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GEOMETRY OF GENERALIZED VIRTUAL POLYHEDRA

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Partial generalizations of the theory of virtual polyhedra (sometimes under different names) appeared recently in the theory of torus manifolds look very different from the original theory of virtual polyhedra. Such generalizations are based on simple arguments from homotopy theory while the original theory is based on integration over the Euler characteristic. We explain how these generalizations are related to the classical theory of convex bodies and the original theory of virtual polyhedra. The paper basically contains no proofs: all proofs and details can be found in the cited literature. Bibliography: 10 titles. Illustrations: 3 figures.

Dedicated to the 85th anniversary of my beloved teacher Vladimir Igorevich Arnold

1 Introduction. Virtual Convex Polyhedra and Their Polynomial Measures

Convex polyhedra in the linear space $\mathbb{R}^n$ form a convex cone in the following way. One can multiply a convex polyhedron Δ by any nonnegative real number λ (i.e., take its dilatation $\lambda\Delta$ centered at the origin with the factor λ) and add two convex polyhedra Δ_1 and Δ_2 in the Minkowski sense. Recall that the *Minkowski sum* of $\Delta_1, \Delta_2 \subset \mathbb{R}^n$ is the set Δ of points z representable in the form $z = x + y$, where $x \in \Delta_1$ and $y \in \Delta_2$.

A *convex chain* is a function on $\mathbb{R}^n$ representable as a finite linear combination of the characteristic functions of closed convex polyhedra (of different dimensions) with real coefficients.

Convex chains form a real vector space in a natural way. One can further define the product $f * g$ of two chains f and g as follows. If f and g are the characteristic functions of closed convex polyhedra $\Delta_1, \Delta_2 \subset \mathbb{R}^n$, then the chain $f * g$ is the characteristic function of $\Delta = \Delta_1 + \Delta_2$ by definition (the addition is understood in the Minkowski sense). This product can be extended to the space of convex chains by linearity.

It is not obvious at all that the above product is well defined. Indeed, a convex chain can be represented as a linear combination of characteristic functions in many different ways, and the independence of the product $f * g$ of such representations of f and g is not obvious. Using integration over the Euler characteristic [1], one can prove [2] that the product is well defined.

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Convex chains in $\mathbb{R}^n$ with the multiplication $*$ form a real algebra with the identity element $\mathbf{1}$ which is the characteristic function of the origin in $\mathbb{R}^n$. The characteristic function χ_Δ of a closed convex polyhedron $\Delta \subset \mathbb{R}^n$ is invertible in the algebra of convex chains. More precisely, the following theorem holds.

Theorem 1.1. *Let $\Delta \subset \mathbb{R}^n$ be a convex polyhedron, and let $-\Delta_0$ be the set of interior points* (*in the intrinsic topology of* Δ) *of the polyhedron $-\Delta$ symmetric to Δ with respect to the origin. Then*

$$(-1)^{\dim \Delta} \chi_{-\Delta_0} * \chi_\Delta = \mathbf{1}.$$

In other words, the convex chain $(-1)^{\dim \Delta} \chi_{-\Delta_0}$ is inverse to Δ with respect to the addition in the Minkowski sense (*extended to the space of convex chains*).

The algebra of convex chains contains the multiplicative subgroup generated by the characteristic functions of closed convex polyhedra. Elements of this group are called *virtual polyhedra* in $\mathbb{R}^n$.

Let us fix closed convex polyhedra $\Delta_1, \dots, \Delta_k \subset \mathbb{R}^n$. For any k-tuple of nonnegative integral numbers $\mathbf{n} = (n_1, \dots, n_k)$ one can define the polyhedron

$$\Delta(\mathbf{n}) = \sum n_i \Delta_i.$$

The following sentence can be considered as the slogan of the theory of virtual polyhedra: The natural continuation of the function $\Delta(\mathbf{n})$ (whose values are convex polyhedra) to k-tuples $\mathbf{n} = (n_1, \dots, n_k)$ of integral numbers (some of which can be negative) is the convex chain $\widetilde{\Delta}(\mathbf{n})$ defined by the following formula:

$$\widetilde{\Delta}(\mathbf{n}) = \chi_{\Delta_1}^{n_1} * \dots * \chi_{\Delta_k}^{n_k}.$$

This slogan can be justified as follows. The value of a polynomial measure (see an example of such a measure below) on a chain $\widetilde{\Delta}(\mathbf{n})$ is a polynomial in $\mathbf{n}$. Generalizations of the theory of virtual polyhedra suggest other families of cycles depending on parameters satisfying the following condition: for any differential form with polynomial coefficients its integrals over cycles from each such a family depend polynomially on paramaters.

We present an example of a polynomial measure on convex polyhedra with integral vertices and justify the slogan of the theory of virtual polyhedra.

Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree m. With P one can associate the following measure μ on convex polyhedra Δ with integral vertices:

$$\mu(\Delta) = \sum_{x \in \mathbb{Z}^n \cap \Delta} P(x).$$

One can prove that the function $\mu(\Delta(\mathbf{n}))$ is a polynomial in k-tuples $\mathbf{n}$ of nonnegative integral numbers of degree $\leqslant (n+m)$.

