logarithm $\log x$, and the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\arcsin x$, $\arccos x$, and $\arctan x$.)

Remark. We often denote the germ of an analytic function (generally speaking, multivalued) by the same letter as the function itself, specifying neither the point nor the specific germ at that point provided that this information is clear from the context.

2. Formulas, their Multigerms, Analytic Continuations, and Riemann Surfaces

We consider classes of analytic function germs representable by formulas that involve the above-mentioned operations and the operation of solving a system of holonomic equations (see Sec. 3). For each formula of this kind, one can define a multigerm containing the germs of all functions participating in the formula (see Sec. 4).

We can speak about analytic continuation of a multigerm of a formula along a curve (which is essentially the analytic continuation along various curves of all germs occurring in the formula and related by the formula). We can also define the Riemann surface of a formula, introduce the $\mathcal{S}$-property of a formula, and so on. We shall discuss these definitions in detail for the simplest formula $y = f \circ G$. In order to avoid cumbersome details, we do not give similar definitions for more complicated formulas. The main ideas are clear from the example below.

Consider the composition of a germ of an analytic map G of a smooth analytic manifold M into $\mathbb{C}^n$ and an analytic function germ $f \colon \mathbb{C}^n \to \mathbb{C}$. A *multigerm of the formula* $y = f \circ G$ is a triple of germs $\{y_b \,|\, G_b, f_a\}$, where y_b and G_b are germs at the point $b \in M$ of an analytic function y and the analytic map $G \colon (M, b) \to (\mathbb{C}^n, b)$, respectively, f_a is a germ at a point $a \in \mathbb{C}^n$ of the analytic function f, and $y_b = f_a \circ G_b$.

Let $\gamma \colon [0,1] \to M$, $\gamma(0) = b$, be a parametrized curve in M. Consider the parametrized curve $G_{\gamma(t)} \circ \gamma \colon [0,1] \to \mathbb{C}^n$ in $\mathbb{C}^n$ taking each point t, $0 \leqslant t \leqslant 1$, to $G_{\gamma(t)} \circ \gamma(t)$, where $G_{\gamma(t)}$ is the analytic continuation of G_b along the curve $\gamma \colon [0,t] \to M$. The *analytic continuation of a multigerm* $\{y_{b_1} | G_{b_1}, f_{a_1}\}$ *of the formula* $y = f \circ G$ along the curve $\gamma \colon [0,1] \to M$, $\gamma(0) = b_1$, $\gamma(1) = b_2$, is the triple $\{y_{b_2} | G_{b_2}, f_{a_2}\}$, where y_{b_2} and G_{b_2} are the germs obtained by the analytic continuation of y_{b_1} and G_{b_1} along γ and f_{a_2} is the germ obtained by the analytic continuation of f_{a_1} along the curve $G_{\gamma(t)} \circ \gamma \colon [0,1] \to \mathbb{C}^n$. Obviously, these germs are related by the equation $y_{b_2} = f_{a_2} \circ G_{b_2}$.

We say that two multigerms of the formula $y = f \circ G$ are equivalent if one of them can be obtained from the other by analytic continuation along some curve. As a set of points, the *Riemann surface* R of the formula $y = f \circ G$ is the union of all multigerms equivalent to a given multigerm $\{y_b \,|\, G_b, f_a\}$. The *natural projection* $\pi \colon R \to M$ of a Riemann surface of the formula $y = f \circ G$ to the manifold M takes a germ $\{y_{b_1} \,|\, G_{b_1}, f_{a_1}\}$ to the point $b_1 \in M$. A small neighborhood U of a point b_1 in M determines a neighborhood $\widetilde{U}$ of a multigerm $\tilde{b}_1 = \{y_{b_1} \,|\, G_{b_1}, f_{a_1}\}$ in the Riemann surface R provided that U belongs to some coordinate neighborhood of b_1 in M, the Taylor series of $G_{b_1} \colon M \to \mathbb{C}^n$ converges to some map $\widetilde{G} \colon U \to \mathbb{C}^n$ in U, and the image $\widetilde{G}(U) \subset \mathbb{C}^n$ of U under $\widetilde{G}$ belongs to the domain of convergence of the Taylor series of f_{a_1}. If these conditions are satisfied, then the neighborhood $\widetilde{U}$ of the multigerm $\tilde{b}_1$ on the Riemann surface R is defined as the set of multigerms $\{y_{b_2} \,|\, G_{b_2}, f_{a_2}\}$ such that $b_2 \in U$, G_{b_2} is the germ at b_2 of the map $\widetilde{G}$, f_{a_2} is the germ at $a_2 = \widetilde{G}(b_2)$ of the function $\tilde{f}$ that is equal to the sum of the Taylor series of the germ f_{b_1}, and $y_{b_2} = f_{a_2} \circ G_{b_2}$.

