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Completions of Convex Families of Convex Bodies

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Abstract—The paper discusses the existence of a continuous extension of functions that are defined on subsets of $\mathbb{R}^n$ and whose values are convex bodies in $\mathbb{R}^n$. This problem arose in convex geometry in connection with the notion, recently introduced in algebraic geometry, of convex Newton–Okounkov bodies.

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The exponential defined at the rational points $x = p/q$ by the formula $\exp x = \sqrt[q]{e^p}$ can be extended to the entire real line by continuity. This is the simplest example of the phenomenon dealt with in the present paper. Let there be a compact convex body $\Delta_m \subset \mathbb{R}^k$ assigned to each integer point m in the positive octant $\mathbb{R}^n_{\geq 0}$, and assume that the mapping $m \to \Delta_m$ has the following properties:

1) The body $\Delta_{m_1+m_2}$ contains the Minkowski sum $\Delta_{m_1} + \Delta_{m_2}$ of Δ_{m_1} and Δ_{m_1}.

2) One has $\Delta_{qm} = q\Delta_m$ for $q \geq 0$ and $m \in \mathbb{R}^n_{\geq 0}$.

3) There exists a continuous function ϕ homogeneous of degree k on $\mathbb{R}^n_{\geq 0}$ and positive on $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ such that the k-volume of Δ_m is $\phi(m)$. (A function ϕ with this property will be called a *volume function*.)

We ask whether the mapping that takes each integer point $m \in \mathbb{R}^k_{\geq 0}$ to the body Δ_m can under these conditions be extended by continuity to the entire octant $\mathbb{R}^n_{\geq 0}$. More precisely, *does there exist a closed convex cone $K \subset \mathbb{R}^n + \mathbb{R}^k$ such that $\Delta_m = \pi^{-1}(m) \cap K$ for each integer point $m \in \mathbb{R}^n_+$, where $\pi\colon (\mathbb{R}^n + \mathbb{R}^k) \to \mathbb{R}^n$ is the natural projection*?

We show that the answer is "yes" (see Section 8). Here the auxiliary property (3) is important; without this property, the answer is "no." We also obtain a similar but more cumbersome description of the mappings $m \to \Delta_m$ satisfying properties (1) and (2) alone.

The question is motivated by algebraic geometry and related to the recently introduced notion of convex Newton–Okounkov bodies. (See the paper [1], where the construction of Newton–Okounkov bodies and references to other papers on the topic can be found.) The present paper is neither based on algebraic geometry nor uses it in any way. To make the picture complete, the next paragraph, which the readers may well skip without any consequences for their understanding of the paper, briefly comments on the algebraic-geometric setting in which the question arises.

Let X be an arbitrary k-dimensional irreducible complex algebraic variety, let $\mathbb{C}(X)$ be the field of rational functions on X, and let $\mathbf{K}_{\mathrm{rat}}(X)$ be the set of all nontrivial finite-dimensional spaces over $\mathbb{C}$ of rational functions on X. The set $\mathbf{K}_{\mathrm{rat}}(X)$ is supplied with a natural multiplication that makes is a commutative semigroup. Take an arbitrary $\mathbb{Z}^k$-valued valuation v on $\mathbb{C}(X)$ such that every point $q \in \mathbb{Z}^k$ can be represented in the form $q = v(f)$, where $f \in \mathbb{C}(X)$ is a nonzero function. The paper [1]

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describes a construction that takes each space $L \in \mathbf{K}_{\mathrm{rat}}(X)$ to the corresponding convex Newton–Okounkov body $\Delta(L) \subset \mathbb{R}^k$. (The body $\Delta(L)$ depends on the valuation v.) The volume $V(\Delta(L))$ of $\Delta(L)$ multiplied by $k!$ gives the asymptotics of the dimension over $\mathbb{C}$ of powers of L; i.e.,

$$\lim_{N\to\infty} \frac{\dim_{\mathbb{C}} L^N}{N^k} = k!V(\Delta(L)).$$

Moreover, the following conditions hold:

1) $\Delta(L_1 L_2) \supset \Delta(L_1) + \Delta(L_2)$;

2) $\Delta(L^q) = q\Delta(L)$ for $q \geq 0$.

Let $L_1, \dots, L_n \in \mathbf{K}_{\mathrm{rat}}(X)$ be arbitrary elements, let $m = (m_1, \dots, , m_n) \in \mathbb{R}^n_{\geq 0}$ be an integer point, and let $L(m) = L^{m_1} \cdots L^{m_n}$. Then the mapping $m \to \Delta(m) = \Delta(L(m))$ satisfies conditions (1) and (2) in the first paragraph of the present paper. Moreover, one can see from [1] that the function $\phi(m) = V(\Delta(L(m)))$ on the set of integer points of $\mathbb{R}^n_{\geq 0}$ is a homogeneous polynomial of degree k. There arises a natural problem: what can one say about the body $\Delta(m)$ as a function of the point m? It is this problem that motivated the question discussed in the present paper.

The paper develops a technique related to the question considered. We introduce the notion of *convexity* of a family of convex bodies located on a given family of parallel affine spaces in $\mathbb{R}^N$. Let $V \subset \mathbb{R}^N$ be a vector subspace, and let $\{V_\alpha\}$ be a set of affine spaces parallel to V. It is convenient to describe $\{V_\alpha\}$ as follows. Let $\mathbb{R}^N = \mathbb{R}^n + V$, and let $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$ be the projection along V. To describe $\{V_\alpha\}$, it suffices to specify the projection $\pi(M) \subset \mathbb{R}^n$ of the set $M = \cup V_\alpha$. It is this description that will be used throughout the paper. Given a convex (possibly empty) set Δ_α in each V_α, we say that $\{\Delta_\alpha\}$ is a *convex family of convex sets* if there exists a convex set $\Delta \subset \mathbb{R}^N$ such that $\Delta_\alpha = \Delta \cap V_\alpha$.

We discuss the following questions. Is a given family of convex sets $\Delta_\alpha \subset V_\alpha$ convex? If the answer is "yes," what can one say about a convex set Δ for which $\Delta_\alpha = \Delta \cap V_\alpha$? For example, what Δ can one choose if the fibers Δ_α are closed?

The outline of the paper is as follows. Section 1 presents some classical definitions and theorems of convex geometry.

The notions of convexity, $\mathbb{F}$-convexity, and $\mathbb{Q}$-convexity of a set over a projection of itself, which are versions of the notion of convexity for a family of sets, are discussed in Sections 2 and 3. The characteristic cone, which is an invariant permitting one to distinguish bounded and unbounded convex bodies, is discussed in Section 5.

Sections 4, 6, and 7 contain proofs of results needed for solving our problem. It is shown in Section 4 that, for a convex set Δ, the operations of closure and taking the intersection with an affine subspace V commute if V meets the interior of Δ. Section 6 deals with sections of a convex body Δ by a family of parallel affine spaces. The continuous dependence of the section on the secant space is discussed. Section 7 gives a classification of all convex sets Δ whose projection is a given polytope P. (In the present paper, by a polytope we mean a closed bounded convex polyhedron.) A sufficient condition for the compactness of Δ in terms of the volume of its sections is given.

In Sec. 8, we solve the problem of describing *semigroups of convex bodies over a semigroup T in* $\mathbb{R}^n$ for the cases in which $T = \mathbb{Z}^n_{\geq 0}$ and $T = \mathbb{F}^n_{\geq 0}$, where $\mathbb{F}$ is a subfield of the reals. For $T = \mathbb{Z}^n_{\geq 0}$, this problem coincides with the above-stated problem arising from algebraic geometry. The solution is based on the results of the preceding sections.

The methods of the present paper are illustrated by yet another example in Section 9, where we define convex functions on a set $X \subseteq \mathbb{R}^n$ and discuss the continuity of such functions and a construction of their convex extension to the convex hull of X. The simplest example of the construction in Section 9 is the continuation of the exponential function from rational to real numbers.

Each of the nine sections is supplemented by a brief introduction. The title of each assertion states what it is about. All assertions are numbered consecutively throughout the paper.

1. GENERAL PROPERTIES OF CONVEX SETS

In this section, we present some classical definitions and theorems of convex geometry (e.g., see [2]).

