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THE RESULTANT OF DEVELOPED SYSTEMS OF LAURENT POLYNOMIALS

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To the memory of Vladivir Igorevich Arnold

ABSTRACT. Let $R_\Delta(f_1, \dots, f_{n+1})$ be the Δ-*resultant* (defined in the paper) of $(n+1)$-tuple of Laurent polynomials. We provide an algorithm for computing R_Δ assuming that an n-tuple $(f_2, \dots, f_{n+1})$ is *developed.* We provide a relation between the product of f_1 over roots of $f_2 = \dots = f_{n+1} = 0$ in $(\mathbb{C}^*)^n$ and the product of f_2 over roots of $f_1 = f_3 = \dots = f_{n+1} = 0$ in $(\mathbb{C}^*)^n$ assuming that the n-tuple $(f_1 f_2, f_3, \dots, f_{n+1})$ is developed. If all n-tuples contained in $(f_1, \dots, f_{n+1})$ are developed we provide a signed version of Poisson formula for R_Δ. In our proofs we use topological arguments and topological version of the Parshin reciprocity laws.

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1. INTRODUCTION

To a Laurent polynomial f in n variables one associates its Newton polyhedron $\Delta(f)$, which is a convex lattice polyhedron in $\mathbb{R}^n$ (through all of this paper by polyhedron we will mean compact convex polyhedron with integer vertices). A system of n equations $f_1 = \dots = f_n = 0$ in $(\mathbb{C}^*)^n$ is called *developed* if (roughly speaking) their Newton polyhedra $\Delta(f_i)$ are located generically enough with respect to each other. The exact definition (see also Sec. 6) is as follows: a collection of n polyhedra $\Delta_1, \dots, \Delta_n \subset \mathbb{R}^n$ is called *developed* if for any covector $v \in (\mathbb{R}^n)^*$ there is i such that on the polyhedron Δ_i the inner product with v attains its biggest value precisely at a vertex of Δ_i.

A developed system resembles an equation in one unknown. A polynomial in one variable of degree d has exactly d roots counting with multiplicity. The number of roots in $(\mathbb{C}^*)^n$ counting with multiplicities of a developed system is always determined by the Bernstein–Koushnirenko formula (see [B]) (if the system is not developed this formula holds only for generic systems with fixed Newton polyhedra).

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As in the one-dimensional case, one can explicitly compute the sum of values of any Laurent polynomial over the roots of a developed system [GKh], [GKh1] and the product of all of the roots of the system regarded as elements in the group $(\mathbb{C}^*)^n$ [Kh1]. These results can be proved topologically [GKh1], using the topological identity between certain homology cycles related to developed system (see Sec. 7), the Cauchy residues theorem, and a topological version of the Parshin reciprocity laws (see Sec. 8).

To an $(n+1)$-tuple $A = (\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$ of finite subsets in $\mathbb{Z}^n$ one associates the A-*resultant* R_A. It is a polynomial defined up to sign in the coefficients of Laurent polynomials $f_1, \ldots, f_{n+1}$ whose supports belong to $A_1, \ldots, A_{n+1}$ respectevely. The A-resultant is equal to ± 1 if the codimension of the variety of consistent systems in the space of all systems with supports in A is greater than 1. Otherwise, R_A is a polynomial vanishing on the variety of consistent systems and such that the degree of R_A in the coefficients of the i-th polynomial is equal to the generic number of roots of the system $f_1 = \ldots = \hat{f}_i = \ldots = f_{n+1} = 0$ (in which the equation $f_i = 0$ is removed).

The notion of A-resultant was introduced and studied in [GKZ] under the following assumption on A: the lattice generated by the differences $a - b$ for all couples $a, b \in \mathcal{A}_i$ and all $0 \leqslant i \leqslant n+1$ is $\mathbb{Z}^n$. Under this assumption the resultant R_A is an irreducible polynomial (which was used in the definition of R_A in [GKZ]). Later in [Est] and [DS] it was shown that in the general case (i.e when the differences from $\mathcal{A}_i$'s do not generate the whole lattice) R_A is some power of an irreducible polynomial. The power is equal to the generic number of roots of a corresponding consistent system (see Sec. 13 for more details).

To an $(n+1)$-tuple $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ of Newton polyhedra one associates the $(n+1)$-tuple A_Δ of finite subsets $(\Delta_1 \cap \mathbb{Z}^n, \ldots, \Delta_{n+1} \cap \mathbb{Z}^n)$ in $\mathbb{Z}^n$. We define the Δ-*resultant* as A-resultant for $A = A_\Delta$. In the paper we deal with Δ-resultants only. If a property of Δ-resultant is a known property of A-resultants for $A = A_\Delta$ we refer to a paper where the property of A-resultants is proven (without mentioning that the paper deals with A-resultants and not with Δ-resultants). Dealing with Δ-resultants only we lose nothing: A-resultants can be reduced to Δ-resultants. One can check that $R_A(f_1, \ldots, f_{n+1})$ for $A = (A_1, \ldots, A_{n+1})$ is equal to $R_\Delta(f_1, \ldots, f_{n+1})$ for $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$, where $\Delta_1, \ldots, \Delta_{n+1}$ are the convex hulls of the sets $A_1, \ldots, A_{n+1}$.

A collection Δ is called i-*developed* if its subcollection obtained by removing the polyhedron Δ_i is developed. Using the Poisson formula (see [PSt], [DS], and Sec. 13.3) one can show that for i-developed Δ the identity

$$R_\Delta = \pm \Pi_\Delta^{[i]} M_i \tag{1.1}$$

holds, where $\Pi_\Delta^{[i]}$ is the product of f_i over the common zeros in $(\mathbb{C}^*)^n$ of f_j, for $j \neq i$, and M_i is an explicit monomial in the vertex coefficients (i.e., the coefficient of f_j in front of a monomial corresponding to a vertex of Δ_j) of all the Laurent polynomials f_j with $j \neq i$.

We provide an explicit algorithm for computing the term $\Pi_\Delta^{[i]}$ using the summation formula over the roots of a developed system (Corollary 8). Hence we get *an* *explicit algorithm for computing the resultant* R_Δ for an i-developed collection Δ. This algorithm heavily uses the Poisson formula (1).

If $(n+1)$-tuple Δ is i-developed and j-developed for some $i \neq j$ the identity

$$\Pi_\Delta^{[i]} = \Pi_\Delta^{[j]} M_{i,j} s_{i,j} \tag{1.2}$$

holds, where $M_{i,j}$ is an explicit monomial in the coefficients of Laurent polynomials $f_1, \ldots, f_{n+1}$ and $s_{i,j} = (-1)^{f_{i,j}}$ is an explicitly defined sign (Corollary 4).

Our proof of the identity (1.2) is topological. We use the topological identity between cycles related to a developed system (see Sec. 7) and a topological version of the Parshin reciprocity laws. The identity (1.2) generalizes the formula from [Kh1] for the product in $(\mathbb{C}^*)^n$ of all roots of a developed system of equations.

An $(n+1)$-tuple Δ is called *completely developed* if it is i-developed for every $1 \leqslant i \leqslant n+1$. For completely developed Δ the identity

$$\Pi_\Delta^{[1]} M_1 s_1 = \ldots = \Pi_\Delta^{[n+1]} M_{n+1} s_{n+1} \tag{1.3}$$

holds, where $(M_1, \ldots, M_{n+1})$ and $\left(\Pi_\Delta^{[1]}, \ldots, \Pi_\Delta^{[n+1]}\right)$ are monomials and products appearing in (1.1) and $(s_1, \ldots, s_{n+1})$ is an $(n+1)$-tuple of signs such that $s_i s_j = s_{i,j}$, where $s_{i,j}$ are the explicit signs from identity (1.2).

Our proof of the identities (1.3) uses the identity (1.2) and does not rely on the theory of resultants. Using one general fact from this theory (Theorem 19) one can see that the quantities in the identities (1.3) are equal to the Δ-resultant, i.e., to $\pm R_\Delta$. Thus the identities (1.3) can be considered as a signed version of the Poisson formula for completely developed systems.

To make the paper more accessible we first describe all of the results in the classical one dimensional case.

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2. Resultants in Dimension One

2.1. Sylvester's formula for resultant. Let $P_1 = a_0 + \cdots + a_k z^k$, $P_2 = b_0 + \cdots + b_n z^n$ be polynomials in one complex variable of degrees $\leqslant k$ and $\leqslant n$. Sylvester defined resultants $R^{[1]}$ and $R^{[2]}$ which are equal up to a sign. $R^{[1]}$ and $R^{[2]}$ are defined as the determinant of the $(n+k) \times (n+k)$ matrices M_1, M_2 respectively, where,

$$M_1(P_1, P_2) = \begin{pmatrix} a_k & a_{k-1} & \ldots & a_0 & 0 & 0 \\ 0 & \ddots & \ddots & \ldots & \ddots & 0 \\ 0 & 0 & a_k & a_{k-1} & \ldots & a_0 \\ b_n & b_{n-1} & \ldots & b_0 & 0 & 0 \\ 0 & \ddots & \ddots & \ldots & \ddots & 0 \\ 0 & 0 & b_n & b_{n-1} & \ldots & b_0 \end{pmatrix}$$

and $M_2(P_1, P_2) = M_1(P_2, P_1)$.

One can see that: 1) $R^{[1]} = (-1)^{kn} R^{[2]}$; 2) resultants are polynomials in the coefficients of P_1 and P_2 with degrees n and k in coefficients of P_1 and P_2 respectively; 3) the polynomials $R^{[1]}$ and $R^{[2]}$ have integer coefficients; 4) the coefficient of the monomial $a_k^n b_0^k$ in $R^{[1]}$ is $+1$ and, hence, the coefficient of the same monomial in $R^{[2]}$ is $(-1)^{kn}$, 5) the coefficients of $R^{[1]}$ and $R^{[2]}$ are coprime integers (this follows from 3) and 4)).