Neighborhoods $\widetilde{U}$ of this form determine a topology on the Riemann surface R. In this topology, the natural projection $\pi \colon R \to M$ is a local homeomorphism of R into M. The local homeomorphism π induces the structure of a complex-analytic manifold on R, since this structure by definition exists on M.

The Riemann surface R of the formula $y = f \circ G$ plays exactly the same role as the Riemann surface of an analytic function. Namely, a multigerm $\{y^*_{\tilde{b}} \,|\, G^*_{\tilde{b}}, f_a\}$ of $y^* = f \circ G^*$, where $G^* = \pi^* G$, admits a unique extension to the entire Riemann surface R, and R is the maximal manifold possessing this property (this means that if $\pi_1 \colon R_1 \to M$ is another manifold R_1 equipped with a local homeomorphism π_1 to M possessing the same property, then there exists an embedding $j \colon R_1 \to R$ commuting with the projections, i.e., satisfying $\pi_1 = \pi \circ j$).

A point $b_2 \in M$ is said to be *singular* for a multigerm $\{y_{b_1} \,|\, G_{b_1}, f_{a_1}\}$ of the formula $y = f \circ G$ if there is a curve $\gamma \colon [0,1] \to M$, $\gamma(0) = b_1$, $\gamma(1) = b_2$, such that the multigerm cannot be continued regularly along this curve but admits a regular continuation along the shorter curve $\gamma \colon [0,t] \to M$ for each t in the interval $0 \leqslant t < 1$. Equivalent multigerms have the same sets of singular points. We say that the formula $y = f \circ G$ possesses the $\mathcal{S}$*-property* if the set of singular points of any of its multigerms is thin (see [6]).

It is also convenient to consider other sets, different from the set of singular points, such that a multigerm of a formula is endlessly continuable outside these sets. A thin set A is called a *forbidden set* for a multigerm of a formula if the multigerm admits a regular extension along each curve $\gamma(t)$, $\gamma(0) = a$, intersecting A at most at the initial point.

The following theorem be proved in the same way as the similar theorem for $\mathcal{S}$-functions in the one-dimensional case; see [2, 3].

Theorem (about a forbidden set (cf. [6])). *A thin set is a forbidden set for a multigerm of a formula if and only if it contains the set of its singular points. In particular, a multigerm of a formula admits a forbidden set if and only if the formula possesses the $\mathcal{S}$-property.*

3. The Class of $\mathcal{SC}$-Germs and its Closedness with Respect to Natural Operations

The definition below plays the key role in what follows.

Definition 1. A germ f_a of an analytic function f at a point $a \in \mathbb{C}^n$ is an *$\mathcal{SC}$-germ* if the following condition is satisfied: for each connected complex-analytic manifold M, each analytic map $G\colon M \to \mathbb{C}^n$, and each preimage b of a ($G(b) = a$) there is a thin set $A \subset M$ such that for any curve $\gamma\colon [0, 1] \to M$ issuing from b ($\gamma(0) = b$) and intersecting A at most at the initial point ($\gamma(t) \notin A$ for $t > 0$) the germ f_a admits an analytic continuation along the curve $G \circ \gamma\colon [0, 1] \to \mathbb{C}^n$.

