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Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory

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To the memory of Vladimir Igorevich Arnold

Abstract

Generalizing the notion of Newton polytope, we define the Newton-Okounkov body, respectively, for semigroups of integral points, graded algebras and linear series on varieties. We prove that any semigroup in the lattice $\mathbb{Z}^n$ is asymptotically approximated by the semigroup of all the points in a sublattice and lying in a convex cone. Applying this we obtain several results. We show that for a large class of graded algebras, the Hilbert functions have polynomial growth and their growth coefficients satisfy a Brunn-Minkowski type inequality. We prove analogues of the Fujita approximation theorem for semigroups of integral points and graded algebras, which imply a generalization of this theorem for arbitrary linear series. Applications to intersection theory include a far-reaching generalization of the Kushnirenko theorem (from Newton polytope theory) and a new version of the Hodge inequality. We also give elementary proofs of the Alexandrov-Fenchel inequality in convex geometry and its analogue in algebraic geometry.

Contents

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Introduction

This paper is dedicated to a generalization of the notion of Newton polytope (of a Laurent polynomial). We introduce the notion of the *Newton-Okounkov body* and prove a series of results about it. It is a completely expanded and revised version of the second part of the preprint [KK]. A revised and extended version of the first part can be found in [KK10a]. Nevertheless, the present paper is totally independent and self-contained. Here we develop a geometric approach to semigroups in $\mathbb{Z}^n$ and apply the results to graded algebras, intersection theory and convex geometry.

A generalization of the notion of Newton polytope was started by the pioneering works of A. Okounkov [Oko96], [Oko03]. A systematic study of the Newton-Okounkov bodies was introduced about the same time in the papers [LM09] and [KK]. Recently the Newton-Okounkov bodies (which Lazarsfeld-Mustata call Okounkov bodies) have been explored and used in the papers of several authors among which we can mention [Yua09], [Nys], [Jow10], [BC11], [And] and [Pet]. There is also the nice recent paper [KLM12] which describes these bodies in some interesting cases. The papers [Kho11] and [Kho12] are also closely related.

First we briefly discuss the results we need from [KK10a], and then we explain the results of the present paper in more details. For the sake of simplicity, throughout the introduction we may use slightly simplified notation compared to the rest of the paper.

The remarkable Bernstein-Kushnirenko theorem computes the number of solutions of a system of equations $P_1 = \cdots = P_n = 0$ in $(\mathbb{C}^*)^n$, where each P_i is a general Laurent polynomial taken from a nonzero finite dimensional subspace

L_i spanned by Laurent monomials. The answer is given in terms of the mixed volumes of the Newton polytopes of the polynomials P_i. (The Kushnirenko theorem deals with the case where the Newton polytopes of all the equations are the same; the Bernstein theorem concerns the general case.)

In [KK10a] a much more general situation is addressed. Instead of $(\mathbb{C}^*)^n$ one takes any irreducible n-dimensional algebraic variety X, and instead of the finite dimensional subspaces L_i spanned by monomials one takes arbitrary nonzero finite dimensional subspaces of rational functions on X. We denote the collection of all the nonzero finite dimensional subspaces of rational functions on X by $\mathbf{K}_{\mathrm{rat}}(X)$. For an n-tuple $L_1, \ldots, L_n \in \mathbf{K}_{\mathrm{rat}}(X)$, we define the intersection index $[L_1, \ldots, L_n]$ as the number of solutions in X of a system of equations $f_1 = \cdots = f_n = 0$, where each f_i is a general element in L_i. In counting the number of solutions one neglects the solutions at which all the functions from a subspace L_i, for some i, are equal to 0, and the solutions at which at least one function in L_i, for some i, has a pole. One shows that this intersection index is well defined and has all the properties of the intersection index of divisors on a complete variety. There is a natural multiplication in the set $\mathbf{K}_{\mathrm{rat}}(X)$. For $L, M \in \mathbf{K}_{\mathrm{rat}}(X)$, the product LM is the span of all the functions fg, where $f \in L$, $g \in M$. With this product, the set $\mathbf{K}_{\mathrm{rat}}(X)$ is a commutative semigroup. Moreover, the intersection index is multi-additive with respect to this product and hence can be extended to the Grothendieck group of $\mathbf{K}_{\mathrm{rat}}(X)$, which we denote by $\mathbf{G}_{\mathrm{rat}}(X)$ (see Section 4.2). If X is a normal projective variety, the group of (Cartier) divisors on X can be embedded as a subgroup in the group $\mathbf{G}_{\mathrm{rat}}(X)$. Under this embedding, the intersection index in the group of divisors coincides with the intersection index in the group $\mathbf{G}_{\mathrm{rat}}(X)$. Thus the intersection index in $\mathbf{G}_{\mathrm{rat}}(X)$ can be considered as a generalization of the classical intersection index of divisors, which is birationally invariant and can be applied to noncomplete varieties also. (As discussed in [KK10a] all the properties of this generalized intersection index can be deduced from the classical intersection theory of divisors.)

Now about the contents of the present article: We begin with proving general (and not very hard) results regarding a large class of semigroups of integral points. The origin of our approach goes back to [Kho92]. Let us start with a class of semigroups with a simple geometric construction: for an integer $0 \leq q < n$, let L be a $(q+1)$-dimensional rational subspace in $\mathbb{R}^n$, C a $(q+1)$-dimensional closed convex cone in L with apex at the origin, and G a subgroup of full rank $q+1$ in $L \cap \mathbb{Z}^n$.[1] The set $\tilde{S} = G \cap C$ is a semigroup with respect to

[1]A linear subspace of $\mathbb{R}^n$ is called *rational* if it can be spanned by rational vectors (equivalently integral vectors). An affine subspace is said to be *rational* if it is a rational subspace after being shifted to pass through the origin.

addition. (After a linear change of coordinates, we can assume that the group G coincides with $L \cap \mathbb{Z}^n$ and hence $\tilde{S} = C \cap \mathbb{Z}^n$.) In addition, assume that the cone C is strongly convex, that is, C does not contain any line. Let $M_0 \subset L$ be a rational q-dimensional linear subspace that intersects C only at the origin. Consider the family of rational q-dimensional affine subspaces in L parallel to M_0 such that they intersect the cone C as well as the lattice G. Let M_k denote the affine subspace in this family that has distance k from the origin. Let us normalize the distance k so that as values it takes all the nonnegative integers. Then this family of parallel affine subspaces can be enumerated as $M_0, M_1, M_2, \ldots$. It is not hard to estimate the number $H_{\tilde{S}}(k)$ of points in the set $\tilde{S}_k = M_k \cap \tilde{S}$. For sufficiently large k, $H_{\tilde{S}}(k)$ is approximately equal to the (normalized in the appropriate way) q-dimensional volume of the convex body $C \cap M_k$. This idea, which goes back to Minkowski, shows that $H_{\tilde{S}}(k)$ grows like $a_q k^q$, where the *q-th growth coefficient* a_q is equal to the (normalized) q-dimensional volume of the convex body $\Delta(\tilde{S}) = C \cap M_1$.[2]

We should point out that the class of semigroups $\tilde{S}$ above already has a rich and interesting geometry even when C is just a simplicial cone. For example, it is related to a higher dimensional generalization of continued fractions originating in the work of V. I. Arnold [Arn98].

Now let us discuss the case of a general semigroup of integral points. Let $S \subset \mathbb{Z}^n$ be a semigroup. Let G be the subgroup of $\mathbb{Z}^n$ generated by S, L the subspace of $\mathbb{R}^n$ spanned by S and C the closure of the convex hull of $S \cup \{0\}$, that is, the smallest closed convex cone (with apex at the origin) containing S. Clearly, G and C are contained in the subspace L. We define the *regularization* $\tilde{S}$ of S to be the semigroup $C \cap G$.[3] From the definition, $\tilde{S}$ contains S. We prove that *the regularization $\tilde{S}$ asymptotically approximates the semigroup S*. We call this the *approximation theorem*. More precisely,

THEOREM 1. *Let $C' \subset C$ be a closed strongly convex cone that intersects the boundary* (*in the topology of the linear space L*) *of the cone C only at the origin. Then there exists a constant $N > 0$* (*depending on C'*) *such that any point in the group G that lies in C' and whose distance from the origin is bigger than N belongs to S.*

[2]For a function f, we define the *q-th growth coefficient* a_q to be the limit $\lim_{k\to\infty} f(k)/k^q$ (whenever this limit exists).

[3]There is also the closely related notion of *saturation* of a semigroup S. The saturation of S is the semigroup of all $x \in \mathbb{Z}^n$ for which $kx \in S$ for some positive integer k. When S is finitely generated, the saturation is the semigroup $C \cap \mathbb{Z}^n$. Note that even when S is finitely generated, the saturation of S can be different from the regularization of S, as the group G can be strictly smaller than $\mathbb{Z}^n$.

Now, in addition, assume that the cone C constructed from S is strongly convex. Let $\dim L = q+1$. Fix a rational q-dimensional subspace $M_0 \subset L$ intersecting C only at the origin and, as above, let M_k, $k \in \mathbb{Z}_{\geq 0}$, be the family of q-dimensional affine subspaces parallel to M_0. That is, each M_k intersects the cone C as well as the group G. Let $H_S(k)$ and $H_{\tilde{S}}(k)$ be the number of points in the levels $S_k = S \cap M_k$ and $\tilde{S}_k = \tilde{S} \cap M_k$ respectively. The function H_S is called the *Hilbert function* of the semigroup S.

Let $\Delta(S) = C \cap M_1$. One observes that it is a convex body.[4] We call it the *Newton-Okounkov body of the semigroup S*. Note that $\dim \Delta(S) = q$. By the above discussion (Minkowski's observation) the Hilbert function $H_{\tilde{S}}(k)$ grows like $a_q k^q$ where a_q is the (normalized) q-dimensional volume of $\Delta(S)$. But, by the approximation theorem, the Hilbert functions $H_S(k)$ and $H_{\tilde{S}}(k)$ have the same asymptotic, as k goes to infinity. It thus follows that the volume of $\Delta(S)$ is responsible for the asymptotic of the Hilbert function H_S as well; i.e.,

THEOREM 2. *The function $H_S(k)$ grows like $a_q k^q$ where q is the dimension of the convex body $\Delta(S)$, and the q-th growth coefficient a_q is equal to the* (*normalized in the appropriate way*) *q-dimensional volume of $\Delta(S)$.*

More generally, we extend the above theorem to the sum of values of a polynomial on the points in the semigroup S (Theorem 1.14).

Next, we describe another result about the asymptotic behavior of a semigroup S. With each nonempty level $S_k = C \cap M_k$ we can associate a subsemigroup $\widehat{S}_k \subset S$ generated by this level. It is nonempty only at the levels kt, $t \in \mathbb{N}$. Consider the Hilbert function $H_{\hat{S}_k}(kt)$ equal to the number of points in the level kt, of the semigroup $\hat{S}_k$. Then if k is sufficiently large, $H_{\hat{S}_k}(kt)$, regarded as a function of $t \in \mathbb{N}$, grows like $a_{q,k} t^q$ where the q-th growth coefficient $a_{q,k}$ depends on k. We show that

THEOREM 3. *The growth coefficient $a_{q,k}$ for the function $H_{\widehat{S}_k}$, considered as a function of k, has the same asymptotic as the Hilbert function $H_S(k)$ of the original semigroup S.*

Now we explain the results in the paper on graded algebras. Let F be a finitely generated field of transcendence degree n over $\mathbf{k} = \mathbb{C}$.[5] Let $F[t]$ be the algebra of polynomials over F. We will be concerned with the graded $\mathbf{k}$-subalgebras of $F[t]$ and their Hilbert functions. In order to apply the results about the semigroups to graded subalgebras of $F[t]$ one needs a valuation v_t on the algebra $F[t]$. Let I be an ordered abelian group. An I-valued

[4]A *convex body* is a convex compact subset of $\mathbb{R}^n$.

[5]For simplicity, here in the introduction we take the ground field $\mathbf{k}$ to be $\mathbb{C}$, although throughout the paper, most of the results are stated for a general algebraically closed field $\mathbf{k}$.

valuation on an algebra A is a map from $A \setminus \{0\}$ to I that respects the algebra operations. (See Section 2.2 for the precise definition.) We construct a $\mathbb{Z}^{n+1}$-valued valuation v_t on $F[t]$ by extending a valuation v on F. We also require v to be *faithful*; i.e., it takes all the values in $\mathbb{Z}^n$. It is well known how to construct many such valuations v. We present main examples in Section 2.2.

The valuation v_t maps the set of nonzero elements of a graded subalgebra $A \subset F[t]$ to a semigroup of integral points in $\mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$. This gives a connection between the graded subalgebras of $F[t]$ and semigroups in $\mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$.

The following types of graded subalgebras in $F[t]$ will play the main roles for us:

– The algebra $A_L = \bigoplus_{k \geq 0} L^k t^k$, where L is a nonzero finite dimensional subspace of F over $\mathbf{k}$. Here $L^0 = \mathbf{k}$, and for $k > 0$, the space L^k is the span of all the products $f_1 \cdots f_k$ with $f_1, \ldots, f_k \in L$. It is a graded algebra generated by $\mathbf{k}$ and finitely many degree 1 elements.
– An *algebra of integral type* is a graded subalgebra A that is a finite module over some algebra A_L, equivalently, a graded subalgebra that is finitely generated and a finite module over the subalgebra generated by A_1.
– An *algebra of almost integral type* is a graded subalgebra A that is contained in an algebra of integral type, equivalently, a graded subalgebra that is contained in some algebra A_L.

Let X be an n-dimensional irreducible variety over $\mathbf{k}$ with $F = \mathbf{k}(X)$ its field of rational functions. To a subspace $L \in \mathbf{K}_{\mathrm{rat}}(X)$ one associates the Kodaira rational map $\Phi_L : X \dashrightarrow \mathbb{P}(L^*)$, where $\mathbb{P}(L^*)$ is the projectivization of the dual space to L. Take a point $x \in X$ such that all the $f \in L$ are defined at x and not all are zero at x. To x there corresponds a functional ξ_x on L given by $\xi_x(f) = f(x)$. The Kodaira map Φ_L sends x to the image of this functional in the projective space $\mathbb{P}(L^*)$. Let $Y_L \subset \mathbb{P}(L^*)$ be the closure of the image of X under the map Φ_L. The algebra A_L, in fact, can be identified with the homogeneous coordinate ring of $Y_L \subset \mathbb{P}(L^*)$. Algebras of integral type are related to the rings of sections of ample line bundles, and algebras of almost integral type to the rings of sections of arbitrary line bundles (see Theorems 3.7 and 3.8).

By the Hilbert-Serre theorem on finitely generated modules over a polynomial ring, it follows that the Hilbert function $H_A(k)$ of an algebra A of almost integral type does not grow faster than k^n. From this one can then show that the cone C associated to the semigroup $S(A) = v_t(A \setminus \{0\})$ is strongly convex. Let $\Delta(A)$ denote the Newton-Okounkov body of the semigroup $S(A)$. We call $\Delta(A)$ the *Newton-Okounkov body of the algebra A*. Applying Theorem 2 above we prove

THEOREM 4. (1) *After appropriate rescaling of the argument k, the Hilbert function $H_A(k)$ grows like $a_q k^q$, where q is an integer between 0 and n.* (2) *Moreover, the degree q is equal to the dimension of $\Delta(A)$, and a_q is the (normalized in the appropriate way) q-dimensional volume of $\Delta(A)$.*

When A is of integral type, again by the Hilbert-Serre theorem, the Hilbert function becomes a polynomial of degree q for large values of k and the number $q!a_q$ is an integer. When A is of almost integral type, the Hilbert function H_A is not in general a polynomial for large k and a_q can be transcendental. It seems to the authors that the result above on the polynomial growth of the Hilbert function of algebras of almost integral type is new.

The Fujita approximation theorem in the theory of divisors states that the so-called *volume of a big divisor* can be approximated by the self-intersection numbers of ample divisors (see [Fuj94], [Laz04, §11.4]). In this paper, we prove an abstract analogue of the Fujita approximation theorem for algebras of almost integral type. This is done by reducing it, via the valuation v_t, to the corresponding result for the semigroups (Theorem 3 above). With each nonempty homogeneous component A_k of the algebra A one associates the graded subalgebra $\widehat{A}_k$ generated by this component. For fixed large enough k, the Hilbert function $H_{\widehat{A}_k}(kt)$ of the algebra $\widehat{A}_k$ grows like $a_{q,k}t^q$.

THEOREM 5. *The q-th growth coefficient $a_{q,k}$ of the Hilbert function $H_{\widehat{A}_k}$, regarded as a function of k, has the same asymptotic as the Hilbert function $H_A(k)$ of the algebra A.*

Hilbert's theorem on the dimension and degree of a projective variety yields an algebro-geometric interpretation of the above results. Consider the algebra A_L associated to a subspace $L \in \mathbf{K}_{\mathrm{rat}}(X)$, and let Y_L denote the closure of the image of the Kodaira map Φ_L. Then by Hilbert's theorem we see that *the dimension of the variety Y_L is equal to the dimension q of the body $\Delta(A_L)$, and the degree of Y_L (in the projective space $\mathbb{P}(L^*)$) is equal to $q!$ times the q-dimensional (normalized in the appropriate way) volume of $\Delta(A_L)$.*

One naturally defines a componentwise product of graded subalgebras (see Definition 2.22). Consider the class of graded algebras of almost integral type such that, for large enough k, all their k-th homogeneous components are nonzero. Let A_1, A_2 be algebras of such kind and put $A_3 = A_1A_2$. It is easy to verify the inclusion

$$\Delta_0(A_1) + \Delta_0(A_2) \subset \Delta_0(A_3),$$

where $\Delta_0(A_i)$ is the Newton-Okounkov body for the algebra A_i projected to $\mathbb{R}^n$ (via the projection on the first factor $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$). Using the previous result on the n-th growth coefficient $a_n(A_i)$ of the Hilbert function of the algebra

A_i and the classical Brunn-Minkowski inequality, we then obtain the following inequality.

THEOREM 6. $a_n^{1/n}(A_1) + a_n^{1/n}(A_2) \leq a_n^{1/n}(A_3)$.

The results about graded subalgebras of polynomials in particular apply to the ring of sections of a divisor. In Section 3.2 we see that the ring of sections of a divisor is an algebra of almost integral type. Applying the above results to this algebra we recover several well-known results regarding the asymptotic theory of divisors and linear series. Moreover, we obtain some new results about the case when the divisor is not a so-called *big divisor*. As a corollary of our Theorem 5 we generalize the interesting Fujita approximation result in [LM09, Th. 3.3]. The result in [LM09] applies to the big divisors (or more generally big graded linear series) on a projective variety. Our generalization holds for any divisor (more generally any graded linear series) on any complete variety (Corollary 3.11). The point is that beside following the ideas in [LM09], we use results that apply to *arbitrary* semigroups of integral points. Another difference between the approach in the present paper and that of [LM09] is that we use abstract valuations on algebras, as opposed to a valuation on the ring of sections of a line bundle and coming from a flag of subvarieties. On the other hand, the use of special valuations with algebro-geometric nature is helpful to get more concrete information about the Newton-Okounkov bodies in special cases.

Let us now return to the subspaces of rational functions on a variety X. Let $L \in \mathbf{K}_{\mathrm{rat}}(X)$. If the Kodaira map $\Phi_L : X \dashrightarrow \mathbb{P}(L^*)$ is a birational isomorphism between X and its image Y_L, then the degree of Y_L is equal to the self-intersection index $[L, \ldots, L]$ of the subspace L. We can then apply the results above to the intersection theory on $\mathbf{K}_{\mathrm{rat}}(X)$. Let us call a subspace L a *big subspace* if for large k, Φ_{L^k} is a birational isomorphism between X and Y_{L^k}.

With a space $L \in \mathbf{K}_{\mathrm{rat}}(X)$, we associate two graded algebras: the algebra A_L and its integral closure $\overline{A_L}$ in the field of fractions of the polynomial algebra $F[t]$. The algebra A_L is easier to define and fits our purposes best when the subspace L is big. On the other hand, the second algebra $\overline{A_L}$ is a little bit more complicated to define (it involves the integral closure) but leads to more convenient results for any $L \in \mathbf{K}_{\mathrm{rat}}(X)$ (Theorem 7 below). The algebraic construction of going from A_L to its integral closure $\overline{A_L}$ can be considered as the analogue of the geometric operation of taking the convex hull of a set of points.