A *convex subset* of a vector space is a set that contains a segment $[A, B]$ whenever it contains the points A and B. A convex subset may be nonclosed and unbounded. The intersection of convex sets is convex. The least convex set $\mathbb{L}Y$ containing a given set Y is called the *convex hull* of Y. The set Δ_* of *interior points* of a convex set Δ is defined as the set of interior points of Δ with respect to the topology of the minimal affine space containing Δ. For example, the set I_* for a segment $I \subset \mathbb{R}^n$ is the set of all points of I except for the endpoints.

First separation theorem. *Let $\Delta \subset \mathbb{R}^n$ be a closed convex subset (possibly unbounded), and let $a \in \mathbb{R}^n \setminus \Delta$. Then there exists a nonzero linear function $L\colon \mathbb{R}^n \to \mathbb{R}$ such that $L(x) < L(a)$ for each $x \in \Delta$.*

Theorem on the boundary of a convex set. *Every boundary point of a convex set $X \subset \mathbb{R}^n$ is a boundary point of the set of interior points of the complement $\mathbb{R}^n \setminus X$.*

Second separation theorem. *Let $\Delta \subset \mathbb{R}^n$ be a convex subset (possibly unbounded and nonclosed), and let $a \in \mathbb{R}^n \setminus \Delta$. Then there exists a nonzero linear function $L\colon \mathbb{R}^n \to \mathbb{R}$ such that $L(x) \le L(a)$ for each $x \in \Delta$.*

The second separation theorem follows from the first separation theorem and the theorem on the boundary of a convex set. We will use the second separation theorem in the form of the following corollary.

Corollary 1 (on separation from a subspace)**.** *Let $\Delta \subset \mathbb{R}^n$ be a convex subset (possibly unbounded and nonclosed), and let $M \subset \mathbb{R}^n$ be an affine subspace that does not meet Δ. Then there exists a nonzero linear function $L\colon \mathbb{R}^n \to \mathbb{R}$ such that $L(x) \le L(y)$ for every pair of points $x \in \Delta$ and $y \in M$.*

Proof. Consider the quotient space $\mathbb{R}^n/\overline{M}$, where $\overline{M}$ is the vector subspace obtained by a parallel translation of M. Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/\overline{M}$ be the natural projection, let $\overline{\Delta} \subset \mathbb{R}^n/\overline{M}$ be the image of Δ, and let $a \in \mathbb{R}^n/\overline{M}$ be the image of M. By the second separation theorem, there exists a nonzero linear function $\overline{L}\colon \mathbb{R}^n/\overline{M} \to \mathbb{R}$ such that $\overline{L}(\overline{x}) \le \overline{L}(a)$ for every $\overline{x} \in \mathbb{R}^n/\overline{M}$. The function $\pi^* L\colon \mathbb{R}^n \to \mathbb{R}$ has the desired property by construction. □

We also need the following classical Carathéodory theorem.

Carathéodory theorem. *The convex hull of a subset $X \subset \mathbb{R}^N$ is the union of tetrahedra of dimension $\le N$ whose vertices are contained in X.*

2. SETS CONVEX OVER PROJECTIONS OF THEMSELVES

In this section, we define the convexity of a set over a projection of itself. This definition is equivalent to that of the convexity of a family of sets and simplifies the latter in the spirit of the Carathéodory theorem (see Theorem 3 on the convex hull). Theorem 5 relates the property of Δ to be dense in its convex hull to the same property of the projection of Δ.

The first question of interest to us can be stated as follows. Let $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$ be the standard projection of $\mathbb{R}^N$ onto a coordinate space $\mathbb{R}^n$, let $Y \subset \mathbb{R}^N$, and let $X = \pi(Y) \subset \mathbb{R}^n$ be the projection of Y.

Question 1. Is it true or false that there exists a convex set $R \subset \mathbb{R}^N$ such that $R \cap \pi^{-1}(X) = Y$?

To answer this question, we need the following definition.

Definition. A set Y is said to be *convex over its projection* $X = \pi(Y)$ if the following assertions hold:

1) The preimage $\pi^{-1}(x) \cap Y$ of an arbitrary point $x \in X$ is convex.

2) If the projections $\pi(a_1), \dots, \pi(a_k)$ of points $a_1, \dots, a_k \in Y$ are affinely independent (i.e., are not contained in an affine space of dimension $< k-1$), then the point $\Gamma \cap \pi^{-1}(x)$, where Γ is the tetrahedron with vertices $a_1, \dots, a_k$ and $x \in X \cap \pi(\Gamma)$ is arbitrary, belongs to Y.

Theorem 2 (answer to Question 1). *The answer is "yes" if and only if Y is convex over the projection of itself.*

It is obvious that if there exists a desired convex body R, then Y is convex over the projection of itself.

If the desired set R exists, then for R one can take the convex hull $\mathbb{L}Y$ of Y. Indeed, the inclusions $Y \subset \mathbb{L}Y \subset R$ and the relation $\pi^{-1}(x) \cap Y = \pi^{-1}(x) \cap R$ imply that $\pi^{-1}(x) \cap Y = \pi^{-1}(x) \cap \mathbb{L}V$. This reduces Theorem 2 to the following theorem on the convex hull.

Theorem 3 (on the convex hull). *Assume that a subset Y of the space $\mathbb{R}^N$ equipped with a projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$ is convex over $X = \pi(Y)$. Then the intersection of the convex hull $\mathbb{L}Y$ of Y with $\pi^{-1}(X)$ coincides with Y.*

To prove Theorem 3, we need a lemma on sections of convex polytopes. For a convex polytope Δ and a point $A \in \Delta$, let Δ_A be the face of Δ containing A as an interior point. (It may happen that $\Delta_A = \Delta$.)

Lemma 4 (on the vertices of a section of a convex polytope). *Let Δ be a convex polytope in $\mathbb{R}^N$, let $\Gamma = M \cap \Delta$ be the section of Δ by an affine subspace M of dimension n, and let A be a vertex of the polytope Γ. Then the least affine space L_A containing Δ_A meets M only in the point A. In particular, the dimension of L_A does not exceed $N - n$.*

Proof. If the dimension of $L_A \cap M$ is positive, then A is an interior point of $L_A \cap M$, because A is an interior point of Δ_A. Consequently, A cannot be a vertex of $\Delta \cap M$. This is a contradiction, which proves the lemma. □

Proof of Theorem 3. By the Carathéodory theorem, to obtain the convex hull of Y, it suffices to take the union of tetrahedra Γ of dimension $\le N$ with vertices in Y. We need to show that, for each $x \in X$, any tetrahedron Γ whose vertices lie in Y meets $\pi^{-1}(x)$ in a subset if the convex set $Y \cap \pi^{-1}(x)$. By the lemma on the vertices of a section of a convex polytope, every vertex A of the polytope $\Gamma \cap \pi^{-1}(x)$ is the intersection of the space $\pi^{-1}(x)$ with the face Γ_A, which is mapped one-to-one onto the image of itself by the projection π. The dimension of Γ_A does not exceed n, and its vertices lie in Y. Since Y is convex over $\pi(Y)$, it follows that each vertex A of $\Gamma \cap M$ belongs to Y. Since the set $Y \cap \pi^{-1}(x)$ is convex, we see that $\Gamma \cap \pi^{-1}(x)$ is contained in Y, as desired. □

Thus, we have proved Theorem 3 and simultaneously Theorem 2.

Theorem 5 (on the property of being dense). *Assume that a subset Y of the space $\mathbb{R}^N$ equipped with a projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$ is convex over $X = \pi(Y)$. If X is everywhere dense in the convex hull $\mathbb{L}X$, then Y is everywhere dense in the convex hull $\mathbb{L}Y$.*

Proof. Without loss in generality, we can assume that $\mathbb{L}X$ is n-dimensional. Every point of $\mathbb{L}Y$ is contained in some simplex Γ with vertices in Y, which is mapped one-to-one onto the image of itself under the projection π. Since X is dense in the n-dimensional set $\mathbb{L}X$, it follows that the simplex Γ is a face of some n-dimensional simplex $\widetilde{\Gamma}$ with vertices in Y, which is mapped one-to-one onto the image of itself under the projection π. Since X is dense in $\mathbb{L}X$, it follows that $X \cap \pi(\widetilde{\Gamma})$ is dense in $\pi(\widetilde{\Gamma})$. Hence $\widetilde{\Gamma}$ lies in the closure of Y. □

3. SETS F-CONVEX OVER PROJECTIONS OF THEMSELVES

Let $\mathbb{F}$ be a subfield of reals. The elements of the set $\mathbb{F}^n \subset \mathbb{R}^n$ will be called $\mathbb{F}$-*points in* $\mathbb{R}^n$. In this section, we simplify the definition of convexity of a set over a projection of itself for the case where the projection is the set of all $\mathbb{F}$-points of a convex set. This definition can be simplified further for $\mathbb{F} = \mathbb{Q}$.