The following theorem justifies the slogan of the theory of virtual polyhedra.

Theorem 1.2. *Let P be a polynomial of degree m, and let $\widetilde{F}(\mathbf{n})$ be the function on k-tuples $\mathbf{n} = (n_1, \dots, n_k)$ of integral numbers* (*which can be negative*) *defined by the formula*

$$\widetilde{F}(\mathbf{n}) = \sum_{x \in \mathbb{Z}^n} \chi_{\Delta_1}^{n_1}(x) * \dots * \chi_{\Delta_k}^{n_k}(x) P(x).$$

Then $\widetilde{F}(\mathbf{n})$ is a polynomial in k-tuples $\mathbf{n}$ of degree $\leqslant (n+m)$ that coincides with $F(\mathbf{n})$ on k-tuples with nonnegative components.

Due to the theory of virtual polyhedra, it becomes possible to develop the general theory of polynomial finite additive measures on convex polyhedra (see [2]) which contains many generalizations of Theorem 1.2.

The theory of virtual polyhedra was motivated by cohomology theory of complete toric varieties with coefficients in sheaves invariant under the torus action. In particular, it provides a combinatorial version of the Riemann–Roch theorem for such varieties [3], which also can be considered as a multi-dimensional version of the classical Euler–MacLuren formula (see [3]).

The general theory is applicable to singular polynomial measures on polyhedra (such as the measure associating with a polyhedron the number of integral points in it) which can take nonzero values on polyhedra Δ with $\dim \Delta < n$. However, if one is interested in nonsingular polynomial measures that vanish on polyhedra and have dimension smaller than n, then one can totally neglect all polyhedra of dimension $< n$ in convex chains. This leads to a significant simplification of the theory of virtual polyhedra, which captures smooth polynomial measures (and which is not appropriate for studying singular measures).

The simplified theory is still useful. In particular, it allows us to provide a topological proof of the Bernstein–Koushnirenko–Khovanskii (BKK) theorem. More generally, using a description of algebras with Poincaré duality (see, for example, [4, Section 6]), one can describe the cohomology ring $H^*(M, \mathbb{Z})$ of a smooth complete toric variety M in terms of the volume function on virtual integral convex polyhedra (the so-called *Khovanskii–Pukhlikov description of the ring* $H^*(M, \mathbb{Z})$).

In this paper, we consider simplified versions of the theory of virtual polyhedra which deal only with nonsingular measures as well as its generalizations. We also mention some topological applications of these generalizations. We start with the geometric meaning of a virtual convex body and its volume for the difference of two strictly convex bodies with smooth boundaries. We also present some applications of mixed volume and virtual polyhedra in algebra.

2 Virtual Strictly Convex Bodies and Their Volumes

A *formal virtual convex body* is a formal difference of compact convex bodies (which, in general, are not polyhedra).

Similar to polyhedra, compact convex bodies in $\mathbb{R}^n$ form a convex cone with respect to the Minkowski addition and dilation with positive factors centered at the origin. Moreover, the addition of convex bodies satisfies the cancelation property, i.e., for a convex body Δ the identity $\Delta_1 + \Delta = \Delta_2 + \Delta$ implies $\Delta_1 = \Delta_2$. Hence one can generate a group by formal differences of convex bodies with $\Delta_1 - \Delta_2 = \Delta_3 - \Delta_4$ whenever $\Delta_1 + \Delta_4 = \Delta_3 + \Delta_2$.

By the Minkowski theorem, the volume is a homogeneous polynomial of degree n on the cone of convex bodies. More precisely, if Δ_1 and Δ_2 are convex bodies and $\lambda, \mu \geqslant 0$, then the volume $\mathrm{Vol}(\lambda\Delta_1 + \mu\Delta_2)$ is a homogeneous polynomail in (λ, μ) of degree n. Therefore, the volume can be extended to the linear space of formal differences of convex bodies as a homogeneous polynomial of degree n. In Section 4, we give a geometric interpretation of virtual convex bodies as well as their volumes.

Since the volume is a homogeneous polynomial of degree n on the cone of convex bodies in $\mathbb{R}^n$, it admits the polarization $\mathrm{Vol}(\Delta_1, \dots, \Delta_n)$, i.e., $\mathrm{Vol}(\Delta_1, \dots, \Delta_n)$ is a unique function of n-tuples of convex bodies $\Delta_1, \dots, \Delta_n$ with the following properties:

1) $\mathrm{Vol}(\Delta_1, \dots, \Delta_n)$ is linear in each argument with respect to the Minkowski addition,
2) $\mathrm{Vol}(\Delta_1, \dots, \Delta_n)$ is symmetric,
3) $\mathrm{Vol}(\Delta, \dots, \Delta) = \mathrm{Vol}(\Delta)$ on the diagonal.

The polarization of a volume polynomial is called the *mixed volume*. By multi-linearity, the mixed volume can be extended to n-tuples of virtual convex bodies.