Under the assumption $a_n \neq 0$, $b_k \neq 0$ the resultant $R^{[1]}$ (the resultant $R^{[2]}$) is equal to zero if and only if the polynomials P_1 and P_2 have common root. One can show that the variety of pairs of polynomials P_1 and P_2 having a common root is an irreducible quasi projective variety X (a simple proof of a multidimensional version of this statement can be found in [GKZ]). Let D be an irreducible polynomial equal to zero on X. According to the previous statement $R^{[1]}$ and $R^{[2]}$ are equal to the same power of D, multiplied by some coefficients.

The Sylvester resultants can be generalized for Laurent polynomials. Consider two Laurent polynomials

$$f_1 = a_k z^k + \cdots + a_n z^n, \qquad f_2 = b_l z^l + \cdots + b_m z^m, \tag{2.1}$$

on $\mathbb{C}^*$ whose Newton polyhedra belong to the segments Δ_1, Δ_2 defined by inequalities $k \leqslant x \leqslant n$, $l \leqslant x \leqslant m$. With f_1, f_2 let us associate the pair of polynomials P_1, P_2, where

$$P_1 = z^{-k} f_1 = a_k + \cdots + a_n z^{n-k}, \qquad P_2 = z^{-l} f_2 = b_l + \cdots + b_m z^{m-l}. \tag{2.2}$$

For $\Delta = (\Delta_1, \Delta_2)$ we define resultants $R_\Delta^{[1]}$, $R_\Delta^{[2]}$ as follows:

$$R_\Delta^{[1]}(f_1, f_2) = (-1)^{n(m-l)} R^{[1]}(P_1, P_2), \qquad R_\Delta^{[2]}(f_1, f_2) = (-1)^{l(n-k)} R^{[2]}(P_1, P_2).$$

The definitions of Δ-resultants $R_\Delta^{[1]}$ and $R_\Delta^{[2]}$ are made in such a way that they coincide with the product resultants, which are defined in the next section (see Theorem 3).

3. Product Formula in Dimension One

For Laurent polynomials f_1, f_2 as in (2.1), let $\Pi_\Delta^{[1]} = \prod_{y_j \in Y} f_1^{m_{y_j}}(y_j)$ and $\Pi_\Delta^{[2]} = \prod_{x_i \in X} f_2^{m_{x_i}}(x_i)$, where $X = \{x_i\}$ and $Y = \{y_j\}$ are the sets of non-zero roots of f_1 and f_2 and m_{y_j}, m_{x_i} are their multiplicities.

Theorem 1. *If $a_k a_n b_l b_m \neq 0$ then the following identity holds*:

$$b_l^{-k} b_m^n \Pi_\Delta^{[1]} = (-1)^{kl+nm} a_k^{-l} a_n^m \Pi_\Delta^{[2]}. \tag{3.1}$$

Let us recall an elementary proof of this classical theorem.

Proof. We have $f_1(z) = a_n (z - x_1)^{m_{x_1}} \cdot \cdots \cdot (z - x_s)^{m_{x_s}} z^k$, so

$$\Pi_\Delta^{[1]} = \prod_{y_j \in Y} \Big[a_n y_j^k \prod_{x_i \in X} (y_j - x_i)^{m_{x_i}} \Big]^{m_{y_j}} = a_n^{m-l} \big[(-1)^{m-l} (b_l / b_m)\big]^k \Pi_\Delta^{[1,2]},$$

where $\Pi^{[1,2]}_\Delta = \prod_{x_i\in X, y_j\in Y}(y_j - x_i)^{m_{x_i}m_{y_j}}$. Here we used the Vieta relation

$$\prod_{y_j\in Y} y_j^{m_{y_j}} = (-1)^{m-l} b_l/b_m.$$

Thus we proved the identity $b_l^{-k} b_m^n \Pi^{[1]}_\Delta = a_n^{m-l} b_m^{n-k}\Pi^{[1,2]}_\Delta(-1)^{mk-lk}$.

In a similar way

$$a_k^{-l} a_n^m \Pi^{[2]}_\Delta = a_n^{m-l} b_m^{n-k}\Pi^{[2,1]}_\Delta(-1)^{nl-kl},$$

where $\Pi^{[2,1]}_\Delta = \prod_{x_i\in X, y_j\in Y}(x_i - y_j)^{m_{x_i}m_{y_j}}$. But $\Pi^{[1,2]}_\Delta = \Pi^{[2,1]}_\Delta(-1)^{(m-l)(n-k)}$. Theorem 1 is proved. □

For Laurent polynomials f_1, f_2 as in (2.1) let us define their product resultants $R^{[1]}_{\Pi,\Delta}(f_1, f_2)$ and $R^{[2]}_{\Pi,\Delta}(f_1, f_2)$ by the formulas

$$R^{[1]}_{\Pi,\Delta} = b_l^{-k} b_m^n \Pi^{[1]}_\Delta, \qquad R^{[2]}_{\Pi,\Delta} = a_k^{-l} a_n^m \Pi^{[2]}_\Delta. \tag{3.2}$$

Theorem 2. (1) *The product resultants are polynomials in the coefficients of f_1 and f_2 (the expressions in (3.2) themselves could have removable singularities at the hyperplanes where the extreme coefficients vanish);*

(2) *If the extreme coefficients are nonzero, i.e., $a_k a_n b_l b_m \neq 0$, the product resultants equal to zero exactly on the pairs of Laurent polynomials having common root in $\mathbb{C}^*$;*

(3) *The product resultants have degrees $(m-l)$ and $(n-k)$ in the coefficients of f_1 and f_2 correspondingly;*

(4) *The coefficient of the monomial $a_k^{m-l} b_m^{n-k}$ in $R^{[1]}_{\Pi,\Delta}$ and $R^{[2]}_{\Pi,\Delta}$ is equal to $(-1)^{k(m-l)}$ and $(-1)^{l(n-k)}$ respectively.*

Proof. The expression $b_l^{-k} b_m^n \Pi^{[1]}_\Delta$ obviously is a polynomial of degree $m-l$ in coefficients of f_1 and the expression $a_k^{-l} a_n^m \Pi^{[1]}_\Delta$ is obviously a polynomial in coefficients of f_2 of degree $n-k$. Since two expressions are equal up to sign we have proved (1) and (3).

It is clear that $R^{[1]}_{\Pi,\Delta}$ vanishes if and only if $f_1(z) = 0$ for some root z of f_2, so z is a common root. The same is true for $R^{[2]}_{\Pi,\Delta}$.

For part (4) let us note that the values of the monomial $a_k^{m-l} b_m^{n-k}$ in $b_l^{-k} b_m^n \Pi^{[1]}_\Delta$ come from multiplying the term $a_k z^k$ over roots of f_2. Using the Vieta formula we have: $b_l^{-k} b_m^n \prod a_k z^k = (-1)^{k(m-l)} a_k^{m-l} b_m^{n-k}$. □

Theorem 3. *Assume that $k = l = 0$. Then the product resultants coincide with the Sylvester resultants:*

$$R^{[1]}_\Delta = R^{[1]}_{\Pi,\Delta} = (-1)^{kl+nm} R^{[2]}_{\Pi,\Delta} = (-1)^{kl+nm} R^{[2]}_\Delta.$$

Proof. Both functions $R^{[1]}_\Delta$ and $R^{[1]}_{\Pi,\Delta}$ are polynomials in the coefficients of f_1, f_2 of the same degree. They both vanish on the set of pairs of polynomials having a common root in $\mathbb{C}^*$. Since the set of pairs of polynomials having a common root is irreducible the polynomials $R^{[1]}_\Delta$ and $R^{[1]}_{\Pi,\Delta}$ are proportional. They have the same coefficient in front of the monomial $a_k^{m-l}b_m^{n-k}$, so they are equal. A similar argument works for $R^{[2]}_{\Delta}$ and $R^{[2]}_{\Pi,\Delta}$. □

Keeping in mind multidimensional generalizations, in Section 4 we will present another topological proof of Theorem 1 and in Section 5 we will present an algorithm for computing the product resultant that does not rely on the Sylvester determinant.

4. Weil Reciprocity Law

4.1. Weil symbol and Weil law. Let f and g be two meromorphic functions on a compact Riemann surface S. About each point $p \in S$ one can choose a local parameter u such that $u(p) = 0$ and consider the Laurent expansions $f = c_1u^{k_1} + \dots$, $g = c_2u^{k_2} + \dots$ of f and g, with dots are standing for higher order terms. One can check that the expression

$$\{f,\, g\}_p = (-1)^{k_1k_2}c_1^{-k_2}c_2^{k_1}$$

is independent of the choice of u. The number $\{f,\, g\}_p$ is called the *Weil symbol* of f and g at the point p.

Example 1. If p is a zero of g of multiplicity m_p and $f(p) \neq 0, \infty$, then $\{f,\, g\}_p = f^{-m_p}(p)$. If p is a zero of f of multiplicity m_p and $g(p) \neq 0, \infty$, then $\{f,\, g\}_p = g(p)^{m_p}$.

Example 2. Consider f_1, f_2 from (2.1) as the functions on $\mathbb{C}P^1$. The main terms of Laurent expansions of f_1, f_2 at 0 are a_kz^k, b_lz^l respectively, so $\{f_1,\, f_2\}_0 = (-1)^{kl}a_k^{-l}b_l^k$.

Let $w = 1/z$ be the local parameter on $\mathbb{C}P^1$ at ∞. Then the main terms of Laurent expansions of f_1, f_2 at ∞ are a_nw^{-n}, b_mw^{-m} respectively, so $\{f_1,\, f_2\}_\infty = (-1)^{nm}a_n^mb_m^{-n}$.

Let $D \subset S$ be a finite set containing all points where f or g is equal to 0 or to ∞ (we assume that each function f,g is not identically equal to zero at each connected component of S).

Theorem 4 (Weil reciprocity law)**.** *For any couple of meromorphic functions f and g the following relation holds:*

$$\prod_{p\in D}\{f,\, g\}_p = 1. \tag{4.1}$$

A compact Riemann surface S equipped with its field of meromorphic functions can be considered as an algebraic curve equipped with its field of rational functions. Under such consideration Theorem 4 becomes purely algebraic.