Definition. One says that a set $Y \subset \mathbb{R}^N$ is $\mathbb{F}$-*convex over its projection* $\pi(Y)$ if the following assertions hold:

1) The projection $\pi(Y)$ consists of all $\mathbb{F}$-points of some convex set in $\mathbb{R}^n$.

2) The set $Y_x = \pi^{-1}(x) \cap Y$ is convex for every $x \in \pi(Y)$.

3) If two points $a, b \in Y$ have distinct projections, $\lambda \in \mathbb{F}$, and $0 \le \lambda \le 1$, then $\lambda a + (1-\lambda)b \in Y$.

Theorem 6 (on $\mathbb{F}$-convexity). *Let Y be a subset of the space $\mathbb{R}^N$ equipped with a projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$. Then the following assertions hold*:

1) *If Y is $\mathbb{F}$-convex over the projection of itself, then Y is convex over the projection of itself.*

2) *If Y is convex over the projection of itself and its image $\pi(Y)$ consists of all $\mathbb{F}$-points of some convex set, then Y is $\mathbb{F}$-convex over the projection of itself.*

To prove the theorem, we need the following lemma.

Lemma 7 (on a simplex). *Let $x \in \mathbb{R}^n$ be an $\mathbb{F}$-point lying in the interior of a $(k-1)$-dimensional simplex whose vertices $A_1, \dots, A_k$ are $\mathbb{F}$-points, and let $k > 2$. Then the edge $[A_1, A_2]$ contains an $\mathbb{F}$-point B such that x is an interior point of the $(k-2)$-dimensional simplex with vertices $B, A_3, \dots, A_k$.*

Proof of Lemma 7. For B one should take the point of intersection of the edge $[A_1, A_2]$ with the $(k-2)$-dimensional affine space spanned by $x, A_3, \dots, A_k$. □

Proof of Theorem 6. To prove (1), it suffices to verify that if the images $\pi(a_1), \dots, \pi(a_k)$ of the set of k points $a_1, \dots, a_k \in Y$ are affinely independent, then $\Gamma \cap \pi^{-1}(x) \in Y$, where Γ is the tetrahedron with vertices $a_1, \dots, a_k$ and $x \in \pi(Y) \cap \pi(\Gamma)$ is arbitrary. For $k = 2$, this property is included in the definition of a set $\mathbb{F}$-convex over the projection of itself. Assume that the claim has already been proved for all sets of less than k points of Y. By assumption, the points $\pi(a_1), \dots, \pi(a_k)$, as well as x, are $\mathbb{F}$-points. By Lemma 7, the segment $[\pi(a_1), \pi(a_2)]$ contains an $\mathbb{F}$-point B such that x is an interior point of the simplex with vertices $B, \pi(a_3), \dots, \pi(a_k)$. By the definition of a set $\mathbb{F}$-convex over the projection of itself, the unique point $\overline{B}$ of intersection of the segment $[a_1, a_2]$ with the space $\pi^{-1}(B)$ lies in the convex body $Y \cap \pi^{-1}(B)$. By the inductive assumption, the unique point of intersection of the simplex $\overline{\Gamma}$ with vertices $\overline{B}, a_3, \dots, a_k$ and the space $\pi^{-1}(x)$ lies in the convex set $Y_x = \pi^{-1}(x) \cap Y$. To complete the proof of (1) it suffices to notice that $\overline{\Gamma}$ is contained in Γ and that $\Gamma \cap \pi^{-1}(x) = \overline{\Gamma} \cap \pi^{-1}(x)$. Claim (2) of the theorem is obvious. □

Among the subfields $\mathbb{F}$ of the reals, the fields $\mathbb{R}$ and $\mathbb{Q}$ are distinguished. By Theorem 6, every convex set in $\mathbb{R}^N$ is $\mathbb{R}$-convex over the projection of itself. The definition of $\mathbb{Q}$-convexity over the projection of itself can be slightly simplified.

Statement 8 (on $\mathbb{Q}$-convexity). *For the field $\mathbb{F} = \mathbb{Q}$, property* (3) *in the definition of $\mathbb{F}$-convexity can be replaced by the following property*:

(3′) *If two points $a, b \in Y$ have distinct projections and n is a positive integer, then the inclusion $a/n + (1 - 1/n)b \in Y$ holds.*

Proof. We need to show that if $a, b \in Y$ have distinct projections and $0 \le p/q \le 1$, then we have $pa/q + (1 - p/q)b \in Y$. Let $c_i \in \mathbb{R}^N$, $0 \le i \le p$, be the points determined by

$$c_0 = b, \qquad c_{j+1} = \frac{a}{q-j} + \left(1 - \frac{1}{q-j}\right)c_j.$$

One can readily verify that

$$c_i = \frac{i}{q}a + \left(1 - \frac{i}{q}\right)b.$$

The points c_i lie in Y. Indeed, $c_0 = b \in Y$. If $c_j \in Y$, then $c_{j+1} \in Y$ by the recursion relation. Hence $c_p \in Y$. The proof is complete. □

4. CLOSURE OF A SECTION OF A CONVEX SET

Here we show that, for a convex set Δ, the operations of intersection with an affine subspace and taking the closure commute if the affine subspace meets the interior of Δ (see Theorem 9).

Let Y be a subset of the space $\mathbb{R}^N$ equipped with the standard projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$, and let $X \subset \mathbb{R}^n$ be the image of Y under π. We are interested in the following version of Question 1 in Section 2.

Question 2. Is it true or false that there exists a closed convex set $V \subset \mathbb{R}^N$ such that $V \cap \pi^{-1}(X) = Y$?

It is required in Question 2 that V be closed. In Question 1, R is not required to be closed.

If the answer to Question 2 for a set Y is "yes," then, obviously, for V *one can always take the set* $\overline{\mathbb{L}Y}$*, i.e., the closure of the convex hull* $\mathbb{L}Y$ *of* Y. If the answer to Question 2 for a set Y is "yes," then, in particular, so is the answer to Question 1 for the same set. Hence *the answer to Question 2 can only be positive for sets* Y *satisfying the following conditions*:

1) *The set* Y *is convex over the projection of itself.*

2) *All sets* $Y_x = \pi^{-1}(x) \cap Y$ *are closed.*

These conditions are not sufficient for the answer to Question 2 to be "yes": the set $\pi^{-1}(a) \cap \overline{\mathbb{L}Y}$ may prove to be strictly larger than $\pi^{-1}(a) \cap Y$ if a is a boundary point of the set $X = \pi(Y)$ (see Example 1 below). However, *under these conditions one has* $\pi^{-1}(a) \cap \overline{\mathbb{L}Y} = \pi^{-1}(a) \cap Y$ *for all interior points* a *of* $\pi(Y)$. This follows from Theorem 9 on the closure of a section of a convex set (see below).

Example 1. Let $\mathbb{R}^2 \to \mathbb{R}^1$ be the standard projection of the plane $\mathbb{R}^2$ on the horizontal line $\mathbb{R}^1$, and let $Y \subset \mathbb{R}^2$ be the subset defined as $T \setminus (l_1 \cap l_2) \cup \{A\}$, where T is the set of interior points of a trapezoid whose bases l_1 and l_2 are vertical segments and A is the midpoint of l_1. The set Y is convex over the projection of itself and has closed fibers. The intersection of the line $\pi^{-1}(\pi(A))$ with the closure of Y is the base l_1, and the intersection of that line with Y is the point A on l_1.

Let Δ be a convex subset (possibly unbounded and nonclosed) of the space equipped with the standard projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$. The image $\pi(\Delta)$ of Δ is a convex set in $\mathbb{R}^n$.