3 Volume and Mixed Volume in Algebra

In this section, we briefly recall the relation of mixed volumes of virtual polytopes with algebraic geometry. Let $\Delta_1, \dots, \Delta_n$ be a collection of convex polyhedra with integral vertices.

The following question was originated by V. I. Arnold in the middle of the 1970s: *Let $P_1, \dots, Pn$ be a generic n-tuple of Laurent polynomials with given Newton polyhedra $\Delta(P_i) = \Delta_i$. How many roots does the system of equations $P_1 = \cdots = P_n = 0$ have in $(\mathbb{C}^*)^n$?*

The answer is given by the BKK theorem which was originally proved by A. G. Koushnirenko and D. N. Bernstein. In the later work, I found many generalizations and different proofs of that result.

Theorem 3.1 (BKK theorem). *The number of solutions is equal to $n! \,\mathrm{Vol}(\Delta_1, \dots, \Delta_n)$.*

One generalization of the BKK theorem comes if we consider rational functions on $(\mathbb{C}^*)^n$ instead of Laurent polynomials. Let $\dfrac{P_1}{Q_1}, \dots, \dfrac{P_n}{Q_n}$ be a generic n-tuple of rational functions with given Newton polyhedra $\Delta(P_i) = \Delta_i$ and $\Delta(Q_i) = \Delta_i'$. Then the intersection number in $(\mathbb{C}^*)^n$ of the principal divisors of these rational functions is equal to the mixed volume of the virtual polyhedra $\widetilde{\Delta}_i = \Delta_i - \Delta_i'$ multiplied by $n!$, i.e., $n! \,\mathrm{Vol}(\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_n)$ (see [5] for details).

4 Geometric Meaning of Virtual Strictly Convex Bodies

First, recall that the support function H_Δ of a compact convex body $\Delta \subset \mathbb{R}^n$ is the function on the dual space $(\mathbb{R}^n)^*$ defined by the following formula:

$$\mathrm{H}_\Delta(\xi) = \max_{x \in \Delta} \langle \xi, x \rangle.$$

One can further associate the support function to a virtual convex body. Indeed, the support function depends linearly on the convex body. Thus, it can be naturally extended to differences of convex bodies: $\mathrm{H}_{\Delta_1 - \Delta_2} = \mathrm{H}_{\Delta_1} - \mathrm{H}_{\Delta_2}$. The support function H_Δ of a (virtual) convex body Δ is a homogeneous function of degree one. More precisely, for $\lambda \geqslant 0$ the following relation holds: $\mathrm{H}_\Delta(\lambda \xi) = \lambda \mathrm{H}_\Delta(\xi)$.

In what follows, we assume that an Euclidian metric in $\mathbb{R}^n$ is fixed, which allows us to identify $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$. Assume further that Δ has smooth boundary and is strictly convex. Then for ξ not equal to zero the inner product $\langle \xi, x \rangle$ attains maxima at one point a of $\partial\Delta$ only and this point $a(\xi)$ is equal to $\mathrm{grad}\, \mathrm{H}_\Delta(\xi)$.

Lemma 4.1. *The vector-valued function* $\mathrm{grad}\, \mathrm{H}_\Delta(\xi)$ *restricted to the unit sphere S^{n-1} defines a map from S^{n-1} to the boundary $\partial\Delta$ of a strictly convex body Δ. Moreover, this map is inverse to the Gauss map $g : \partial\Delta \to S^{n-1}$.*

To a virtual convex body Δ with smooth support function H_Δ on $\mathbb{R}^n \setminus \{0\}$ one can associate the image $\mathrm{grad}\ \mathrm{H}_\Delta(S^{n-1})$ of the unit sphere under the map $\mathrm{grad}\ \mathrm{H}_\Delta : S^{n-1} \to \mathbb{R}^n$. This image has a natural parametrization by the sphere S^{n-1}. The correspondence $\Delta \to \mathrm{grad}\ \mathrm{H}_\Delta$ provides a map from the space of virtual convex bodies with smooth support function H_Δ to the linear space of gradient mappings from S^{n-1} to $\mathbb{R}^n$.

We consider an $(n-1)$-form $\omega = x_1 dx_2 \wedge \cdots \wedge dx_n$ on $\mathbb{R}^n$. Note that the differential $d\omega$ is the standard volume form on $\mathbb{R}^n$. The following statement is a direct corollary of Lemma 4.1 and the Stokes formula.