One can then associate to L two convex bodies $\Delta(A_L)$ and $\Delta(\overline{A_L})$. In general, $\Delta(A_L) \subseteq \Delta(\overline{A_L})$, while for a big subspace L we have $\Delta(A_L) = \Delta(\overline{A_L})$.

The following generalization of the Kushnirenko theorem gives a geometric interpretation of the self-intersection index of a subspace L.

THEOREM 7. *For any n-dimensional irreducible algebraic variety X and for any $L \in \mathbf{K}_{\text{rat}}(X)$, we have*

$$[L, \dots, L] = n!\mathrm{Vol}(\Delta(\overline{A_L})). \tag{1}$$

The Kushnirenko theorem is a special case of formula (1). The Newton polytope of the product of two Laurent polynomials is equal to the sum of the corresponding Newton polytopes. This additivity property of the Newton polytope and multi-additivity of the intersection index in $\mathbf{K}_{\text{rat}}(X)$ give the Bernstein theorem as a corollary of the Kushnirenko theorem.

Both of the bodies $\Delta(A_L)$ and $\Delta(\overline{A_L})$ satisfy superadditivity property; that is, *the convex body associated to the product of two subspaces contains the sum of the convex bodies corresponding to the subspaces.*

Formula (1) and the superadditivity of the Newton-Okounkov body $\Delta(\overline{A_L})$, together with the classical Brunn-Minkowski inequality for convex bodies, then imply an analogous inequality for the self-intersection index.

THEOREM 8. *Let $L_1, L_2 \in \mathbf{K}_{\text{rat}}(X)$, and put $L_3 = L_1L_2$. We have*

$$[L_1, \dots, L_1]^{1/n} + [L_2, \dots, L_2]^{1/n} \le [L_3, \dots, L_3]^{1/n}.$$

For an algebraic surface X, i.e., for $n = 2$, this inequality is equivalent to the following analogue of the Hodge inequality (from the Hodge index theorem):

$$[L_1, L_1][L_2, L_2] \le [L_1, L_2]^2. \tag{2}$$

The Hodge index theorem holds for smooth irreducible projective (or compact Kähler) surfaces. Our inequality (2) holds for any irreducible surface, not necessarily smooth or complete, and hence is easier to apply. In contrast to the usual proofs of the Hodge inequality, our proof of inequality (2) is completely elementary.

Using properties of the intersection index in $\mathbf{K}_{\text{rat}}(X)$ and using inequality (2) one can easily prove the algebraic analogue of Alexandrov-Fenchel inequality (see Theorem 4.19). The classical Alexandrov-Fenchel inequality (and its many corollaries) in convex geometry follows easily from its algebraic analogue via the Bernstein-Kushnirenko theorem. These inequalities from intersection theory and their application to deduce the corresponding inequalities in convex geometry, are known (see [Kho88], [Tei79]). A contribution of the present paper is an elementary proof of the key inequality (2) which makes all the chain of arguments involved elementary and more natural.

This paper stems from an attempt to understand the right definition of the Newton polytope for actions of reductive groups on varieties. Unexpectedly, we found that one can define many convex bodies (i.e., Newton-Okounkov bodies) analogous to the Newton polytope and their definition, in general, is not related with the group action. It is unlikely that one can completely understand the shape of a Newton-Okounkov body in the general situation. (See [KLM12] for some results in this direction.)

In [KK12], we return to reductive group actions and consider the Newton-Okounkov bodies associated to invariant subspaces of rational functions on varieties with a reductive group action and constructed via special valuations. A case of special interest is when the variety is a spherical variety. The Newton-Okounkov bodies in this case are convex polytopes, and at least for horospherical spaces these polytopes can be completely described (in a fashion similar to Newton polytopes for toric varieties) and the results of the present paper become more concrete (see [KK11] and also [KK10b]).

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1. **Part I: Semigroups of integral points**

In this part we develop a geometric approach to the semigroups of integral points in $\mathbb{R}^n$. The origin of this approach goes back to the paper [Kho92]. We show that a semigroup of integral points is sufficiently close to the semigroup of all points in a sublattice and lying in a convex cone in $\mathbb{R}^n$. We then introduce the notion of Newton-Okounkov body for a semigroup, which is responsible for the asymptotic of the number of points of the semigroup in a given (co)direction. Finally, we prove a theorem that compares the asymptotic of a semigroup and that of its subsemigroups. We regard this as an abstract version of the Fujita approximation theorem in the theory of divisors. Later in the paper, the results of this part will be applied to graded algebras and to intersection theory.

1.1. *Semigroups of integral points and their regularizations.* Let S be an additive semigroup in the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. In this section we will define the *regularization of* S, a simpler semigroup with more points constructed out of the semigroup S. The main result is the approximation theorem (Theorem 1.6) which states that the regularization of S asymptotically approximates S. Exact definitions and statement will be given below.

To a semigroup S we associate the following basic objects.

Definition 1.1. (1) The *subspace generated by the semigroup* S is the real span $L(S) \subset \mathbb{R}^n$ of the semigroup S. By definition, the linear space $L(S)$ is spanned by integral vectors and thus *the rank of the lattice* $L(S) \cap \mathbb{Z}^n$ *is equal to* $\dim L(S)$.

(2) The *cone generated by the semigroup* S is the closed convex cone $\mathrm{Con}(S) \subset L(S)$ that is the closure of the set of all linear combinations $\sum_i \lambda_i a_i$ for $a_i \in S$ and $\lambda_i \geq 0$.

(3) The *group generated by the semigroup* S is the group $G(S) \subset L(S)$ generated by all the elements in the semigroup S. The group $G(S)$ consists of all the linear combinations $\sum_i k_i a_i$ where $a_i \in S$ and $k_i \in \mathbb{Z}$.

Definition 1.2. The *regularization of a semigroup* S is the semigroup

$$\mathrm{Reg}(S) = G(S) \cap \mathrm{Con}(S).$$

Clearly the semigroup S is contained in its regularization.

The *ridge* of a closed convex cone with apex at the origin is the biggest linear subspace contained in the cone. A cone is called *strictly convex* if its ridge contains only the origin. The *ridge* $L_0(S)$ of a semigroup S is the ridge of the cone $\mathrm{Con}(S)$.

First we consider the case of finitely generated semigroups. The following statement is obvious.

PROPOSITION 1.3. *Let $A \subset \mathbb{Z}^n$ be a finite set generating a semigroup S, and let $\Delta(A)$ be the convex hull of A. Then* (1) *The space $L(S)$ is the smallest subspace containing the polytope $\Delta(A)$.* (2) *The cone $\mathrm{Con}(S)$ is the cone with the apex at the origin over the polytope $\Delta(A)$.* (3) *If the origin O belongs to $\Delta(A)$, then the ridge $L_0(S)$ is the space generated by the smallest face of the polytope $\Delta(A)$ containing O, otherwise $L_0(S) = \{O\}$.*

The following statement is well known in toric geometry (conductor ideal). For the sake of completeness we give a proof here.

THEOREM 1.4. *Let $S \subset \mathbb{Z}^n$ be a finitely generated semigroup. Then there is an element $g_0 \in S$ such that $\mathrm{Reg}(S) + g_0 \subset S$; i.e., for any element $g \in \mathrm{Reg}(S)$, we have $g + g_0 \in S$.*

Proof. Let A be a finite set generating S, and let $P \subset \mathbb{R}^n$ be the set of vectors x that can be represented in the form $x = \sum \lambda_i a_i$, where $0 \leq \lambda_i < 1$ and $a_i \in A$. The set P is bounded, and hence $Q = P \cap G(S)$ is finite. For each $q \in Q$, fix a representation of q in the form $q = \sum k_i(q) a_i$, where $k_i(q) \in \mathbb{Z}$ and $a_i \in A$. Let $g_0 = \sum_{a_i \in A} m_i a_i$, with $m_i = 1 - \min_{q \in Q}\{k_i(q)\}$. Each vector $g \in \mathrm{Reg}(S) \subset \mathrm{Con}(S)$ can be represented in the form $g = \sum \lambda_i a_i$, where$\lambda_i \geq 0$ and $a_i \in A$. Let $g = x + y$, with $x = \sum [\lambda_i] a_i$ and $y = \sum (\lambda_i - [\lambda_i]) a_i$. Clearly $x \in S \cup \{0\}$ and $y \in P$. Let us verify that $g + g_0 \in S$. In fact $g + g_0 = x + (y + g_0)$. Because $g \in \mathrm{Reg}(S)$, we have $y \in Q$. Now $y + g_0 = \sum k_i(y) a_i + \sum m_i a_i = \sum (k_i(y) + m_i) a_i$. By definition, $k_i(y) + m_i \geq 1$ and so $(y + g_0) \in S$. Thus $g + g_0 = x + (y + g_0) \in S$. This finishes the proof. $\square$

Fix any Euclidean metric in $L(S)$.

Corollary 1.5. *Under the assumptions of Theorem* 1.4, *there is a constant* $N > 0$ *such that any point in* $G(S) \cap \mathrm{Con}(S)$ *whose distance to the boundary of* $\mathrm{Con}(S)$ (*as a subset of the topological space* $L(S)$) *is bigger than or equal to* N *is in* S.

Proof. It is enough to take N to be the length of the vector g_0 from Theorem 1.4. □

Now we consider the case where the semigroup S is not necessarily finitely generated. Let $S \subset \mathbb{Z}^n$ be a semigroup, and let Con be a closed strongly convex cone inside $\mathrm{Con}(S)$ that intersects the boundary of $\mathrm{Con}(S)$ (as a subset of $L(S)$) only at the origin. We then have

Theorem 1.6 (Approximation of a semigroup by its regularization). *There is a constant* $N > 0$ (*depending on the choice of* $\mathrm{Con} \subset \mathrm{Con}(S)$) *such that each point in the group* $G(S)$ *that lies in* Con *and whose distance from the origin is bigger than* N *belongs to* S.

We will need a simple lemma.

Lemma 1.7. *Let* $C' \subset C \subset \mathbb{R}^n$ *be closed convex cones with apex at the origin. Moreover, assume that the boundaries of* C *and* C' (*in the topologies of their linear spans*) *intersect only at the origin. Take* $x_0 \in \mathbb{R}^n$. *Then the shifted cone* $x_0 + C$ *contains all the points in* C' *that are far enough from the origin.*

Proof. Consider $B' = \{x \in C' \mid |x| = 1\}$. Then as the boundaries of C and C' intersect only at the origin, B' is a compact subset of C that lies in the interior of C. Thus there exist $R > 0$ such that for any $r > R$, we have $(x_0/r) + B' \subset C$. But since C is a cone, we conclude that $x_0 + rB' \subset C$, which proves the claim. □

Proof of Theorem 1.6. Fix a Euclidean metric in $L(S)$, and equip $L(S)$ with the corresponding topology. We will only deal with $L(S)$, and the ambient space $\mathbb{R}^n$ will not be used in the proof below. Let us enumerate the points in the semigroup S, and let A_i be the collection of the first i elements of S. Denote by S_i the semigroup generated by A_i. There is $i_0 > 0$ such that for $i > i_0$ the set A_i contains a set of generators for the group $G(S)$. If $i > i_0$, then the group $G(S_i)$ generated by the semigroup S_i coincides with $G(S)$ and the space $L(S_i)$ coincides with $L(S)$.

Fix any linear function $\ell : L(S) \to \mathbb{R}$ that is strictly positive on $\mathrm{Con} \setminus \{0\}$. Let $\Delta_\ell(\mathrm{Con}(S))$ and $\Delta_\ell(\mathrm{Con})$ be the closed convex sets obtained by intersecting

$\mathrm{Con}(S)$ and Con by the hyperplane $\ell = 1$ respectively. By definition, $\Delta_\ell(\mathrm{Con})$ is bounded and is strictly inside $\Delta_\ell(\mathrm{Con}(S))$.

The convex sets $\Delta_\ell(\mathrm{Con}(S_i))$, obtained by intersecting $\mathrm{Con}(S_i)$ with the hyperplane $\ell = 1$, form an increasing sequence of closed convex sets in this hyperplane. The closure of the union of the sets $\Delta_\ell(\mathrm{Con}(S_i))$ is, by construction, the convex set $\Delta_\ell(\mathrm{Con}(S))$. So there is an integer i_1 such that for $i > i_1$, the set $\Delta_\ell(\mathrm{Con})$ is strictly inside $\Delta_\ell(\mathrm{Con}(S_i))$. Take any integer j bigger than i_0 and i_1. By Theorem 1.4, for the finitely generated semigroup S_j, there is a vector g_0 such that any point in $G(S) \cap (g_0 + \mathrm{Con}(S_j))$ belongs to S. The convex cone Con is contained in $\mathrm{Con}(S_j)$, and their boundaries intersect only at the origin. Now by Lemma 1.7 the shifted cone $g_0 + \mathrm{Con}(S_j)$ contains all the points of Con that are far enough from the origin. This finishes the proof of the theorem. □

Example 1.8. In $\mathbb{R}^2$ with coordinates x and y, consider the domain U defined by the inequality $y \geq F(x)$ where F is an even function, i.e., $F(x) = F(-x)$, such that $F(0) = 0$ and F is concave and increasing on the ray $x \geq 0$. The set $S = U \cap \mathbb{Z}^2$ is a semigroup. The group $G(S)$ associated to this semigroup is $\mathbb{Z}^2$. The cone $\mathrm{Con}(S)$ is given by the inequality $y \geq c|x|$ where $c = \lim_{x\to\infty} F(x)/x$ and the regularization $\mathrm{Reg}(S)$ is $\mathrm{Con}(S) \cap \mathbb{Z}^2$. In particular, if $F(x) = |x|^\alpha$ where $0 < \alpha < 1$, then $\mathrm{Con}(S)$ is the half-plane $y \geq 0$ and $\mathrm{Reg}(S)$ is the set of integral points in this half-plane. Here the distance from the point $(x, 0) \in \mathrm{Con}(S)$ to the semigroup S goes to infinity as x goes to infinity.

1.2. *Rational half-spaces and admissible pairs*. In this section we discuss admissible pairs consisting of a semigroup and a half-space. We define the Newton-Okounkov body and the Hilbert function for an admissible pair.

Let L be a linear subspace in $\mathbb{R}^n$ and M a half-space in L with boundary ∂M. A half-space $M \subset L$ is *rational* if the subspaces L and ∂M can be spanned by integral vectors, i.e., are rational subspaces.

With a rational half-space $M \subset L$ one can associate $\partial M_{\mathbb{Z}} = \partial M \cap \mathbb{Z}^n$ and $L_{\mathbb{Z}} = L \cap \mathbb{Z}^n$. Take the linear map $\pi_M : L \to \mathbb{R}$ such that $\ker(\pi_M) = \partial M$, $\pi_M(L_{\mathbb{Z}}) = \mathbb{Z}$ and $\pi_M(M \cap \mathbb{Z}^n) = \mathbb{Z}_{\geq 0}$, the set of all nonnegative integers. The linear map π_M induces an isomorphism from $L_{\mathbb{Z}}/\partial M_{\mathbb{Z}}$ to $\mathbb{Z}$.

Now we define an admissible pair of a semigroup and a half-space.

Definition 1.9. A pair (S, M) where S is a semigroup in $\mathbb{Z}^n$ and M a rational half-space in $L(S)$ is called *admissible* if $S \subset M$. We call an admissible pair (S, M) *strongly admissible* if the cone $\mathrm{Con}(S)$ is strictly convex and intersects the space ∂M only at the origin.

With an admissible pair (S, M) we associate the following objects:

– $\mathrm{ind}(S, \partial M)$, the index of the subgroup $G(S) \cap \partial M$ in the group $\partial M_{\mathbb{Z}}$.

– $\mathrm{ind}(S, M)$, the index of the subgroup $\pi_M(G(S))$ in the group $\mathbb{Z}$. (We will usually denote $\mathrm{ind}(S, M)$ by the letter m.)
– S_k, the subset $S \cap \pi_M^{-1}(k)$ of the points of S at level k.

Definition 1.10. The *Newton-Okounkov convex set* $\Delta(S, M)$ of an admissible pair (S, M), is the convex set $\Delta(S, M) = \mathrm{Con}(S) \cap \pi_M^{-1}(m)$ where $m = \mathrm{ind}(S, M)$. It follows from the definition that the convex set $\Delta(S, M)$ is compact (i.e., is a convex body) if and only if the pair (S, M) is strongly admissible. In this case we call $\Delta(S, M)$ the *Newton-Okounkov body* of (S, M).

We now define the Hilbert function of an admissible pair (S, M). It is convenient to define it in the following general situation. Let T be a commutative semigroup and $\pi : T \to \mathbb{Z}_{\geq 0}$ a homomorphism of semigroups.

Definition 1.11. (1) The *Hilbert function* H of (T, π) is the function $H : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, defined by $H(k) = \#\pi^{-1}(k)$. The *support* $\mathrm{supp}(H)$ of the Hilbert function is the set of $k \in \mathbb{Z}_{\geq 0}$ at which $H(k) \neq 0$. (2) The Hilbert function of an admissible pair (S, M) is the Hilbert function of the semigroup S and the homomorphism $\pi_M : S \to \mathbb{Z}_{\geq 0}$. That is, $H(k) = \#S_k$, for any $k \in \mathbb{Z}_{\geq 0}$.

The following is easy to verify.

PROPOSITION 1.12. *Let T and π be as above.* (1) *The support* $\mathrm{supp}(H)$ *of the Hilbert function H is a semigroup in $\mathbb{Z}_{\geq 0}$.* (2) *If the semigroup T has the cancellation property, then the set $H^{-1}(\infty)$ is an ideal in the semigroup* $\mathrm{supp}(H)$, *i.e., if $x \in H^{-1}(\infty)$ and $y \in \mathrm{supp}(H)$, then $x + y \in H^{-1}(\infty)$.* (3) *Let m be the index of the subgroup generated by* $\mathrm{supp}(H) \subset \mathbb{Z}_{\geq 0}$ *in $\mathbb{Z}$. Then* $\mathrm{supp}(H)$ *is contained in $m\mathbb{Z}$ and there is a constant N_1 such that for $mk > N_1$, we have $mk \in$* $\mathrm{supp}(H)$. (4) *If the semigroup T has the cancellation property and $H^{-1}(\infty) \neq \emptyset$, then there is N_2 such that for $mk > N_2$, we have $H(mk) = \infty$.*

Proof. (1) and (2) are obvious. (3) follows from Theorem 1.6 applied to the semigroup $\mathrm{supp}(H) \subset \mathbb{Z}$. Finally (4) follows from (2) and (3). □

In particular, if the Hilbert function of an admissible pair (S, M) is equal to infinity for at least one k, then for sufficiently large values of k, Proposition 1.12(4) describes this function completely. Thus in what follows we will assume that the Hilbert function always takes finite values.

1.3. *Hilbert function and volume of the Newton-Okounkov convex set.* In this section we establish a connection between the asymptotic of the Hilbert function of an admissible pair and its Newton-Okounkov body.

First let us define the notion of integral volume in a rational affine subspace.

Definition 1.13 (Integral volume). Let $L \subset \mathbb{R}^n$ be a rational linear subspace of dimension q. The *integral measure* in L is the translation invariant Euclidean measure in L normalized such that the smallest measure of a q-dimensional parallelepiped with vertices in $L \cap \mathbb{Z}^n$ is equal to 1. Let E be a rational affine subspace of dimension q and parallel to L. The *integral measure* on E is the integral measure on L shifted to E. The measure of a subset $\Delta \subset E$ will be called its *integral volume* and denoted by $\mathrm{Vol}_q(\Delta)$.

For the rest of the paper, unless otherwise stated, Vol_q refers to the integral volume.

Now let (S, M) be an admissible pair with $m = \mathrm{ind}(S, M)$. Put $q = \dim \partial M$. We denote the integral measure in the affine space $\pi_M^{-1}(m)$ by $d\mu$. Take a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree d, and let $f = f^{(0)} + f^{(1)} + \cdots + f^{(d)}$ be its decomposition into homogeneous components.