Theorem 9 (on the closure of a section of a convex set))**.** *Let* x *be an interior point of* $\pi(\Delta)$*. Then the closure of the convex set* $\Delta_x = \pi^{-1}(x) \cap \Delta$ *coincides with the intersection of the affine subspace* $\pi^{-1}(x)$ *with the closure of* Δ*.*

Remark 1. The theorem can be restated as follows. *Assume that the section* Δ_M *of a convex set* Δ *by an affine subspace* M *contains an interior point of* Δ*. Then the closure of* Δ_M *coincides with the section of the closure of* Δ *by* M*.*

Remark 2. It is assumed in the theorem that x is an interior point of $\pi(Y)$. The theorem is not true without this assumption (see Example 1 above). In Section 6, we state conditions under which the theorem remains valid for the boundary points $x \in \partial(\pi(\Delta))$.

Proof of Theorem 9. Let c be a point that belongs to $\pi^{-1}(x)$ and does not lie in the closure of Δ_x. By the first separation theorem, there exists a hyperplane $V_x \subset \pi^{-1}(x)$ such that c lies on one side of V_x and the closure of Δ_x lies on the other side. The affine space $V_x \subset \mathbb{R}^N$ entirely lies in $\pi^{-1}(x)$ and does not meet Δ. By Corollary 1, there exists a hyperplane $V \subset \mathbb{R}^N$ containing V_x such that Δ lies in one of the two closed half-spaces of $\mathbb{R}^N$ with common boundary V.

Let us show that $V \cap \pi^{-1}(x) = V_x$. Since $V_x \subset V$ by assumption, we should verify that V cannot contain $\pi^{-1}(x)$. Indeed, otherwise the image of V under π would pass through the interior point x of the body $\pi(\Delta)$. By construction, $\pi(\Delta)$ should lie on one side of this section, but x is an interior point of $\pi(\Delta)$ by assumption. This is a contradiction, which shows that $\pi^{-1}(x) \cap V = V_x$.

The point c cannot lie in the closure of Δ. Indeed, it belongs to the interior of a half-space with boundary V, while Δ lies in the closure of the other half-space. The proof of the theorem is complete. $\square$

5. THE CHARACTERISTIC CONE OF A NONCOMPACT CONVEX BODY

Here we define the characteristic cone, which is a simple invariant of convex bodies permitting one to distinguish bounded and unbounded bodies. To each point x of a convex body Δ (unbounded in general), we assign the following set $K(x, \Delta)$: a vector v belongs to $K(x, \Delta)$ if $x + \lambda v \in \Delta$ for each $\lambda \geq 0$. The following assertion is a straightforward consequence of the convexity of Δ.

Statement 10 (on the characteristic cone). *If Δ is a convex set, then $K(x, \Delta)$ is a convex cone for each $x \in \Delta$. The cones $K(x_1, \Delta)$ and $K(x_2, \Delta)$ of distinct interior points $x_1, x_2 \in \Delta_*$ coincide.*

The *characteristic cone of a convex body* Δ is defined as the cone $K(\Delta) = K(x, \Delta)$ of its arbitrary interior point x.

Statement 11 (criterion for boundedness). *The cone $K(\Delta)$ does not coincide with the point* 0 *if and only if the set Δ is unbounded. The cone $K(\Delta)$ of a convex set is always closed.*

Proof. Without loss in generality, we can assume that Δ is of full dimension. For an interior point $x \in \Delta_*$, there exists a ball B_r centered at x and contained in Δ. If Δ is unbounded, then there exists a sequence $y_i \in \Delta$ such that $\|y_i\| \to \infty$ and the sequence $(y_i - x)/\|(y_i - x)\|$ of unit vectors converges to some vector v. One has $v \in K(x, \Delta)$. Indeed, Δ contains the convex hull Y_i of the union $B_r \cap \{y_i\}$. Let l_i be the intersection of Y_i with the ray l formed by the points $x + \lambda v$, $\lambda \geq 0$. The sets l_i lie in Δ, and their union covers the entire l. Hence $v \in K(x, \Delta)$.

Let v be a limit point of unit vectors $v_i \in K(x, \Delta)$. The body Δ contains the cylindrical bodies $B_r + L_i$, where L_i is the ray formed by the points λv, $\lambda \geq 0$. The union of the sets $B_r + L_i$ contains the ray $x + \lambda v$, $\lambda \geq 0$. Hence $K(x, \Delta)$ is closed. $\square$

Corollary 12 (on boundedness). *Let Δ be a convex subset of the space $\mathbb{R}^N$ equipped with the projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$. Let the image $\pi(\Delta) \subset \mathbb{R}^n$ be a bounded set. The set $\Delta \subset \mathbb{R}^N$ is bounded if and only if the set $\Delta_x = \Delta \cap \pi^{-1}(x)$ is bounded for some interior point $x \in \pi(\Delta)$.*

Proof. Let us show that if Δ_x is bounded, then so is Δ. Indeed, if Δ is unbounded, then the cone $K(a, \Delta)$, $a \in (\Delta_x)_*$, contains nonzero vectors. Since the closed cone $a + K(a, \Delta)$ is contained in Δ and $\pi(\Delta)$ is bounded, it follows that the cone $(a + K(x, \Delta)) \cap \pi^{-1}(x)$ contains vectors other than a. Hence Δ_x is unbounded. This contradiction proves the lemma. $\square$

6. CONTINUITY OF SECTIONS AS FUNCTIONS OF THE PARAMETER

Let Δ be a closed convex subset of the space $\mathbb{R}^N$ equipped with the standard projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$. Consider the section $\Delta_x = \pi^{-1}(x) \cap \Delta$ of Δ by the affine space $\pi^{-1}(x)$ as a function of a point $x \in \pi(\Delta)$. In this section, we discuss the following question: is it true that Δ_x continuously depends on x?

First, note that the answer is negative if one does not introduce additional conditions (even if the restriction of π to Δ is a proper mapping). The section may experience a jump near a boundary point $a \in \partial(\pi(\Delta))$.

Example 2. Let $\Delta \subset \mathbb{R}^3$ be a cone over the disk B_2 lying on the horizontal plane $\mathbb{R}^2$, and let the vertex O of the cone lie over some point A of the boundary circle ∂B_2. The image of Δ under the standard projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is the disk B_2. If $x \in \partial B_2$ and $x \neq A$, then $\Delta_x = \{x\}$. The section Δ_A is the segment $[O, A]$. The dependence of the sections on the point of B_2 in a neighborhood of $A \in B_2$ is discontinuous.

In the example, $\pi(\Delta)$ is a disk. If $\pi(\Delta)$ is a polytope, then Δ_x continuously depends on $x \in \pi(\Delta)$.

Theorem 13 (on the continuity of sections as functions of the parameter). *Let $\Delta \subset \mathbb{R}^N$ be closed and convex, and let $\pi(\Delta) \subset \mathbb{R}^n$ be a polytope. Assume that the section Δ_x is bounded for some interior point $x \in \pi(\Delta)$. Then the sections Δ_x depend continuously in the Hausdorff metric on the point $x \in \pi(\Delta)$.*

Proof. Under the assumptions of the theorem, the boundedness of Δ_x implies the compactness of the closure of Δ by corollary 12. To prove that the function Δ_x is continuous at a point $a \in \pi(\Delta)$, we proceed as follows. First, for each x in a neighborhood U of a, we construct a convex set $V_x \subset \Delta_x$ that is a continuous (and even piecewise linear) function of x. Second, we show that the function $x \mapsto \Delta_x$ is upper semicontinuous.

(1) *Lower bound.* Let us construct the family V_x. Take a triangulation of $\pi(\Delta)$ such that a is one of the triangulation vertices, $a = A_1$. The closure $\overline{U}$ of the star U of a with respect to this triangulation is a union of finitely many simplices containing the vertex $A_1 = a$. Let us define a piecewise linear mapping F of $\overline{U}$ into the set of convex subsets of Δ such that $F(x) \subseteq \Delta_x$. First, we define F at the triangulation vertices A_i. For the vertex $A_1 = a$, set $F(A_1) = \Delta_a$. For the other vertices A_i, $i > 1$, set $F(A_i) = C_i$, where C_i is an arbitrary point in the fiber Δ_{A_i}. Next, we define F by linearity in each simplex of the triangulation: if a point x lies in a simplex of $\overline{U}$ with vertices $A_1 = a, A_2, \dots, A_k$ and

$$x = \lambda_1 A_1 + \lambda_1 A_1 + \dots + \lambda_k A_k,$$

where $\sum \lambda_i = 1$ and $\lambda_i \geq 0$, then

$$F(x) = \lambda_1 \Delta_a + \lambda_2 C_2 + \dots + \lambda_k C_k.$$

For $x \in \overline{U}$, set $V_x = F(x)$. The family V_x has the desired properties.