Corollary 4.1. *The volume of a convex body Δ with smooth strictly convex boundary $\partial\Delta$ is equal to*

$$\int\limits_{S^{n-1}} f^*\omega,$$

where f is the restriction of $\mathrm{grad}\ \mathrm{H}_\Delta$ *to the sphere S^{n-1}.*

Corollary 4.1 provides a proof of the Minkowski theorem for convex bodies with smooth strictly convex boundaries. Indeed,

$$(\mathrm{grad}\ \mathrm{H}_{\lambda\Delta_1+\mu\Delta_2})^*\omega = (\lambda \cdot \mathrm{grad}\ \mathrm{H}_{\Delta_1} + \mu \cdot \mathrm{grad}\ \mathrm{H}_{\Delta_2})^*\omega$$

is an $(n-1)$-form whose coefficients are homogeneous polynomials in (λ, μ) of degree n. Moreover, since the above formula for the volume is written in terms of support functions, it is applicable to virtual convex bodies. More concretely, for a virtual convex body $\Delta = \Delta_1 - \Delta_2$, where Δ_1 and Δ_2 are strictly convex bodies with smooth boundaries, let f be $\mathrm{grad}\ \mathrm{H}_\Delta = \mathrm{grad}\,(\mathrm{H}_{\Delta_1} - \mathrm{H}_{\Delta_2})$ restricted to the unit sphere. Then

$$\mathrm{Vol}(\Delta) = \int\limits_{S^{n-1}} f^*\omega.$$

Now, we give a different presentation for the volume of virtual convex bodies which is applicable to the case of generalized virtual polyhedra. Let $f : S^{n-1} \to \mathbb{R}^n$ be a smooth mapping of the unit sphere to $\mathbb{R}^n$. The image $f(S^{n-1})$ of the unit sphere S^{n-1} cuts the space $\mathbb{R}^n$ into a collection of connected open bodies.

Definition 4.1. The *winding number* $W_f(U)$, where U is an open connected component of $\mathbb{R}^n \setminus f(S^{n-1})$, is the mapping degree of the map $\tau_a : S^{n-1} \to S^{n-1}$, where

$$\tau(\xi) = \frac{f(\xi) - a}{|f(\xi) - a|}, \quad \xi \in S^{n-1},$$

and a is any point in U.

Informally, the number $W_f(U)$ shows how many times the image $f(S^{n-1})$ of the sphere S^{n-1} rotates around U.

Definition 4.2. Let $\mathrm{H}(\xi)$ be a smooth function on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree one. Then the *virtual convex body with support function* H is defined as the chain

$$\sum_U W_f(U) U,$$

where $f = \mathrm{grad}\ \mathrm{H}$ and the sum is taken over all bounded connected components of the complement $\mathbb{R}^n \setminus f(S^{n-1})$.

Theorem 4.1. *The volume of a virtual convex body with smooth support function* H *on* $\mathbb{R}^n \setminus \{0\}$ *is equal to the integral of the volume form over the chain* $\sum W_f(U)U$ *associated with the virtual convex body. In other words, the volume of the virtual bogy is equal to*

$$\sum_U W_f(U)\,\mathrm{Vol}(U),$$

where $\mathrm{Vol}(U)$ *is the volume of* U.

The proof follows from the formula for the volume of a virtual convex body and the Stokes formula. Theorem 4.1 has the following automatic generalization.

Theorem 4.2. *The integral of a polynomial* P *of degree* m *over a virtual convex body with smooth support function* H *on* $\mathbb{R}^n \setminus \{0\}$ *is equal to the integral of the polynomial* P *over the chain, associated with this virtual convex body, i.e., is equal to*

$$\sum_U W_f(U) \int\limits_U P dx_1 \wedge \cdots \wedge dx_n.$$

Proof. This theorem can be proved in the same way as Theorem 4.1. It is enough to replace the form $\omega = x_1 dx_2 \wedge \cdots \wedge dx_n$ with the form $Q dx_2 \wedge \cdots \wedge dx_n$, where Q is a polynomial of degree $m+1$ satisfying

$$\frac{\partial Q}{\partial x_1} = P.$$

The theorem is proved. □

One can generalize the above theorems in the following directions.

1. Instead of the unit sphere S^{n-1} and its gradient mappings to $\mathbb{R}^n$, one can take any piecewise smooth $(n-1)$-cycle Γ and consider the space of piecewise smooth mappings $f : \Gamma \to \mathbb{R}^n$. The integral of the form $f^*\omega$ over Γ, where ω is a fixed $(n-1)$ form with polynomial coefficients on $\mathbb{R}^n$, is a polynomial on the space of maps f from Γ to $\mathbb{R}^n$. The same polynomial on the space of mappings f can be obtained by integrating the n-form $d\omega$ over the chain $\sum W_f(U)U$, where U are connected components of $\mathbb{R}^n \setminus f(\Gamma)$ and $W_f(U)$ is the mapping degree of the map $\tau : \Gamma \to S^{n-1}$, where

$$\tau(x) = \frac{f(x) - a}{|f(x) - a|} \in S^{n-1},$$

where $x \in \Gamma$ and a is any point in U. The chain $W_f(U)U$ is an analog of the chain associated with a virtual convex body.

2. Let Γ be an $(n-1)$-cycle as above, and let $M(\Gamma, L)$ be the space of piecewise linear mappings of Γ to a real linear space L. With a fixed $(n-1)$ form ω with polynomial coefficients on the space L one can associate a polynomial function on $M(\Gamma, L)$ whose value on $f \in M(\Gamma, L)$ is equal to

$$\int\limits_\Gamma f^*\omega.$$

In such a generalization, one has integrals depending on the parameters in a polynomial way (but in such a generalization there are no chains analogous to the chains associated with virtual convex polyhedra).