Theorem 1.14. *Let (S, M) be a strongly admissible pair. Then*

$$\lim_{k\to\infty} \frac{\sum_{x \in S_{mk}} f(x)}{k^{q+d}} = \frac{\int_{\Delta(S,M)} f^{(d)}(x) d\mu}{\mathrm{ind}(S, \partial M)}.$$

Let M be the positive half-space $x_{q+1} \geq 0$ in $\mathbb{R}^{q+1}$. Take a $(q+1)$-dimensional closed strongly convex cone $C \subset M$ that intersects ∂M only at the origin. Let $S = C \cap \mathbb{Z}^{q+1}$ be the semigroup of all the integral points in C. Then (S, M) is a strongly admissible pair. For such kind of a saturated semigroup S, Theorem 1.14 is relatively easy to show. We restate the above theorem in this case as it will be needed in the proof of the general case. Results of such kind have origins in the classical work of Minkowski.

Theorem 1.15. *Let $S = C \cap \mathbb{Z}^{q+1}$ and $\Delta = C \cap \{x_{q+1} = 1\}$. Then*

$$\lim_{k\to\infty} \frac{\sum_{x \in S_k} f(x)}{k^{q+d}} = \int_\Delta f^{(d)}(x) d\mu.$$

Here S_k is the set of all the integral points in $C \cap \{x_{q+1} = k\}$ and $d\mu$ is the Euclidean measure at the hyperplane $x_{q+1} = 1$.

Theorem 1.15 can be easily proved by considering the Riemann sums for the integrals of the homogeneous components of f over Δ.

Proof of Theorem 1.14. The theorem follows from Theorem 1.6 (approximation theorem) and Theorem 1.15. Firstly, one reduces to the case where $L(S) = \mathbb{R}^n$, $q + 1 = n$, M is given by the inequality $x_{q+1} \geq 0$, $G(S) = \mathbb{Z}^{q+1}$, $\mathrm{ind}(S, \partial M) = \mathrm{ind}(S, M) = 1$ and $\Delta(S, M)$ is a q-dimensional convex body in the hyperplane $x_{q+1} = 1$, as follows. Choose a basis $e_1, \ldots, e_q, e_{q+1}, \ldots e_n$ in $\mathbb{R}^n$ such that $e_1, \ldots, e_q$ generate the group $G(S) \cap \partial M$ and the vectors $e_1, \ldots, e_{q+1}$ generate the group $G(S)$ (no condition on the rest of vectors in the basis).

This choice of basis identifies the spaces $L(S)$ and ∂M with $\mathbb{R}^{q+1}$ and $\mathbb{R}^q$ respectively. We will not deal with the vectors outside $\mathbb{R}^{q+1}$, and hence we can assume $q+1=n$. Under such choice of a basis the lattice $L(S)_\mathbb{Z}$ identifies with a lattice $\Lambda \subset \mathbb{R}^{q+1}$ that may contain nonintegral points. Also the lattice $\partial M_\mathbb{Z}$ identifies with a lattice $\Lambda \cap \mathbb{R}^q$. The index of the subgroup $\mathbb{Z}^q$ in the group $\Lambda \cap \mathbb{R}^q$ is equal to $\mathrm{ind}(S, \partial M)$. The coordinate x_{q+1} of the points in the lattice $\Lambda \subset \mathbb{Z}^{q+1}$ is proportional to the number $1/m$ where $m = \mathrm{ind}(S, M)$. The map $\pi_M : L(S)_\mathbb{Z}/M_\mathbb{Z} \to \mathbb{Z}$ then coincides with the restriction of the map mx_{q+1} to the lattice Λ. The semigroup S becomes a subsemigroup in the lattice $\mathbb{Z}^q$ and the level set S_k is equal to $S \cap \{x_{q+1} = k\}$. Also the measure $d\mu$ is given by $d\mu = \rho d\mathbf{x} = \rho dx_1 \wedge \cdots \wedge x_q$, where $\rho = \mathrm{ind}(S, \partial M)$. Thus with the above choice of basis the theorem is reduced to this particular case.

To prove that the limit exists and is equal to $\int_{\Delta(S,M)} f^{(d)}(x)d\mathbf{x}$, it is enough to show that any limit point of the sequence $\{g_k\}$, $g_k = \sum_{x \in S_k} f(x)/k^{q+d}$, lies in arbitrarily small neighborhoods of $\int_{\Delta(S,M)} f^{(d)}(x)d\mathbf{x}$. Take a convex body Δ in the hyperplane $x_{q+1} = 1$ that lies strictly inside the Newton-Okounkov body $\Delta(S, M)$. Consider the convex bodies $k\Delta(S, M)$ and $k\Delta$ in the hyperplane $x_{q+1} = k$. Let S'_k and S''_k be the sets $k\Delta \cap \mathbb{Z}^{q+1}$ and $k\Delta(S, M) \cap \mathbb{Z}^{q+1}$ respectively. By Theorem 1.6, for large values of k, we have $S'_k \subset S_k \subset S''_k$. Also, by Theorem 1.15,

$$\lim_{k\to\infty} \frac{\sum_{x \in S'_k} f(x)}{k^{q+d}} = \int_\Delta f^{(d)}(x)d\mathbf{x},$$
$$\lim_{k\to\infty} \frac{\sum_{x \in S''_k} f(x)}{k^{q+d}} = \int_{\Delta(S,M)} f^{(d)}(x)d\mathbf{x},$$
$$\lim_{k\to\infty} \frac{\#(S''_k \setminus S'_k)}{k^q} = \mathrm{Vol}_q(\Delta(S, M) \setminus \Delta).$$

Since (S, M) is strongly admissible, one can find a constant $N > 0$ such that for any point $x \in \mathrm{Con}(S)$ with $x_{q+1} \geq 1$, we have $|f(x)|/x^d_{q+1} < N$ and $|f^{(d)}(x)|/x^d_{q+1} < N$. This implies that for large values of k, we have

$$\frac{\sum_{x \in (S''_k \setminus S'_k)} |f(x)|}{k^{q+d}} \leq \tilde{N}\mathrm{Vol}_q(\Delta(S, M) \setminus \Delta),$$
$$\int_{\Delta(S,M)\setminus\Delta} |f^{(d)}(x)|d\mathbf{x} < \tilde{N}\mathrm{Vol}_q(\Delta(S, M) \setminus \Delta),$$

where $\tilde{N}$ is any constant bigger than N. Thus

$$\left| \frac{\sum_{x \in S_k} f(x)}{k^{q+d}} - \int_{\Delta(S,M)} f^{(d)}(x)d\mathbf{x} \right| < 2\tilde{N}\mathrm{Vol}_q(\Delta(S, M) \setminus \Delta).$$

For any given $\varepsilon > 0$, we may choose the convex body Δ such that we have $\mathrm{Vol}_q(\Delta(S, M) \setminus \Delta) < \varepsilon/2\tilde{N}$. This shows that for any $\varepsilon > 0$, all the limit points of the sequence $\{g_k\}$ belong to the ε-neighborhood of the number $\int_{\Delta(S,M)} f^{(d)}(x)d\mathbf{x}$, which finishes the proof. $\square$

Corollary 1.16. *With the assumptions as in Theorem* 1.14, *the following holds*:

$$\lim_{k\to\infty} \frac{\#S_{mk}}{k^q} = \frac{\mathrm{Vol}_q(\Delta(S,M))}{\mathrm{ind}(S,\partial M)}.$$

Proof. Apply Theorem 1.14 to the polynomial $f = 1$. $\square$

Definition 1.17. Let (S,M) be an admissible pair with $m = \mathrm{ind}(S,M)$ and $q = \dim \partial M$. We say that S has *bounded growth* with respect to the half-space M if there exists a sequence $k_i \to \infty$ of positive integers such that the sets S_{mk_i} are finite and the sequence of numbers $\#S_{mk_i}/k_i^q$ is bounded.

Theorem 1.18. *Let (S,M) be an admissible pair. The semigroup S has bounded growth with respect to M if and only if the pair (S,M) is strongly admissible. In fact, if (S,M) is strongly admissible, then S has polynomial growth.*

Proof. Let us show that if S has bounded growth, then (S,M) is strongly admissible. Suppose the statement is false. Then the Newton-Okounkov convex set $\Delta(S,M)$ is an unbounded convex q-dimensional set and hence has infinite q-dimensional volume. Assume that P is a constant such that for any i, $\#S_{mk_i}/k_i^q < P$. Choose a convex body Δ strictly inside $\Delta(S,M)$ in such a way that the q-dimensional volume of Δ is bigger than mP. Let Con be the cone over the convex body Δ with the apex at the origin. By Theorem 1.6, for large values of k_i, the set S_{mk_i} contains the set $S'_{mk_i} = \mathrm{Con} \cap G(S) \cap \pi_M^{-1}(mk_i)$. Then by Corollary 1.16,

$$\lim_{k_i\to\infty} \frac{\#S'_{mk_i}}{k_i^q} = \frac{\mathrm{Vol}_q(\Delta)}{\mathrm{ind}(S,\partial M)} > P.$$

The contradiction proves the claim. The other direction, namely if (S,M) is strongly admissible then it has polynomial growth (and hence bounded growth), follows immediately from Corollary 1.16. $\square$

Theorem 1.19. *Let (S,M) be an admissible pair and assume that the sets S_k, $k \in \mathbb{Z}_{\geq 0}$, are finite. Let H be the Hilbert function of (S,M), and put* $\dim \partial M = q$. *Then*

(1) *The limit*

$$\lim_{k\to\infty} \frac{H(mk)}{k^q}$$

exists (*possibly infinite*), *where* $m = \mathrm{ind}(S,M)$.

(2) *This limit is equal to the volume* (*possibly infinite*) *of the Newton-Okounkov convex set $\Delta(S,M)$ divided by the integer* $\mathrm{ind}(S,\partial M)$.

Proof. First assume that $H(mk)/k^q$ does not approach infinity (as k goes to infinity). Then there is a sequence $k_i \to \infty$ with $k_i \in \mathbb{Z}_{\geq 0}$ such that the sets S_{mk_i} are finite and the sequence $\#S_{mk_i}/k_i^q$ is bounded. But this means that the semigroup S has bounded growth with respect to the half-space M. Thus by Theorem 1.18 the cone $\mathrm{Con}(S)$ is strictly convex and intersects ∂M only at the origin. In this case the theorem follows from Corollary 1.16. Now if $\lim_{k\to\infty} H(mk)/k^q = \infty$, then the conditions in Theorem 1.14 cannot be satisfied. Hence the convex set $\Delta(S, M)$ is unbounded and thus has infinite volume. This shows that Theorem 1.19 is true in this case as well. □

Example 1.20. Let S be the semigroup in Example 1.8 where $F(x) = |x|^{1/n}$ for some natural number $n > 1$. Also let M be the half-space $y \geq 0$. Then the pair (S, M) is admissible. Its Newton-Okounkov set $\Delta(S, M)$ is the line $y = 1$, and its Hilbert function is given by $H(k) = 2k^n + 1$. Thus in spite of the fact that the dimension of the Newton-Okounkov convex set $\Delta(S, M)$ is 1, the Hilbert function grows like k^n. This effect is related to the fact that the pair (S, M) is not strongly admissible.

1.4. *Nonnegative semigroups and approximation theorem.* In $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ there is a natural half-space $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$, consisting of the points whose last coordinate is nonnegative. In this section we will deal with semigroups that are contained in this fixed half-space of full dimension. For such semigroups we refine the statements of theorems proved in the previous sections.

We start with definitions. A *nonnegative* semigroup of integral points in $\mathbb{R}^{n+1}$ is a semigroup $S \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ that is not contained in the hyperplane $x_{n+1} = 0$. With a nonnegative semigroup S we can associate an admissible pair $(S, M(S))$ where $M(S) = L(S) \cap (\mathbb{R}^n \times \mathbb{R}_{\geq 0})$. We call a nonnegative semigroup, *strongly nonnegative* if the corresponding admissible pair is strongly admissible. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}$ be the projection on the $(n+1)$-th coordinate. We can associate all the objects defined for an admissible pair to a nonnegative semigroup:

– $\mathrm{Con}(S)$, the cone of the pair $(S, M(S))$;
– $G(S)$, the group generated by the semigroup S;
– H_S, the Hilbert function of the pair $(S, M(S))$;
– $\Delta(S)$, the Newton-Okounkov convex set of the pair $\Delta(S, M(S))$;
– $G_0(S) \subset G(S)$, the subgroup $\pi^{-1}(0) \cap G(S)$;
– S_k, the subset $S \cap \pi^{-1}(k)$ of points in S at level k;
– $\mathrm{ind}(S)$, the index of the subgroup $G_0(S)$ in $\mathbb{Z}^n \times \{0\}$, i.e., $\mathrm{ind}(S, \partial M(S))$;
– $m(S)$, the index $\mathrm{ind}(S, M(S))$.

We now give a more refined version of the approximation theorem for the nonnegative semigroups. We will need the following elementary lemma.

Lemma 1.21. *Let B be a ball of radius $\sqrt{n}$ centered at a point a in the Euclidean space $\mathbb{R}^n$, and let $A = B \cap \mathbb{Z}^n$. Then* (1) *the point a belongs to the convex hull of A.* (2) *The group generated by $x - y$ where $x, y \in A$ is $\mathbb{Z}^n$.*

Proof. For $n = 1$, the statement is obvious. For $n > 1$, the lemma follows from the one-dimensional case and the fact that the ball B contains the product of closed intervals of radius 1 centered at the projections of the point a on the coordinate lines. □

Remark 1.22. K. A. Matveev (an undergraduate student at the University of Toronto) has shown that the smallest radius for which the above proposition holds is $\sqrt{n+3}/2$.

Let us now proceed with the refinement of the approximation theorem for nonnegative semigroups. Let $\dim L(S) = q + 1$ and let $\mathrm{Con} \subset \mathrm{Con}(S)$ be a closed strongly convex $(q+1)$-dimensional cone that intersects the boundary (in the topology of $L(S)$) of $\mathrm{Con}(S)$ only at the origin 0.

Theorem 1.23. *There is a constant $N > 0$* (*depending on the choice of* Con) *such that for any integer $p > N$ that is divisible by $m(S)$, we have* (1) *The convex hull of the set S_p contains the set $\Delta(p) = \mathrm{Con} \cap \pi^{-1}(p)$.* (2) *The group generated by the differences $x - y$, $x, y \in S_p$ is independent of p and coincides with the group $G_0(S)$.*

Proof. By a linear change of variables we can assume that $L(S)$ is $\mathbb{R}^{q+1}$ (whose coordinates we denote by $x_1, \ldots, x_{q+1}$), $M(S)$ is the positive half-space $x_{q+1} \geq 0$, $G(S)$ is $\mathbb{Z}^{q+1}$ and the index $m(S)$ is 1. To make the notation simpler denote $\mathrm{Con}(S)$ by Con_2. Take any $(q+1)$-dimensional closed convex cone Con_1 such that (1) $\mathrm{Con} \subset \mathrm{Con}_1 \subset \mathrm{Con}_2$ and (2) Con_1 intersects the boundaries of the cones Con and Con_2 only at the origin. Consider the sections $\Delta(p) \subset \Delta_1(p) \subset \Delta_2(p)$ of the cones $\mathrm{Con} \subset \mathrm{Con}_1 \subset \mathrm{Con}_2$ (respectively) by the hyperplane $\pi^{-1}(p)$ for some positive integer p. Take $N_1 > 0$ large enough so that for any integer $p > N_1$, a ball of radius $\sqrt{q}$ centered at any point of the convex body $\Delta(p)$ is contained in $\Delta_1(p)$. Then by Lemma 1.21 the convex body $\Delta_1(p)$ is contained in the convex hull of the set of integral points in $\Delta_1(p)$. Also by Theorem 1.6 (approximation theorem) there is $N_2 > 0$ such that for $p > N_2$, the semigroup S contains all the integral points in $\Delta_1(p)$. Thus if $p > N = \max\{N_1, N_2\}$, the convex hull of the set S_p contains the convex body $\Delta(p)$. This proves Part (1). Moreover, since for $p > N$, $\Delta_1(p)$ contains a ball of radius $\sqrt{q}$ and S_p contains all the integral points in this ball, by Lemma 1.21 the differences of the integral points in S_p generates the group $\mathbb{Z}^q = \mathbb{Z}^{q+1} \cap \pi^{-1}(0)$. This proves Part (2). □

1.5. *Hilbert function of a semigroup S and its subsemigroups $\widehat{S}_p$.* Let S be a strongly nonnegative semigroup with the Hilbert function H_S. For an integer p in the support of H_S, let $\widehat{S}_p$ denote the subsemigroup generated by $S_p = S \cap \pi^{-1}(p)$. In this section we compare the asymptotic of H_S with the asymptotic, as $p \to \infty$, of the Hilbert functions of the semigroups $\widehat{S}_p$.

In Sections 2.4 and 3.2 we will apply the results here to prove a generalization of the Fujita approximation theorem (from the theory of divisors). Thus we consider the main result of this section (Theorem 1.27) as an analogue of the Fujita approximation theorem for semigroups.

We will follow the notation introduced in Section 1.4. In particular, $\Delta(S)$ is the Newton-Okounkov body of the semigroup S, $q = \dim \Delta(S)$ its dimension, and $m(S)$ and $\mathrm{ind}(S)$, the indices associated to S. Also $\mathrm{Con}(\widehat{S}_p)$, $G(\widehat{S}_p)$, $H_{\widehat{S}_p}$, $\Delta(\widehat{S}_p)$, $G_0(\widehat{S}_p)$, $\mathrm{ind}(\widehat{S}_p)$, $m(\widehat{S}_p)$, denote the corresponding objects for the semigroup $\widehat{S}_p$. If $S_p = \emptyset$, put $\widehat{S}_p = \Delta(\widehat{S}_p) = \widehat{G}_0(S_p) = \emptyset$ and $H_{\widehat{S}_p} \equiv 0$.

The next proposition is straightforward to verify.

Proposition 1.24. *If the set S_p is not empty, then $m(\widehat{S}_p) = p$, $\Delta(\widehat{S}_p)$ is the convex hull of S_p, the cone $\mathrm{Con}(\widehat{S}_p)$ is the cone over $\Delta(\widehat{S}_p)$, $G(\widehat{S}_p)$ is the group generated by the set S_p and $G_0(\widehat{S}_p) = G(\widehat{S}_p) \cap \pi^{-1}(0)$ is the group generated by the differences $a - b$, $a, b \in S_p$. Also $\mathrm{Con}(\widehat{S}_p) \subset \mathrm{Con}(S)$. If p is not divisible by $m(S)$, then $S_p = \emptyset$.*

Below we deal with functions defined on a nonnegative semigroup $T \subset \mathbb{Z}_{\geq 0}$. A semigroup $T \subset \mathbb{Z}_{\geq 0}$ contains any large enough integer divisible by $m = m(T)$. Let $O_m : \mathbb{Z} \to \mathbb{Z}$ be the scaling map given by $O_m(k) = mk$. For any function $f : T \to \mathbb{R}$ and for sufficiently large p, the pull-back $O_m^*(f)$ is defined by $O_m^*(f)(k) = f(mk)$.

Definition 1.25. Let φ be a function defined on a set of sufficiently large natural numbers. The q-th *growth coefficient* $a_q(\varphi)$ is the value of the limit $\lim_{k\to\infty} \varphi(k)/k^q$ (whenever this limit exists).

The following is a reformulation of Corollary 1.16.

Theorem 1.26. *The q-th growth coefficient of the function $O_m^*(H_S)$, i.e.,*

$$a_q(O_m^*(H_S)) = \lim_{k\to\infty} \frac{H_S(mk)}{k^q},$$

exists and is equal to $\mathrm{Vol}_q(\Delta(S))/\mathrm{ind}(S)$.

For large enough p divisible by $m(S)$, $S_p \neq \emptyset$ and the subsemigroups $\widehat{S}_p$ are defined. The following theorem holds.

Theorem 1.27. *For p sufficiently large and divisible by $m = m(S)$, we have*

(1) $\dim \Delta(\widehat{S}_p) = \dim \Delta(S) = q$.
(2) $\mathrm{ind}(\widehat{S}_p) = \mathrm{ind}(S)$.