In other words, a germ f_a at $a \in \mathbb{C}^n$ is an $\mathcal{SC}$-germ if for each analytic map $G\colon M \to \mathbb{C}^n$ and each point $b \in M$ such that $G(b) = a$ the multigerm $\{y_b \,|\, G_b, f_a\}$ of the formula $y = f \circ G$ possesses the $\mathcal{S}$-property on M.

Proposition 1. *Each germ of an $\mathcal{S}$-function f of one variable is an $\mathcal{SC}$-germ.*

Proof. If $G\colon M \to \mathbb{C}^1$ is a constant map, then the function $f \circ G$ is constant. If the map is not constant, then it suffices to take the thin set A in the form $A = G^{-1}(O)$, where O is the set of singularities of f.

Proposition 2. *If $f_1, \dots, f_m$ are $\mathcal{SC}$-germs at $a \in \mathbb{C}^n$ and g is an $\mathcal{SC}$-germ at the point $(f_1(a), \dots, f_m(a)) \in \mathbb{C}^m$, then $g(f_1, \dots, f_m)$ is an $\mathcal{SC}$-germ at a.*

Proof. Let $G\colon M \to \mathbb{C}^n$ be an analytic map of a connected complex manifold M into $\mathbb{C}^n$, and let $b \in M$ be a point such that $G(b) = a$. Since the germs $f_1, \dots, f_m$ at $a \in \mathbb{C}^n$ are $\mathcal{SC}$-germs, it follows that for each $i = 1, \dots, m$ there is a thin set $A_i \in M$ forbidden for the germ of the formula $y_i = f_i \circ G$. A forbidden set for the multigerm of the formula $\mathbf{z} = \mathbf{f} \circ G$, where $\mathbf{f} = (f_1, \dots, f_m)$ is a vector function germ at the point $a \in \mathbb{C}^n$, can be taken in the form $A = \bigcup_{i=1}^m A_i$. Let $\pi\colon R \to M$ be the natural projection of the Riemann surface R of the formula $\mathbf{z} = \mathbf{f} \circ G$, and let $\tilde{b}$ the point of R corresponding to the multigerm $\{\mathbf{z}_b \,|\, G_b, \mathbf{f}_a\}$. The germ of g at the point $c = (f_1(a), \dots, f_m(a))$ of $\mathbb{C}^m$ is an $\mathcal{SC}$-germ. Therefore, there is a thin subset $B \subset R$ forbidden for the multigerm $\{w_{\tilde{b}} \,|\, (\mathbf{f} \circ G \circ \pi)_{\tilde{b}}, g_c\}$ of the formula $w = g \circ (\mathbf{f} \circ G \circ \pi)$. Then the thin set $A \cup \pi(B)$ is a forbidden set for the multigerm $\{u_b \,|\, (\mathbf{f} \circ G)_b, g_c\}$ of the formula $u = g \circ (\mathbf{f} \circ G)$.

Definition 2. An operation $\aleph$ that takes an analytic vector function germ $\mathbf{f}$ at a point $a \in \mathbb{C}^n$ to an analytic function germ $\varphi = \aleph(\mathbf{f})$ at the same point a is called an *operation with controllable singularities* if the germ $\pi^*\varphi$, where $\pi\colon R \to M$ is the natural projection of the Riemann surface R of the germ $\mathbf{f}$, has a forbidden closed analytic subset $A \subset R$. (In other words, the germ $\pi^*\varphi$ admits an analytic continuation along each curve $\gamma\colon [0, 1] \to R$, $\gamma(0) = \tilde{a}$, where $\tilde{a} \in R$ is the point corresponding to the germ $\mathbf{f}$, such that γ intersects A at most at the initial point, i.e., $\gamma(t) \notin A$ for $0 < t \leqslant 1$.)