5 Analogous Virtual Polyhedra and Their Volume

Let us return to the original definition of virtual polyhedra. With any given convex polyhedron Δ_0 one associates a subgroup of virtual polyhedra majorized by Δ_0. In this section, we first recall this construction and then describe a simplified theory of virtual polyhedra.

First, recall that each convex polyhedron Δ defines the dual fan $\Delta^{\perp}$ in the following way. Two covectors are said to be Δ-*equivalent* if they attain maxima at the same face of Δ. The set of all Δ-equivalent covectors forms a cone (which is open in the intrinsic topology). The closures of such cones form the dual fan $\Delta^{\perp}$ for Δ.

Definition 5.1. Two polyhedra Δ_1 and Δ_2 are called *analogous* if their dual fans coincide. In particular, for each facet of Δ_1 there is exactly one facet of Δ_2 parallel to it and having the same coorientation.

The following lemma is straightforward to show.

Lemma 5.1. *Let Δ_1 and Δ_2 be convex polyhedra analogous to Δ_0. Then $\Delta_1 + \Delta_2$ is also analogous to Δ_0.*

If a virtual polyhedron Δ is representable as the difference $\Delta_1 - \Delta_2$ of polyhedra analogous to Δ_0, then we say that the virtual polyhedron Δ is *majorized* by Δ_0. In other words, a virtual polyhedron majorized by Δ_0 if the corresponding convex chain is representable in the form $\chi_{\Delta_1} * \chi_{\Delta_2}^{-1}$, where χ_{Δ_i} is the characteristic function of Δ_i. We note that the virtual polyhedron $\Delta_1 - \Delta_2$ depends on its support function (and is independent of the representation of this function in the form $\mathrm{H}_{\Delta_1} - \mathrm{H}_{\Delta_2}$).

Simplified version of theory of analogous convex polyhedra

If one is interested only in nonsingular measures of a virtual polyhedron, one can neglect polyhedra of dimension $< n$ in the convex chain associated with a virtual polyhedron majorized by Δ_0. This leads to a simplified theory of virtual polyhedra which can be described by using support functions in a way similar to the above-presented description of virtual convex bodies with smooth boundaries.

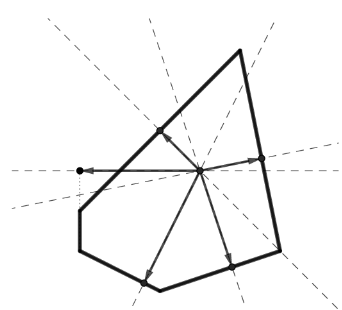

Figure 1. Dual fan to a convex 5-gon.

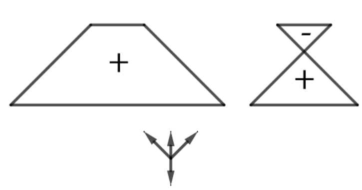

Figure 2. Trapezoid, its dual fan, and a virtual 4-gon majorized by it.

Convex polyhedra are not strictly convex, and the Gauss map from the unit sphere to the boundary of a convex polyhedron is not defined. But one can define (up to a homotopy) an analog of the Gauss map from the boundary of one polyhedron to the boundary of an analogous polyhedron.

Let us fix a convex polyhedron Δ_0. In what follows, it will play a role of the unit sphere in our construction.

To each polyhedron Δ analogous to Δ_0 we associate the union L_Δ of affine hyperplanes L_{Γ_i} which are affine spans of the facets Γ_i of Δ (i.e., faces of Δ having dimension $(n-1)$).

Definition 5.2. A continuous map $f_\Delta : \partial\Delta_0 \to L_\Delta$ is a *Gauss type map* if the following condition holds: If $x \in \partial\Delta_0$ belongs to the closure of an $(n-1)$-dimensional face Γ_i^0 of Δ_0, then $f(x)$ has to belong to L_{Γ_i}, where Γ_i and Γ_i^0 are parallel faces of Δ and Δ_0 having the same coorientation.

Lemma 5.2. 1. *For any Δ analogous to Δ_0 there exists a piecewise smooth Gauss type map f_Δ.*

2. *Moreover, f_Δ can be defined in such a way that it linearly depends on Δ, i.e.,*

$$f_{\lambda\Delta_1+\mu\Delta_2} = \lambda f_{\Delta_1} + \mu f_{\Delta_2}.$$

3. *Any two Gauss type maps from $\partial\Delta_0$ to L_Δ are homotopies equivalent to each other.*

Now, we are ready to define the volume of a virtual polyhedron and the integral of a polynomial form over a virtual polyhedron.

First, let us associate a collection of cooriented affine hyperplanes to a virtual polyhedron Δ majorized by Δ_0. Let H be the support function of Δ. Then H is a piecewise linear function on $\mathbb{R}^n$ which is linear on each cone of the dual fan $\Delta_0^\perp$ of Δ_0. Then H defines the collection $L(\mathrm{H})$ of cooriented hyperplanes which is in one-to-one correspondence with the collection of facets of Δ_0. The hyperplane $L_{\Gamma_i}(H) \in L(H)$ corresponding to a facet $\Gamma_i \subset \Delta_0$ is parallel to Γ_i and has the same coorientation.