(2) *Upper semicontinuity.* For a positive number ρ, let B^1_ρ and B^2_ρ be the closed balls of radius ρ centered at the origin in $\mathbb{R}^n$ and in the kernel $\mathbb{R}^{N-k}$ of the projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$, respectively. The boundary Γ_ε of the set $\Delta_a + B^2_\varepsilon$, where $\varepsilon > 0$, is a compact set that does not meet the closed set Δ. Hence there exists a $\delta > 0$ such that the set $\Gamma_\varepsilon + B^1_\delta + B^2_\delta$ does not meet Δ. For each point x in the ball $a + B^1_\delta$ of radius δ centered at a, the section Δ_x is contained in the set $\Delta_a + B^2_\delta$ translated into the space $\pi^{-1}(x)$ (i.e., in the set $\Delta_a + B^2_\delta + (x - a)$). This proves that Δ_x is an upper semicontinuous function of $x \in \pi(\Delta)$.

The theorem follows from (1) and(2). $\square$

We say that a set $X \subset \mathbb{R}^n$ is *polyhedral near a point* $a \in X$ if there exists a polytope P and a neighborhood $U \subset \mathbb{R}^n$ of a such that $U \cap X = U \cap P$. Any set X is polyhedral near its arbitrary interior point a. A closed polyhedral cone is polyhedral near any of its points. Theorem 13 readily implies the following corollary.

Corollary 14 (on polyhedrality and continuity). *Let $\Delta \subset \mathbb{R}^N$ be closed and convex, and let $\pi(\Delta) \subset \mathbb{R}^n$ be polyhedral near some point $a \in \pi(\Delta)$. Assume that the section Δ_b is bounded for some interior point $b \in \pi(\Delta)$. Then the sections Δ_b depend continuously in the Hausdorff metric on the point $x \in \pi(\Delta)$ in a neighborhood of a.*

7. CONVEX SETS THAT ARE PROJECTED ONTO A POLYTOPE

Here we classify all convex sets Δ projected onto a given convex polytope P and such that the preimage in Δ of every point $x \in P$ is a compacts set. The classification is given in Theorems 15 and 16 below. Corollary 18 gives a sufficient condition for the compactness of Δ in terms of the volume of its sections.

Associated with each polytope is the set of its faces, which includes the polytope itself.

Theorem 15 (on bodies convex over a polytope). *Let $\Delta \subset \mathbb{R}^N$ be a convex subset, and let $\pi(\Delta) \subset \mathbb{R}^n$ be a polytope. Assume that the set $\Delta_x = \Delta \cap \pi^{-1}(x)$ is closed for all $x \in \pi(\Delta)$ and bounded for some interior point $x \in \pi(\Delta)$. Then the following assertions hold*:

1) *The closure Δ_Γ of the preimage $\pi^{-1}(\Gamma)$ of a face $\Gamma \subset \pi(\Delta)$ is a convex compact set.*

2) *If a face Γ_1 is contained in a face Γ_2, then $\Delta_{\Gamma_1} \subset \Delta_{\Gamma_2}$.*

3) *If x is an interior point of a face Γ, then $\pi^{-1}(x) \cap \Delta_\Gamma = \Delta_x$.*

Proof. Under the assumptions of the theorem, the boundedness of Δ_x implies the compactness of the closure of Δ by Corollary 12. By Theorem 9, $\Delta_x = \overline{\Delta} \cap \pi^{-1}(x)$ for every interior point x of $\pi(\Delta)$. Likewise, $\Delta_x = \Delta_\Gamma \cap \pi^{-1}(x)$ for each interior point x of a face $\Gamma \subset \pi(\Delta)$. This proves the theorem. $\square$

Theorem 15 admits a converse. Let $\mathbb{R}^N$ be equipped with the standard projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$. The following theorem is obvious.

Theorem 16 (converse of Theorem 15). *Let $P \subset \mathbb{R}^n$ be a polytope. Assume that, for each face $\Gamma \subseteq P$, there is given a convex compact set $\Delta_\Gamma \subset R^N$ such that $\pi(\Delta_\Gamma) = \Gamma$ and if $\Gamma_1 \subseteq \Gamma_2$, then $\Delta_{\Gamma_1} \subset \Delta_{\Gamma_2}$. Then the set $\Delta = \bigcup(\Delta_\Gamma \cap \pi^{-1}(\Gamma)_*)$, where Γ_* is the set of interior points of a convex face Γ, is convex. All fibers $\Delta_x = \pi^{-1}(x) \cap \Delta$ are closed. If x is an interior point of a face Γ, then $\Delta_x = \pi^{-1}(x) \cap \Delta_\Gamma$.*

The volume V is a continuous function of the space of bounded convex sets equipped with the Hausdorff metric. The volume has the following property of strict monotonicity. Let $\Delta_2 \supset \Delta_1$ be distinct closed convex bodies. If $V(\Delta_2) > 0$, then $V(\Delta_2) > V(\Delta_1)$. (If $V(\Delta_2) = 0$, then $V(\Delta_2) = V(\Delta_1)$.)

Under the assumptions of Theorem 15, the function ϕ taking each point $x \in \pi(\Delta)$ to the $(N-n)$-volume $\phi(x)$ of the section Δ_x is defined on the polytope $\pi(\Delta)$. Theorem 16 implies the following theorem.

Theorem 17 (on the volume of a fiber). *The restriction ϕ_{Γ_*} of the function ϕ to the set Γ_* of interior points of a face $\Gamma \subset \pi(\Delta)$ is a continuous function.* (*In particular, the restriction ϕ_{Δ_*} of ϕ_Δ to the set of interior points of Δ is continuous.*) *The function ϕ_{Γ_*} extends to a continuous function ϕ_Γ on the entire face Γ. If $\Gamma_1 \subset \Gamma_2$, then $\phi_{\Gamma_1} \leq \phi_{\Gamma_2}$. If $\phi_\Gamma(a) = \phi(a)$ for each face Γ containing a point a and $\phi(a) > 0$, then the function that takes each point $x \in \pi(\Delta)$ to the section Δ_x is continuous at the point a.*

Corollary 18 (on the continuity of the fiber and the volume). *Under the assumptions of Theorem 16, suppose that the volume function ϕ is continuous and positive on the entire polytope P. Then Δ is a compact set, and the fiber Δ_x continuously depends on the point $x \in P$.*

Definition. A set $X \subset P$ is said to be *dense in a polytope P and its faces* if the vertices of P belong to X and the intersection $X \cap \Gamma$ is dense in Γ for each face $\Gamma \subseteq P$ (including $\Gamma = P$).

Example 3. Let $P \subset \mathbb{R}^n$ be a polytope all of whose vertices are $\mathbb{F}$-points in $\mathbb{R}^n$. Then the set X of all $\mathbb{F}$-points of P is dense in P and the faces of P.

Our immediate problem is to classify the sets $\Delta \subset \mathbb{R}^N$ that are convex over a given set $X \subset P$ dense in a polytope P and its faces and have compact fibers Δ_x over all points $x \in X$. The theorems stated below reduce this problem to the classification problem for $X = P$, which has been solved above.

Let a set $\Delta \subset \mathbb{R}^N$ be convex with respect to a projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$ over a set X dense in a polytope P and its faces. To each face $\Gamma \subseteq P$, we assign the *set $\widetilde{\Delta}_{\Gamma_*}$ defined as the intersection of the closure of $\pi^{-1}(\Gamma_* \cap \Delta)$ with $\pi^{-1}(\Gamma_*)$*, where Γ_* is the interior of Γ.

Definition. The set $\cup\widetilde{\Delta}_{\Gamma_*} \subset \mathbb{R}^N$ equal to the union of the sets Δ_{Γ_*} over all faces $G \subseteq P$ of P (including $G = P$) is called the *closure of Δ over the faces of P*.

Assume that a set $\Delta \subset \mathbb{R}^N$ is convex with respect to a projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$ over a polytope P and all sections $\Delta_x = \Delta \cap \pi^{-1}(x)$, $x \in P$, are compact.