Corollary 1. *Consider f_1, f_2 from (2.1) as the functions on $\mathbb{C}P^1$. Let X, Y be sets of non-zero roots of f_1, f_2. Assume that $X \cap Y = \varnothing$ and roots $x_i \in X$, $y_j \in Y$ have multiplicities m_{x_i}, m_{y_j}. Then according to the Weil reciprocity law and examples 1 and 2*

$$b_l^{-k}b_m^n\prod_{y_j\in Y}f_1^{m_{y_j}}(y_j) = (-1)^{kl+nm}a_k^{-l}a_n^m\prod_{x_i\in X}f_2^{m_{x_i}}(x_i).$$

Thus Theorem 1 could be considered as a corollary of the Weil reciprocity law. On the other hand Theorem 1 provides an elementary proof of the Weil reciprocity law in the case under consideration. In the general case the Weil reciprocity law also can be reduced using Newton polygons to similar elementary arguments [Kh1].

4.2. Topological extension of the Weil reciprocity law. Let S be a Riemann surface (not necessary compact) and let $D \subset S$ be a discrete subset. The *Leray coboundary operator* δ associates to every point $p \in D$ an element $\delta(p) \in H_1(S \setminus D, \mathbb{Z})$ represented by a small circle centered at p with the counterclockwise orientation. Let M be the multiplicative group of meromorphic functions on S which are regular and nonzero on $S \setminus D$.

Theorem 5. *To each couple $f, g \in M$ one can associate a map*

$$\{f, g\} \colon H^1(S \setminus D, \mathbb{Z}) \to \mathbb{C}^*$$

such that the following properties hold:

(1) *for each $p \in D$ the image $\{f, g\}(\delta(p))$ of the cycle $\delta(p)$ under the map $\{f, g\}$ is equal to the Weil symbol $\{f, g\}_p$*;

(2) $\{f, g\} = \{g, f\}^{-1}$;

(3) *for any triple $f, g, \phi \in M$ the identity $\{f, g\phi\} = \{f, g\}\{f, \phi\}$ holds.*

A simple proof of Theorem 5 can be found in [Kh3]. If the surface S is compact then the following relation between the cycles $\delta(p)$ holds:

Lemma 1. *The element $\sum_{p \in D} \delta(p) \in H_1(S \setminus D, \mathbb{Z})$ is equal to zero.*

Proof. Indeed the cycle $-\sum_{p\in D} \delta(p)$ is the boundary of $S \setminus \bigcup_{p\in D} B_p$, where B_p is the open ball centered in p with the boundary $\delta(p)$. □

The Weil reciprocity law follows from Theorem 5 and Lemma 1: $\prod_{p\in D}\{f, g\}_p = 1$ because in $H_1(S \setminus D)$ the identity $\sum_{p\in D} \delta(p) = 0$ holds.

Let us reformulate Lemma 1 in the case related to the torus $\mathbb{C}^* = \mathbb{C}P^1 \setminus \{0, \infty\}$. We will work with $S = \mathbb{C}P^1$ and $D = \{0, \infty\} \cup D'$, where D' is a finite set containing the sets X, Y of non-zero roots of the functions f_1, f_2 from (2.1), i.e., containing non-zero roots of $P = f_1 f_2$. Let Δ be the Newton polyhedron of P. Then Δ is the segment with vertices $A_0 = k + l$ and $A_\infty = n + m$. Let $T^1 \subset \mathbb{C}^*$ be the circle $|z| = 1$ orientated by the form $d(\arg z)$. Let $T^1_{A_0}, T^1_{A_\infty}$ be the cycles in $\mathbb{C}^* \setminus D'$ given by $\frac{1}{\lambda}T^1$, λT^1, where $|\lambda|$ is big enough. Let $k_{A_0} = 1$ and $k_{A_\infty} = -1$.

Theorem 6 (one dimensional topological theorem)**.** *In the notations above the identity $\sum_{p\in D'} \delta(p) = -(k_{A_0} T^1_{A_0} + k_{A_\infty} T^1_{A_\infty})$ holds.*

Proof. Theorem 6 immediately follows from Lemma 1 because $T^1_{A_0} = \delta(0)$ and $T^1_{A_\infty} = -\delta(\infty)$. □

The identity $\prod_{p\in D}\{f_1, f_2\}_p^{-1} = \{f_1, f_2\}_0\{f_1, f_2\}_\infty$ follows from Theorems 5 and 6. It can be rewritten as $\Pi^{[1]}_\Delta / \Pi^{[2]}_\Delta = (-1)^{-kl+nm} a_k^{-l} b_l^k a_n^m b_m^{-n}$. Thus we obtained a topological proof of Theorem 1.

5. Sums over Roots of Laurent Polynomial and Elimination Theory

5.1. Sums over roots of Laurent polynomial. Let $z \in \mathbb{C}^*$ be a root of multiplicity $\mu(z)$ of the Laurent polynomial P. Then for any Laurent polynomial f the following theorem holds.

Theorem 7. *The sum $\sum f(z)\mu(z)$ over all roots $z \in \mathbb{C}^*$ of P is equal to $-(\operatorname{res}_0 \omega + \operatorname{res}_\infty \omega)$, where $\omega = f\frac{dP}{P} = \frac{\partial P}{\partial z} \cdot \frac{fz}{P} \cdot \frac{dz}{z}$.*

Proof. Theorem 7 follows from the Cauchy residue formula since the residue of the form ω at a root z is equal to $f(z)\mu(z)$. □

Theorem 7 provides an explicit formula for the sum $\sum f(z)\mu(z)$: in contrast to the roots of P, the points $0, \infty$ are independent of the coefficients of P and therefore the residues of ω at 0 and ∞ could be explicitly computed.

5.2. Elimination theory related to the one-dimensional case. Let P and f be Laurent polynomials as above. Here we explain how to find any symmetric function of the set of values $\{f(z)\}$ over all roots $z \in \mathbb{C}^*$ (each root z is taken with multiplicity $\mu(z)$).

Denote by $f^{(k)}$ the number $f^{(k)} = \sum_z f^k(z)\mu(z)$. By Theorem 7 one can calculate $f^{(k)}$ for any k explicitly. The power sum symmetric polynomials form a generating set for the ring of symmetric polynomials.

Corollary 2. *One can find explicitly all symmetric functions of $\{f(z)\}$, construct a monic polynomial whose roots are $\{f(z)\}$, and eliminate z from the definition of the set $\{f(z)\}$. In particular, one can compute the products $\Pi_\Delta^{[1]}, \Pi_\Delta^{[2]}$ defined in Section 3.*

6. Developed Systems and Combinatorial Coefficients

Let $\Delta = (\Delta_1, \dots, \Delta_n)$ be an n-tuple of convex polyhedra in $\mathbb{R}^n$, and let $\sum \Delta_i = \Delta_1 + \dots + \Delta_n$ be their Minkowski sum. Each face Γ of the polyhedron $\sum \Delta_i$ can be uniquely represented as a sum $\Gamma = \Gamma_1 + \dots + \Gamma_n$, where Γ_i is a face of Δ_i.

An n-tuple Δ is called *developed* if for each face Γ of the polyhedron $\sum \Delta_i$, at least one of the terms Γ_i in its decomposition is a vertex.

The system of equations $f_1 = \dots = f_n = 0$ on $(\mathbb{C}^*)^n$, where $f_1, \dots, f_n$ are Laurent polynomials, is called developed if the n-tuple $(\Delta_1, \dots, \Delta_n)$ of their Newton polyhedra is developed.

For a developed n-tuple of polyhedra Δ, a map $h\colon \partial \sum \Delta_i \to \partial \mathbb{R}^n_+$ of the boundary $\partial \sum \Delta_i$ of $\sum \Delta_i$ into the boundary of the positive octant is called *characteristic* if the component h_i of the map $h = (h_1, \dots, h_n)$ vanishes precisely on the faces Γ for which the i-th term Γ_i in the decomposition is a point (a vertex of the polyhedron Δ_i). One can show that the space of characteristic maps is nonempty and connected. The preimage of the origin under a characteristic map is precisely the set of all vertices of the polyhedron $\sum \Delta_i$.

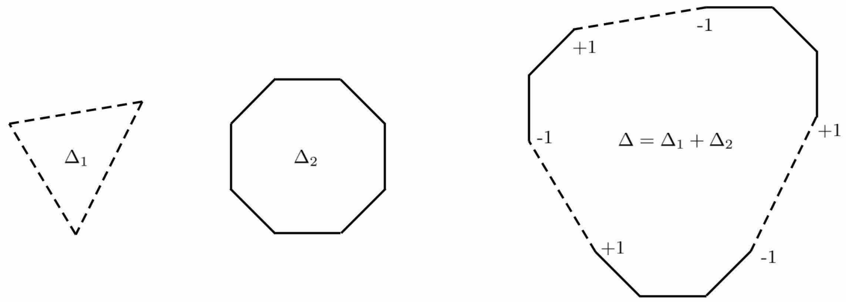

FIGURE 1. Combinatorial coefficient in dimension 2

The *combinatorial coefficient* k_A of a vertex A of $\sum \Delta_i$ is the local degree of the germ

$$h\colon (\partial \sum \Delta_i,\, A) \to (\partial \mathbb{R}^n_+,\, 0)$$

of a characteristic map restricted to the boundary $\partial \sum \Delta_i$ of $\sum \Delta_i$. The combinatorial coefficient is independent of a choice of a characteristic map, but it depends on the choice of the orientation of $\sum \Delta_i$ and $\mathbb{R}^n_+$. The first one is given by the orientation of the space of characters on $(\mathbb{C}^*)^n$. The second is defined by an ordering of the polyhedra $\Delta_1, \ldots, \Delta_n$ in n-tuple Δ. Both orientations are an arbitrary choice, and after changing each of them the combinatorial coefficient will change sign. For more detailed discussion of combinatorial coefficients see [GKh1], [Sop], [Kh1].