To each facet Γ_i of Δ_0 one associates the dual ray $l(\Gamma_i)$ in the dual fan $\Delta_0^\perp$ to Δ_0.

The collection $L(H)$ is defined as follows.

Definition 5.3. For each facet Γ_i of Δ_0 the *hyperplane* $L_{\Gamma_i}(H) \in L(H)$ is defined by the equation $\langle e_i, x\rangle = H(e_i)$, where e_i is any nonzero vector in the ray $l(\Gamma_i)$. The coorientation of $L_{\Gamma_i}(H)$ is defined by the covector e_i.

It is easy to check the following lemma.

Lemma 5.3. *If $\Gamma_i \bigcap \Gamma_j = F$ is a nonempty face of Δ_0, then $L_{\Gamma_i}(\mathrm{H}) \bigcap L_{\Gamma_j}(\mathrm{H})$ is an affine space parallel to F.*

Definition 5.4. A *Gauss type map* f_H for a virtual polyhedron with support function H is a map $f_\mathrm{H} : \partial\Delta_0 \to L(\mathrm{H})$ which maps the face $F = \cap\Gamma_{i_j}$ of Δ_0 to the affine space $L_{\mathrm{H}(F)} = \cap L_{\Gamma_{i_j}(\mathrm{H})}$.

The statement of Lemma 5.2 also holds for virtual polyhedra. More precisely, one gets the following lemma.

Lemma 5.4. 1. *For any* H *which is linear on each cone of $\Delta_0^\perp$ there exists a Gauss type map f_H.*

2. *Moreover, f_{H} can be defined in such a way that it linearly depends on* H, *i.e.,*

$$f_{\lambda \mathrm{H}_1+\mu \mathrm{H}_2} = \lambda f_{\mathrm{H}_1} + \mu f_{\mathrm{H}_2}.$$

3. *Any two Gauss type maps from $\partial\Delta_0$ to $L(\mathrm{H})$ are homotopies equivalent to each other.*

Definition 5.5. The *winding number* $W_{f_{\mathrm{H}}}(U)$, where U is an open connected component of $\mathbb{R}^n \setminus L(\mathrm{H})$, is the mapping degree of the map $\tau : \partial\Delta_0 \to S^{n-1}$, where

$$\tau(x) = \frac{f_{\mathrm{H}}(\xi) - a}{||f_{\mathrm{H}}(\xi) - a||}, \quad x \in \partial\Delta_0,$$

and a is any point in U.

As in the case of virtual convex bodies with smooth boundaries, to a virtual polyhedron with support function H one can associate the chain

$$\sum_U W_{f_{\mathrm{H}}}(U)U,$$

where the sum is taken over open connected components of $\mathbb{R}^n \setminus L(\mathrm{H})$.

One can prove the following theorem.

Theorem 5.1. *The chain $\sum W_{f_{\mathrm{H}}}(U)U$ can be obtained from virtual polyhedra with support function* H *by neglecting all polyhedra in the chain whose dimension is smaller than n.*

Thus, the integral of any n-form with polynomial coefficients over a virtual polyhedron can be obtained by integrating this form over the chain $\sum W_{f_{\mathrm{H}}}(U)U$. One can deal with integrals of such type using simple arguments which we applied above to similar integrals over virtual convex bodies with smooth boundaries (and the technique of integrating over the Euler characteristic is not needed here).

Let P be a polynomial of degree m, and let Q be a polynomial of degree $m+1$ such that $P = \dfrac{\partial Q}{\partial dx_1}$.

Theorem 5.2. *The values of the integrals*

$$\int\limits_{\partial\Delta_0} f_\Delta^*(x_1 dx_2 \wedge \cdots \wedge dx_n), \quad \int\limits_{\partial\Delta_0} f_\Delta^*(Q dx_2 \wedge \cdots \wedge dx_n)$$

are equal to

$$\sum W_{f_{\mathrm{H}}}(U) \int\limits_U dx_1 \wedge \cdots \wedge x_n, \quad \sum W_{f_{\mathrm{H}}}(U) \int\limits_U P dx_1 \wedge \cdots \wedge x_n$$

correspondingly.

With a convex support function H linear on each cone of $\Delta^\perp$ one associates an oriented polyhedron $\Delta(\mathrm{H})$ with support function H. If one is interested in integrals of polynomial differential forms over a chain, then the natural continuation of the functor $\mathrm{H} \to \Delta(\mathrm{H})$ to nonconvex support functions linear on each cone of $\Delta^{\perp}$ is the functor $\mathrm{H} \to \sum W_{f_{\mathrm{H}}}(U)U$. Below, we discuss a wide generalization of the above construction.

The following formulation allows even wider generalizations.

Let $\mathrm{H} \to f_{\mathrm{H}}(\partial\Delta_0) \in H_{n-1}(L(\mathrm{H}))$ *be a functor associating to* H *the homotopy class in* $H_{n-1}(L(\mathrm{H}))$ *which is the image of the fundamental class of* $\partial\Delta_0$ *under the map* f_{H}. *That functor has a generalization to the case where, instead of the union of hyperplanes, one consider the union of affine spaces.*

6 Theory of Generalized Virtual Polyhedra

We generalize the above construction in the following directions.