Proposition 3. 1) *For each $i = 1, \dots, n$ the differentiation operation that takes an analytic function germ f at a point $a \in \mathbb{C}^n$ to the germ $\partial f/\partial x_i$ at the same point is an operation with controllable singularities.*

2) *The integration operation that takes a vector function germ $\mathbf{f} = (f_1, \dots, f_n)$ at a point $a \in \mathbb{C}^n$ to an analytic function germ φ at the same point such that the identity $d\varphi = f_1\,dx_1 + \dots + f_n\,dx_n$ holds is an operation with controllable singularities.*

Proof. If the function germ f (or the 1-form germ $\alpha = f_1\,dx_1 + \dots + f_n\,dx_n$) admits an analytic continuation along some curve in $\mathbb{C}^n$, then the partial derivatives of f (respectively, the indefinite integral of α) admit analytic continuation along the same curve. Therefore, the partial derivative (the indefinite integral) has no singularities on the Riemann surface of f (respectively, $\mathbf{f}$) at all.

Proposition 4. *The operation of solving an algebraic equation, which takes a vector function germ* $\mathbf{f} = (f_0, \dots, f_k)$ *at* $a \in \mathbb{C}^n$, *where* $f_0 \not\equiv 0$, *to a germ* y *at the same point* $a \in \mathbb{C}^n$ *such that* $f_0 y^k + \dots + f_k = 0$, *is an operation with controllable singularities.*

Proof. Consider the field K generated over $\mathbb{C}$ by the germs $f_0, \dots, f_k$. By definition, y satisfies the algebraic equation $f_0 y^k + \dots + f_k = 0$ over K; however, this equation may reducible. We take an irreducible equation

$$Q_0 y^l + \dots + Q_l = 0 \qquad (*)$$

over K satisfied by y. We can assume that the coefficients $Q_0, \dots, Q_l$ belong to the ring over $\mathbb{C}$ generated by $f_0, \dots, f_k$. (If this is not the case, we just multiply the coefficients of this equation by their common denominator.) The coefficients $Q_0, \dots, Q_l$ admit a single-valued continuation to the Riemann surface R of $\mathbf{f}$.

Let $D(Q_0, \dots, Q_l)$ be the discriminant of Eq. $(*)$. It does not vanish identically on R, since Eq. $(*)$ is irreducible. Let $\Sigma_D \subset R$ be the analytic set of zeros of $D(Q_0, \dots, Q_l)$ and let $\Sigma_0 \subset R$ be the analytic set of zeros of the coefficient Q_0. We can take $\Sigma = \Sigma_0 \cup \Sigma_D$.

Recall that a system

$$L_j(y) = \sum a^j_{i_1,\dots,i_n} \frac{\partial^{i_1+\dots+i_n} y}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} = 0, \qquad j = 1, \dots, N, \qquad (**)$$

of N linear partial differential equations for an unknown function y whose coefficients $a^j_{i_1,\dots,i_n}$ are analytic functions of n complex variables $x_1, \dots, x_n$ is said to be *holonomic* if the space of germs of its solutions at each point of $\mathbb{C}^n$ is finite-dimensional.

Definition. The *operation of solving a holonomic system of equations* is the operation that takes a vector function germ $\mathbf{a} = (a^j_{i_1,\dots,i_n})$ at a point a whose components are the coefficients of a holonomic system $(**)$ numbered in an arbitrary order to the germ y at a of some solution of this system.

Proposition 5. *The operation of solving a holonomic system of equations is an operation with controllable singularities.*

This statement follows from general theorems about holonomic systems.

Theorem 1. *Let* $\mathbf{f}$ *be an analytic vector function germ at a point* $a \in \mathbb{C}^n$, $\mathbf{f} = (f_1, \dots, f_N)$, *whose components* $f_1, \dots, f_N$ *are* $\mathcal{SC}$*-germs. Suppose that a germ* φ *at* $a \in \mathbb{C}^n$ *is obtained by an application of an operation with controllable singularities to* $\mathbf{f}$. *Then* φ *is an* $\mathcal{SC}$*-germ.*