Let us discuss combinatorial coefficients in the two-dimensional case. Two polygons $\Delta_1, \Delta_2 \subset \mathbb{R}^2$ are developed if and only if they do not have parallel sides with the same direction of the outer normals (see figure 1). If Δ_1, Δ_2 are developed each side of $\Delta_1 + \Delta_2$ comes either from Δ_1 or from Δ_2. That is, each side of $\Delta_1 + \Delta_2$ is either the sum of a side of Δ_1 and a vertex of Δ_2, or the sum of a vertex of Δ_1 and a side of Δ_2.

The two types of sides are labeled by 2 (dashed in the picture) and 1 (solid in the picture) respectively. Giving $\mathbb{R}^2$ and the positive octant the standard orientations we can find the local degree of a characteristic map at a vertex. It is equal to 0 if neighbouring edges have the same label, to $+1$ if the label at A is changing from 2 to 1 in the counter clockwise direction, and to -1 if it is changing from 1 to 2.

So the combinatorial coefficient k_A of a vertex $A \in \Delta$ is equal to 0 if neighbouring edges have the same label, and $+1$ or -1 (depending on the orientation) if the labeling changes at A. The only possible value of combinatorial coefficient in dimension 2 is -1, 0 or $+1$ because the local mapping degree of one dimensional manifolds could take only these values. The combinatorial coefficient in dimension $\geqslant 3$ could be any integer number.

7. Topological Theorem

7.1. Grothendieck cycle. Let z be an isolated root of a system $f_1 = \ldots = f_n = 0$ on $(\mathbb{C}^*)^n$, where $f_1, \ldots, f_n$ are Laurent polynomials. The Grothendieck cycle γ_z is a class in the group of n-dimensional homologies of the complement $(U \setminus \Gamma)$ of a small neighborhood U of the point z, of the hyperplane Γ defined by the equation $P = f_1 \ldots f_n = 0$. For almost all small enough $\varepsilon = (\varepsilon_1, \ldots \varepsilon_n) \in \mathbb{R}^n_+$, the subset $\gamma_{z,\varepsilon}$ defined by $|f_i| = \varepsilon_i$ is a smooth compact real submanifold of $(U \setminus \Gamma)$. The Grothendieck cycle γ_z is the cycle of the submanifold $\gamma_{z,\varepsilon}$ for small enough ε oriented by the form $d(\arg f_1) \wedge \ldots \wedge d(\arg f_n)$. The orientation of the Grothendieck cycle depends on the order of the equations $f_1 = 0, \ldots, f_n = 0$.

7.2. The cycle related to a vertex of the Newton polyhedron. Let $\Gamma \subset (\mathbb{C}^*)^n$ be a hypersurface $P = 0$, where P is a Laurent polynomial with Newton polyhedron $\Delta(P)$. Let $T^n \subset (\mathbb{C}^*)^n$ be the torus $|z_1| = \ldots = |z_n| = 1$ orientated by $\omega = d(\arg z) \wedge \ldots \wedge d(\arg(z_n)$. The sign of ω depends on the order of variables, so the sign of a cycle T^n depends on the orientation of the space $\mathbb{R}^n$ of characters on $(\mathbb{C}^*)^n$.

For every vertex A of $\Delta(P)$ we will assign an n-dimensional cycle T^n_A in $(\mathbb{C}^*)^n \setminus \Gamma$ defined up to homological equivalence. For this denote by $\xi_A = (\xi_1, \ldots, \xi_n)$ an integer covector such that the inner product of $x \in \sum \Delta_1$ with ξ_A attains its maximum value at A. Consider the 1-parameter subgroup $\lambda(t) = (t^{\xi_1}, \ldots, t^{\xi_n})$ of $(\mathbb{C}^*)^n$. For t with large enough absolute value $|t|$ the translation $\lambda(t)T^n$ of T^n by the subgroup $\lambda(t)$ does not intersect the hypersurface Γ.

Let us define T^n_A as a cycle $\lambda(t)T^n$ with $|t|$ large enough. The definition makes sense because the homology class of $\lambda(t)T^n$ in $(\mathbb{C}^*)^n \setminus \Gamma$ does not depend on the choice of ξ_A and t, provided $|t|$ is large enough.

7.3. Topological theorem for n Laurent polynomials. Let $f_1, \ldots, f_n$ be Laurent polynomials with developed Newton polyhedra $\Delta_1, \ldots, \Delta_n$. Let Γ be a hypersurface in $(\mathbb{C}^*)^n$ defined by the equation $P = f_1 \ldots f_n = 0$. In [GKh],[GKh1] the following theorem is proved.

Theorem 8. *In $(\mathbb{C}^*)^n \setminus \Gamma$ the sum of the Grothendieck cycles γ_z over all roots z of the system $f_1 = \ldots = f_n = 0$ is homologous to the cycle $(-1)^n \sum k_A T^n_A$, where the sum is taken over all vertices A of $\sum \Delta_i$ and k_A is the combinatorial coefficient at the vertex A.*

The signs in the topological theorem depend on the choice of order of variables $z_1, \ldots, z_n$ and on the choice of order of functions $f_1, \ldots, f_n$. In the statement these orders are fixed in an arbitrary way. Changing the order of the variables changes the sign of cycle at vertices T^n_A and all of the combinatorial coefficients. Choosing a different order for the equations will change the signs of all the Grothendieck cycles γ_z, and all of the combinatorial coefficients as well.

7.4. Topological theorem for $(n+1)$ Laurent polynomials. We will say that the collection of polyhedra $\Delta_1, \ldots, \Delta_{n+1}$ is *i-developed* if the collection with Δ_i removed is developed. In this section we present a version of the topological theorem applicable for a collection of $n+1$ Laurent polynomials which is i-developed and j-developed for some $1 \leqslant i < j \leqslant n$

Let i, j be indexes such that $1 \leqslant i < j \leqslant n+1$. We will associate with i, j the permutation $\{k_1, \ldots, k_{n+1}\}$ of $\{1, \ldots, n+1\}$ defined by the relations $k_1 = i$, $k_2 = j$, and $k_3 < \cdots < k_{n+1}$.

The following lemma is obvious.

Lemma 2. *A collection of $(n+1)$ polyhedra $\Delta_1, \ldots, \Delta_{n+1}$ in $\mathbb{R}^n$ is i-developed and j-developed for some $i < j$ if and only if the collection $(\Delta_i + \Delta_j)$, $\Delta_{k_3}, \ldots, \Delta_{k_{n+1}}$ is developed.*

Let $\Delta_1, \ldots, \Delta_{n+1}$ be an i-developed and j-developed collection of Newton polyhedra. Let us denote by $k_A^{i,j}$ the combinatorial coefficient of a vertex $A \in \sum \Delta_i$ associated with the the collection $\Delta_i + \Delta_j$, $\Delta_{k_3}, \ldots, \Delta_{k_{n+1}}$. Let $f_1, \ldots, f_{n+1}$ be Laurent polynomials with Newton polyhedra $\Delta_1, \ldots, \Delta_{n+1}$ such that the system

$$f_1 = \cdots = f_{n+1} = 0 \tag{7.1}$$

is not consistent in $(\mathbb{C}^*)^n$. Denote by X_i the set of all roots x of the system

$$f_1 = \cdots = \hat{f}_i = \cdots = f_{n+1} = 0, \tag{7.2}$$

where the equation $f_i = 0$ is removed. Denote by X_j the set of all roots y of the system

$$f_1 = \cdots = \hat{f}_j = \cdots = f_{n+1} = 0, \tag{7.3}$$

where the equation $f_j = 0$ is removed.

Theorem 9. *Assume that the Newton polyhedra $\Delta_1, \ldots, \Delta_{n+1}$ of the Laurent polynomials $f_1, \ldots, f_{n+1}$ are i-developed and j-developed and that the system (7.1) is not consistent in $(\mathbb{C}^*)^n$. Then in the group $H_n((\mathbb{C}^*)^n \setminus \Gamma, \mathbb{Z})$ the identity*

$$(-1)^{j-2} \sum_{x \in X_i} \gamma_x + (-1)^{i-1} \sum_{y \in X_j} \gamma_y = (-1)^n \sum k_A^{i,j} T_A^n$$

holds, where γ_x and γ_y are the Grothendieck cycles of the roots x and y of the systems (7.2), (7.3) and the summation on the right is taken over all vertices A of $\Delta = \Delta_1 + \cdots + \Delta_n$.

Proof. According to the topological theorem the sum of the Grothendieck cycles γ_z over the set $X_{i,j}$ of all roots z of the system

$$f_i f_j = f_{k_1} = \cdots = f_{k_{n+1}} = 0$$

is equal to $(-1)^n k_A^{i,j} T_A$. The set $X_{i,j}$ is equal to $X_i \cup X_j$, where X_i is the set of roots x of the system $f_j = f_{k_1} = \cdots = f_{k+1} = 0$ and X_j is the set of roots y of the system $f_i = f_{k_1} = \cdots = f_{k+1} = 0$. If $z = x \in X_i$ then the cycle γ_z is equal to the cycle $(-1)^{j-2}\gamma_x$ for the system (7.2). The sign $(-1)^{j-2}$ in the identity appears because of the change of the order equations from $f_j = f_{k_1} = \cdots = f_{k_{n+1}} = 0$ to $f_1 = \cdots = \hat{f}_i = \cdots = f_{n+1} = 0$. In a similar way if $z = y \in X_j$ then the cycle γ_z is equal to the cycle $(-1)^{i-1}\gamma_y$ for the system (7.3). □

8. Parshin Reciprocity Laws

8.1. Analog of the determinant of $n+1$ vectors in n-dimesional space over $\mathbb{F}_2$. A determinant of n vectors in n-dimensional space L_n over the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ is unique non-zero multilinear function on n-tuples of vectors in L_n which is invariant under the $\mathrm{GL}(n, \mathbb{F}_2)$ action and which has value 0 if the n-tuple is dependent.

It turns out that there exists unique function on $(n+1)$-tuples of vectors in L_n having exactly the same properties (see [Kh1], [Kh4]).