(3) *Let the function* φ *be defined by*

$$\varphi(p) = \lim_{t\to\infty} \frac{H_{\widehat{S}_p}(tp)}{t^q}.$$

That is, φ *is the* q*-th growth coefficient of* $O_p^*(H_{\widehat{S}_p})$*. Then the* q*-th growth coefficient of the function* $O_m^*(\varphi)$*, i.e.,*

$$a_q(O_m^*(\varphi)) = \lim_{k\to\infty} \frac{\varphi(mk)}{k^q},$$

exists and is equal to $a_q(O_m^*(H_S)) = \mathrm{Vol}_q(\Delta(S))/\mathrm{ind}(S)$.

Proof. (1) follows from Theorem 1.23(1). (2) follows from Theorem 1.23(2). (3) By Theorem 1.26, applied to the semigroup $\widehat{S}_p$, we have

$$\varphi(p) = \frac{\mathrm{Vol}_q(\Delta(\widehat{S}_p))}{\mathrm{ind}(S)}.$$

Now we use Theorem 1.23 to estimate the quantity $\mathrm{Vol}_q(\Delta(\widehat{S}_p))$. Let Con_0 be a $(q+1)$-dimensional closed cone contained in $\mathrm{Con}(S)$ that intersects its boundary (in the topology of the space $L(S)$) only at the origin. Then for sufficiently large p and divisible by m, the volume $\mathrm{Vol}_q(\Delta(\widehat{S}_p))$ satisfies the inequalities

$$\mathrm{Vol}_q(\mathrm{Con}_0 \cap \pi^{-1}(p)) < \mathrm{Vol}_q(\Delta(\widehat{S}_p)) < \mathrm{Vol}_q(\mathrm{Con}(S) \cap \pi^{-1}(p)).$$

Let $p = km$. Dividing the inequalities above by $k^q\mathrm{ind}(S)$, we obtain

$$\frac{\mathrm{Vol}_q(\mathrm{Con}_0 \cap \pi^{-1}(m))}{\mathrm{ind}(S)} < \frac{\varphi(mk)}{k^q} < \frac{\mathrm{Vol}_q(\mathrm{Con}(S) \cap \pi^{-1}(m))}{\mathrm{ind}(S)} = a_q(O_m^*(H_S)).$$

Since we can choose Con_0 as close as we want to $\mathrm{Con}(S)$, this proves Part (3). □

1.6. *Levelwise addition of semigroups.* In this section we define the levelwise addition of nonnegative semigroups, and we consider a subclass of semigroups for which the n-th growth coefficient of the Hilbert function depends on the semigroup in a polynomial way.

Let $\pi_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the projections on the first and second factors respectively. Define the operation of levelwise addition $\oplus_t$ on the pairs of points with the same last coordinate by

$$(\mathbf{x}_1, h) \oplus_t (\mathbf{x}_2, h) = (\mathbf{x}_1 + \mathbf{x}_2, h),$$

where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$, $h \in \mathbb{R}$. In other words, if $\mathbf{e}$ is the $(n+1)$-th standard basis vector in $\mathbb{R}^n \times \mathbb{R}$ and $\mathbf{y}_1 = (\mathbf{x}_1, h)$, $\mathbf{y}_2 = (\mathbf{x}_2, h) \in \mathbb{R}^n \times \mathbb{R}$, then we have $\mathbf{y}_1 \oplus_t \mathbf{y}_2 = \mathbf{y}_1 + \mathbf{y}_2 - h\mathbf{e}$.

Next we define the operation of levelwise addition between any two subsets. Let X, $Y \subset \mathbb{R}^n \times \mathbb{R}$. Then $X \oplus_t Y = Z$ where Z is the set such that for any $h \in \mathbb{R}$, we have

$$\pi_1(Z \cap \pi^{-1}(h)) = \pi_1(X \cap \pi^{-1}(h)) + \pi_1(Y \cap \pi^{-1}(h)).$$

(By convention the sum of the empty set with any other set is the empty set.)

The following proposition can be easily verified.

Proposition 1.28. *For any two nonnegative semigroups S_1, S_2, the set $S = S_1 \oplus_t S_2$ is a nonnegative semigroup and the following hold*:

(1) $L(S) = L(S_1) \oplus_t L(S_2)$.
(2) $M(S) = M(S_1) \oplus_t M(S_2)$.
(3) $\partial M(S) = \partial M(S_1) \oplus_t \partial M(S_2)$.
(4) $G(S) = G(S_1) \oplus_t G(S_2)$.
(5) $G_0(S) = G_0(S_1) + G_0(S_2)$.

Let us say that a nonnegative semigroup has *almost all levels* if $m(S) = 1$. Also for a nonnegative semigroup S, let $\Delta_0(S)$ denote its Newton-Okounkov convex set shifted to level 0, i.e., $\Delta_0(S) = \pi_1(\Delta(S))$.

Proposition 1.29. *For nonnegative semigroups S_1, S_2 and $S = S_1 \oplus_t S_2$, the following relations hold*:

(1) *The cone* $\mathrm{Con}(S)$ *is the closure of the levelwise addition* $\mathrm{Con}(S_1) \oplus_t \mathrm{Con}(S_2)$ *of the cones* $\mathrm{Con}(S_1)$ *and* $\mathrm{Con}(S_2)$.
(2) *If the semigroups S_1, S_2 have almost all levels, then the Newton-Okounkov set $\Delta(S)$ is the closure of the levelwise addition $\Delta(S_1) \oplus_t \Delta(S_2)$ of the Newton-Okounkov sets $\Delta(S_1)$ and $\Delta(S_2)$.*

(*In fact, since in this case the Newton-Okounkov convex sets live in the level* 1, *we have that $\Delta_0(S)$ is the closure of the Minkowski sum $\Delta_0(S_1) + \Delta_0(S_2)$.*)

Proof. (1) It is easy to see that $S_1 \oplus_t S_2 \subset \mathrm{Con}(S_1) \oplus_t \mathrm{Con}(S_2) \subset \mathrm{Con}(S)$ and the set $\mathrm{Con}(S_1) \oplus_t \mathrm{Con}(S_2)$ is dense in $\mathrm{Con}(S)$. Note that the set $\mathrm{Con}(S_1) \oplus_t \mathrm{Con}(S_2)$ may not be closed (see Example 1.31 below). (2) follows from Part (1). Note that the Minkowski sum of closed convex subsets may not be closed (see Example 1.30). □

Example 1.30. Let Δ_1, Δ_2 be closed convex sets in $\mathbb{R}^2$ with coordinates (x, y) defined by $\{(x, y) \mid xy \geq 1, y > 0\}$ and $\{(x, y) \mid -xy \geq 1, y > 0\}$ respectively. Then the Minkowski sum $\Delta_1 + \Delta_2$ is the open upper half-plane $\{(x, y) \mid y > 0\}$.

Example 1.31. Let Δ_1, Δ_2 be the sets from Example 1.30, and let $\Delta_1 \times \{1\}$, $\Delta_2 \times \{1\}$ in $\mathbb{R}^2 \times \mathbb{R}$ (with coordinates (x, y, z)) be the shifted copies of these sets to the plane $z = 1$. Let Con_1 and Con_2 be the closures of the cones over these sets. Then $\mathrm{Con}_1 \oplus_t \mathrm{Con}_2$ is a nonclosed cone that is the union of the set $\{(x, y, z) \mid 0 \le z, 0 < y\}$ and the line $\{(x, y, z) \mid y = z = 0\}$.

PROPOSITION 1.32. *Let S_1 be a strongly nonnegative semigroup and S_2 a nonnegative semigroup. Let $S = S_1 \oplus S_2$. Then* $\mathrm{Con}(S) = \mathrm{Con}(S_1) \oplus_t \mathrm{Con}(S_2)$ *and* $\mathrm{Reg}(S) = \mathrm{Reg}(S_1) \oplus_t \mathrm{Reg}(S_2)$. *If in addition, S_1, S_2 have almost all levels, then $\Delta(S) = \Delta(S_1) \oplus_t \Delta(S_2)$. (In other words, $\Delta_0(S) = \Delta_0(S_1) + \Delta_0(S_2)$.)*

Proof. Let D be the set of pairs $(\mathbf{y}_1, \mathbf{y}_2) \in \mathrm{Con}(S_1) \times \mathrm{Con}(S_2)$ defined by the condition $\pi(\mathbf{y}_1) = \pi(\mathbf{y}_2)$. Let us show that the map $F : D \to \mathbb{R}^n \times \mathbb{R}$ given by $F(\mathbf{y}_1, \mathbf{y}_2) = \mathbf{y}_1 \oplus_t \mathbf{y}_2$ is proper. Consider a compact set $K \subset \mathbb{R}^n \times \mathbb{R}$. The function x_{n+1} is bounded on the compact set K, i.e., there are constants N_1, N_2 such that $N_1 \le x_{n+1} \le N_2$. The subset K_1 in the cone $\mathrm{Con}(S_1)$ defined by the inequalities $N_1 \le x_{n+1} \le N_2$ is compact. Consider the set K_2 consisting of the points $\mathbf{y}_2 \in \mathrm{Con}(S_2)$ for which there is $\mathbf{y}_1 \in K_1$ such that $\mathbf{y}_1 \oplus_t \mathbf{y}_2 \in K$. The compactness of K and K_1 implies that K_2 is also compact and hence the map F is proper. The properness of F implies that the sum $\mathrm{Con}(S_1) \oplus_t \mathrm{Con}(S_2)$ is closed, which proves $\mathrm{Con}(S) = \mathrm{Con}(S_1) \oplus_t \mathrm{Con}(S_2)$. The other statements follow from this and Proposition 1.29. □

Finally, let us define $\mathcal{S}(n)$ to be the collection of all strongly nonnegative semigroups $S \subset \mathbb{Z}^n \times \mathbb{Z}_{\ge 0}$ with almost all levels, i.e., $m(S) = 1$, and $\mathrm{ind}(S) = 1$. The set $\mathcal{S}(n)$ is a (commutative) semigroup with respect to the levelwise addition.

Let $f : \mathcal{S} \to \mathbb{R}$ be a function defined on a (commutative) semigroup $\mathcal{S}$. We say that f is a *homogeneous polynomial of degree d* if for any choice of the elements $a_1, \dots, a_r \in \mathcal{S}$, the function $F(k_1, \dots, k_r) = f(k_1 a_1 + \cdots + k_r a_r)$, where $k_1, \dots, k_r \in \mathbb{Z}_{\ge 0}$, is a homogeneous polynomial of degree d in the k_i.

THEOREM 1.33. *The function on $\mathcal{S}(n)$ that associates to a semigroup $S \in \mathcal{S}(n)$, the n-th growth coefficient of its Hilbert function, is a homogeneous polynomial of degree n. The value of the polarization of this polynomial on an n-tuple $(S_1, \dots, S_n)$ is equal to the mixed volume of the Newton-Okounkov bodies $\Delta(S_1), \dots, \Delta(S_n)$.*

Proof. According to Theorem 1.26, the n-th growth coefficient of a semigroup $S \in \mathcal{S}(n)$ exists and is equal to the n-dimensional volume of the convex body $\Delta(S)$. By Proposition 1.32, the Newton-Okounkov bodies are added under the levelwise addition of semigroups. Thus the n-th growth coefficient is a homogeneous polynomial of degree n and the value of its polarization is the mixed volume. (See Section 4.1 for a review of the mixed volume.) □

2. **Part II: Valuations and graded algebras**

In this part we consider the graded subalgebras of a polynomial ring in one variable with coefficients in a field F of transcendence degree n over a ground field $\mathbf{k}$. For a large class of graded subalgebras (which are not necessarily finitely generated), we prove the polynomial growth of the Hilbert function, a Brunn-Minkowski inequality for their growth coefficients and an *abstract* version of the Fujita approximation theorem. We obtain all these from the analogous results for the semigroups of integral points. The conversion of problems about algebras into problems about semigroups is made possible via a faithful $\mathbb{Z}^n$-valued valuation on the field F. Two sections of this part are devoted to valuations.

2.1. *Prevaluation on a vector space.* In this section we define a prevaluation and discuss its basic properties. A prevaluation is a weaker version of a valuation that is defined for a vector space (while a valuation is defined for an algebra).

Let V be a vector space over a field $\mathbf{k}$ and I a totally ordered set with respect to some ordering $<$.

Definition 2.1. A *prevaluation on V with values in I* is a function $v : V \setminus \{0\} \to I$ satisfying the following:

(1) For all $f, g \in V$ with $f, g, f+g \neq 0$, we have $v(f+g) \geq \min(v(f), v(g))$.
(2) For all $0 \neq f \in V$ and $0 \neq \lambda \in \mathbf{k}$, $v(\lambda f) = v(f)$.

Example 2.2. Let V be a finite dimensional vector space with a basis $\{e_1, \ldots, e_n\}$ and $I = \{1, \ldots, n\}$, ordered with the usual ordering of numbers. For $f = \sum_i \lambda_i e_i$, define

$$v(f) = \min\{i \mid \lambda_i \neq 0\}.$$

Then v is a prevaluation on V with values in I.

Let $v : V \setminus \{0\} \to I$ be a prevaluation. For $\alpha \in I$, let $V_\alpha = \{f \in V \mid v(f) \geq \alpha$ or $f = 0\}$. It follows immediately from the definition of a prevaluation that V_α is a subspace of V. The *leaf* $\widehat{V}_\alpha$ above the point $\alpha \in I$ is the quotient vector space $V_\alpha / \bigcup_{\alpha < \beta} V_\beta$.

PROPOSITION 2.3. *Let $P \subset V$ be a set of vectors. If the prevaluation v sends different vectors in P to different points in I, then the vectors in P are linearly independent.*

Proof. Let $\sum_{i=1}^s \lambda_i w_i = 0$, $\lambda_i \neq 0$, be a nontrivial linear relation between the vectors in P. Let $\alpha_i = v(w_i)$, $i = 1, \ldots, s$, and without loss of generality assume $\alpha_1 < \cdots < \alpha_s$. We can rewrite the linear relation in the form $\lambda_1 w_1 = -\sum_{i=2}^s \lambda_i w_i$. But this cannot hold since $\lambda_1 w_1 \notin V_{\alpha_2}$ while $\sum_{i>1} \lambda_i w_i \in V_{\alpha_2}$. □

Proposition 2.4. *Let V be finite dimensional. Then for all but a finite set of $\alpha \in I$, the leaf $\widehat{V}_\alpha$ is zero, and we have*

$$\sum_{\alpha \in I} \dim \widehat{V}_\alpha = \dim V.$$

Proof. From Proposition 2.3 it follows that $v(V \setminus \{0\})$ contains no more than $\dim V$ points. Let $v(V \setminus \{0\}) = \{\alpha_1, \ldots, \alpha_s\}$ where $\alpha_1 < \cdots < \alpha_s$. We have a filtration $V = V_{\alpha_1} \supset V_{\alpha_2} \supset \cdots \supset V_{\alpha_s}$ and $\dim V$ is equal to $\sum_{k=1}^{s-1} \dim(V_{\alpha_k}/V_{\alpha_{k+1}}) = \sum_{k=1}^{s-1} \dim \widehat{V}_{\alpha_k}$. ☐

Let $W \subset V$ be a nonzero subspace. Let $J \subset I$ be the image of $W \setminus \{0\}$ under the prevaluation v. The set J inherits a total ordering from I. The following is clear.

Proposition 2.5. *The restriction $v_{|W} : W \setminus \{0\} \to J$ is a prevaluation on W. For each $\alpha \in J$, we have* $\dim \widehat{V}_\alpha \geq \dim \widehat{W}_\alpha$.

A prevaluation v is said to have *one-dimensional leaves* if for every $\alpha \in I$ the dimension of the leaf $\widehat{V}_\alpha$ is at most 1.

Proposition 2.6. *Let V be equipped with an I-valued prevaluation v with one-dimensional leaves. Let $W \subset V$ be a nonzero subspace. Then the number of elements in $v(W \setminus \{0\})$ is equal to* $\dim W$.

Proof. Let $J = v(W \setminus \{0\})$. From Proposition 2.5, v induces a J-valued prevaluation with one-dimensional leaves on the space W. The proposition now follows from Proposition 2.4 applied to W. ☐

Example 2.7 (Schubert cells in Grassmannian). Let $\mathbf{k}$ be an arbitrary field. Let $\mathrm{Gr}(n, k)$ be the Grassmannian of k-dimensional planes in $\mathbf{k}^n$. Take the prevaluation v in Example 2.2 for $V = \mathbf{k}^n$ and the standard basis. Under this prevaluation each k-dimensional subspace $L \subset \mathbf{k}^n$ goes to a subset $J \subset I$ with k elements. The set of all k-dimensional subspaces that are mapped onto J forms the *Schubert cell* X_J in the Grassmannian $\mathrm{Gr}(n, k)$.

In a similar fashion to Example 2.7, the Schubert cells in the variety of complete flags can also be recovered from the above prevaluation v on $\mathbf{k}^n$.

2.2. *Valuations on algebras.* In this section we define a valuation on an algebra and describe its basic properties. It will allow us to reduce the properties of the Hilbert functions of graded algebras to the corresponding properties of semigroups. We will present several examples of valuations.

An *ordered abelian group* is an abelian group Γ equipped with a total order $<$ that respects the group operation; i.e., for $a, b, c \in \Gamma$, $a < b$ implies $a + c < b + c$.

Definition 2.8. Let A be an algebra over a field $\mathbf{k}$ and Γ an ordered abelian group. A prevaluation $v : A \setminus \{0\} \to \Gamma$ is a *valuation* if, in addition, it satisfies the following. For any $f, g \in A$ with $f, g \neq 0$, we have $v(fg) = v(f) + v(g)$. The valuation v is called *faithful* if its image is the whole Γ.

For the rest of the paper by a valuation we will mean a valuation with one-dimensional leaves.

Example 2.9. Let $\mathbf{k}$ be an algebraically closed field and X an irreducible curve over $\mathbf{k}$. As the algebra, take the field of rational functions $\mathbf{k}(X)$ and $\Gamma = \mathbb{Z}$ (with the usual ordering of numbers). Let $a \in X$ be a smooth point. Then the map

$$v(f) = \mathrm{ord}_a(f)$$

defines a faithful $\mathbb{Z}$-valued valuation (with one-dimensional leaves) on $\mathbf{k}(X)$.

The following proposition is straightforward.

Proposition 2.10. *Let A be an algebra over $\mathbf{k}$ together with a Γ-valued valuation $v : A \setminus \{0\} \to \Gamma$. (1) For each subalgebra $B \subset A$, the set $v(B \setminus \{0\})$ is a subsemigroup of Γ. (2) For subspaces $L_1, L_2 \subset A$, put $D_1 = v(L_1 \setminus \{0\})$, $D_2 = v(L_2 \setminus \{0\})$ and $D = v(L_1 L_2 \setminus \{0\})$. We then have $D_1 + D_2 \subset D$.*

In general, it is not true that $D = D_1 + D_2$, as the following example shows.

Example 2.11. Let $\mathbf{k}$ be an arbitrary field, $F = \mathbf{k}(t)$, the field of rational functions in one variable, $\Gamma = \mathbb{Z}$ (with the usual ordering of numbers) and v the valuation which associates to a polynomial its order of vanishing at the origin. Let $L_1 = \mathrm{span}\{1, t\}$ and $L_2 = \mathrm{span}\{t, 1 + t^2\}$. Then $D_1 = D_2 = \{0, 1\}$. The space $L_1 L_2$ is spanned by the polynomials t, $1 + t^2$, t^2, $t + t^3$, and hence by 1, t, t^2 and t^3. We have $D = \{0, 1, 2, 3\}$, while $D_1 + D_2$ is $\{0, 1, 2\}$.

We will work with valuations with values in the group $\mathbb{Z}^n$ (equipped with some total ordering). One can define orderings on $\mathbb{Z}^n$ as follows. Take n independent linear functions $\ell_1, \ldots, \ell_n$ on $\mathbb{R}^n$. For $p, q \in \mathbb{Z}^n$, we say that $p > q$ if for some $1 \leq r < n$, we have $\ell_i(p) = \ell_i(q)$, $i = 1 \ldots, r$, and $\ell_{r+1}(p) > \ell_{r+1}(q)$. This is a total ordering on $\mathbb{Z}^n$ that respects the addition.