1. Instead of the union $L(\mathrm{H})$ of hyperplanes parallel to the faces of a convex polyhedron Δ, we consider the union X of arbitrary affine subspaces of any dimension in an affine space.
2. Instead of the image of $\partial\Delta_0$ in $L(\mathrm{H})$, we consider arbitrary cycles in X. We identify the homology groups of the unions X_1 and X_2 of different collections of affine subspaces under some combinatorial assumptions.
3. If the affine subspaces are hyperplanes in $\mathbb{R}^n$ and the cycle has dimension $(n-1)$, then the above generalization can be modified as follows: Instead of $(n-1)$-dimensional cycles in the union of hyperplanes in $\mathbb{R}^n$, we consider n-dimensional chains in $\mathbb{R}^n$ whose boundaries are the above-mentioned cycles. In the particular case of analogous simplified virtual polyhedra, such chains coincide with the chains $\sum W_{f_{\mathrm{H}}}(U)U$ discussed above.

In this section, we deal with ordered sets of affine subspaces L_i indexed by the same set I.

Definition 6.1. The set $X = \bigcup_{i \in I} L_i$ has the natural covering by the spaces L_i. The *nerve* K_X of the natural covering of X is the following simplicial complex:

1) the set of vertices of K_X is the set I of indices i,

2) the set $J \subset I$ of vertices belongs to one simplex if and only if $\bigcap_{i \in J} L_i \neq \varnothing$.

Definition 6.2. Let $X_1 = \bigcup L_i$ and $X_2 = \bigcup M_i$ be the unions of affine subspaces of spaces L and M indexed by the same set $\{i\} = I$. We say that

1) X_1 *dominates* X_2 if the nerve K_{X_1} is a subcomplex of the nerve K_{X_2},

2) X_1 and X_2 are *equivalent* if $K_{X_1} = K_{X_2}$.

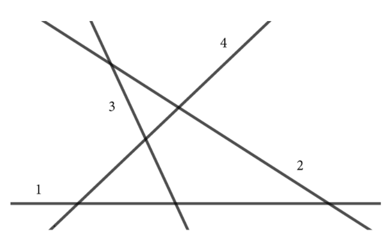

Figure 3. An ordered set of four lines on a plane.

Let BK_X be the barycentric subdivision of the nerve K_X. For each $i \in I$ let B_iK_X be the union of all (closed) simplices in BK_X which contain the vertex A_i (corresponding to the space L_i in the covering of X).

Lemma 6.1. *The nerve of the covering of BK_X by the closed sets B_iK_X coincides with the original nerve K_X.*

Definition 6.3. A map $g : K_{X_1} \to X_2$ is *compatible with coverings* if for any $i \in I$ and $x \in BX_i$ the image $g(x)$ belongs to M_i.

Theorem 6.1. 1. *A map $g : K_{X_1} \to X_2$ compatible with coverings exists if and only if $K_{X_1} \subset K_{X_2}$.*

2. *All maps from K_{X_1} to X_2 compatible with coverings are homotopies equivalent to each other.*

3. *If $K_{X_1} = K_{X_2}$, then the map $g : K_{X_1} \to X_2$ provides a homotopy equivalence between these spaces.*

Theorem 6.1 implies that all cycles of $H_*(X)$ can be seen in the homology group of the nerve K_X of the covering of X. Moreover, if K_{X_1} is a subcomplex of K_{X_2}, then each cycle in $H_*(K_{X_1})$ has the natural image in $H_*(X_2)$.

We consider a collection of affine k-dimensional subspaces $\{L_i\}$ in a vector space L with $i \in I$. For each i we denote by Y_i the factor space $L/\widetilde{L}_i$, where $\widetilde{L}_i$ is the vector subspace parallel to L_i.

Definition 6.4. Vectors $y_i \in Y_i$ are *compatible with the nerve* of $X = \bigcup L_i$ if the following condition holds: if $L_{i_1} \cap \cdots \cap L_{i_m} \neq \varnothing$, then $(L_{i_1} + y_{i_1}) \cap \cdots \cap (L_{i_m} + y_{i_m}) \neq \varnothing$.

Let Y be the space of all I-tuples $y_1, \dots, y_{|I|}$ compatible with the nerve of $X = \bigcup L_i$.

Definition 6.5. To each point $\mathbf{y} \in Y$ one associates the *collection* $\{L_i(y)\}$, where $L_i(y) = L_i + y_i$.

The set Y parametrizes translations of the subspaces L_i which preserve the existed intersections. More precisely, for a generic point $\mathbf{y} \in Y$ the collections $\{L_i(\mathbf{y})\}$ have the same nerve, denoted by K_X. There is a subset Σ in Y of a smaller dimension than the dimension of Y such that the nerve of $\bigcap L_i(\mathbf{y})$ contains K_X as a proper subcomplex.

We consider $K_X \times Y$ and $L \times Y$ as fiber bundles over the base Y.