Theorem 10. *There exists a unique non-zero function D on $(n+1)$-tuples of vectors in L_n satisfying the following properties*:

(i) *D is $\mathrm{GL}(n, \mathbb{F}_2)$ invariant, i.e., for any $A \in \mathrm{GL}(n, \mathbb{F}_2)$ the equality*

$$D(k_1, \ldots, k_{n+1}) = D(A(k_1), \ldots, A(k_{n+1}))$$

holds;

(ii) *if the rank of $k_1, \ldots, k_{n+1}$ is $< n$ then $D(k_1, \ldots, k_{n+1}) = 0$;*

(iii) *D is multilinear.*

Let us present two explicit formulas for D:

I) If the rank of $k_1, \ldots, k_{n+1}$ is $< n$ then $D(k_1, \ldots, k_{n+1}) = 0$ (see (ii)). If the rank is n then there is a unique non-zero collection $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{F}_2$ such that $\lambda_1 k_1 + \cdots + \lambda_{n+1} k_{n+1} = 0$ and . In this case $D(k_1, \ldots, k_{n+1}) = 1 + \lambda_1 + \cdots + \lambda_{n+1}$.

II) If on the space L_n the coordinates are fixed, then $D(k_1, \ldots, k_{n+1}) = \sum_{j>i} \Delta_{ij}$, where Δ_{ij} is the determinant of $(n \times n)$ matrix whose first $n-1$ columns are coordinates of vectors $k_1, \ldots, k_{n+1}$ with vectors k_i, k_j removed and the last column is the coordinatewise product of vectors k_i and k_j.

For a vector $v \in \mathbb{Z}^n$ let $\tilde{v} \in \mathbb{F}_2^n$ be its mod 2 reduction. For an $(n+1)$-tuple of vectors $v_1 \ldots v_{n+1} \in \mathbb{Z}^n$ we define $D(v_1 \ldots v_{n+1})$ as $D(\tilde{v}_1 \ldots \tilde{v}_{n+1})$.

The determinant of a matrix A over $\mathbb{R}$ is the volume of the oriented parallelepiped spanned by the columns of A. It turns out that the function D also computes the volume of some figure (see [Kh2]). This property of D allows to fit it into the topological version of Parshin reciprocity laws (see [Kh4] and Section 8.3).

8.2. Parshin symbols of monomials. Consider $n+1$ monomials $c_1\mathbf{z}^{\mathbf{k}_1}, \ldots, c_{n+1}\mathbf{z}^{\mathbf{k}_{n+1}}$ with nonzero coefficients $c_i \in \mathbb{C}^*$ in n complex variables $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{k}_i \in (\mathbb{Z})^n$, $\mathbf{k}_i = (k_{i,1}, \ldots, k_{i,n})$, $c_i\mathbf{z}^{\mathbf{k}_i} = c_i z_1^{k_{i,1}} \cdot \cdots \cdot z_n^{k_{i,n}}$. The *Parshin Symbol* $[c_1\mathbf{z}^{\mathbf{k}_1}, \ldots, c_{n+1}\mathbf{z}^{\mathbf{k}_{n+1}}]$ of the sequence $c_1\mathbf{z}^{\mathbf{k}_1}, \ldots, c_{n+1}\mathbf{z}^{\mathbf{k}_{n+1}}$ is equal by definition to

$$(-1)^{D(\mathbf{k}_1,\ldots,\mathbf{k}_{n+1})} c_1^{-\det(\mathbf{k}_2,\ldots,\mathbf{k}_{n+1})} \ldots c_{n+1}^{(-1)^{n+1}\det(\mathbf{k}_1,\ldots,\mathbf{k}_n)}$$
$$= (-1)^{D(\mathbf{k}_1,\ldots,\mathbf{k}_{n+1})} \exp\left(-\det \begin{pmatrix} \ln c_1 & k_{1,1} & \ldots & k_{1,n+1} \\ \ldots & \ldots & \ldots & \ldots \\ \ln c_{n+1} & k_{n+1,1} & \ldots & k_{n+1,n+1} \end{pmatrix} \right),$$

where $D\colon (\mathbb{Z}^n)^{n+1} \to \mathbb{Z}/2\mathbb{Z}$ is the function defined in the previous section.

Example 3. The Parshin symbol $[c_1z^{k_1},\, c_2z^{k_2}]$ of functions $c_1z^{k_1},\, c_2z^{k_2}$ in one variable z is equal to $(-1)^{k_1k_2}c_1^{-k_2}c_1^{k_1}$, thus it is equal to the Weil symbol $\{c_1z^{k_1},\, c_2z^{k_2}\}_0$ of these functions at the origin $z=0$.

By definition, the Parshin symbol is skew-symmetric, so for example,

$$[c_1\mathbf{z}^{\mathbf{k}_1},\, c_2\mathbf{z}^{\mathbf{k}_2},\, \dots,\, c_{n+1}\mathbf{z}^{\mathbf{k}_{n+1}}] = [c_2\mathbf{z}^{\mathbf{k}_2},\, c_1\mathbf{z}^{\mathbf{k}_1},\, \dots,\, c_{n+1}\mathbf{z}^{\mathbf{k}_{n+1}}]^{-1},$$

and multiplicative, so for example, if $c_1\mathbf{z}^{\mathbf{k}_1} = a_1b_1\mathbf{z}^{\mathbf{l}_1+\mathbf{m}_1}$, then

$$[c_1\mathbf{z}^{\mathbf{k}_1},\, \dots,\, c_{n+1}\mathbf{z}^{\mathbf{k}_{n+1}}] = [a_1\mathbf{z}^{\mathbf{l}_1},\, \dots,\, c_{n+1}\mathbf{z}^{\mathbf{l}_{n+1}}][b_1\mathbf{z}^{\mathbf{m}_1},\, \dots,\, c_{n+1}\mathbf{z}^{\mathbf{m}_{n+1}}].$$

8.3. Topological version of Parshin laws. The Parshin reciprocity laws (see [P], [FP]) are applicable to $(n+1)$ rational functions on an n-dimensional algebraic variety over an algebraically closed field of any characteristic. They contain several general relations between the Parshin symbols of these $(n+1)$ functions analogous to the relation between the Weil symbols of two functions given in the Weil reciprocity law for an algebraic curve. We will need a topological version of Parshin's laws over $\mathbb{C}$ in the special situation that the algebraic variety is $(\mathbb{C}^*)^n$ and the $(n+1)$ functions are Laurent polynomials $f_1,\, \dots,\, f_{n+1}$. Let us state needed facts for that special situation (for general case see [Kh4]).

Let Γ be a hypersurface in $(\mathbb{C}^*)^n$ defined by the equation $P = f_1 \dots f_n = 0$. According to the topological version of the Parshin reciprocity laws there is a map $[f_1,\, \dots,\, f_{n+1}]\colon H^n((\mathbb{C}^*)^n \setminus \Gamma,\, \mathbb{Z}) \to \mathbb{C}^*$ having the following properties:

1) The map $[f_1,\, \dots,\, f_n]$ depends skew symmetrically on the components f_i, so for example $[f_1,\, f_2,\, \dots,\, f_{n+1}] = [f_2,\, f_1,\, \dots,\, f_{n+1}]^{-1}$.

2) Let $A = A_1 + \ldots + A_{n+1}$ be a vertex of $\Delta_1 + \cdots + \Delta_{n+1}$, where A_i is a vertex in Δ_i. Let $c_i\mathbf{z}^{\mathbf{k}_i}$ be the monomial with the coefficient c_i in f_i corresponding to $A_i \in \Delta_i$. Then $[f_1,\, \dots,\, f_{n+1}](T^n_A) = [c_1\mathbf{z}^{\mathbf{k}_1},\, \dots,\, c_{n+1}\mathbf{z}^{\mathbf{k}_{n+1}}]$, where T^n_A is the cycle corresponding to the vertex A.

3) Let z be a root of multiplicity $\mu(z)$ of the system $f_1 = \ldots = \hat{f}_i = \cdots = f_{n+1} = 0$, where the equation $f_i = 0$ has been removed. Let γ_z be the corresponding Grothendieck cycle. Assume that $f_i(z) \neq 0$. Then $[f_1,\, f_2,\, \dots,\, f_{n+1}](\gamma_z) = f_i(z)^{(-1)^i\mu(z)}$.

9. Product over Roots of a System of Equations

Let $\Delta = (\Delta_1,\, \dots,\, \Delta_{n+1})$ be $(n+1)$ Newton polyhedra in the lattice $\mathbb{Z}^n$ and let Ω_Δ be the space of $(n+1)$-tuples of Laurent polynomials $(f_1,\, \dots f_{n+1})$ such that the Newton polyhedron of f_i is contained in Δ_i. We will define a rational function $\Pi^{[i]}_\Delta$ on the space Ω_Δ, which we will call *the product of f_i over the common zeros of f_j for $j \neq i$*. Let $U^i_\Delta \subset \Omega_\Delta$ be the Zariski open set defined by the following condition: $(f_1,\, \dots,\, f_{n+1}) \in U^i_\Delta$ if and only if the set $Y^i_\Delta \subset (\mathbb{C}^*)^n$ of common zeros of f_j for $j \neq i$ is finite and the number of points in Y^i_Δ (counting with multiplicities) is equal to $n!\,\mathrm{Vol}(\Delta_1,\, \dots,\, \hat{\Delta}_i,\, \dots,\, \Delta_{n+1})$ (the polyhedron Δ_i is omitted in this mixed volume).

Definition 1. We define the function $\Pi_\Delta^{[i]}$ on U_Δ^i as follows. If the set $Y_\Delta^i \subset (\mathbb{C}^*)^n$ is empty (i.e., if $n! \operatorname{Vol}(\Delta_1, \dots, \hat{\Delta}_i, \dots, \Delta_{n+1}) = 0$) then $\Pi_\Delta^{[i]} \equiv 1$. Otherwise

$$\Pi_\Delta^{[i]}(f_1, \dots, f_{n+1}) = \prod_{x \in Y_\Delta^i} f_i^{m_x}(x),$$

where m_x is the multiplicity of the common zero $x \in Y_\Delta^i$ of the Laurent polynomials f_j for $j \neq i$.