Definition. The set $\Delta \cap \pi^{-1}(X)$ is called the *restriction of Δ to X*.

The following theorem shows that the operation of closure over the faces of P and the operator of restriction to X establish a one-to-one correspondence between the sets Δ convex over a set X dense in a polytope P and its faces and the similar sets for $X = P$.

Theorem 19 (on the convexity over a subset of a polytope)**.** (1) *If Δ is convex over a set X dense in a polytope P and its faces, and Δ has compact fibers, then the closure of Δ over the faces of P is convex over P and has compact fibers.*

(2) *If Δ is convex over P and has compact fibers, then its restriction $\Delta \cap \pi^{-1}(X)$ to a set X dense in a polytope P and its faces is convex over X and has compact fibers.*

(3) *If Δ is convex over a set X dense in a polytope P and its faces, then $\left(\bigcup \widetilde{\Delta}_{\Gamma_*}\right) \cap \pi^{-1}(X) = \Delta$.*

(4) *If Δ is convex over P and has compact fibers, then $\cup\widetilde{\Omega}_{\Gamma_*} = \Delta$, where $\Omega = \Delta \cap \pi^{-1}(X)$.*

Proof. (1) Let us verify that the set $\bigcup \widetilde{\Delta}_{\Gamma_*}$ is convex. To each face $\Gamma \subseteq P$, we assign the closure in $\mathbb{R}^N$ of the linear span of the set $\pi^{-1}(\Gamma) \cap \Delta$. By Theorem 5, this set coincides with the closure $\widetilde{\Delta}_\Gamma$ of the set $\pi^{-1}(\Gamma) \cap \Delta$. Consequently, the closed set $\widetilde{\Delta}_\Gamma$ is convex. If $\Gamma_1 \subset \Gamma_2$, then $\widetilde{\Delta}_{\Gamma_1} \subset \widetilde{\Delta}_{\Gamma_2}$. Hence the set $\bigcup \widetilde{\Delta}_{\Gamma_*}$ is convex.

(2) This is obvious.

(3) The set $\cup\widetilde{\Delta}_{\Gamma_*}$ coincides with the intersection of the closed convex hull of the set $\pi^{-1}(\Gamma) \cap \Delta$ with the set Δ (cf. the proof of claim (1)). Now the desired assertion follows by applying the theorem on the closure of a section to the convex hull of the set $\pi^{-1}(\Gamma) \cap \Delta$.

(4) This follows from claim (3) in Theorem 15. $\square$

8. HOMOGENEOUS SEMIGROUPS OF CONVEX BODIES

In this section, we solve the problem arisen from algebraic geometry and stated in the introduction. We also solve the problem of describing semigroups of convex bodies over the semigroup $\mathbb{F}^n_{\geq 0}$ in $\mathbb{R}^n$. All these results are a straightforward consequence of the preceding sections.

Let $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$ be the standard projection, and let there be given an additive semigroup $T \subset \mathbb{R}^n$ containing the point 0. The following semigroups T will be of interest to us: the semigroup $\mathbb{Z}^n_{\geq 0}$ of points with nonnegative integer coordinates; the semigroup $\mathbb{F}^n_{\geq 0}$ of $\mathbb{F}$-points with nonnegative coordinates in $\mathbb{R}^n$; in particular, the semigroups $\mathbb{Q}^n_{\geq 0}$ and $\mathbb{R}^n_{\geq 0}$.

Let $G \subset \mathbb{R}^N$ be a subset whose image under π coincides with T. For each $a \in T$, let

$$G_a = \pi^{-1}(a) \cap G.$$

We say that the set G is a *homogeneous semigroup of convex bodies over the semigroup T* with respect to the projection π (or simply a *semigroup over T*) if the following conditions are satisfied:

1) The set G_a is convex for each $a \in T$.

2) For the point $0 \in T$, the set G_0 consists of the point $0 \in \mathbb{R}^N$.

3) If $a \in T$ and $b = \lambda a$, where $\lambda \geq 0$, then $G_b = \lambda G_a$.

4) If $a, b \in T$ and $a + b = c$, then $G_a + G_b \subset G_c$. (In other words, the set G is a subgroup in $\mathbb{R}^N$ with respect to addition.)

Our aim is to describe homogeneous semigroups of convex bodies over $\mathbb{Z}^n_{\geq 0}$ and $\mathbb{F}^n_{\geq 0}$ (in particular, over $\mathbb{Q}^n_{\geq 0}$ and $\mathbb{R}^n_{\geq 0}$). The description of semigroups over $\mathbb{Z}^n_{\geq 0}$ can be reduced to that of semigroups over $\mathbb{Q}^n_{\geq 0}$.

Definition. (1) Let a set $G \subset \mathbb{R}^N$ be a semigroup over $\mathbb{Z}^n_{\geq 0}$ with respect to a projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$. The set $\mathbb{Q}G \subset \mathbb{R}^N$ determined by the condition that $y \in \mathbb{Q}G$ if and only if there exists a positive integer k such that $ky \in G$ is called the *extension of G by homogeneity*.

(2) Let a set $G \subset \mathbb{R}^N$ be a semigroup over $\mathbb{Q}^n_{\geq 0}$ with respect to a projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$. The set $G\mathbb{Z} \subset \mathbb{R}^N$ equal to $G \cap \pi^{-1}(\mathbb{Z}^n_{\geq 0})$ is called the *restriction of G to $\mathbb{Z}^n_{\geq 0}$*.

The following lemma shows that the operation of extension by homogeneity and the operator of restriction to $\mathbb{Z}^n_{\geq 0}$ establish a one-to-one correspondence between semigroups over $\mathbb{Z}^n_{\geq 0}$ and semigroups over $\mathbb{Q}^n_{\geq 0}$.

Lemma 20 (on semigroups over $\mathbb{Z}^n_{\geq 0}$ and over $\mathbb{Q}^n_{\geq 0}$)**.** (1) *If G is a semigroup over $\mathbb{Z}^n_{\geq 0}$, then its extension $\mathbb{Q}G$ by homogeneity is a semigroup over $\mathbb{Q}^n_{\geq 0}$.*

(2) *If G is a semigroup over $\mathbb{Q}^n_{\geq 0}$, then its restriction $G\mathbb{Z}$ to $\mathbb{Z}^n_{\geq 0}$ is a semigroup over $\mathbb{Z}^n_{\geq 0}$.*

(3) *If G is a semigroup over $\mathbb{Z}^n_{\geq 0}$, then $(\mathbb{Q}G)\mathbb{Z} = G$.*

(4) *If G is a semigroup over $\mathbb{Q}^n_{\geq 0}$, then $\mathbb{Q}(G\mathbb{Z}) = G$.*

The proof is by a straightforward verification, and we omit it.

Lemma 21 (semigroups over $\mathbb{F}^n_{\geq 0}$ and $\mathbb{F}$-convexity)**.** *A set $G \subset \mathbb{R}^N$ is a semigroup over $\mathbb{F}^n_{\geq 0}$ if and only if the following conditions hold*:

1) *The set G is $\mathbb{F}$-convex over $\pi(G)$.*

2) $\pi(G) = \mathbb{F}^n_{\geq 0}$.

3) $\pi^{-1}(0) \cap G = 0 \in \mathbb{R}^N$.

4) *If* $\lambda \in \mathbb{F}$, $\lambda \geq 0$, *and* $x \in G$, *then* $\lambda x \in G$.

Proof. Let $G \subset \mathbb{R}^N$ be a semigroup of convex bodies over the semigroup $\mathbb{F}^n_{\geq 0}$. By definition, G satisfies (2)–(4). Let us verify that G satisfies (1). Let $x, y \in \mathbb{F}^n_{\geq 0}$ be distinct points, and let a point $z \in \mathbb{F}^n_{\geq 0}$ lie in the interior of the segment $[x, y]$. Let $A \in G_x$ and $B \in G_y$ be arbitrary points, and let C be the point of intersection of the segment $[A, B]$ with the space $\pi^{-1}(z)$. Let us prove that $C \in G_z$. Let $\lambda \in \mathbb{F}_+$ be the number defined by $z = \lambda x + (1-\lambda)y$. By homogeneity, one has $\lambda A \in G_{\lambda x}$ and $(1-\lambda)B \in G_{(1-\lambda)y}$. By the definition of a semigroup of convex bodies, the point $C = \lambda A + (1-\lambda)B$ lies in $G_z = G_{\lambda x+(1-\lambda)y}$.