We are essentially interested in orderings on $\mathbb{Z}^n$ whose restriction to the semigroup $\mathbb{Z}^n_{\geq 0}$ is a well-ordering. This holds for the above ordering if the following properness condition is satisfied: there is $1 \leq k \leq n$ such that $\ell_1, \ldots, \ell_k$ are nonnegative on $\mathbb{Z}^n_{\geq 0}$ and the map $\ell = (\ell_1, \ldots, \ell_k)$ is a proper map from $\mathbb{Z}^n_{\geq 0}$ to $\mathbb{R}^k$.

Let us now define the *Gröbner valuation* on the algebra $A = \mathbf{k}[[x_1, \ldots, x_n]]$ of formal power series in the variables $x_1, \ldots, x_n$ and with coefficients in a field $\mathbf{k}$. Fix a total ordering on $\mathbb{Z}^n$ (respecting the addition) that restricts to a well-ordering on $\mathbb{Z}^n_{\geq 0}$. For $f \in A$, let $cx_1^{a_1} \cdots x_n^{a_n}$ be the term in f with the smallest exponent $(a_1, \ldots, a_n)$ with respect to this ordering. It exists since $\mathbb{Z}^n_{\geq 0}$ is well ordered. Define $v(f) = (a_1, \ldots, a_n)$. We extend v to the field of fractions K of A by defining $v(f/g) = v(f) - v(g)$ for any $f, g \in A$, $g \neq 0$. One verifies that v is a faithful $\mathbb{Z}^n$-valued valuation (with one-dimensional leaves) on the field K.

The following fact is important for us: *any field of transcendence degree n over a ground filed* $\mathbf{k}$ *has faithful $\mathbb{Z}^n$-valued valuations* (cf. [Jac80, Chap. 9]). For all our purposes such valuations can be realized as restrictions of the above Gröbner valuation to subfields of K.

Example 2.12. Let X be an irreducible n-dimensional variety over an arbitrary field $\mathbf{k}$. First assume that there is a smooth point p in X over $\mathbf{k}$. Let $u_1, \ldots, u_n$ be regular functions at p that form a system of local coordinates at p. Then the field of rational function on X is naturally embedded in the field of fractions K of the algebra of formal power series $A = \mathbf{k}[[u_1, \ldots, u_n]]$. The restriction of the above Gröbner valuation v to $\mathbf{k}(X)$ gives a $\mathbb{Z}^n$-valued valuation. Since $\mathbf{k}(X)$ contains $u_1, \ldots, u_n$ this valuation is faithful. In general, X may have no smooth points over $\mathbf{k}$, but almost every point in X over the algebraic closure $\tilde{\mathbf{k}}$ is a smooth point. Take a smooth point p in X over $\tilde{\mathbf{k}}$. Without loss of generality we can assume that X is an affine variety contained in an affine space $\mathbb{A}^N$. Moreover, we can assume that the projection of X to the coordinate plane with coordinates $x_1, \ldots, x_n$ is nondegenerate at the point p. Then the functions $u_i = x_i - \alpha_i$ where $\alpha_i = x_i(p)$, $i = 1, \ldots, n$, form a local system of coordinates at p. Consider a $\mathbb{Z}^n$-valued faithful valuation v on $\tilde{\mathbf{k}}(X)$ constructed as above from a Gröbner valuation on the algebra $\tilde{\mathbf{k}}[[u_1, \ldots, u_n]]$. For each $\alpha_i = x_i(p)$, choose a polynomial g_i in x_i with coefficients in $\mathbf{k}$ that has α_i as a root. The vectors $v(u_i)$ generate the lattice $\mathbb{Z}^n$ and we have $v(g_i) = \mathrm{ord}_{\alpha_i}(g_i)v(u_i)$. The image of $\mathbf{k}(X) \setminus \{0\}$ under the valuation v contains the vectors $v(g_i)$ and hence is a sublattice Λ of rank n in $\mathbb{Z}^n$. The restriction of the valuation v to $\mathbf{k}(X)$ is a faithful Λ-valued valuation, and Λ is isomorphic to $\mathbb{Z}^n$ as a group.

Let Y be an irreducible variety birationally isomorphic to X. Then a valuation v on the field of rational functions on Y (e.g., the faithful $\mathbb{Z}^n$-valued valuation in Example 2.12) automatically gives a valuation on the field of rational functions on X. The following is an example of this kind of valuation defined in terms of the variety X, although it indeed corresponds to a system of parameters at a smooth point on some birational model Y of X (at least when the ground field $\mathbf{k}$ is algebraically closed and has characteristic 0).

Example 2.13 (Valuation constructed from a Parshin point on X). Let X be an irreducible n-dimensional variety over an algebraically closed field $\mathbf{k}$.

Consider a sequence of maps

$$X_\bullet : \quad \{a\} = X_0 \overset{\pi_0}{\to} X_1 \overset{\pi_1}{\to} \cdots \overset{\pi_{n-1}}{\to} X_n \overset{\pi_n}{\to} X,$$

where each X_i is a normal irreducible variety of dimension i, the map $X_i \overset{\pi_i}{\to} X_{i+1}$ for $i = 0, \dots, n-1$ is the normalization map for the image $\pi_i(X_i) \subset X_{i+1}$ and π_n is the normalization map for X. We call such a sequence $X_\bullet$ a *Parshin point* on the variety X. We say that a collection of rational functions $f_1, \dots, f_n$ is *a system of parameters* about $X_\bullet$ if, for each, i, the function $\pi_i^* \circ \cdots \circ \pi_n^*(f_i)$ vanishes at order 1 along the hypersurface $\pi_{i-1}(X_{i-1})$ in the normal variety X_i. Given a Parshin point $X_\bullet$ together with a system of parameters, one can associate an *iterated Laurent series* to each rational function g on X (see [Par83], [Oko03]). An iterated Laurent series is defined inductively (on the number of parameters). It is a usual Laurent series $\sum_k c_k f_n^k$ with a finite number of terms with negative degrees in the variable f_n, and every coefficient c_k is an iterated Laurent series in the variables $f_1, \dots, f_{n-1}$. An iterated Laurent series has a monomial $cf_1^{k_1} \cdots f_n^{k_n}$ of smallest degree with respect to the lexicographic order in the degrees $(k_1, \dots, k_n)$ (where we order the parameters by $f_n > f_{n-1} > \cdots > f_1$). The map that assigns to a Laurent series its smallest monomial defines a faithful valuation (with one-dimensional leaves) on the field of rational functions on X.

Finally let us give an example of a faithful $\mathbb{Z}^n$-valued valuation on a field of transcendence degree n over a ground field $\mathbf{k}$ that is not finitely generated over $\mathbf{k}$.

Example 2.14. Let $\mathbf{k}$ be an arbitrary field. As above, let K denote the field of fractions of the algebra of formal power series $\mathbf{k}[[x_1, \dots, x_n]]$, and let K' be the subfield consisting of the elements that are algebraic over the field of rational functions $\mathbf{k}(x_1, \dots, x_n)$. This subfield has transcendence degree n over $\mathbf{k}$ but is not finitely generated over $\mathbf{k}$. The restriction of the above Gröbner valuation on K to K' gives a faithful valuation on K'

2.3. *Graded subalgebras of the polynomial ring $F[t]$.* In this section we introduce certain large classes of graded algebras and discuss their basic properties.

Let F be a field containing a field $\mathbf{k}$, which we take as the ground field. A homogeneous element of degree $m \geq 0$ in $F[t]$ is an element $a_m t^m$ where $a_m \in F$. (For any m, the element $0 \in F$ is a homogeneous element of degree m.) Let M be a linear subspace of $F[t]$. For any $k \geq 0$, the collection M_k of homogeneous elements of degree k in M is a linear subspace over $\mathbf{k}$ called the *k-th homogeneous component of M*. Similarly the linear subspace $L_k \subset F$ consisting of those a such that $at^k \in M_k$ is called the *k-th subspace of M*. A linear subspace $M \subset F[t]$ over $\mathbf{k}$ is called a *graded space* if it is the direct sum of its homogeneous components. A subalgebra $A \subset F[t]$ is called *graded* if it is graded as a linear subspace of $F[t]$.

We now define three classes of graded subalgebras that will play main roles later:

(1) To each nonzero finite dimensional linear subspace $L \subset F$ over $\mathbf{k}$ we associate the graded algebra A_L defined as follows. Its zero-th homogeneous component is $\mathbf{k}$ and for each $k > 0$, its k-th subspace is L^k, the subspace spanned by all the products $f_1 \cdots f_k$ with $f_i \in L$. That is,
$$A_L = \bigoplus_{k \geq 0} L^k t^k.$$
The algebra A_L is a graded algebra generated by $\mathbf{k}$ and finitely many elements of degree 1.

(2) We call a graded subalgebra $A \subset F[t]$ an *algebra of integral type* if there is an algebra A_L for some nonzero finite dimensional subspace L over $\mathbf{k}$ such that A is a finitely generated A_L-module (and equivalently, if A is finitely generated over $\mathbf{k}$ and is a finite module over the subalgebra generated by A_1).

(3) We call a graded subalgebra $A \subset F[t]$ an *algebra of almost integral type* if there is an algebra $A' \subset F[t]$ of integral type such that $A \subset A'$ (and equivalently, if $A \subset A_L$ for some finite dimensional subspace $L \subset F$).

As mentioned in the introduction, algebras A_L are related to the homogeneous coordinate rings of projective varieties, algebras of integral type are related to the rings of sections of ample line bundles and algebras of almost integral type to the rings of sections of arbitrary line bundles (see Theorems 3.7 and 3.8).

As the following shows, the class of algebras of almost integral type already contains the class of finitely generated graded subalgebras. However, in general, an algebra of almost integral type may not be finitely generated.

Proposition 2.15. *Let A be a finitely generated graded subalgebra of $F[t]$ (over $\mathbf{k}$). Then A is an algebra of almost integral type.*

Proof. Let $f_1 t^{d_1}, \ldots, f_r t^{d_r}$ be a set of homogeneous generators for A. Let L be the subspace spanned by 1 and all the f_i. Then A is contained in the algebra A_L and hence is of almost integral type. □

The following proposition is easy to show.

Proposition 2.16. *Let $M \subset F[t]$ be a graded subspace, and write $M = \bigoplus_{k \geq 0} L_k t^k$, where L_k is the k-th subspace of M. Then M is a finitely generated module over an algebra A_L if and only if there exists $N > 0$ such that for any $m \geq N$ and $\ell > 0$, we have $L_{m+\ell} = L_m L^\ell$.*

Let A be a graded subalgebra of $F[t]$. Let us denote the integral closure of A in the field of fractions $F(t)$ by $\overline{A}$.[6] It is a standard result that $\overline{A}$ is contained in $F[t]$ and is graded (see [Eis95, Ex. 4.21]).

The following is a corollary of the classical theorem of Noether on finiteness of integral closure.

THEOREM 2.17. *Let F be a finitely generated field over a field* $\mathbf{k}$ *and A a graded subalgebra of $F[t]$.* (1) *If A is of integral type, then $\overline{A}$ is also of integral type.* (2) *If A is of almost integral type, then $\overline{A}$ is also of almost integral type.*

Let $L \subset F$ be a linear subspace over $\mathbf{k}$. Let $P(L) \subset F$ denote the field consisting of all the elements f/g where $f, g \in L^k$ for some $k > 0$ and $g \neq 0$. We call $P(L)$ the *subfield associated to* L and its transcendence degree over $\mathbf{k}$ the *projective transcendence degree* of the subspace L.

Definition 2.18. The *Hilbert function* of a graded subspace $M \subset F[t]$ is the function H_M defined by $H_M(k) = \dim M_k$ (over $\mathbf{k}$), where M_k is the k-th homogeneous component of M. We put $H_M(k) = \infty$ if M_k is infinite dimensional.

The theorem below is a corollary of the so-called Hilbert-Serre theorem on Hilbert function of a finitely generated module over a polynomial ring. Algebraic and combinatorial proofs of this theorem can be found in [ZS60, Chap. VII, §12], [Kho95] and [CK06].

THEOREM 2.19. *Let $L \subset F$ be a finite dimensional subspace over* $\mathbf{k}$, *and let q be its projective transcendence degree. Let $M \subset F[t]$ be a finitely generated graded module over A_L. Then for sufficiently large values of k, the Hilbert function $H_M(k)$ of M coincides with a polynomial $\tilde{H}_M(k)$ of degree q. The leading coefficient of this polynomial multiplied by $q!$ is a positive integer.*

Definition 2.20. The polynomial $\tilde{H}_M$ in Theorem 2.19 is called the *Hilbert polynomial* of the graded module M.

Two numbers appear in Theorem 2.19: the degree q of the Hilbert polynomial and its leading coefficient multiplied by $q!$. When $M = A_L$, both of these numbers have geometric meanings (see Section 3.1).

Assume that a graded algebra $A \subset F[t]$ has at least one nonzero homogeneous component of positive degree. Then the set of k for which the homogeneous component A_k is not 0 forms a nontrivial semigroup $T \subset \mathbb{Z}_{\geq 0}$.

[6]Let $A \subset B$ be commutative rings. An element $f \in B$ is called *integral* over A if f satisfies an equation $f^m + a_1 f^{m-1} + \cdots + a_m = 0$ for $m > 0$ and $a_i \in A$, $i = 1, \ldots, m$. The integral closure $\overline{A}$ of A in B is the collection of all the elements of B that are integral over A. It is a ring containing A.

Let $m(A)$ be the index of the group $G(T)$ in $\mathbb{Z}$. When k is sufficiently large, the homogeneous component A_k is nonzero (and hence $H_A(k)$ is nonzero) if and only if k is divisible by $m(A)$. It follows from definition that when $A \subset F[t]$ is of integral type, we have $m(A) = 1$.

Next we define the componentwise product of graded spaces. Recall that for two subspaces L_1, $L_2 \subset F$, the product L_1L_2 denotes the $\mathbf{k}$-linear subspace spanned by all the products fg, where $f \in L_1$, $g \in L_2$.

Definition 2.21. The collection of all the nonzero finite dimensional subspaces of F is a (commutative) semigroup with respect to this product. We will denote it by $\mathbf{K}(F)$.

Definition 2.22. Let M', M'' be graded spaces with k-th subspaces L'_k, L''_k respectively. The *componentwise product* of spaces M' and M'' is the graded space $M = M'M''$ whose k-th subspace L_k is $L'_kL''_k$.

In particular, the componentwise product can be applied to graded subalgebras of $F[t]$. The following can be easily verified.

PROPOSITION 2.23. (1) *The componentwise product of graded algebras is a graded algebra.* (2) *Let $L', L'' \subset F$ be two nonzero finite dimensional subspaces over* $\mathbf{k}$, *and let $L = L'L''$. Then $A_L = A_{L'}A_{L''}$.* (3) *Let M', M'' be two finitely generated modules over $A_{L'}$ and $A_{L''}$ respectively. Then $M = M'M''$ is a finitely generated module over A_L where $L = L'L''$.* (4) *If A', A'' are algebras of integral type* (*respectively of almost integral type*), *then $A = A'A''$ is also of integral type* (*respectively of almost integral type*).

COROLLARY 2.24. (1) *The map $L \mapsto A_L$ is an isomorphism between the semigroup $\mathbf{K}(F)$ of nonzero finite dimensional subspaces in F and the semigroup of subalgebras A_L with respect to the componentwise product.* (2) *The collection of algebras of almost integral type in $F[t]$ is a semigroup with respect to the componentwise product of subalgebras.*

2.4. *Valuations on graded algebras and semigroups.* In this section, given a valuation on the field F, we construct a valuation on the ring $F[t]$. Using this valuation we will deduce results about the graded algebras of almost integral type from the analogous results for the strongly nonnegative semigroups.

It will be easier to prove the statements in this section if, in addition, F is assumed to be finitely generated over $\mathbf{k}$. One knows that *a field extension $F/\mathbf{k}$ is finitely generated if and only if it is the field of rational functions of an irreducible algebraic variety over* $\mathbf{k}$. Moreover, *the transcendence degree of $F/\mathbf{k}$ is the dimension of the variety X.* The following simple proposition justifies that the general case can be reduced to the case where F is finitely generated over $\mathbf{k}$.

Proposition 2.25. *Let $A_1, \dots, A_k \subset F[t]$ be algebras of almost integral type over $\mathbf{k}$. Then there exists a field $F_0 \subset F$ finitely generated over $\mathbf{k}$with $A_1, \dots, A_k \subset F_0[t]$. If F has finite transcendence degree over $\mathbf{k}$, then the field F_0 can be chosen to have the same transcendence degree.*

Therefore, to prove a statement about a finite collection of subalgebras $A_1, \dots, A_k \subset F[t]$ of almost integral type over $\mathbf{k}$, it is enough to prove it for the case where F is finitely generated over $\mathbf{k}$.

To carry out our constructions we need a faithful valuation $v : F \setminus \{0\} \to \mathbb{Z}^n$ with one-dimensional leaves (where it is understood that $\mathbb{Z}^n$ is equipped with a total order $<$ respecting addition). By the above proposition, we can assume that F is finitely generated over $\mathbf{k}$, i.e., is the field of rational functions on a variety. In this case, one can construct many such valuations (see Examples 2.12 and 2.13).

Using v on F we define a $\mathbb{Z}^n \times \mathbb{Z}$-valued valuation v_t on the algebra $F[t]$. Consider the total ordering $\prec$ on the group $\mathbb{Z}^n \times \mathbb{Z}$ given by the following. Let $(\alpha, n), (\beta, m) \in \mathbb{Z}^n \times \mathbb{Z}$.

(1) If $n > m$, then $(\alpha, n) \prec (\beta, m)$.
(2) If $n = m$ and $\alpha < \beta$, then $(\alpha, n) \prec (\beta, m)$.

Definition 2.26. Define $v_t : F[t] \setminus \{0\} \to \mathbb{Z}^n \times \mathbb{Z}$ as follows. Let $P(t) = a_n t^n + \cdots + a_0$, $a_n \neq 0$, be a polynomial in $F[t]$. Then

$$v_t(P) = (v(a_n), n).$$

It is easy to verify that v_t is a valuation (extending v on F) where $\mathbb{Z}^n \times \mathbb{Z}$ is equipped with the total ordering $\prec$. The extension of v_t to the field of fractions $F(t)$ is faithful and has one-dimensional leaves.

Let $A \subset F[t]$ be a graded subalgebra. Then

$$S(A) = v_t(A \setminus \{0\})$$

is a nonnegative semigroup (see Proposition 2.10). We will use the following notations:

- $\mathrm{Con}(A)$, the cone of the semigroup $S(A)$;
- $G(A)$, the group generated by the semigroup $S(A)$;
- $G_0(A)$, the subgroup $G_0(S(A))$;
- H_A, the Hilbert function of the graded algebra A;
- $\Delta(A)$, the Newton-Okounkov convex set of the semigroup $S(A)$;
- $m(A)$, $\mathrm{ind}(A)$, the indices $m(S(A))$, $\mathrm{ind}(S(A))$ for the semigroup $S(A)$ respectively.

Proposition 2.27. *The Hilbert function $H_{S(A)}$ of the nonnegative semigroup $S(A)$ coincides with the Hilbert function H_A of the algebra A.*

Proof. Follows from Proposition 2.6. □

Now we show that when A is an algebra of almost integral type then the semigroup $S(A)$ is strongly nonnegative.