Lemma 3. *The function $\Pi_\Delta^{[i]}$ is regular on U_Δ^i. It can be extended to a rational function on Ω_Δ.*

Proof. Let $\tilde{U}_\Delta^i \subset U_\Delta^i$ be the Zariski open set in which common zeros of f_j with $j \neq i$ have multiplicity one. The function $\Pi_\Delta^{[i]}$ is obviously regular on $\tilde{U}_\Delta^i$. By the removable singularity theorem it is regular in U_Δ^i. By definition $\Pi_\Delta^{[i]}$ is an algebraic single-valued function on U_Δ^i. Thus, it is rational function on Ω_Δ. □

Note that even if $(f_1, \dots, f_{n+1}) \notin U_\Delta^i$ then the product of f_i over the common roots of f_j for $j \neq i$ is well defined if the set of common roots is finite. But this product is not necessarily equal to $\Pi_\Delta^{[i]}(f_1, \dots, f_{n+1})$.

Assume that the collection $\Delta = (\Delta_1, \dots, \Delta_{n+1})$ is i-developed. Consider the space Ω_Δ of $(n+1)$-tuples $(f_1, \dots, f_{n+1})$ of Laurent polynomials whose Newton polyhedra are contained correspondingly in $(\Delta_1, \dots, \Delta_{n+1})$. Denote by Ω_Δ^i the open subset in Ω_Δ defined by the condition that the Newton polyhedron of f_j is Δ_j for $j \neq i$.

Theorem 11. *The function $\Pi_\Delta^{[i]}$ is regular on Ω_Δ^i. Moreover, there exists a monomial M_i in vertex coefficients of all the f_j for $j \neq i$ such that the product $M_i \Pi_\Delta^{[i]}$ is a polynomial.*

Proof. Since the system $f_1 = \ldots = \hat{f}_i = \cdots = f_{n+1} = 0$ is developed, the number of roots counting with multiplicities is constant on Ω_Δ^i and $\Omega_\Delta^i \subset U_\Delta^i$. Therefore the function $\Pi_\Delta^{[i]}$ is regular on Ω_Δ^i. Thus there exists a monomial M_i in the vertex coefficients of f_j for $j \neq i$ such that the product $M_i \Pi_\Delta^{[i]}$ is a polynomial on Ω_Δ. □

In Section 12.4 we will present an algorithm for computing the function $\Pi_\Delta^{[i]}$ for i-developed systems.

10. Identity for i- and j- Developed System

Theorem 12. *Assume that $\Delta = (\Delta_1, \dots, \Delta_{n+1})$ is i-developed and j-developed for some $i \neq j$. Assume also that $(f_1, \dots, f_{n+1}) \in \Omega_\Delta$ satisfies the assumptions of Theorem 9. Then*

$$\Pi_\Delta^{[i]}(f_1, \dots, f_{n+1})(\Pi_\Delta^{[j]}(f_1, \dots, f_{n+1}))^{-1} = \prod [f_1, \dots, f_{n+1}]_A^{(-1)^{n+i+j} k_A^{i,j}},$$

where the product on the right is taken over the vertices A of $\sum \Delta_i$.

Proof. Let us apply the element $[f_1, \ldots, f_{n+1}] \in H^n((\mathbb{C}^*)^n \setminus \Gamma, \mathbb{C}^*)$ to the identity from Theorem 9. According to Section 8.3 we have

$$\Pi_\Delta^{[i]}(f_1, \ldots, f_{n+1})^{(-1)^i(-1)^{j-2}}\Pi_\Delta^{[j]}(f_1, \ldots, f_{n+1}))^{(-1)^j(-1)^{i-1}} = \prod [f_1, \ldots, f_{n+1}]_A^{(-1)^n k_A^{i,j}}.$$

To complete the proof it is enough to raise each side of this identity to the power $(-1)^{i+j}$. □

Theorem 12 contains the formula from [Kh1] for the product in $(\mathbb{C}^*)^n$ of all the roots of a developed system of n equations. To find such a product it is enough to compute the product over all roots of any monomial $\mathbf{z}^\mathbf{m}$: taking the coordinate functions $z_1, \ldots, z_n$ as such monomials one obtains all coordinates of the product of all roots. Assume that $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ is, say, 1-developed and that $\Delta_1 = \{m\}$ is a single point. Consider an $(n+1)$ tuple of Laurent polynomials $f_1, \ldots, f_{n+1}$ with Newton polyhedra $\Delta_1, \ldots, \Delta_{n+1}$.

Then: 1) f_1 is the monomial $\mathbf{z}^\mathbf{m}$ with a nonzero coefficient c, i.e., $f_1 = c\mathbf{z}^\mathbf{m}$; 2) Δ is j-developed for any $1 < j \leqslant (n+1)$ because the collection Δ with Δ_j skipped contains the point Δ_1.

Let us apply Theorem 12 to the case under consideration with $i = 1$, $j = 2$. We have: a) $\Pi_\Delta^{[2]} = 1$ because the system $cx^m = f_3 = \cdots = f_{n+1} = 0$ has no roots in $(\mathbb{C}^*)^n$; b) $\Pi_\Delta^{[1]}$ is equal to the product of $c\mathbf{z}^\mathbf{m}$ over all roots of the system $f_2 = \cdots = f_{n+1} = 0$, i.e., is equal to $c^{n!\,\mathrm{Vol}(\Delta_2,\ldots,\Delta_{n+1})}$ multiplied by the product of $\mathbf{z}^\mathbf{m}$ over all roots of the system.

Corollary 3. *With the assumptions of Theorem 12 for $i = 1$, $j = 2$ and $f_1 = c\mathbf{z}^\mathbf{m}$ the product of $\mathbf{z}^\mathbf{m}$ over the roots of the system $f_2 = \cdots = f_{n+1} = 0$ multiplied by $c^{n!\,\mathrm{Vol}(\Delta_2,\ldots,\Delta_{n+1})}$ is equal to $\prod [f_1, \ldots, f_{n+1}]_A^{(-1)^{n+1}k_A^{1,2}}$.*

Corollary 4. *If $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ is i-developed and j-developed then on Ω_Δ the relation*

$$\Pi_\Delta^{[i]}/\Pi_\Delta^{[j]} = M_{i,j}s_{i,j} \tag{10.1}$$

holds, where $M_{i,j}$ is an explicit monomial in the vertex coefficients of all f_k and $s_{i,j} = \prod_A (-1)^{D(A_1,\ldots,A_{n+1})k_A^{i,j}}$, where $A_1, \ldots, A_{n+1}$ are vertices of $\Delta_1, \ldots, \Delta_{n+1}$ such that $A_1 + \cdots + A_{n+1} = A$.

Proof. By definition $[f_1, \ldots, f_{n+1}]_A$ is an explicit monomial in coefficients of $f_1, \ldots, f_{n+1}$ corresponding to the vertices $A_1, \ldots, A_{n+1}$ multiplied by $(-1)^{D(A_1,\ldots,A_{n+1})}$. □

11. Identities for a Completely Developed System

Definition 2. A collection $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ of polyhedra is called *completely developed* if it is i-developed for all $1 \leqslant i \leqslant n+1$.

Theorem 13. *For a completely developed Δ there is a $(n+1)$-tuple $(M_1, \dots, M_{n+1})$, where M_k is a monomial depending on the vertex coefficients of $(f_1, \dots, f_{n+1}) \in \Omega_\Delta$ with f_k removed and an $(n+1)$-tuple $(s_1, \dots, s_{n+1})$, where $s_i = \pm 1$, such that*

$$\Pi_\Delta^{[1]} M_1 s_1 = \dots = \Pi_\Delta^{[n+1]} M_{n+1} s_{n+1}. \tag{11.1}$$

The $(n+1)$-tuple $(M_1, \dots, M_{n+1})$ of monomials is unique and the $(n+1)$-tuple of signs $(s_1, \dots, s_{n+1})$ is unique up to simultaneous multiplication by -1. Moreover the relation $s_i s_j = s_{i,j}$ holds.

Proof. To prove existence we will use the identities (10.1) for $j > i = 1$. Let us represent each monomial $M_{1,j}$ as a product $\prod_{1\leqslant k\leqslant n+1} m_{1,j}^{(k)}$, where $m_{1,j}^{(k)}$ is a monomial depending on the vertex coefficients of f_k only. Denote $m = \prod_{j\neq 1} m_{1,j}^{(j)}$ and divide each identity (10.1) for $j > i = 1$ by m. We obtain a needed representation with $(M_1, \dots, M_{n+1}) = (m^{-1}, M_{1,2}m^{-1}, \dots, M_{1,n+1}m^{-1})$, and $(s_1, \dots, s_{k+1}) = (1, s_{1,2}, \dots, s_{1,n+1})$.

To show uniqueness assume that $(M'_1, \dots, M'_{n+1})$ and $(s'_1, \dots, s'_{n+1})$ are another pair of $(n+1)$-tuples of monomials and signs such that

$$\Pi_\Delta^{[1]} M'_1 s'_1 = \dots = \Pi_\Delta^{[n+1]} M'_{n+1} s'_{n+1}.$$

For any i the ratio M_i/M'_i is a monomial which does not depend on coefficients of f_i . But since $M_1 s_1/M'_1 s'_1 = \dots = M_{n+1}s_{n+1}/M'_{n+1}s'_{n+1}$, the ratio M_i/M'_i is equal to 1 and collections of signs are proportional. The relation $s_i s_j = s_{i,j}$ follows from (10.1). □

Remark 1. Let $G = (\mathbb{Z}/2\mathbb{Z})^{n+1}/D$ be the factor group of $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ by the diagonal subgroup $D = \{(1, \dots, 1), (-1, \dots, -1)\}$. Assigning to a collection of completely developed polyhedra $\Delta_1, \dots, \Delta_{n+1}$ the collection of signs $(s_1, \dots, s_{n+1})$ defined up to simultaneous multiplication by -1 gives a map to G. This map is a coordinatewise homomorphism with respect to Minkowski sum, for example the relation

$$\phi(\Delta_1 + \Delta'_1, , \dots, \Delta_{n+1}) = \phi(\Delta_1, \dots, \Delta_{n+1})\phi(\Delta'_1, \dots, \Delta_{n+1}),$$

holds for any completely developed collections $(\Delta_1, \Delta_2, \dots, \Delta_{n+1})$ and $(\Delta'_1, \Delta_2, \dots, \Delta_{n+1})$.