Proof. Let $\pi\colon R \to \mathbb{C}^n$ be the natural projection of the Riemann surface R of $\mathbf{f}$, and let $\tilde{a} \in R$ be the marked point of R corresponding to this germ, $\pi(\tilde{a}) = a$. By definition, the germ $\pi^*\varphi$ at the point $\tilde{a} \in R$ admits an analytic continuation along any curve in R intersecting some analytic subset $\Sigma \subset R$ at most at the initial point. We take a Whitney stratification of the pair (R, Σ) such that the closure of each stratum is a closed complex-analytic set. We shall be interested only in those strata whose closures contain the marked point $\tilde{a} \in R$. Let $\overline{\Sigma}_1$ be the closure of one of these strata Σ_1, and let Σ_1^0 be the union of all strata except for Σ_1 contained into $\overline{\Sigma}_1$. According to the result in [4], the germ $\pi^*\varphi$ admits an analytic continuation along each curve $\gamma\colon [0,1] \to \Sigma_1$, $\gamma(0) = \hat{a}$, intersecting Σ_1^0 at most at the initial point. Now the theorem follows.

Indeed, let $G\colon M \to \mathbb{C}^n$ be an analytic map of a connected complex manifold M into $\mathbb{C}^n$, and let $b \in M$ be a point such that $G(b) = a$. Since all components of $\mathbf{f}$ are $\mathcal{SC}$-germs, it follows that there is a thin set $A \subset M$ forbidden for the multigerm $\{\mathbf{y}_b \,|\, G_b, \mathbf{f}_a\}$ of the formula $\mathbf{y} = \mathbf{f} \circ G$. Let $\pi_1\colon R_1 \to M$ be the natural projection of the Riemann surface R_1 of this formula, and let $\tilde{b} \in R_1$ be the marked point in M_1 corresponding to this germ. The map germ $\pi^{-1} G \pi_1\colon R_1 \to R$ at the point $\tilde{b} \in R_1$ taking $\tilde{b}$ to $\tilde{a}$ admits an analytic continuation to R_1, which will be denoted by $\widetilde{G}\colon R_1 \to R$.

In the Whitney stratification of the pair (R, Σ), consider all strata whose closures contain the image $\widetilde{G}(R_1)$ of R_1. Let $\overline{\Sigma}_1$ be the minimal of all these closures. Let Σ_1^0 be the union of all strata except for Σ_1 contained in $\overline{\Sigma}_1$. The set $B \subset R_1$, where $B = \widetilde{G}^{-1}(\Sigma_1^0)$, is a proper analytic subset of R_1. According to [4], the germ φ_a admits an analytic continuation along the image $G \circ \pi_1 \circ \gamma\colon [0,1] \to \mathbb{C}^n$ of each curve $\gamma\colon [0,1] \to M_1$, $\gamma(0) = \tilde{b}$, intersecting B at most at the initial point. Therefore, the set $A \cup \pi_1(B) \subset M$ is a forbidden set for the multigerm $\{y_c \,|\, G_c, \varphi_a\}$ of the formula $y = \varphi \circ G$. This completes the proof of the theorem.

Corollary 1. *Suppose that the set of singularities of a multivalued analytic function in $\mathbb{C}^n$ is a closed analytic set. Then each germ of this function is an $\mathcal{SC}$-germ.*

Proof. By definition, a germ of such a function at a point $a \in \mathbb{C}^n$ can be obtained by an application of an operation with controllable singularities to the germ at a of the vector function $\mathbf{x} = (x_1, \dots, x_n)$ whose components are the coordinate functions.

Theorem 2 (about the closedness of the class of $\mathcal{SC}$-germs). *The class of $\mathcal{SC}$-germs contains all germs of $\mathcal{S}$-functions of one variable and all germs of $\mathcal{S}$-functions of several variables with analytic sets of singular points.*

The class of $\mathcal{SC}$-germs in $\mathbb{C}^n$ is closed under the operations of superposition with $\mathcal{SC}$-germs of functions of m variables, differentiation, integration, solving algebraic equations, and solving holonomic systems of linear differential equations.