LEMMA 2.28. *Let A be an algebra of integral type. Assume that the rank of $G(A) \subset \mathbb{Z}^n \times \mathbb{Z}$ is equal to $n+1$. Then the semigroup $S(A)$ is strongly nonnegative.*

Proof. It is obvious that the semigroup $S(A)$ is nonnegative. Let A be a finitely generated module over some algebra A_L. Since $P(L) \subset F$, the projective transcendence degree of L cannot be bigger than n. By Theorem 2.19 (Hilbert-Serre theorem), for large values of k, the Hilbert function of the algebra A is a polynomial in k of degree $\leq n$. Thus by Theorem 1.18, the semigroup $S(A)$ is strongly nonnegative. □

LEMMA 2.29. *Let A be an algebra of integral type. Then there exists an algebra of integral type B containing A such that the group $G(B)$ is the whole $\mathbb{Z}^n \times \mathbb{Z}$.*

Proof. By assumption v is faithful, and thus we can find elements $f_1, \ldots, f_n \in F$ such that $v(f_1), \ldots, v(f_n)$ is the standard basis for $\mathbb{Z}^n$. Consider the space L spanned by 1 and $f_1, \ldots, f_n$ and take its associated graded algebra A_L. The semigroup $S(A_L)$ contains the basis $\{\mathbf{e}_{n+1}, \mathbf{e}_1 + \mathbf{e}_{n+1}, \ldots, \mathbf{e}_n + \mathbf{e}_{n+1}\}$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$ is the standard basis in $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. Hence $G(A_L) = \mathbb{Z}^n \times \mathbb{Z}$. Let $B = A_L A$ be the componentwise product of A and A_L. One sees that $G(B) = \mathbb{Z}^n \times \mathbb{Z}$. Since $1 \in L$, we have $A \subset B$. □

THEOREM 2.30. *Let $A \subset F[t]$ be an algebra of almost integral type. Then $S(A)$ is a strongly nonnegative semigroup, and hence its Newton-Okounkov convex set $\Delta(A)$ is a convex body.*

Proof. By definition, the algebra A is contained in some algebra of integral type, and moreover by Lemma 2.29, it is contained in an algebra B of integral type such that $G(B) = \mathbb{Z}^n \times \mathbb{Z}$. By Lemma 2.28, $S(B)$ is strongly nonnegative. Since $A \subset B$, we have $S(A) \subset S(B)$, which shows that $S(A)$ is also strongly nonnegative. □

Using Theorem 2.30 we can translate the results in Part I about the Hilbert function of strongly nonnegative semigroups to results about the Hilbert function of algebras of almost integral type.

Let A be an algebra of almost integral type with the Newton-Okounkov body $\Delta(A)$. Put $m = m(A)$ and $q = \dim \Delta(A)$. The Hilbert function H_A vanishes at those p not divisible by m. Recall that O_m denotes the scaling map $O_m(k) = mk$. For a function f, $O_m^*(f)$ is the pull-back of f defined by

$O_m^*(f)(k) = f(mk)$ for all k. Also Vol_q denotes the integral volume (Definition 1.13).

Theorem 2.31. *The q-th growth coefficient of the function $O_m^*(H_A)$, i.e.,*

$$a_q(O_m^*(H_A)) = \lim_{k\to\infty} \frac{H_A(mk)}{k^q},$$

exists and is equal to $\mathrm{Vol}_q(\Delta(A))/\mathrm{ind}(A)$.

Proof. This follows from Theorems 2.30 and 1.26. □

The semigroup associated to an algebra of almost integral type has the following superadditivity property with respect to the componentwise product.

Proposition 2.32. *Let A', A'' be algebras of almost integral type and $A = A'A''$. Put $S = v_t(A \setminus \{0\})$, $S' = v_t(A' \setminus \{0\}))$ and $S'' = v_t(A'' \setminus \{0\})$. Then $S' \oplus_t S'' \subset S$. Moreover, if $m(A') = m(A'') = 1$, then*

$$\Delta(A') \oplus_t \Delta(A'') \subset \Delta(A).$$

(*In other words*, $\Delta_0(A') + \Delta_0(A'') \subset \Delta_0(A)$, *where Δ_0 is the Newton-Okounkov body projected to the level* 0 *and* + *is the Minkowski sum.*)

Proof. If L_k, L_k' and L_k'' are the k-th subspaces corresponding to A, A' and A'' respectively, then by definition $L_k = L_k' L_k''$. According to Proposition 2.10, we have $v(L_k' \setminus \{0\}) + v(L_k'' \setminus \{0\}) \subset v(L_k \setminus \{0\})$. The proposition follows from this inclusion. □

Next we prove a Brunn-Minkowski type inequality for the n-th growth coefficients of Hilbert functions of algebras of almost integral type where, as usual, n is the transcendence degree of F over $\mathbf{k}$. This is a generalization of the corresponding inequality for the volume of big divisors (see Corollary 3.11(3) and Remark 3.12).

Theorem 2.33. *Let A_1, A_2 be algebras of almost integral type, and let $A_3 = A_1 A_2$ be their componentwise product. Moreover assume $m(A_1) = m(A_2) = 1$. Then the n-growth coefficients ρ_1, ρ_2 and ρ_3 of the Hilbert functions of the algebras A_1, A_2, A_3 respectively satisfy the following Brunn-Minkowski type inequality*:

$$\rho_1^{1/n} + \rho_2^{1/n} \le \rho_3^{1/n}. \tag{3}$$

Proof. By Proposition 2.32 applied to the valuation v_t, we have $S(A_1) \oplus_t S(A_2) \subset S(A_3)$ and $\Delta(A_1) \oplus_t \Delta(A_2) \subset \Delta(A_3)$. From the classical Brunn-Minkowski inequality (Theorem 4.2) we then get

$$\mathrm{Vol}_n^{1/n}(\Delta(A_1)) + \mathrm{Vol}_n^{1/n}(\Delta(A_2)) \le \mathrm{Vol}_n^{1/n}(\Delta(A_3)). \tag{4}$$

For $i = 1, 2, 3$, we have $\rho_i = \mathrm{Vol}_n(\Delta(A_i))/\mathrm{ind}(A_i)$ (Theorem 2.31). Since $S(A_1) \oplus_t S(A_2) \subset S(A_3)$, the index $\mathrm{ind}(A_3)$ is less than or equal to both of the indices $\mathrm{ind}(A_1)$ and $\mathrm{ind}(A_2)$. From this and (4) the required inequality (3) follows. □

Let $A \subset F[t]$ be an algebra of almost integral type. For an integer p in the support of the Hilbert function H_A, let $\widehat{A}_p$ be the graded subalgebra generated by the p-th homogeneous component A_p of A. We wish to compare the asymptotic of H_A with the asymptotic, as p tends to infinity, of the growth coefficients of the Hilbert functions of the algebras $\widehat{A}_p$.

To every p in the support of H_A we associate two semigroups: (1) the semigroup $\widehat{S}_p(A)$ generated by the set $S_p(A)$ of points at level p in $S(A)$, and (2) the semigroup $S(\widehat{A}_p)$ associated to the algebra $\widehat{A}_p$.

Theorem 2.34. *Let A be an algebra of almost integral type and p any integer in the support of H_A. Then the semigroup $S(\widehat{A}_p)$ satisfies the inclusions*

$$\widehat{S}_p(A) \subset S(\widehat{A}_p) \subset S(A).$$

Proof. The inclusion $S(\widehat{A}_p) \subset S(A)$ follows from $\widehat{A}_p \subset A$. By definition, the set of points at level p in the semigroups $\widehat{S}_p(A)$ and $S(\widehat{A}_p)$ coincide. Denote this set by S_p. For any $k > 0$, the set of points in $\widehat{S}_p(A)$ at the level kp is equal to $k * S_p = S_p + \cdots + S_p$ (k-times), and the set $S_{kp}(\widehat{A}_p)$ is equal to $v_t(A_p^k \setminus \{0\})$. By Proposition 2.32 we get $k * S_p \subset v_t(A_p^k \setminus \{0\})$, i.e., $k * S_p \subset S_{kp}(\widehat{A}_p)$, which implies the required inclusion. □

Let A be an algebra of almost integral type with index $m = m(A)$. Any positive integer p that is sufficiently large and is divisible by m lies in the support of the Hilbert function H_A, and hence the subalgebra $\widehat{A}_p$ is defined. To this subalgebra there corresponds its Hilbert function $H_{\widehat{A}_p}$, the semigroup $S(\widehat{A}_p)$, the Newton-Okounkov body $\Delta(\widehat{A}_p)$ and the indices $m(\widehat{A}_p)$, $\mathrm{ind}(\widehat{A}_p)$.

The following can be considered as a generalization of the Fujita approximation theorem (regarding the volume of big divisors) to algebras of almost integral type.

Theorem 2.35. *For p sufficiently large and divisible by $m = m(A)$, we have*

(1) $\dim \Delta(\widehat{A}_p) = \dim \Delta(A) = q$.
(2) $\mathrm{ind}(\widehat{A}_p) = \mathrm{ind}(A)$.
(3) *Let the function φ be defined by*

$$\varphi(p) = \lim_{t \to \infty} \frac{H_{\widehat{A}_p}(tp)}{t^q}.$$

That is, φ is the q-th growth coefficient of $O^*_p(H_{\widehat{A}_p})$. Then the q-th growth coefficient of the function $O^*_m(\varphi)$, i.e.,

$$a_q(O^*_m(\varphi)) = \lim_{k\to\infty} \frac{\varphi(mk)}{k^q},$$

exists and is equal to $a_q(O^*_m(H_A)) = \mathrm{Vol}_q(\Delta(A))/\mathrm{ind}(A)$.

Proof. This follows from Theorems 2.34 and 2.31. ☐

When A is an algebra of integral type, Theorem 2.35 can be refined using the Hilbert-Serre theorem (Theorem 2.19). Note that when A is of integral type, we have $m(A) = 1$.

THEOREM 2.36. *Let A be an algebra of integral type and, as in Theorem 2.35, let $\varphi(p)$ be the q-th growth coefficient of $O^*_p(H_{\widehat{A}_p})$. Then for sufficiently large p, the number $\varphi(p)/p^q$ is independent of p, and we have*

$$\frac{\varphi(p)}{p^q} = \frac{\mathrm{Vol}_q(\Delta(A))}{\mathrm{ind}(A)} = a_q(H_A).$$

Proof. This follows from Theorem 2.19. Let $\tilde{H}_A(k) = a_q k^q + \cdots + a_0$ be the Hilbert polynomial of the algebra A. From Proposition 2.16 it follows that if p is sufficiently large then, for any $k > 0$, the (kp)-th homogeneous component of the algebra $\widehat{A}_p$ coincides with the (kp)-th homogeneous component of the algebra A, and hence the dimension of the (kp)-th homogeneous component of $\widehat{A}_p$ is equal to $\tilde{H}_A(kp)$. Thus the q-th growth coefficient of the function $O^*_p(H_{\widehat{A}_p})$ equals $p^q a_q$, which proves the theorem. ☐

3. Part III: Projective varieties and algebras of almost integral type

The famous Hilbert theorem computes the dimension and degree of a projective subvariety of projective space by means of the asymptotic growth of its Hilbert function. The constructions and results in the previous parts relate the asymptotic of Hilbert function with the Newton-Okounkov body. In this part we use Hilbert's theorem to give geometric interpretations of these results. We will take the ground field $\mathbf{k}$ to be algebraically closed.

3.1. *Dimension and degree of projective varieties.* In this section we give a geometric interpretation of the dimension and degree of (the closure of) the image of an irreducible variety under a rational map to projective space.

Let X be an irreducible algebraic variety over $\mathbf{k}$ of dimension n, and let $F = \mathbf{k}(X)$ denote the field of rational functions on X. Recall from the introduction that to each nonzero finite dimensional subspace $L \subset F$ we associate the Kodaira rational map $\Phi_L : X \dashrightarrow \mathbb{P}(L^*)$. Let Y_L denote the closure of the image of X under the Kodaira map in $\mathbb{P}(L^*)$ (more precisely, the image of a Zariski open subset of X where Φ_L is defined).

Consider the algebra A_L associated to L. For large values of k, the Hilbert function $H_{A_L}(k)$ coincides with the Hilbert polynomial $\tilde{H}_{A_L}(k) = a_q k^q + \cdots + a_0$. The following is the celebrated Hilbert theorem on the dimension and degree of a projective subvariety, customized for the purposes of this paper (see [Har77, §I.7]).

THEOREM 3.1 (Hilbert). *The degree q of the Hilbert polynomial $\tilde{H}_{A_L}$ is equal to the dimension of the variety Y_L, and its leading coefficient a_q multiplied by $q!$ is equal to the degree of the subvariety Y_L in the projective space $\mathbb{P}(L^*)$.*

Fix a faithful $\mathbb{Z}^n$-valued valuation v on the field of rational functions $F = \mathbf{k}(X)$. The extension v_t of v to $F[t]$ associates to any algebra A of almost integral type the strongly nonnegative semigroup $S(A) \subset \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$. Comparing Theorem 1.19 and Theorem 3.1 (Hilbert's theorem) we obtain that for $A = A_L$, the Newton-Okounkov body $\Delta(A_L)$ is responsible for the dimension and degree of the variety Y_L.

COROLLARY 3.2. *The dimension q of the Newton-Okounkov body $\Delta(A_L)$ is equal to the dimension of the variety Y_L, and its q-dimensional integral volume $\mathrm{Vol}_q(\Delta(A_L))$ multiplied by $q!/\mathrm{ind}(A_L)$ is equal to the degree of Y_L.*

Let A be an algebra of almost integral type in $F[t]$. Let L_k be the k-th subspace of the algebra A. To each nonzero subspace L_k we can associate the following objects: the Kodaira map $\Phi_{L_k} : X \dashrightarrow \mathbb{P}(L_k^*)$, the variety $Y_{L_k} \subset \mathbb{P}(L_k^*)$ (i.e., the closure of the image of Φ_{L_k}) and its dimension and degree. Recall that for a sufficiently large integer p divisible by $m = m(A)$, the space L_p is nonzero. As before, let O_m be the scaling map $O_m(k) = mk$ and O_m^* the pull-back given by $O_m^*(f)(k) = f(mk)$. We have the following

THEOREM 3.3. *If p is sufficiently large and divisible by $m = m(A)$, the dimension of the variety Y_{L_p} is independent of p and is equal to the dimension q of the Newton-Okounkov body $\Delta(A)$. Let* deg *be the function given by* $\deg(p) = \deg Y_{L_p}$. *Then the q-th growth coefficient of the function $O_m^*(\deg)$, i.e.,*

$$a_q(O_m^*(\deg)) = \lim_{k\to\infty} \frac{\deg Y_{L_{mk}}}{k^q},$$

exists and is equal to $q!a_q(O_m^*(H_A))$, which in turn equals $q!\mathrm{Vol}_q(\Delta(A))/\mathrm{ind}(A)$.

Proof. Follows from Theorem 2.35 and Hilbert's theorem. □

When A is an algebra of integral type, Theorem 3.3 can be refined. Note that in this case $m(A) = 1$.

Theorem 3.4. *Let A be an algebra of integral type. Then for sufficiently large p, the dimension q of the variety Y_{L_p}, as well as the degree of the variety Y_{L_p} divided by p^q, are independent of p. Moreover, the dimension of Y_{L_p} is equal to the dimension of the Newton-Okounkov body $\Delta(A)$ and its degree is given by*

$$\deg Y_{L_p} = q!p^q a_q(O_m^*(H_A)) = \frac{q!p^q \mathrm{Vol}_q(\Delta(A))}{\mathrm{ind}(A)}.$$

Proof. Follows from Theorem 2.36, Hilbert's theorem and Theorem 3.3. □

3.2. *Algebras of almost integral type associated to linear series.* In this section we apply the results on graded algebras to the rings of sections of divisors and more generally to linear series. One of the main results is a generalization of the Fujita approximation theorem (for a big divisor) to any divisor on a complete variety.

Let X be an irreducible variety of dimension n over an algebraically closed field $\mathbf{k}$, and let D be a Cartier divisor on X. To D one associates the subspace $\mathcal{L}(D)$ of rational functions defined by

$$\mathcal{L}(D) = \{f \in \mathbf{k}(X) \mid (f) + D \geq 0\}.$$

Let $\mathcal{O}(D)$ denote the line bundle corresponding to D. When X is normal, the elements of the subspace $\mathcal{L}(D)$ are in one-to-one correspondence with the sections in $H^0(X, \mathcal{O}(D))$. One also knows that for a complete variety X the dimension of $H^0(X, \mathcal{O}(D))$ is finite (see [Har77, Chap. II, Th. 5.19]). Thus whenever X is normal and complete, the vector space $\mathcal{L}(D)$ is finite dimensional.

Let D, E be divisors, and let $f \in \mathcal{L}(D)$, $g \in \mathcal{L}(E)$. From the definition it is clear that $fg \in \mathcal{L}(D+E)$. Thus multiplication of functions gives a map

$$\mathcal{L}(D) \times \mathcal{L}(E) \to \mathcal{L}(D+E). \tag{5}$$

In general, *this map is not surjective.*

To a divisor D we associate a graded subalgebra $\mathcal{R}(D)$ of the ring $F[t]$ of polynomials in t with coefficients in the field of rational functions $F = \mathbb{C}(X)$ as follows.

Definition 3.5. Define $\mathcal{R}(D)$ to be the collection of all the polynomials $f(t) = \sum_k f_k t^k$ with $f_k \in \mathcal{L}(kD)$ for all k. In other words,

$$\mathcal{R}(D) = \bigoplus_{k=0} \mathcal{L}(kD)t^k.$$

From (5) it follows that $\mathcal{R}(D)$ is a graded subalgebra of $F[t]$.

Remark 3.6. One can find example of a divisor D such that the algebra $\mathcal{R}(D)$ is not finitely generated. See, for example, [Laz04, §2.3].

THEOREM 3.7. *For any Cartier divisor D on a complete variety X, the algebra $\mathcal{R}(D)$ is of almost integral type.*

To prove Theorem 3.7 we need some preliminaries, which we recall here. When D is a very ample divisor, the following well-known result describes $\mathcal{R}(D)$ (see [Har77, Chap. II, Ex. 5.14]).

THEOREM 3.8. *Let X be a normal projective variety and D a very ample divisor. Let $L = \mathcal{L}(D)$ be the finite dimensional subspace of rational functions associated to D, and let $A_L = \bigoplus_{k\geq 0} L^k t^k$ be the algebra corresponding to L. Then* (1) *$\mathcal{R}(D)$ is the integral closure of A_L in its field of fractions.* (2) *$\mathcal{R}(D)$ is a graded subalgebra of integral type.*

It is well known that very ample divisors generate the group of all Cartier divisors (see [Laz04, Ex. 1.2.10]). More precisely,

THEOREM 3.9. *Let X be a projective variety. Let D be a Cartier divisor and E a very ample divisor. Then for large enough k, the divisor $D + kE$ is very ample. In particular, D can be written as the difference of two very ample divisors $D + kE$ and kE.*

Finally we need the following statement, which is an immediate corollary of Chow's lemma and the normalization theorem.

LEMMA 3.10. *Let X be any complete variety. Then there exists a normal projective variety X' and a morphism $\pi : X' \to X$ that is a birational isomorphism.*

Proof of Theorem 3.7. Let $\pi : X' \to X$ be as in Lemma 3.10. Let $D' = \pi^*(D)$ be the pull-back of D to X'. Then $\pi^*(\mathcal{R}(D)) \subset \mathcal{R}(D')$. Thus replacing X with X', it is enough to prove the statement when X is normal and projective. Now by Theorem 3.9 we can find very ample divisors D_1 and D_2 with $D = D_1 - D_2$. Moreover, we can take D_2 to be an effective divisor. It follows that $\mathcal{R}(D) \subset \mathcal{R}(D_1)$. By Theorem 3.8, $\mathcal{R}(D_1)$ is of integral type and hence $\mathcal{R}(D)$ is of almost integral type. □

We can now apply the results of Section 2.4 to the graded algebra $\mathcal{R}(D)$ and derive some results on the asymptotic of the dimensions of the spaces $\mathcal{L}(kD)$.

Let us recall some terminology from the theory of divisors and linear series (see [Laz04, Chap. 2]). These are special cases of the corresponding general definitions for graded algebras in Part II of this article.

A graded subalgebra W of $\mathcal{R}(D)$ is usually called a *graded linear series for* D. Since $\mathcal{R}(D)$ is of almost integral type, then any graded linear series W for D is also an algebra of almost integral type. Let us write $W = \bigoplus_{k\geq 0} W_k = \bigoplus_{k\geq 0} L_k t^k$, where W_k (respectively L_k) is the k-th homogeneous component (respectively k-th subspace) of the graded subalgebra W.