The multihomomorphism ϕ is closely related to the resultants. Take any collection of Laurent polynomials $f = (f_1, \dots, f_{n+1})$ with completely developed collection of Newton polyhedra $\Delta = (\Delta_1, \dots, \Delta_{n+1})$ such that all vertex coefficients of f_i's are equal to 1. Then the values of the $n+1$ product resultants on the collection f would coincide up to a sign, and so $\phi(\Delta) = (R_{\Pi\Delta}^{[1]}, \dots, R_{\Pi\Delta}^{[n+1]})$ up to multiplication by common factor.

In a contrast to the Δ-resultants, the map ϕ in general is not translation invariant (although, it is invariant under simultaneous translation of Δ_i's) and is not symmetric.

Theorems 12 and 13 allow to describe the monomials $M_1, \dots, M_{n+1}$ in terms of Parshin symbols. Now we will describe these monomials in terms of Newton polyhedra. Let us introduce some notation.

For an i-developed collection $\Delta_1, \ldots, \Delta_{n+1}$ denote by $\tilde{\Delta}_i$ the sum $\Delta_1+\cdots+\hat{\Delta}_i+\cdots+\Delta_{n+1}$, where Δ_i is removed. For each facet $\Gamma \subset \tilde{\Delta}_i$ denote by v_Γ an irreducible integral covector such that the inner product with v_Γ attains its maximum value on $\tilde{\Delta}_i$ at the facet Γ. With v_Γ one associates the value $H_{\Delta_i}(v_\Gamma)$ of the support function of Δ_i on v_Γ, the faces $\Delta_j^{v_\Gamma}$ of Δ_j at which the inner product with v_Γ attains the maximal value. The facet Γ is *essential* if among the faces $\Delta_j^{v_\Gamma}$ with $j \neq i$ exactly one face $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ is a vertex. With an essential facet Γ one associates a coefficient $a_{j(v_\Gamma)}$ of the Laurent polynomial $f_{j(v_\Gamma)}$ at the vertex $\Delta_{j(v_\Gamma)}^{v_\Gamma}$, and the integral mixed volume $V(v_\Gamma)$ of the collection of polyhedra $\{\Delta_j^{v_\Gamma}\}$ in which the polyhedra $\Delta_i^{v_\Gamma}$ and $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ are removed.

Let $L(\Gamma)$ be a linear subspace parallel to the minimal affine subspace containing Γ. We define the integral volume on $L(\Gamma)$ as the translation invariant volume normalized by the following condition: for any $v_1, \ldots, v_{n-1}$ the generators of the lattice $L(\Gamma) \cap \mathbb{Z}^n$, the volume the parallelepiped with sides $v_1, \ldots, v_{n-1}$ is equal to 1.

Any polyhedron in the collection $\{\Delta_j^{v_\Gamma}\}$ in which the polyhedra $\Delta_i^{v_\Gamma}$ and $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ are removed could be translated to $L(\Gamma)$. By $V(v_\Gamma)$ we mean the integral mixed volume of these translations (note that $V(v_\Gamma)$ could vanish for some Γ).

Theorem 14. *For the monomial M_i the following formula holds*

$$M_i = \prod a_{j(v_\Gamma)}^{(n-1)! H_{\Delta_i}(v_\Gamma) V(v_\Gamma)},$$

where the product is taken over all essential facets Γ of $\tilde{\Delta}_i$.

Proof. Let us sketch a proof for M_1. In the proof of Theorem 13 we represented M_1 in the form $M_1 = \left(\prod_{j\neq 1} m_{1,j}^{(j)}\right)^{-1}$. One can deal with each factor $m_{1,j}^{(j)}$ separately. We will show that $m_{1,2}^{(2)} = \prod_\Gamma a_{2(v_\Gamma)}^{d(v_\Gamma)}$, where $a_{2(v_\Gamma)}$ is the coefficient of f_2 at the vertex $\Delta_2^{v_\Gamma}$, $d(v_\Gamma) = (n-1)! V(v_\Gamma) H_{\Delta_1}(v_\Gamma)$, and the product is taken over all facets Γ of $\Delta_{1,2} = \Delta_3 + \cdots + \Delta_{n+1}$.

Let $C \subset (\mathbb{C}^*)^n$ be the curve defined by the system $f_3 = \ldots = f_{n+1} = 0$ (we assume that this system is generic enough). The normalization $\tilde{C}$ of C has a very explicit description: it can be obtained as the closure of C in the toric comactification X of $(\mathbb{C}^*)^n$ associated with the polyhedron Δ_{12}. In particular, each facet Γ of Δ_{12} corresponds to a codimension 1 orbit X_Γ in X. The equality $\tilde{C} \setminus C = \bigcup_\Gamma (\tilde{C} \cap X_\Gamma)$ holds. Moreover the number of points in $\tilde{C} \cap X_\Gamma$ is equal to $(n-1)! V(v_\Gamma)$, (see [Kh] for details).

By (10.1) we have $M_{1,2} = \pm \Pi_\Delta^{[1]} / \Pi_\Delta^{[2]}$. By definition $M_{1,2} = m_{1,2}^{(1)} m_{1,2}^{(2)} F$, where $F = \prod_{k>2} m_{1,2}^{(k)}$ is independent of f_1, f_2. On the other hand $\Pi_\Delta^{[1]} / \Pi_\Delta^{[2]}$ is equal to the product of $\{f_1, f_2\}_p^{-1}$ over all zeros p of $f_1 f_2$ on the curve C. By Weil's theorem this product is equal to $\prod_{q \in (\tilde{C} \setminus C)} \{f_1, f_2\}_q$.

Explicit calculations show that for any $g \in \tilde{C} \cap X_\Gamma$ the following identity holds: $\{f_1, f_2\}_p = a_{1(v_\Gamma)}^{H_{\Delta_2}(v_\Gamma)} a_{2(v_\Gamma)}^{-H_{\Delta_1}(v_\Gamma)} G$, where G is independent of f_1, f_2). The number of points in $\tilde{C} \cap X_F$ is equal to $(n-1)! \operatorname{Vol}(\Delta_3^v, \ldots, \Delta_{n+1}^v)$. Putting everything together we get the needed identity $m_{1,2}^{(2)} = \prod_\Gamma a_{2(v_\Gamma)}^{d(v_\Gamma)}$. □

Definition 3. For completely developed $(n+1)$-tuple Δ and $1 \leqslant i \leqslant n+1$ we define the *i-th product resultant* $R^{[i]}_{\Pi\Delta}$ on Ω_Δ as $R^{[i]}_{\Pi\Delta} = \Pi^{[i]}_\Delta M_i$. By (11.1) all the product resultants $R^{[i]}_{\Pi\Delta}$ are equal up to sign.

Theorem 15. *Let $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ be a completely developed collection. Then*:

(1) *Each product resultant $R^{[i]}_{\Pi\Delta}$ is a polynomial on Ω_Δ. The degree of $R^{[i]}_{\Pi\Delta}$ in the coefficients of f_j is equal to the number of roots of the generic system $f_1 = \ldots = f_{n+1} = 0$ with f_j skipped (i.e is equal to $n!\,\mathrm{Vol}(\Delta_1, \ldots, \hat{\Delta}_j, \ldots, \Delta_{n+1})$).*

(2) *The function $R^{[i]}_{\Pi\Delta}$ is equal to zero at $(f_1, \ldots, f_{n+1}) \in \Omega^{[i]}_\Delta$ if and only if the system $f_1 = \ldots = f_{n+1} = 0$ has a root in $(\mathbb{C}^*)^n$.*

Proof. The expression $M_i\Pi^{[i]}_\Delta = R^{[i]}_{\Pi\Delta}$ is obviously a polynomial of degree $n!\,\mathrm{Vol}(\Delta_1, \ldots, \hat{\Delta}_j, \ldots, \Delta_{n+1})$ in the coefficients of f_i. Since all product resultants are equal up to sign we have proven (1).

Statement (2) is obvious from the definitions. □

Let $m \in \Delta_i \cap \mathbb{Z}^n$. Denote by c the coefficient in front of z^m in the Laurent polynomial f_i with Newton polyhedron Δ_i.

Theorem 16. *In the notations from Theorem* 15*, the degree of $R^{[j]}_{\Pi\Delta}$ in a specific coefficient c of f_i is equal to $n!\,\mathrm{Vol}(\Delta_1, \ldots, \hat{\Delta}_i, \ldots, \Delta_{n+1})$. Each polynomial $R^{[j]}_{\Pi\Delta}$ contains exactly one monomial of the highest degree in c and the coefficient in front of this monomial is ± 1.*

Proof. Without loss of generality we can assume that $i = 1$. Since all product resultants are equal up to sign it is enough to prove the statement for $R^{[1]}_{\Pi\Delta} = M_1\Pi^{[1]}_\Delta$. The monomial M_1 is independent of the coefficients of f_1 and monomial of the highest degree in c comes from multiplying the monomial cz^m over roots of $f_2 = \cdots = f_{n+1} = 0$. Now the theorem follows from Corollary 3. □

12. Sums of Grothendieck Residues over Roots of Developed System

In this section we discuss a formula from [GKh], [GKh1] for the sum of Grothendieck residues over the roots of a developed system. As a corollary we provide an algorithm for computing the product of values of a Laurent polynomial over the roots of a developed system.

12.1. Grothendieck residue. Consider the system $f_1 = \ldots = f_n = 0$ in $(\mathbb{C}^*)^n$ and the hypersurface Γ defined by $f_1 \cdot \ldots \cdot f_n = 0$. Let ω be a holomorphic n-form on $(\mathbb{C}^*)^n \setminus \Gamma$.