Proof. The fact that the germs of $\mathcal{S}$-functions mentioned in the statement of Theorem 2 belong to the class of $\mathcal{SC}$-germs is proved in Proposition 1 and Corollary 1. The closedness of the class of $\mathcal{SC}$-germs with respect to superposition is proved in Proposition 2. The closedness of the class of $\mathcal{SC}$-germs with respect to the other operations follows from Theorem 1 by virtue of Propositions 3–5.

Corollary 2. *If a function germ f can be obtained from germs of $\mathcal{S}$-functions with analytic sets of singularities and from germs of $\mathcal{S}$-functions of one variable by integration, differentiation, arithmetic operations, superpositions, and solving algebraic equations and holonomic systems of linear differential equations, then f is an $\mathcal{SC}$-germ. In particular, a germ that is not an $\mathcal{SC}$-germ cannot be expressed in generalized quadratures.*

4. The Class of Multigerms of Formulas Possessing the $\mathcal{SC}$-Property

Suppose that a class $\mathscr{A}$ of analytic function germs is defined by a set $\mathscr{B}$ of basic germs and a list $\mathscr{D}$ of admissible operations. Suppose that $\mathscr{D}$ contains only the operations mentioned in Sec. 1 together with the operation of solving holonomic equations (see Sec. 2). By definition, each germ of the class $\mathscr{A}$ can be expressed via the basic germs by formulas containing admissible operations. Let us inductively define multigerms of formulas of this kind.

A *multigerm* of the simplest *formula* Ω *that states that a germ* φ *is basic*, by definition, *consists of the germs* φ *and* g, *where* g *is an element of* $\mathscr{B}$, *and the equation* $\varphi = g$; i.e., $\Omega = \{\varphi \,|\, g \,|\, \varphi = g\}$. Suppose that a germ φ at a point $a \in \mathbb{C}^n$ can be expressed via function germs $f_1, \dots, f_m$ at a by one of operations 1)–4) in Definition 1 in Sec. 1 or by solving a system of holonomic equations. Let $\Omega_1, \dots, \Omega_m$ be the multigerms of formulas expressing $f_1, \dots, f_m$ via the basic germs. Then the *multigerm of the formula expressing* φ *is the set consisting of* φ, *the multigerms of all formulas* $\Omega_1, \dots, \Omega_m$, *and the identity corresponding to the operation in question.* For example, if φ is obtained from $f_1, \dots, f_m$ by solving an algebraic equation $\varphi^m + f_1\varphi^{m-1} + \dots + f_m = 0$, then $\Omega = \{\varphi \,|\, \Omega_1, \dots, \Omega_m \,|\, \varphi^m + f_1\varphi^{m-1} + \dots + f_m = 0\}$.

If a germ φ at a point $a \in \mathbb{C}^n$ can be expressed via the function germs $f_1, \dots, f_m$ at a and a function germ g at the point $b = (f_1(a), \dots, f_m(a)) \in \mathbb{C}^m$ by superposition, then the multigerm Ω of the formula expressing φ is $\Omega = \{\varphi \,|\, \Omega_1, \dots, \Omega_m, \Omega_0 \,|\, \varphi = g(f_1, \dots, f_m)\}$, where Ω_i, $i = 1, \dots, m$, is the multigerm of the formula for the germ f_i at a and Ω_0 is the multigerm of the formula for the germ g at b. (Because of the presence of superposition, multigerms of formulas can contain function germs in different spaces.)

For a multigerm of a formula Ω representing a germ φ at a point $a \in \mathbb{C}^n$, the notions of *analytic continuation* and *Riemann surface* are introduced in the same way as in Sec. 2 for the formula $y = f \circ G$. Note that the Riemann surface R of a formula Ω lies over the space $\mathbb{C}^n$ (i.e., the natural projection $\pi\colon R \to \mathbb{C}^n$ is well defined), although the formula can contain germs of functions of different numbers of variables.