(1) The n-th growth coefficient of the algebra W multiplied with $n!$ is called the *volume* of the graded linear series W and denoted by $\mathrm{Vol}(W)$. When $W = \mathcal{R}(D)$, the volume of W is denoted by $\mathrm{Vol}(D)$. In the classical case, i.e., when D is ample, $\mathrm{Vol}(D)$ is equal to its self-intersection number. In the case $\mathbf{k} = \mathbb{C}$ and D very ample, $\mathrm{Vol}(D)$ is the (symplectic) volume of the image of X under the embedding of X into projective space induced by D, and hence the term volume.

(2) The index $m = m(W)$ of the algebra W is usually called the *exponent* of the graded linear series W. Recall that for large enough p and divisible by m, the homogeneous component W_p is nonzero.

(3) The growth degree q of the Hilbert function of the algebra W is called the *Kodaira-Iitaka dimension of* W.

The general theorems proved in Section 2.4 about algebras of almost integral type, applied to a graded linear series W, give the following results.

Corollary 3.11. *Let X be a complete irreducible n-dimensional variety. Let D be a Cartier divisor on X and $W \subset \mathcal{R}(D)$ a graded linear series. Then*
(1) *The q-th growth coefficient of the function $O^*_m(H_W)$, i.e.,*

$$a_q(O^*_m(H_W)) = \lim_{k\to\infty} \frac{\dim W_{mk}}{k^q},$$

exists. Fix a faithful $\mathbb{Z}^n$-valued valuation for the field $\mathbf{k}(X)$. Then the Kodaira-Iitaka dimension q of W is equal to the dimension of the convex body $\Delta(W)$ and the growth coefficient $a_q(O^*_m(H_W))$ is equal to $\mathrm{Vol}_q(\Delta(W))$. *Following the notation for the volume of a divisor, we denote the quantity $q!a_q(O^*_m(H_W))$ by* $\mathrm{Vol}_q(W)$.

(2) (*A generalized version of Fujita approximation*). *For p sufficiently large and divisible by m, let $\varphi(p)$ be the p-th growth coefficient of the graded algebra $A_{L_p} = \bigoplus_k L_p^k t^k$ associated to the q-th subspace L_p of W, i.e., $\varphi(p) = \lim_{t\to\infty} \dim L_p^t/t^q$. Then the q-th growth coefficient of the function $O^*_m(\varphi)$, i.e.,*

$$a_q(O^*_m(\varphi)) = \lim_{k\to\infty} \frac{\varphi(mk)}{k^q},$$

exists and is equal to $\mathrm{Vol}_q(\Delta(W))/\mathrm{ind}(W) = \mathrm{Vol}_q(W)/q!\mathrm{ind}(W)$.

(3) (*Brunn-Minkowski for volume of graded linear series*). *Suppose W_1 and W_2 are two graded linear series for divisors D_1 and D_2 respectively. Also*

assume $m(W_1) = m(W_2) = 1$, *then we have*

$$\mathrm{Vol}^{1/n}(W_1) + \mathrm{Vol}^{1/n}(W_2) \leq \mathrm{Vol}^{1/n}(W_1 W_2),$$

where $W_1 W_2$ *denotes the componentwise product of* W_1 *and* W_2. *In particular, if* $W_1 = \mathcal{R}(D_1)$ *and* $W_2 = \mathcal{R}(D_2)$, *then* $W_1 W_2 \subset \mathcal{R}(D_1 + D_2)$, *and hence*

$$\mathrm{Vol}^{1/n}(D_1) + \mathrm{Vol}^{1/n}(D_2) \leq \mathrm{Vol}^{1/n}(D_1 + D_2).$$

Remark 3.12. The existence of the limit in (1) has been known for the graded algebra $\mathcal{R}(D)$, where D is a so-called *big divisor* (see [Laz04]). A divisor D is big if its volume $\mathrm{Vol}(D)$ is strictly positive. Equivalently, D is big if for some $k > 0$, the Kodaira map of the subspace $\mathcal{L}(kD)$ is a birational isomorphism onto its image.

It seems that for a general graded linear series (and, in particular, the algebra $\mathcal{R}(D)$ of a general divisor D), the existence of the limit in (1) has not previously been known (see [Laz04, Rem. 2.1.39]).

Part (2) above is, in fact, a generalization of the Fujita approximation result of [LM09, Th. 3.3]. Using similar methods, for certain graded linear series of big divisors, Lazarsfeld and Mustata prove a statement very close to statement (2) above.

In [LM09] and [KK, Th. 5.13] the Brunn-Minkowski inequality in (3) is proved with similar methods.

4. Part IV: Applications to intersection theory and mixed volume

In this part we associate a convex body to any nonzero finite dimensional subspace of rational functions on an n-dimensional irreducible variety such that

(1) the volume of the body multiplied by $n!$ is equal to the self-intersection index of the subspace,

2) the body corresponding to the product of subspaces contains the sum of the bodies corresponding to the factors.

This construction allows us to prove that the intersection index enjoys all the main inequalities concerning the mixed volume and also to prove these inequalities for the mixed volume itself.

4.1. *Mixed volume.* In this section we recall the notion of mixed volume of convex bodies and list its main properties (without proofs).

The collection of all convex bodies in $\mathbb{R}^n$ is a cone; that is, we can add convex bodies and multiply a convex body with a positive number. Let Vol denote the n-dimensional volume in $\mathbb{R}^n$ with respect to the standard Euclidean metric. The function Vol is a homogeneous polynomial of degree n on the cone of convex bodies; i.e., its restriction to each finite dimensional section of the cone is a homogeneous polynomial of degree n.

By definition, the *mixed volume* $V(\Delta_1, \dots, \Delta_n)$ of an n-tuple $(\Delta_1, \dots, \Delta_n)$ of convex bodies is the coefficient of the monomial $\lambda_1 \cdots \lambda_n$ in the polynomial

$$P_{\Delta_1,\dots,\Delta_n}(\lambda_1, \dots, \lambda_n) = \mathrm{Vol}(\lambda_1\Delta_1 + \cdots + \lambda_n\Delta_n)$$

divided by $n!$. This definition implies that mixed volume is the *polarization* of the volume polynomial. That is, it is the unique function on the n-tuples of convex bodies satisfying the following:

(i) (Symmetry). V is symmetric with respect to permuting the bodies $\Delta_1, \dots, \Delta_n$.

(ii) (Multi-linearity). It is linear in each argument with respect to the Minkowski sum. The linearity in first argument means that for convex bodies Δ_1', Δ_1'', $\Delta_2, \dots, \Delta_n$, and real numbers $\lambda', \lambda'' \geq 0$, we have

$$V(\lambda'\Delta_1' + \lambda''\Delta_1'', \dots, \Delta_n) = \lambda' V(\Delta_1', \dots, \Delta_n) + \lambda'' V(\Delta_1'', \dots, \Delta_n).$$

(iii) (Relation with volume). On the diagonal it coincides with the volume, i.e., if $\Delta_1 = \cdots = \Delta_n = \Delta$, then $V(\Delta_1, \dots, \Delta_n) = \mathrm{Vol}(\Delta)$.

It is easy to verify that (1) mixed volume is nonnegative and (2) mixed volume is monotone. That is, for two n-tuples of convex bodies $\Delta_1' \subset \Delta_1, \dots, \Delta_n' \subset \Delta_n$, we have $V(\Delta_1', \dots, \Delta_n') \leq V(\Delta_1, \dots, \Delta_n)$.

The following inequality attributed to Alexandrov and Fenchel is important and very useful in convex geometry. All its previously known proofs are rather complicated (see [BZ88]).

THEOREM 4.1 (Alexandrov-Fenchel). *Let $\Delta_1, \dots, \Delta_n$ be convex bodies in $\mathbb{R}^n$. Then*

$$V(\Delta_1, \Delta_1, \Delta_3, \dots, \Delta_n)V(\Delta_2, \Delta_2, \Delta_3, \dots, \Delta_n) \leq V^2(\Delta_1, \Delta_2, \dots, \Delta_n).$$

In dimension 2, this inequality is elementary. We will call it the *generalized isoperimetric inequality* because when Δ_2 is the unit ball, it coincides with the classical isoperimetric inequality.

The celebrated *Brunn-Minkowski inequality* concerns volume of convex bodies in $\mathbb{R}^n$. It is an easy corollary of the Alexandrov-Fenchel inequality.

THEOREM 4.2 (Brunn-Minkowski). *Let Δ_1, Δ_2 be convex bodies in $\mathbb{R}^n$. Then*

$$\mathrm{Vol}^{1/n}(\Delta_1) + \mathrm{Vol}^{1/n}(\Delta_2) \leq \mathrm{Vol}^{1/n}(\Delta_1 + \Delta_2).$$

We used this inequality in the proof of Theorem 2.33. In dimension 2, the Brunn-Minkwoski inequality is equivalent to the generalized isoperimetric inequality (compare with Corollary 4.16).

On the other hand, all the classical proofs of the Alexandrov-Fenchel inequality deduce it from the Brunn-Minkowski inequality. But these deductions are the main and most complicated parts of the proofs ([BZ88]). Interestingly, the main construction in the present paper (using algebraic geometry) allows us to obtain the Alexandrov-Fenchel inequality as an immediate corollary of its simplest case, namely the generalized isoperimetric inequality (that is, when $n = 2$).

4.2. *Semigroup of subspaces and intersection index.* In this section we briefly review some concepts and results from [KK10a]. That is, we discuss the semigroup of subspaces of rational functions, its Grothendieck group and the intersection index on the Grothendieck group. We also recall the key notion of the completion of a subspace.

For the rest of the paper we will take the ground field $\mathbf{k}$ to be the field of complex numbers $\mathbb{C}$. Let F be a field finitely generated over $\mathbb{C}$. Later we will deal with the case where $F = \mathbb{C}(X)$ is the field of rational functions on a variety X over $\mathbb{C}$. Recall (Definition 2.21) that $\mathbf{K}(F)$ denotes the collection of all nonzero finite dimensional subspaces of F over $\mathbb{C}$. Moreover, for $L_1, L_2 \in \mathbf{K}(F)$, the product L_1L_2 is the $\mathbf{k}$-linear subspace spanned by all the products fg where $f \in L_1$ and $g \in L_2$. With respect to this product, $\mathbf{K}(F)$ is a (commutative) semigroup.

In general, the semigroup $\mathbf{K}(F)$ does not have the cancellation property. That is, the equality $L_1M = L_2M$, $L_1, L_2, M \in \mathbf{K}(F)$, does not imply $L_1 = L_2$. Let us say that L_1 and L_2 are *equivalent* and write $L_1 \sim L_2$ if there is $M \in \mathbf{K}(F)$ with $L_1M = L_2M$. Naturally the quotient $\mathbf{K}(F)/\sim$ is a semigroup with the cancellation property and hence can be extended to a group. The *Grothendieck group* $\mathbf{G}(F)$ of $\mathbf{K}(F)$ is the collection of formal quotients L_1/L_2, $L_1, L_2 \in \mathbf{K}(F)$, where $L_1/L_2 = L_1'/L_2'$ if $L_1L_2' \sim L_1'L_2$. There is a natural homomorphism $\phi : \mathbf{K}(F) \to \mathbf{G}(F)$. The Grothendieck group has the following universal property. For any group G' and a homomorphism $\phi' : \mathbf{K}(F) \to G'$, there exists a unique homomorphism $\psi : \mathbf{G}(F) \to G'$ such that $\phi' = \psi \circ \phi$.

Similar to the notion of integrality of an element over a ring, one defines the integrality of an element over a linear subspace.

Definition 4.3. Let L be a $\mathbf{k}$-linear subspace in F. An element $f \in F$ is *integral over* L if it satisfies an equation

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0, \tag{6}$$

where $m > 0$ and $a_i \in L^i$, $i = 1, \ldots, m$. The *completion* or *integral closure* $\overline{L}$ of L in F is the collection of all $f \in F$ that are integral over L.

The facts below about the completion of a subspace can be found, for example, in [ZS60, App. 4]. One shows that $f \in F$ is integral over a subspace

L if and only if $L \sim L + \langle f \rangle$. Moreover, the completion $\overline{L}$ is a subspace containing L, and if L is finite dimensional, then $\overline{L}$ is also finite dimensional.

The completion $\overline{L}$ of a subspace $L \in \mathbf{K}(F)$ can be characterized in terms of the notion of equivalence of subspaces: for every $L \in \mathbf{K}(F)$, $\overline{L}$ is the largest subspace equivalent to L. That is, (1) $L \sim \overline{L}$ and (2) if $L \sim M$, then $M \subset \overline{L}$.

The following standard result shows the connection between the completion of subspaces and integral closure of algebras.

THEOREM 4.4. *Let L be a finite dimensional $\mathbf{k}$-subspace, and let $A_L = \bigoplus_k L^k t^k$ be the corresponding graded subalgebra of $F[t]$. Then the k-th subspace of the integral closure $\overline{A_L}$ is $\overline{L^k}$, the completion of the k-th subspace of A_L. That is,*

$$\overline{A_L} = \bigoplus_k \overline{L^k} t^k.$$

Consider an n-dimensional irreducible algebraic variety X, and let $F = \mathbb{C}(X)$ be its field of rational functions. We denote the semigroup $\mathbf{K}(F)$ of finite dimensional subspaces in F by $\mathbf{K}_{\mathrm{rat}}(X)$.

Next we recall the notion of intersection index of an n-tuple of subspaces. Let $(L_1, \ldots, L_n)$ be an n-tuple of finite dimensional spaces of rational functions on X. Put $\mathbf{L} = L_1 \times \cdots \times L_n$. Let $U_{\mathbf{L}} \subset X$ be the set of all nonsingular points in X at which all the functions from all the L_i are regular, and let $Z_{\mathbf{L}} \subset U_{\mathbf{L}}$ be the collection of all the points x in $U_{\mathbf{L}}$ such that all the functions from some L_i (i depending on x) vanish at x.

Definition 4.5. Let us say that for an n-tuple of subspaces $(L_1, \ldots, L_n)$, the intersection index is defined and equal to $[L_1, \ldots, L_n]$ if there is a proper algebraic subvariety $\mathbf{R} \subset \mathbf{L} = L_1 \times \cdots \times L_n$ such that for each n-tuple $(f_1, \ldots, f_n) \in \mathbf{L} \setminus \mathbf{R}$, the following hold:

(1) The number of solutions of the system $f_1 = \cdots = f_n = 0$ in the set $U_{\mathbf{L}} \setminus Z_{\mathbf{L}}$ is independent of the choice of $(f_1, \ldots, f_n)$ and is equal to $[L_1, \ldots, L_n]$.

(2) Each solution $a \in U_{\mathbf{L}} \setminus Z_{\mathbf{L}}$ of the system $f_1 = \cdots = f_n = 0$ is nondegenerate, i.e., the form $df_1 \wedge \cdots \wedge df_n$ does not vanish at a.

The following is proved in [KK10a, Prop. 5.7].

THEOREM 4.6. *For any n-tuple $(L_1, \ldots, L_n)$ of subspaces $L_i \in \mathbf{K}_{\mathrm{rat}}(X)$, the intersection index $[L_1 \ldots, L_n]$ is defined.*

The following are immediate corollaries of the definition of the intersection index: (1) $[L_1, \ldots, L_n]$ is a symmetric function of $L_1, \ldots, L_n \in \mathbf{K}_{\mathrm{rat}}(X)$, (2) the intersection index is monotone, (i.e., if $L'_1 \subseteq L_1, \ldots, L'_n \subseteq L_n$, then $[L'_1, \ldots, L'_n] \leq [L_1, \ldots, L_n]$ and (3) the intersection index is nonnegative.

The next theorem contains the main properties of the intersection index (see [KK10a, §5]).

THEOREM 4.7. (1) (*Multi-additivity*). *Let* $L_1', L_1'', L_2, \ldots, L_n \in \mathbf{K}_{\mathrm{rat}}(X)$ *and put* $L_1 = L_1' L_1''$. *Then*

$$[L_1, \ldots, L_n] = [L_1', L_2, \ldots, L_n] + [L_1'', L_2, \ldots, L_n].$$

(2) (*Invariance under the completion*). *Let* $L_1 \in \mathbf{K}_{\mathrm{rat}}(X)$, *and let* $\overline{L_1}$ *be its completion. Then for any* $(n-1)$*-tuple* $L_2, \ldots, L_n \in \mathbf{K}_{\mathrm{rat}}(X)$, *we have*

$$[L_1, L_2, \ldots, L_n] = [\overline{L_1}, L_2, \ldots, L_n].$$

Because of the multi-additivity, the intersection index can be extended to the Grothendieck group $\mathbf{G}_{\mathrm{rat}}(X)$ of the semigroup $\mathbf{K}_{\mathrm{rat}}(X)$. The Grothendieck group of $\mathbf{K}_{\mathrm{rat}}(X)$ can be considered as an analogue (for a typically noncomplete variety X) of the group of Cartier divisors on a complete variety, and the intersection index on the Grothendieck group $\mathbf{G}_{\mathrm{rat}}(X)$ can be considered as an analogue of the intersection index of Cartier divisors.

The next proposition relates the self-intersection index of a subspace with the degree of the image of the Kodaira map. It easily follows from the definition of the intersection index.

PROPOSITION 4.8 (Self-intersection index and degree). *Let* $L \in \mathbf{K}_{\mathrm{rat}}(X)$ *be a subspace and* $\Phi_L : X \dashrightarrow Y_L \subset \mathbb{P}(L^*)$ *its Kodaira map.* (1) *If* $\dim X = \dim Y_L$, *then* Φ_L *has finite mapping degree* d *and* $[L, \ldots, L]$ *is equal to the degree of the subvariety* Y_L (*in* $\mathbb{P}(L^*)$) *multiplied with* d. (2) *If* $\dim X > \dim Y_L$, *then* $[L, \ldots, L] = 0$.

4.3. *Newton-Okounkov body and intersection index.* We now discuss the relation between the self-intersection index of a subspace of rational functions and the volume of the Newton-Okounkov body.

Let X be an irreducible n-dimensional variety (over $\mathbb{C}$) and $L \in \mathbf{K}_{\mathrm{rat}}(X)$ a nonzero finite dimensional subspace of rational functions. We can naturally associate two algebras of integral type to L: the algebra A_L and its integral closure $\overline{A_L}$. (Note that by Theorem 2.17, $\overline{A_L}$ is an algebra of integral type.)

Let $F = \mathbb{C}(X)$. As in Section 2.4, let $v : F \setminus \{0\} \to \mathbb{Z}^n$ be a faithful valuation with one-dimensional leaves and v_t its extension to the polynomial ring $F[t]$. Then v_t associates two convex bodies to the space L, namely $\Delta(A_L)$ and $\Delta(\overline{A_L})$.

Since $A_L \subset \overline{A_L}$, then $\Delta(A_L) \subset \Delta(\overline{A_L})$. In general, $\Delta(\overline{A_L})$ can be strictly bigger than $\Delta(A_L)$ (see Example 4.12).

The following two theorems can be considered as far generalizations of the Kushnirenko theorem in Newton polytope theory and toric geometry. Below, Vol_n denotes the standard Euclidean measure in $\mathbb{R}^n$.