Assume that the set G has properties (1)–(4). Let us show that G is a homogeneous semigroup of convex bodies over $\mathbb{F}^n_{\geq 0}$. We should only verify that if $x, y \in G$, then $x + y \in G$. First, let us show that the midpoint $u = (x + y)/2$ of the segment $[x, y]$ belongs to G. Indeed, if $\pi(x) = \pi(y) = a$, then the desired inclusion follows from the convexity of the fiber G_a. If $\pi(x) \neq \pi(y)$, then $\pi(u)$ is the midpoint of the segment $[\pi(x), \pi(y)]$, and $u \in G$, because G is convex over the $\mathbb{F}$-points. Next, $x + y = 2u$, and $x + y \in G$ by homogeneity. □

Corollary 22 (on semigroups over $\mathbb{R}^n_{\geq 0}$). *A set* $G \subset \mathbb{R}^N$ *is a semigroup over* $\mathbb{R}^n_{\geq 0}$ *if and only if the following conditions are satisfied*:

(1)) G *is a convex cone*;

(2)) $\pi(G) = \mathbb{R}^n_{\geq 0}$;

(3)) $\pi^{-1}(0) \cap G = 0 \in \mathbb{R}^N$.

We say that a semigroup G of convex bodies over a semigroup T is a *semigroup with compact fibers* if the set $G_x = G \cap \pi^{-1}(x)$ is compact for each $x \in T$.

Statement 23 (on the description of semigroups over $\mathbb{F}^n_{\geq 0}$). *The description of semigroups G with compact fibers over $\mathbb{F}^n_{\geq 0}$ can be reduced to the description of sets Δ convex over the set of $\mathbb{F}$-points of the standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$.*

Proof. Let $P \in \mathbb{R}^n$ be the standard $(n-1)$-dimensional simplex, whose vertices are the standard basis vectors in $\mathbb{R}^n$. If G is a semigroup over $\mathbb{F}^n_{\geq 0}$, then the set $\Delta = G \cap \pi^{-1}(P)$ is convex over the set of $\mathbb{F}$-points of P. Conversely, let Δ be convex over the set of $\mathbb{F}$-points of P. For a nonzero $a \in \mathbb{F}^n$, set

$$\lambda = (a_1 + \dots + a_n) \qquad \text{and} \qquad G_a = \lambda\pi(a/\lambda) \cap \Delta.$$

Also set $G_0 = 0$. The union G of the sets G_a is a semigroup over $\mathbb{F}^n$. This follows from the homogeneity of G by Lemma 21. Obviously, distinct sets Δ corresponding to distinct semigroups G, and vice versa. A semigroup G is a semigroup with compact fibers if and only if all sets $\Delta_x = \pi^{-1}(x) \cap \Delta$, $x \in P$, are compact. □

The problem of describing semigroups with compact fibers over $\mathbb{Z}^n_{\geq 0}$ and $\mathbb{F}^n_{\geq 0}$ is thereby solved. Let us give a detailed answer for the case of $\mathbb{Z}^n_{\geq 0}$, because it is this case that is of interest in algebraic geometry.

To each subset $J \subset I_n = \{1, 2, \dots, n\}$, we assign the coordinate subspace R_j defined in the space $\mathbb{R}^n$ with coordinates $x_1, \dots, x_n$ by the equations $x_i = 0$, $i \notin J$. Let the space $\mathbb{R}^N$ be equipped with the standard projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^n$.

Definition. Let there be a cone $\Delta_J \subseteq \mathbb{R}^N$ assigned to each subset $J \subseteq I_n$. We say that the family $\{\Delta_J\}$ of cones is *compatible* if the following conditions are satisfied:

1) For each J, the restriction $\pi\colon \Delta_J \to \mathbb{R}_J$ of the projection π to the cone Δ_J is a proper mapping of Δ_J onto the coordinate subspace $\mathbb{R}_J$.

2) If $J_1 \subset J_2$, then $\Delta_{J_1} \subset \Delta_{J_2}$.

Theorem 24 (on semigroups with compact fibers over $\mathbb{Z}^n_{\geq 0}$). (1) *To each compatible family of cones Δ_J, there corresponds a semigroup G with compact fibers over $\mathbb{Z}^n_{\geq 0}$ by the following rule: the fiber G_m of G over a point $m \in \mathbb{Z}^n_{\geq 0}$ is $\pi^{-1}(m) \cap \Delta_J$, where J is the minimum subset such that $m \in \mathbb{R}_J$.*

(2) *The rule in* (1) *defines a one-to-one correspondence between compatible families of cones and semigroups with compact fibers over $\mathbb{Z}^n_{\geq 0}$.*

The semigroups with compact fibers over $\mathbb{R}^n_{\geq 0}$ that are of interest from the viewpoint of algebraic geometry admit a simpler description. This is because the volume of the fiber of such a semigroup over a point $m \neq 0$ is positive, and the function can be extended by continuity to the entire positive octant.

Definition. A semigroup G over $\mathbb{Z}^n_{\geq 0}$ is said to be *controlled* if it has compact fibers and there exists a continuous *control function* ϕ on $\mathbb{R}^n_{\geq 0}$ such that the following conditions hold:

1) If $a \in \mathbb{Z}^n_{\geq 0}$, then $\phi(a)$ is the $(N-n)$-volume of the body G_a.

2) If $x \in \mathbb{R}^n_{\geq 0}$ and $\lambda \geq 0$, then $\phi(\lambda x) = \lambda^{N-n}\phi(x)$;

3) If $x \in \mathbb{R}^n_{\geq 0}$ and $x \neq 0$, then $\phi(x) > 0$.

Definition. Let $\phi\colon \mathbb{R}^N_{\geq 0} \to \mathbb{R}$ be a continuous function homogeneous of degree $N-n$ and positive everywhere except for the point 0. A cone $\Delta \subset \mathbb{R}^N$ is said to be *compatible* with ϕ if the following conditions are satisfied:

1) The projection $\pi\colon \Delta \to \mathbb{R}^n$ is a proper mapping onto $\mathbb{R}^n$.

2) The $(N-n)$-volume of the fiber $\Delta_x = \pi^{-1}(x)$, $x \in \mathbb{R}^N_{\geq 0}$, is $\phi(x)$.

Theorem 25 (on controlled semigroups over $\mathbb{Z}^n_{\geq 0}$). (1) *To each cone $\Delta \subset \mathbb{R}^N$ compatible with ϕ, there corresponds a controlled semigroup G over $\mathbb{Z}^n_{\geq 0}$ with control function ϕ by the following rule: the fiber G_m of G over a point $m \in \mathbb{Z}^n_{\geq 0}$ is $\pi^{-1}(m) \cap \Delta_J$.*

(2) *The rule in* (1) *defines a one-to-one correspondense between families of cones compatible with ϕ and controlled semigroups over $\mathbb{Z}^n_{\geq 0}$ with control function ϕ.*

9. CONVEX FUNCTIONS AND THEIR CONTINUOUS EXTENSIONS

Here we define convex functions on an arbitrary subset $X \subset \mathbb{R}^n$ and discuss issues related to the continuity of such functions and their convex continuation to the convex hull of X.

Let $X \subset \mathbb{R}^n$ be some set, and let $f\colon P \to \mathbb{R}$ be a real function on X. The *epigraph* $\Gamma_{\geq f}$ of f is the subset of $\mathbb{R}^{n+1} = \mathbb{R}^n + \mathbb{R}^1$ defined as follows: $(x,y) \in \Gamma_{\geq f}$ if and only if $x \in P$ and $y \geq f(x)$.

The space $\mathbb{R}^{n+1}$ is equipped with the natural projection $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$.

Definition. A function is said to be *convex on a set* X if the epigraph $\Gamma_{\geq f} \subset \mathbb{R}^{n+1}$ is a convex set over its projection X.

Let $\mathbb{R}^{n+1} = \mathbb{R}^n + \mathbb{R}^1$, and let $Y \subset \mathbb{R}^{n+1}$ be a convex set whose intersection with each line $\pi^{-1}(x)$ is closed and is either the whole line $\pi^{-1}(x)$ or some ray of the form (x,u), $u \geq u_0(x)$.