Repeating the definition in Sec. 3, we say that a multigerm Ω of a formula expressing a function germ φ at a point $a \in \mathbb{C}^n$ via the basic germs possesses the *$\mathcal{SC}$-property* if the following condition is satisfied: for each connected complex-analytic manifold M, each analytic map $G\colon M \to \mathbb{C}^n$, and each preimage b of a ($G(b) = a$) there exists a thin set $A \subset M$ such that for each curve $\gamma\colon [0,1] \to M$ issuing from b ($\gamma(0) = b$) and intersecting A at most at the initial point ($\gamma(t) \notin A$ for $t > 0$) the multigerm Ω can be analytically continued along the curve $G \circ \gamma\colon [0,1] \to \mathbb{C}^n$.

Theorem. 1) *Suppose that a class $\mathscr{A}$ of germs is given by a set $\mathscr{B}$ of basic germs containing only $\mathcal{SC}$-germs and a list of admissible operations $\mathscr{D}$ containing only operations mentioned in Sec.* 1 *and the operation of solving holonomic systems of differential equations. Then for each germ in $\mathscr{A}$ each formula expressing this germ via the basic germs by admissible operations possesses the $\mathcal{SC}$-property.*

2) *If, moreover, the set $\mathscr{B}$ of basic germs is closed with respect to the operation of analytic continuation, then for each germ $\varphi_a \in \mathscr{A}$ at a point $a \in \mathbb{C}^n$ there exists a forbidden set $A \subset \mathbb{C}^n$ such that at each point $b \notin A$ each germ φ_b equivalent to φ_a also belongs to the class $\mathscr{A}$* (*and, in a sense, can be expressed via the basic germs by the same formula as φ*).

Proof. To prove statement 1), it suffices to reproduce the argument in Sec. 3 (replacing function germs by multigerms of formulas). Let us prove statement 2). By 1), the multigerm of the formula Ω that expresses the germ φ_a via the basic germs possesses the $\mathcal{SC}$-property and, in particular, has a thin forbidden set A. Suppose that a germ φ_b can be obtained by analytic continuation of φ_a along a curve γ. We can assume that $\gamma(t)$ is not contained in A for $0 < t \leqslant 1$ (see the theorem about taking a curve off a thin set in [3, 6]). Under the analytic continuation of the multigerm of the formula Ω, we obtain a multigerm of a formula expressing the multigerm φ_b by admissible operations, since the set of basic germs is closed under analytic continuation.

In the assumptions of statement 2) of the theorem, we have the following alternative: For each multivalued analytic function φ, either none of its germs belongs to the class $\mathscr{A}$, or all its germs outside some thin set belong to this class (and can be expressed via the basic germs by "one and the same formula"). In the first case, we say that the function φ *cannot be expressed via the basic germs by admissible operations*, and in the second case we say that *such an expression exists*. In particular, the notions of *representability of a multivalued analytic function in quadratures, k-quadratures, and generalized quadratures* are well defined.

5. Topological Obstructions to the Representability of Functions in Quadratures

We take a nonempty I-almost complete class IM of pairs of groups (see [6]). Let $\widehat{IM}$ be the class of $\mathcal{SC}$-germs of analytic functions (at points of all spaces $\mathbb{C}^n$, $n \geqslant 1$, simultaneously) whose monodromy pair belongs to the class IM.

Main theorem. *The class $\widehat{IM}$ contains the $\mathcal{SC}$-germs of all single-valued functions and is closed with respect to superpositions and differentiations. Moreover,*

1) *if IM contains the additive group $\mathbb{C}$ of complex numbers, then $\widehat{IM}$ is closed with respect to integration*;

2) *if IM contains the permutation group $S(k)$ on k elements, then $\widehat{IM}$ is closed with respect to solving algebraic equations of degree leqk.*

Proof. To prove the theorem, one studies how the monodromy pairs of function germs change under the operations mentioned in the theorem. The argument follows that in the proof of the similar theorem about $\mathcal{S}$-functions of one variable [2, 3]. Therefore, we only mention the differences between the two proofs. First, the theorem about the closedness of the class of $\mathcal{SC}$-germs (see Sec. 3) is more complicated than its one-dimensional counterpart. It is based on the results in [4]. Secondly, the multidimensional composition operation is related to a new operation with pairs of groups, namely, the operation of induced closure. Related issues are described in detail in the paper [6].