THEOREM 4.9. *Let $L \in \mathbf{K}_{\mathrm{rat}}(X)$ with the Kodaira map Φ_L.* (1) *If Φ_L has finite mapping degree, then*

$$[L, \ldots, L] = \frac{n! \deg \Phi_L}{\mathrm{ind}(A_L)} \mathrm{Vol}_n(\Delta(A_L)).$$

Otherwise, both $[L, \ldots, L]$ and $\mathrm{Vol}_n(\Delta(A_L))$ are equal to 0. (2) *In particular, if Φ_L is a birational isomorphism between X and Y_L, then $\deg \Phi_L = \mathrm{ind}(A_L) = 1$ and we obtain*

$$[L, \ldots, L] = n! \mathrm{Vol}_n(\Delta(A_L)).$$

(3) *The correspondence $L \mapsto \Delta(A_L)$ is superadditive, i.e., if L_1, L_2 are finite dimensional subspaces of rational functions. Then*

$$\Delta(A_{L_1}) \oplus_t \Delta(A_{L_2}) \subset \Delta(A_{L_1 L_2}).$$

(*In other words, $\Delta_0(A_{L_1}) + \Delta_0(A_{L_2}) \subset \Delta_0(A_{L_1 L_2})$, where Δ_0 is the Newton-Okounkov body projected to the level* 0 *and* + *is the Minkowski sum.*)

Proof. (1) Follows from Proposition 4.8 and Corollary 3.2. (2) If Φ_L is a birational isomorphism, then it has degree 1. On the other hand, from the birational isomorphism of Φ_L it follows that the subfield $P(L)$ associated to L coincides with the whole field $\mathbb{C}(X)$. Since the valuation v is faithful, we then conclude that the subgroup $G_0(A_L)$ coincides with the whole $\mathbb{Z}^n$ and hence $\mathrm{ind}(A_L) = 1$. Part (2) then follows from (1). (3) We know that $A_{L_1 L_2} = A_{L_1} A_{L_2}$ and $m(A_{L_1}) = m(A_{L_2}) = m(A_{L_1 L_2}) = 1$. Proposition 2.32 now gives the required result. □

THEOREM 4.10. (1) *We have*

$$[L, \ldots, L] = n! \mathrm{Vol}_n(\Delta(\overline{A_L})).$$

(2) *The correspondence $L \mapsto \Delta(\overline{A_L})$ is superadditive*; *i.e., if L_1, L_2 are finite dimensional subspaces of rational functions, then*

$$\Delta(\overline{A_{L_1}}) \oplus_t \Delta(\overline{A_{L_2}}) \subset \Delta(\overline{A_{L_1 L_2}}).$$

(*In other words, $\Delta_0(\overline{A_{L_1}}) + \Delta_0(\overline{A_{L_2}}) \subset \Delta_0(\overline{A_{L_1 L_2}})$.*)

We need the following lemma.

LEMMA 4.11. *Let L be a subspace of rational functions. Suppose $\dim Y_L = n$; i.e., the Kodaira map Φ_L has finite mapping degree. Then there exists $N > 0$ such that the following holds. For any $p > N$, the subfield associated to the completion $\overline{L^p}$ coincides with the whole $\mathbb{C}(X)$.*

Proof. Let $E = P(L) \cong \mathbb{C}(Y_L)$ and $F = \mathbb{C}(X)$. The extension F/E is a finite extension because $\dim Y_L = n$. Clearly for any $p > 0$, $P(\overline{L^p}) \subset F$. We will show that there is $N > 0$ such that for $p > N$, we have $F \subset P(\overline{L^p})$. Let $f_1, \dots, f_r$ be a basis for F/E. Let $f \in \{f_1, \dots f_r\}$. Then f satisfies an equation

$$a_0 f^m + \cdots + a_m = 0, \tag{7}$$

where $a_i = P_i/Q_i$ with $P_i, Q_i \in L^{d_i}$ for some $d_i > 0$. Let $N_f = \sum_{i=0}^m d_i$, and put $Q = Q_0 \cdots Q_m$. Then $Q \in L^{N_f}$. Multiplying (7) with Q, we have $b_0 f^m + \cdots + b_m = 0$, where $b_i = P_i Q/Q_i \in L^{N_f}$. Then multiplying with b_0^{m-1} gives

$$(b_0 f)^m + b_1 (b_0 f)^{m-1} + \cdots + (b_0^{m-1}) b_m = 0,$$

which shows that $b_0 f$ is integral over L^{N_f}. Now $f \in P(\overline{L^{N_f}})$ because $f = b_0 f / b_0$ and $b_0 \in L^{N_f}$. Let N be the maximum of the N_f for $f \in \{f_1, \dots, f_r\}$. It follows that $F \subset P(\overline{L^N})$. It is easy to see that for $p > N$, we have $\overline{L^N} L^{p-N} \subset \overline{L^p}$ and hence $P(\overline{L^N}) \subset P(\overline{L^p})$. Thus $F = P(\overline{L^p})$ as required. $\square$

Proof of Theorem 4.10. (1) Suppose $\dim Y_L < n$. Then from the definition of the self-intersection index it follows that $[L, \dots, L] = 0$. But we know that the n-dimensional volume of $\Delta(\overline{A_L})$ is 0 because $\dim \Delta(\overline{A_L})$ equals the dimension of Y_L and hence is less than n. This proves the theorem in this case. Now suppose $\dim Y_L = n$. Then the Kodaira map Φ_L has finite mapping degree. By Lemma 4.11, the Kodaira map $\Phi_{\overline{L^p}}$ is a birational isomorphism onto its image. Thus the self-intersection index of $\overline{L^p}$ is equal to the degree of the variety $Y_{\overline{L^p}}$. By the main properties of intersection index (Theorem 4.7), we have

$$[\overline{L^p}, \dots, \overline{L^p}] = [L^p, \dots, L^p] = p^n [L, \dots, L].$$

On the other hand, by Theorem 2.36,

$$\deg Y_{\overline{L^p}} = \frac{p^n}{\operatorname{ind}(\overline{A_L})} \operatorname{Vol}_n(\Delta(\overline{A_L})).$$

But since the field $P(\overline{L^p})$ coincides with $\mathbb{C}(X)$, we have $\operatorname{ind}(\overline{A_L}) = 1$, which finishes the proof of (1). To prove (2) first note that we have the inclusion

$$\overline{A_{L_1}}\ \overline{A_{L_2}} \subset \overline{A_{L_1 L_2}}.$$

(This follows from the fact that for any two subspaces L, M, we have $\overline{L}\ \overline{M} \subset \overline{LM}$.) Secondly, since $m(\overline{A_{L_1}}) = m(\overline{A_{L_2}}) = 1$ by Proposition 2.32, we know

$$\Delta(\overline{A_{L_1}}) \oplus_t \Delta(\overline{A_{L_2}}) \subset \Delta(\overline{A_{L_1}}\ \overline{A_{L_2}}) \subset \Delta(\overline{A_{L_1 L_2}}).$$

The theorem is proved. $\square$

Example 4.12. Let X be the affine line $\mathbb{C}$, and let z denote the coordinate function on it. Let $L = \operatorname{span}\{1, z^2\}$. Clearly $[L] = 2$. Now let $v : \mathbb{C}(X) \to \mathbb{Z}$ be the order of vanishing at the point $1 \in X$. Then $S(A_L) = \mathbb{Z}_{\geq 0}$, and hence $\operatorname{ind}(A_L) = 1$. This shows that $\deg(\Phi_L) \gneqq \operatorname{ind}(A_L)$. Also it is easy to see that

$\Delta(A_L)$ is the line segment $[0,1]$. On the other, hand one sees that $\Delta(\overline{A_L})$ is the line segment $[0,2]$.

Next, let us see that the well-known Bernstein-Kushnirenko theorem follows from Theorem 4.9. For this, we take the variety X to be $(\mathbb{C}^*)^n$ and the subspace L a subspace spanned by Laurent monomials.

We identify the lattice $\mathbb{Z}^n$ with the *Laurent monomials* in $(\mathbb{C}^*)^n$: to each integral point $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we associate the monomial $x^a = x_1^{a_1} \ldots x_n^{a_n}$ where $x = (x_1, \ldots, x_n)$. A *Laurent polynomial* $P(x) = \sum_a c_a x^a$ is a finite linear combination of Laurent monomials with complex coefficients. The *support* $\mathrm{supp}(P)$ of a Laurent polynomial P is the set of exponents a for which $c_a \neq 0$. We denote the convex hull of a finite set $I \subset \mathbb{Z}^n$ by $\Delta_I \subset \mathbb{R}^n$. The *Newton polytope* $\Delta(P)$ of a Laurent polynomial P is the convex hull $\Delta_{\mathrm{supp}(P)}$ of its support. With each finite set $I \subset \mathbb{Z}^n$ one associates the linear space $L(I)$ of Laurent polynomials P with $\mathrm{supp}(P) \subset I$.

THEOREM 4.13 (Kushnirenko). *The number of solutions in $(\mathbb{C}^*)^n$ of a general system of Laurent polynomial equations $P_1 = \cdots = P_n = 0$ with $P_1, \ldots, P_n \in L(I)$ is equal to $n!\mathrm{Vol}(\Delta_I)$, i.e.,*

$$[L(I), \ldots, L(I)] = n!\mathrm{Vol}(\Delta_I).$$

Proof. If I, J are finite subsets in $\mathbb{Z}^n$, then $L_I L_J = L_{I+J}$. Consider the graded algebra $A_{L(I)} \subset F[t]$ where F is the field of rational functions on $(\mathbb{C}^*)^n$. Take any valuation v on F coming from the Gröbner valuation on the field of fractions of the algebra of formal power series $\mathbb{C}[[x_1, \ldots, x_n]]$. (See the paragraph before Example 2.12.) From the definition it is easy to see that $v(L(I)\backslash\{0\}) = I$ and more generally $v(L(I)^k\backslash\{0\}) = k{*}I$, where $k{*}I$ is the sum of k copies of the set I. Let $S = S(A_{L(I)})$ be the semigroup associated to the algebra $A_{L(I)}$. Then the cone $\mathrm{Con}(S)$ is the cone in $\mathbb{R}^{n+1}$ over $\Delta_I \times \{1\}$ and the group $G_0(S)$ is generated by the differences $a-b$, $a, b \in I$. Let $I = \{\alpha_1, \ldots, \alpha_r\}$. Then $\{x^{\alpha_1}, \ldots, x^{\alpha_r}\}$ is a basis for $L(I)$ and one verifies that, in the dual basis for $L(I)^*$, the Kodaira map is given by $\Phi_{L(I)}(x) = (x^{\alpha_1} : \cdots : x^{\alpha_r})$. From this it follows that the mapping degree of $\Phi_{L(I)}$ is equal to the index of the subgroup $G_0(S)$, i.e., $\mathrm{ind}(A_{L(I)})$. By Theorem 4.9, we then have

$$[L(I), \ldots, L(I)] = n!\frac{\deg \Phi_{L(I)}}{\mathrm{ind}(A_{L(I)})}\mathrm{Vol}(\Delta_I) = n!\mathrm{Vol}(\Delta_I),$$

which proves the theorem. □

The Bernstein theorem computes the intersection index of an n-tuple of subspaces of Laurent polynomials in terms of the mixed volume of their Newton polytopes.

Theorem 4.14 (Bernstein). *Let $I_1, \dots, I_n \subset \mathbb{Z}^n$ be finite subsets. The number of solutions in $(\mathbb{C}^*)^n$ of a general system of Laurent polynomial equations $P_1 = \cdots = P_n = 0$, where $P_i \in L(I_i)$, is equal to $n!V(\Delta_{I_1}, \dots, \Delta_{I_n})$, i.e.,*

$$[L(I_1), \dots, L(I_n)] = n!V(\Delta_{I_1}, \dots, \Delta_{I_n})$$

(where, as before, V denotes the mixed volume of convex bodies in $\mathbb{R}^n$).

Proof. The Bernstein theorem readily follows from the multi-additivity of the intersection index, Theorem 4.13 (the Kushnirenko theorem) and the observation that for any two finite subsets $I, J \subset \mathbb{Z}^n$, we have $L(I+J) = L(I)L(J)$ and $\Delta_I + \Delta_J = \Delta_{I+J}$. □

The above proofs of the Bernstein and Kushnirenko theorems are in fact very close to the ones in [Kho92].

4.4. *Proof of the Alexandrov-Fenchel inequality and its algebraic analogue.* Finally in this section, using the notion of Newton-Okounkov body, we prove an algebraic analogue of the Alexandrov-Fenchel inequality. From this we deduce the classical Alexandrov-Fenchel inequality in convex geometry.

As be before, X is an n-dimensional irreducible complex algebraic variety. The self-intersection index enjoys the following analogue of the Brunn-Minkowski inequality.

Corollary 4.15. *Let L_1, L_2 be finite dimensional subspaces in $\mathbb{C}(X)$ and $L_3 = L_1L_2$. Then*

$$[L_1, \dots L_1]^{1/n} + [L_2, \dots, L_2]^{1/n} \leq [L_3, \dots, L_3]^{1/n}.$$

Proof. By definition, the Newton-Okounkov body $\Delta(A)$ of an algebra A lives at level 1. For $i = 1, 2, 3$, let Δ_i be the Newton-Okounkov body of the algebra $\overline{A_{L_i}}$ projected to the level 0. Then by Theorem 4.10 we have $[L_i, \dots, L_i] = n!\mathrm{Vol}_n(\Delta_i)$, and $\Delta_1 + \Delta_2 \subset \Delta_3$. Now by the classical Brunn-Minkowski inequality $\mathrm{Vol}_n^{1/n}(\Delta_1) + \mathrm{Vol}_n^{1/n}(\Delta_2) \leq \mathrm{Vol}_n^{1/n}(\Delta_3)$ which proves the corollary (see also Theorem 2.33). □

Surprisingly the most important case of the above inequality is the $n = 2$ case, i.e., when X is an algebraic surface. As we show next, the general case of the above inequality and many other inequalities for the intersection index follow from this $n = 2$ case and the basic properties of the intersection index.

Corollary 4.16 (A version of the Hodge inequality). *Let X be an irreducible algebraic surface, and let L, M be nonzero finite dimensional subspaces of $\mathbb{C}(X)$. Then*

$$[L, L][M, M] \leq [L, M]^2.$$

Proof. From Corollary 4.15, for $n = 2$, we have

$$\begin{aligned}[L, L] + 2[L, M] + [M, M] &= [LM, LM] \\ &\geq ([L, L]^{1/2} + [M, M]^{1/2})^2 \\ &\geq [L, L] + 2[L, L]^{1/2}[M, M]^{1/2} + [M, M],\end{aligned}$$

which readily implies the claim. □

In other words, Theorem 4.10 allowed us to easily reduce the Hodge inequality above to the generalized isoperimetric inequality. We can now give an easy proof of the Alexandrov-Fenchel inequality for the intersection index.

Let us call a subspace $L \in \mathbf{K}_{\mathrm{rat}}(X)$ a *very big subspace* if the Kodaira rational map of L is a birational isomorphism between X and its image. Also we call a subspace *big* if for some $m > 0$, the subspace L^m is very big. It is not hard to show that the product of two big subspaces is again a big subspace and thus the big subspaces form a subsemigroup of $\mathbf{K}_{\mathrm{rat}}(X)$.

Theorem 4.17 (A version of the Bertini-Lefschetz theorem). *Let X be a smooth irreducible n-dimensional variety, and let $L_1, \ldots, L_k \in \mathbf{K}_{\mathrm{rat}}(X)$, $k < n$, be very big subspaces. Then there is a Zariski open set $\mathbf{U}$ in $\mathbf{L} = L_1 \times \cdots \times L_k$ such that for each point $\mathbf{f} = (f_1, \ldots, f_k) \in \mathbf{U}$ the variety $X_{\mathbf{f}}$ defined in X by the system of equations $f_1 = \cdots = f_k = 0$ is smooth and irreducible.*

A proof of the Bertini-Lefschetz theorem can be found in [Har77, Chap. II, Th. 8.18]

One can slightly extend Theorem 4.17. Assume that we are given k very big spaces $L_1, \ldots, L_k \in \mathbf{K}_{\mathrm{rat}}(X)$ and $(n-k)$ arbitrary subspaces $L_{k+1}, \ldots, L_n$. We denote by $[L_{k+1}, \ldots, L_n]_{X_{\mathbf{f}}}$ the intersection index of the restriction of the subspaces $L_{k+1}, \ldots, L_n$ to the subvariety $X_{\mathbf{f}}$. It is easy to verify the following reduction theorem.

Theorem 4.18. *There is a Zariski open subset $\mathbf{U}$ in $L_1 \times \cdots \times L_k$ such that for $\mathbf{f} = (f_1, \ldots, f_k) \in \mathbf{U}$, the system $f_1 = \cdots = f_k = 0$ defines a smooth irreducible subvariety $X_{\mathbf{f}}$ in X and the identity*

$$[L_1, \ldots L_n]_X = [L_{k+1}, \ldots L_n]_{X_{\mathbf{f}}}$$

holds.

Theorem 4.19 (Algebraic analogue of the Alexandrov-Fenchel inequality). *Let X be an irreducible n-dimensional variety, and let $L_1, \ldots, L_n \in \mathbf{K}_{\mathrm{rat}}(X)$. Also assume that $L_3, \ldots, L_n$ are big subspaces. Then the following inequality holds*:

$$[L_1, L_1, L_3, \ldots, L_n][L_2, L_2, L_3, \ldots, L_n] \leq [L_1, L_2, L_3, \ldots, L_n]^2.$$

Proof. Because of the multi-additivity of the intersection index, if the inequality holds for the spaces L_i replaced with ${L_i}^N$, for some N, then it holds for the original spaces L_i. So without loss of generality we can assume that $L_3, \ldots, L_n$ are very big. By Theorem 4.18, for almost all the $(f_3, \ldots, f_n) \in L_3 \times \cdots \times L_n$ and the variety Y defined by the system $f_3 = \cdots = f_n = 0$, we have

$$[L_1, L_2, L_3, \ldots, L_n] = [L_1, L_2]_Y,$$

$$[L_1, L_1, L_3, \ldots, L_n] = [L_1, L_1]_Y,$$

$$[L_2, L_2, L_3, \ldots, L_n] = [L_2, L_2]_Y.$$

Now applying Corollary 4.16 (the Hodge inequality) for the surface Y, we have

$$[L_1, L_1]_Y [L_2, L_2]_Y \leq [L_1, L_2]_Y^2,$$

which proves the theorem. □

Let us introduce notation for repetition of subspaces in the intersection index. Let $2 \leq m \leq n$ be an integer and $k_1 + \cdots + k_r = m$ a partition of m with $k_i \in \mathbb{N}$. Consider the subspaces $L_1, \ldots, L_n \in \mathbf{K}_{\mathrm{rat}}(X)$. Denote by $[k_1 * L_1, \ldots, k_r * L_r, L_{m+1}, \ldots, L_n]$ the intersection index of $L_1, \ldots, L_n$ where L_1 is repeated k_1 times, L_2 is repeated k_2 times, etc., and $L_{m+1}, \ldots, L_n$ appear once.

Corollary 4.20 (Corollaries of the algebraic analogue of the Alexandrov-Fenchel inequality). *Let X be an n-dimensional irreducible variety.* (1) *Let $2 \leq m \leq n$ and $k_1 + \cdots + k_r = m$ with $k_i \in \mathbb{N}$. Take big subspaces of rational functions $L_1, \ldots, L_n \in \mathbf{K}_{\mathrm{rat}}(X)$. Then*

$$\prod_{1 \leq j \leq r} [m * L_j, L_{m+1}, \ldots, L_n]^{k_j} \leq [k_1 * L_1, \ldots, k_r * L_r, L_{m+1}, \ldots, L_n]^m.$$

(2) (*Generalized Brunn-Minkowski inequality*). *For any fixed big subspaces $L_{m+1}, \ldots, L_n \in \mathbf{K}_{\mathrm{rat}}(X)$, the function*

$$F : L \mapsto [m * L, L_{m+1}, \ldots, L_n]^{1/m}$$

is a concave function on the semigroup $\mathbf{K}_{\mathrm{rat}}(X)$.

(1) follows formally from the algebraic analogue of the Alexandrov-Fenchel, the same way that the corresponding inequalities follow from the classical Alexandrov-Fenchel in convex geometry. (2) can be easily deduced from Corollary 4.15 and Theorem 4.18.

We now prove the classical Alexandrov-Fenchel inequality in convex geometry (Theorem 4.1).

Proof of Theorem 4.1. As we saw above, the Bernstein-Kushnirenko theorem follows from Theorem 4.9. Applying Theorem 4.19 (algebraic analogue of the Alexandrov-Fenchel inequality) to the situation considered in the Bernstein-Kushnirenko theorem one proves the Alexandrov-Fenchel inequality for convex polytopes of full dimension and with integral vertices. The homogeneity then implies the Alexandrov-Fenchel inequality for convex polytopes of full dimension and with rational vertices. But since any convex body can be approximated by convex polytopes of full dimension and with rational vertices, by continuity we obtain the Alexandrov-Fenchel inequality in complete generality. □

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