Let $X = \pi(Y)$, and let X_* be the set of interior points of X. Assume that Y_x is a ray for at least one point $x \in X_*$. Then the intersection of the boundary of Y with the set $X_* + \mathbb{R}^1$ is the graph of a continuous convex function on X_*. In other words, the following theorem holds.

Theorem 26 (on a function related to a convex set). *Assume that there exists at least one point $c \in \mathbb{R}^{n+1}$ such that $c \notin Y$ and $\pi(c) \in X_*$. Then, for each $x \in X$, the set of points $(x, y) \in Y$ contains a point with minimum coordinate $y = f(x)$. The function f defined by this relation is continuous on X_*.*

Proof. The assertion of the theorem can be reduced to Theorem 13 on the continuity of sections. That theorem does not apply directly, for the epigraph is not bounded. We need an additional argument.

By the second separation theorem, there exists a hyperplane passing through c such that Y lies in one of the two closed half-spaces for which this hyperplane is the common boundary. This hyperplane cannot be vertical, because $\pi(c)$ is an interior point of X^0. Hence it can be viewed as the graph of a linear function on $\mathbb{R}^n$. The set Y lies above the graph of this linear function. Hence the coordinate y is bounded below on the closed set $\pi^{-1}(x) \cap Y$ for each $x \in X$. Consequently, the function $y = f(x)$ is defined on the entire set X.

Since $a \in X_*$ it follows that there exists a simplex Γ with vertices $a_1, \dots, a_{n+1} \in X_*$ such that a is an interior point of Γ. Let C be the maximum of the numbers $f(a_1), \dots, f(a_k)$. Since Y is convex, it follows that f does not exceed C on Γ.

Define a convex set $Y_{\text{com}} \subset \mathbb{R}^{n+1}$ by setting $Y_{\text{com}} = Y \cap \pi^{-1}(\Gamma) \cap L_C$, where L_C is the half-space formed by the points (x, u), $u \le C$. By construction, Y_{com} is a compact convex set, $\pi(Y_{\text{com}}) = \Gamma$, and the minimum value of y at the points $(x, y) \in \pi^{-1}(x) \cap Y_{\text{com}}$ is equal to $f(x)$ for each $x \in \Gamma$. By Theorem 13, the set $\pi^{-1}(x) \cap Y_{\text{com}}$ continuously depends on $x \in \Gamma_*$. Hence the function f is continuous. □

Theorem 27 (on a continuous extension of a convex function). *Let $X \subset \mathbb{R}^n$ be some set, let $(\mathbb{L}X)_*$ be the set of interior points of the convex hull $\mathbb{L}X$ of X, and let $\phi\colon X \to \mathbb{R}$ be a convex function on X. Then there exists a continuous function $f\colon (\mathbb{L}X)_* \to \mathbb{R}^1$ on $(\mathbb{L}X)_*$ whose restriction to $X \cap (\mathbb{L}X)_*$ coincides with ϕ.*

Proof. If the epigraph $\Gamma_{\ge\phi}$ of ϕ contains some point (x, y), then it contains all points (x, u) with $u \ge y$. It is seen from the Carathéodory theorem that the convex hull $\mathbb{L}\Gamma_{\ge\phi}$ of the epigraph $\Gamma_{\ge\phi}$ has this property as well. Obviously, the closed convex hull $\overline{\mathbb{L}\Gamma}_{\ge\phi}$ of the epigraph has the same property. Let us apply the preceding theorem to the set $\overline{\mathbb{L}\Gamma}_{\ge\phi}$. Let $f\colon X_* \to \mathbb{R}^1$ be the function whose existence and uniqueness are guaranteed by the preceding theorem. By Theorem 9 on the closure of a section of a convex set, $f = \phi$ on the set $X \cap X_*$. The proof of the theorem is complete. □

Theorem 27 only deals with interior points of the convex hull of X and cannot be extended to boundary points.

Example 4. Every function on the circle $X = \partial B_2$ is convex on X, because simplices with vertices on X do not contain any points of X other than vertices. Fortunately, Theorem 27 does not say anything about *arbitrary* functions on the circle, because the set $X \cap X_*$ is empty in this case.

Corollary 28 (on the uniqueness of the extension). *Let $X \subset \mathbb{R}^n$ be a set everywhere dense in its convex hull $\mathbb{L}X$, and let $\phi\colon X \to \mathbb{R}$ be a convex function on X. Then there exists a unique continuous function $f\colon (\mathbb{L}X)_* \to \mathbb{R}^1$ that is on the interior $(\mathbb{L}X)_*$ of the convex hull and whose restriction to $X \cap (\mathbb{L}X)_*$ coincides with ϕ.*

Theorem 29 (on the behavior of the extension on the faces of the boundary). *Let $\phi\colon X \to \mathbb{R}$ be a function convex on a set X dense in a polytope P and in the faces of P. Then, on each face $\Gamma \subset P$ (including the face $\Gamma = P$), there exists a continuous function f_Γ such that the following conditions hold*:

1) *If $x \in X$ is an interior point of Γ, then $f_\Gamma(x) = \phi(x)$.*

2) *If a face Γ_1 is contained in a face Γ_2, then $f_{\Gamma_1} \ge f_{\Gamma_2}$.*

This theorem can be derived from Theorem 19 in the same way as Theorem 26 is derived from Theorem 13.

The definition of convexity can be simplified for functions on the set $\Delta_{\mathbb{F}}$ of all $\mathbb{F}$-points of a convex set $\Delta \subset \mathbb{R}^n$.

Definition. A function $\phi\colon \Delta_{\mathbb{F}} \to \mathbb{R}^1$ is said to be $\mathbb{F}$-*convex* if

$$\phi(\lambda a + (1-\lambda)b) \le \lambda\phi(a) + (1-\lambda)\phi(b)$$

for two arbitrary points $a, b \in \Delta_{\mathbb{F}}$ and an arbitrary number $\lambda \in \mathbb{F}$, $0 \le \lambda \le 1$.

Statement 30 (on $\mathbb{F}$-convexity of functions). *A function ϕ is convex of the set $\Delta_{\mathbb{F}}$ if and only if it is $\mathbb{F}$-convex.*

Proof. The claim follows from Theorem 6. □

The definition of $\mathbb{F}$-convexity can be slightly simplified further for the field $\mathbb{F} = \mathbb{Q}$ of rationals.

Definition. A function $\phi\colon \Delta_{\mathbb{Q}} \to \mathbb{R}^1$ is said to be $\mathbb{Q}$-*convex* if

$$\phi\left(\frac{a}{k} + \frac{(k-1)b}{k}\right) \le \frac{\phi(a)}{k} + \frac{(k-1)\phi(b)}{k}$$

for two arbitrary points $a, b \in X_\Delta$ and an arbitrary positive integer k.

Statement 31 (on $\mathbb{Q}$-convexity of functions). *A function ϕ is convex on $\Delta_{\mathbb{Q}}$ if and only if it is $\mathbb{Q}$-convex.*

Proof. The claim follows from Statement 8. □

Example 5. Let $c > 0$, and let c^r be the function defined on rational numbers by the formula $c^r = \sqrt[q]{c^p}$, where $r = p/q$. For $a, b \in \mathbb{Q}$ and integer $k > 1$, one has the inequality

$$c^{a/k+(k-1)b/k} \le \frac{c^a}{k} + \frac{(k-1)c^b}{k}.$$

Indeed, by dividing by c^b, we reduce the inequality to the form

$$c^{(a-b)/k} \le \frac{c^{a-b}-1}{k} + 1.$$

Set $u = (c^{a-b}-1)/k$. Clearly, $u > -1$. This, we have reduced the inequality to the assertion that $(1+u)^k \ge 1 + ku$ which can be verified automatically by induction over k.

Thus, the function c^r is $\mathbb{Q}$-convex. Hence it can be extended by continuity to the entire real line.

Remark. It is well known that the function c^x is not only continuous but also differentiable. This fact can be explained from the viewpoint of convex geometry as well. *A continuous convex function of one variable has left and right derivatives at every point and is differentiable at all but finitely many points*. This is a consequence of the following geometric fact: *A convex figure on the plane has a tangent cone at each point of the boundary, and this cone is a half-plane at all but finitely many points*. The function c^x has the "same structure," at every point, because $c^{a+x} = c^a c^x$. Hence the function c^x is differentiable everywhere.

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