A result about quadratures. *The monodromy group of a function germ f representable in quadratures is solvable. Moreover, the monodromy group of any function germ f representable via germs of single-valued $\mathcal{S}$-functions with analytic sets of singular points and germs of single-valued $\mathcal{S}$-functions of one variable by integration, differentiation, and superpositions is solvable.*

A result about k-quadratures. *The monodromy pair of a function germ f representable in k-quadratures is k-solvable* (*see* [6]). *Moreover, the monodromy pair of any function germ f representable in terms of germs of single-valued $\mathcal{S}$-functions with analytic sets of singular points and germs of single-valued $\mathcal{S}$-functions of one variable by integration, differentiation, and solving algebraic equations of degree $\leqslant k$ is k-solvable.*

A result about generalized quadratures. *The monodromy pair of a function germ f representable in generalized quadratures is almost solvable* (*see* [6]). *Moreover, the monodromy pair of any function germ f representable in terms of germs of single-valued $\mathcal{S}$-functions with analytic sets of singular points and germs of single-valued $\mathcal{S}$-functions of one variable by integration, differentiation, and solving algebraic equations is almost solvable.*

Proof. The above results follow from the main theorem, since the germs mentioned there are $\mathcal{SC}$-germs (see Sec. 3), and classes of pairs of groups having solvable, k-solvable, and almost solvable monodromy group, respectively, contain the additive group $\mathbb{C}$. The last two classes of pairs of groups contain also the group $S(k)$ and all groups $S(m)$, $0 < m < \infty$, respectively, (see [6]).

Kolchin extended the Picard–Vessiot theory to the case of holonomic systems of linear partial differential equations. Let us state the corollaries to Kolchin's theory related to the solvability of regular holonomic systems in quadratures. By analogy with the one-dimensional case, a holonomic system is said to be *regular* if all solutions of the system grow not faster than polynomially as the argument approaches either the set of singular points of the system or infinity.

Theorem 1. *A regular holonomic system is solvable in quadratures, k-quadratures, or generalized quadratures if its monodromy group is solvable, k-solvable, or almost solvable, respectively.*

Hence Kolchin's theory implies the following two results.

1) *If the monodromy group of a regular holonomic system is solvable* (*k-solvable, almost solvable*), *then this system is solvable in quadratures* (*k-quadratures, generalized quadratures*).

2) *If the monodromy group of a regular holonomic system is unsolvable* (*not k-solvable, not almost solvable*), *then this system cannot be solved in quadratures* (*k-quadratures, generalized quadratures*).

Our theorem allows one to sharpen the negative result 2).

Theorem 2. *If the monodromy group of a holonomic system is unsolvable* (*not k-solvable, not almost solvable*), *then none of the germs of almost any solution of this system can be expressed via germs of single-valued $\mathcal{S}$-functions with analytic sets of singular points and germs of single-valued $\mathcal{S}$-functions of one variable by superposition, integration and differentiation* (*and solving algebraic equations of degree at most k, and solving algebraic equations*).

Galois theory readily implies the following assertion.

Theorem 3. *Solutions of the algebraic equation $y^m + r_1 y^{m-1} + \cdots + r_m = 0$, where r_i are rational functions of n variables, can be expressed by radicals* (*by radicals and solutions of algebraic equations of degree at most k*) *if and only if the monodromy group of the equation is solvable* (*k-solvable*).

Our theorem allows one to sharpen the negative results in Theorem 3. For example, the following version of the classical Abel theorem, which is stronger than any previous result in this direction, is valid.

Theorem 4 (cf. [7, 8]). *For $n \geqslant 5$, none of the germs of a solution of the general algebraic equation $y^n + x_1 y^{n-1} + \cdots + x_n = 0$, where $x_1, \ldots, x_n$ are independent variables, can be expressed* *via germs of elementary functions, germs of single-valued $\mathcal{S}$-functions with analytic sets of singular points, and germs of single-valued $\mathcal{S}$-functions of one variable by superposition, integration, differentiation, and solving algebraic equation of degree less than n.*

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