One can define a map $g_\bullet : K_X \times Y \to L \times Y$ which fixes the base, respects the fibers, and has the following properties. For each point $\mathbf{y} \in Y$ the restriction $g_{\bullet,\mathbf{y}}$ of $g_\bullet$ to the fiber $K_X \times \mathbf{y}$ is compatible with the nerves of the corresponding fibers and depends on $\mathbf{y}$ linearly, i.e.,

$$g_{\bullet,\lambda\mathbf{y}_1+\mu\mathbf{y}_2} = \lambda g_{\bullet,\mathbf{y}_1} + \mu g_{\bullet,\mathbf{y}_2}.$$

For any k-form α on $L \times Y$ with polynomial coefficients and any cycle $\gamma \in H_k(K_X)$ one can consider the following function $F_{\alpha,\gamma}$ on Y:

$$F_{\alpha,\gamma}(\mathbf{y}) = \int\limits_\gamma g^*_{\bullet,\mathbf{y}}\alpha.$$

Theorem 6.2. *The function $F_{\alpha,\gamma}$ is a polynomial function on Y.*

7 Homotopy Type of Union of Affine Subspaces

We know that the homotopy type of $X = \bigcup L_i$ is the same as the homotopy type of its nerve K_X.

For any finite simplicial complex it is easy to construct a collection of affine subspaces whose nerve is homeomorphic to a given complex. However, if affine subspaces have codimension one in L, then their union always has the homotopy type of the wedge of spheres.

Let $\{L_i\}$ be a collection of hyperplanes in L. We denote by $l(\{L_i\})$ the biggest subspace parallel to all these hyperplanes. One can check that $l(\{L_i\})$ is equal to the intersection of linear subspaces parallel to the affine spaces L_i. For a sufficiently general collection of hyperplanes the space $l(\{L_i\})$ is equal to zero.

Theorem 7.1. *The union $X = \bigcup L_i$ of affine hyperplanes under the condition $l(\{L_i\}) = 0$ is a homotopy equivalent to the wedge of $(n-1)$-dimensional spheres, which are in one-to-one correspondence with the boundaries of convex polyhedra that are the closures of connected bounded components of $L \setminus \bigcup L_i$.*

Corollary 7.1. *If $l(\{L_i\})$ has dimension m, then $X = \bigcup L_i$ has the homotopy type of the wedge of spheres of dimension $n - 1 - m$.*

Proof. Indeed, under the assumptions of Corollary 7.1, X is equal to $X \cap l^{\perp} \times l(\{L_i\})$, where the space $l^{\perp}$ is transversal to $l(\{L_i\})$. Theorem 7.1 can be applied to $X \cap l^{\perp}$. □

Under the assumptions of Theorem 7.1, we choose a cycle $\gamma \in H_{n-1}(K_X, \mathbb{Z})$. For a point $\mathbf{y} \in Y$ we consider the map $g_{\mathbf{y}} : K_X \to \bigcup L_i(\mathbf{y}) = L(\mathbf{y})$ compatible with coverings. To this map one can associate the chain $\sum W_\tau(U)U$, where U is a connected component of $L \setminus L(\mathbf{y})$. In this chain, $W_\tau(U)$ is the winding number of the cycle γ under the map $\tau : K_X \to S^{n-1}$, where

$$\tau = \frac{g_{\mathbf{y}} - a}{|g_{\mathbf{y}} - a|},$$

a is a point in U, and S^{n-1} is the unit sphere. This chain can be considered as the generalized virtual polyhedron appeared in the assumptions of Theorems 5.1 and 5.2. In particular, an integral of a form $\omega = P dx_1 \wedge \cdots \wedge dx_n$, where P is a polynomial, over such chains depends polynomially on $\mathbf{y}$.

8 Applications of Generalized Virtual Polyhedra

We finish the paper with a brief description of recent applications of generalized virtual polyhedra. Exact statements and details can be found in [6].

Torus manifolds (see [7, 8]) provide a wide topological generalization of smooth algebraic toric varieties. Such a manifold can be associated with the union $L(y)$ of hyperplanes in $\mathbb{R}^n$ depending on the parameter y and $(n-1)$-dimensional cycle Γ in the nerve of the natural covering of $L(y)$ [9, 10, 6]. Using the results described in Section 7, one can define a homogeneous polynomial in y of degree n that is the volume of the corresponding virtual polyhedron. One can describe the cohomology ring of a torus manifold by using the Khovanskii–Pukhlikov construction known in the theory of toric varieties.

On a torus manifold there is a special collection of characteristic linear bundles which are in one-to-one correspondence with the generalized virtual polyhedra responsible for the torus manifold. The intersection number of n-sections of such bundles is equal to $n!$ multiplied by the mixed volume of the corresponding virtual polyhedron. This theorem generalizes the BKK-theorem for torus manifolds. Moreover, one can describe the cohomology ring of the total space of a fiber bundle whose fibers are torus manifolds in terms of integrals of some polynomials over corresponding virtual polyhedra. This theorem generalizes an analogous result for bundles with toric fibers.

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