Definition 4. The Grothendieck residue of ω at the root z of the system $f_1 = \ldots = f_n = 0$ is defined as the number $\frac{1}{(2\pi i)^n}\int_{\gamma_z}\omega$, where γ_z is the Grothendieck cycle at z.

As ω is automatically closed, the Grothendieck residue at the root z is well defined.

12.2. The residue of the form at a vertex of a polyhedron. For each vertex A of the Newton Polyhedron $\Delta(P)$ of a Laurent polynomial P, we will construct the Laurent series of the function f/P, for any Laurent polynomial f.

Let $q_A \neq 0$ be the coefficient of the monomial in P which corresponds to the vertex A of $\Delta(P)$. The constant term of the Laurent polynomial $\tilde{P} = P/(q_A z^a)$ equals one. We will define the Laurent series of $1/\tilde{P}$ by the formula:

$$1/\tilde{P} = 1 + (1 - \tilde{P}) + (1 - \tilde{P})^2 + \dots .$$

Since each monomial z^b appears only in finitely many summands $(1 - \tilde{P})^k$, the above sum is well defined. The *Laurent series of the rational function f/P at the vertex A of $\Delta(P)$* is the product of the series $1/\tilde{P}$ and the Laurent polynomial $q_A z^a f$.

Consider the n-form $\omega_f = f dz_1 \wedge \dots \wedge dz_n / P z_1 \cdot \dots \cdot z_n$.

Definition 5. The Grothendieck residue $\operatorname{res}_A \omega$ of $\omega = \omega_f$ at the vertex A of $\Delta(P)$ is defined as the number $\frac{1}{(2\pi i)^n} \int_{T_A} \omega_f$, where T_A is the cycle assigned to a vertex A (see Sec. 7.2).

Lemma 4. *The residue* $\operatorname{res}_A \omega_f$ *is equal to the coefficient in from of the monomial* $(z_1 \cdot \dots \cdot z_n)^{-1}$ *in Laurent series of* f/P *at the vertex* A.

We will not prove this simple lemma. See [GKh1] for the details.

12.3. Summation formula. Consider the developed system of equations $f_1 = \dots = f_n = 0$ in $(\mathbb{C}^*)^n$ with Newton polyhedra $\Delta_1, \dots, \Delta_n$. Denote by P the product $f_1 \cdot \dots \cdot f_n$.

Theorem 17. *For any Laurent polynomial f the sum of the Grothendieck residues of the form $\omega_f = f\, dz_1 \wedge \dots \wedge dz_n / P z_1 \cdot \dots \cdot z_n$ over all the roots of the system is equal to $(-1)^n \sum k_A \operatorname{res}_A \omega_f$, where the summation is taken over all vertices A of $\Delta_1 + \dots + \Delta_n$.*

Proof. Theorem 17 follows from Theorem 8 and the results of Sections 12.1, 12.2. □

Corollary 5. *The sum $\sum f(z)\mu(z)$ of the values of any Laurent polynomial f over all roots z of a developed system, counted with multiplicities $\mu(z)$, is equal to $(-1)^n \sum k_A \operatorname{res}_A \omega_\varphi$, where $\varphi = f \det M$ and M is $(n \times n)$-matrix with the entries $M_{i,j} = \partial f_i / \partial z_j$.*

Proof. The Grothendieck residue of $\omega = f df_1 \wedge \dots \wedge df_n / f_1 \cdot \dots \cdot f_n$ at the root z is equal to $f(z)\mu(z)$. It is easy to see that $\omega = \omega_\varphi$. So Corollary 5 follows from Theorem 17. □

12.4. Elimination theory. Here we use notations from the previous section. Let f be a Laurent polynomial. We will explain how to find any symmetric function of the sequence of the numbers $\{f(z)\}$ for all roots z of the system (each root z is taken with multiplicity $\mu(z)$)

Denote by $f^{[k]}$ the number $f^{[k]} = \sum_z f^k(z)\mu(z)$. By Corollary 5 one can calculate $f^{[k]}$ for any k explicitly. The power sum symmetric polynomials form a generating set for the ring of symmetric polynomials.

Corollary 6. *One can find explicitly all symmetric functions of* $\{f(z)\}$ *and construct a monic polynomial whose roots are* $\{f(z)\}$. *In particular one can compute* $\Pi_\Delta^{[i]} = \prod_z f(z)^{\mu(z)}$ *for any* i*-developed system.*

13. Δ-Resultants

13.1. Definition and some properties of Δ-resultant. Following [GKZ] we define the Δ-resultant for a collection $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ of $(n+1)$ Newton polyhedra in $\mathbb{R}^n$. We also generalize this definition to a collection Δ in $\mathbb{R}^N$ with $N \geqslant n$ such that the polyhedron $\Delta_1 + \cdots + \Delta_{n+1}$ has dimension $\leqslant n$.

For $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ with $\Delta_i \subset \mathbb{R}^n$ denote by $X_\Delta \subset \Omega_\Delta$ the quasi-projective set of points $(f_1, \ldots, f_{n+1}) \in \Omega_\Delta$ such the system $f_1 = \cdots = f_{n+1} = 0$ in $(\mathbb{C}^*)^n$ is consistent. The set X_Δ is irreducible, an easy proof of this fact can be found in [GKZ].

Definition 6. The Δ-*resultant* R_Δ is a polynomial on Ω_Δ satisfying the following conditions:

(i) The degree of R_Δ in the coefficients of f_i is equal to the number of roots of the generic system $f_1 = \ldots = f_{n+1} = 0$ with f_i skipped (i.e., it is equal to $n!\,\mathrm{Vol}(\Delta_1, \ldots, \hat{\Delta}_i, \ldots, \Delta_{n+1})$.) The coefficients of R_Δ are coprime integers.

(ii) If the codimension of X_Δ in Ω_Δ is greater then 1 then $R_\Delta \equiv \pm 1$.

(iii) $R_\Delta(f_1, \ldots, f_{n+1}) = 0$ if and only if $(f_1, \ldots, f_{n+1})$ belongs to the closure $\overline{X}_\Delta$ of X_Δ in Ω_Δ.

Theorem 18 [GKZ].[1] *There exists a unique up to sign polynomial* R_Δ *on* Ω_Δ *satisfying the conditions* (i)–(iii).

The Δ-resultant obviously has the following properties: 1) it is independent of the ordering of the polyhedra from the set $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$; 2) it is invariant under translations of the polyhedra from the set Δ; 3) it is invariant under linear transformations of $\mathbb{R}^n$ inducing an automorphism of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Example 4. Suppose $\Delta_1 = \{m\}$ is a one point set, i.e., $f_1 = cz^m$ is a monomial z^m with some coefficient c. Then $R_\Delta = \pm c^{n!\,\mathrm{Vol}(\Delta_2, \ldots, \Delta_{n+1})}$.

Assume that $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ with $\Delta_i \subset \mathbb{R}^N$ satisfying inequality $\dim(\Delta_1 + \cdots + \Delta_{n+1}) \leqslant n$. Choose any linear isomorphism $A : \mathbb{R}^N \to \mathbb{R}^N$ preserving the lattice $\mathbb{Z}^N$ and choose vectors $v_1, \ldots, v_{n+1} \in \mathbb{Z}^N$ such that the polyhedra $\Delta'_i = A(\Delta_i) + v_i$ belong to the n dimensional coordinate subspace $\mathbb{R}^n \subset \mathbb{R}^N$.

Definition 7. The generalized Δ-resultant for Δ as above is defined as the Δ'-resultant for $\Delta' = (\Delta'_1, \ldots, \Delta'_{n+1})$. The generalized Δ-resultant is well defined, i.e., is independent of the choice of A and $v_1, \ldots, v_{n+1}$.

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[1]In [GKZ] the A-resultant is defined, the connection between A-resultants and Δ-resultants is described in introduction.

In [GKZ] A-resultant is defined under some assumption on $(n+1)$-tuple of supports A. The general definition of the A-resultant is given in [Est].

The condition (i) on the degrees of resultant could be replaced by the condition that the resultant is a polynomial which vanishes on X_Δ with multiplicity equal to the generic number of solutions of consistent system (see [DS]).

13.2. Product resultants and Δ-resultant. In this section we show that for a completely developed collection Δ all product resultants are equal up to sign to the Δ-resultant.

The Sylvester formula represents the Δ-resultants of two polynomials in one variable as a determinant of one of two explicitly written matrices (see Sec. 2.1). In [CE] the Sylvester formula was beautifully generalized in the following way. For the collection of $n+1$ Newton polyhedra Δ, Canny and Emiris construct $n+1$ matrices M_i (which coincide with Sylvester's matrix up to permutation of rows in dimension 1) such that the Δ-resultant R_Δ divides their determinants. Moreover, R_Δ is the greatest common divisor of polynomials $\det(M_i)$, thus they obtain a practical algorithm for computing R_Δ. The construction in [CE] heavily uses geometry of Newton polyhedra.

By the extreme monomials of a polynomial P we will mean the monomials corresponding to the vertices of the Newton polyhedron of P. In [St] Sturmfelds generalized the construction in [CE] and using this generalization proved the following theorem.

Theorem 19. *All extreme monomials of the* Δ*-resultant have coefficient* -1 *or* $+1$*.*

Now we are able to prove the following theorem.

Theorem 20. *For a completely developed* Δ *for any* $1 \leqslant i \leqslant n+1$ *the product resultant* $R^{[i]}_{\Pi\Delta}$ *is equal up to sign to the* Δ*-resultant* R_Δ*.*

Proof. Without the loss of generality we can assume that $i = 1$. Both functions R_Δ and $R^{[1]}_{\Pi,\Delta}$ are polynomials in the coefficients of $f_1, \ldots, f_{n+1}$ of the same degrees and they both vanish on the set $\overline{X}_\Delta$ (see Theorem 15). Since the set $\overline{X}_\Delta$ is irreducible polynomials R_Δ and $R^{[1]}_{\Pi,\Delta}$ are proportional. According to Corollary 3 for any chosen coefficient c of f_1, in the polynomial $R^{[1]}_{\Pi\Delta}$ there is a unique monomial having the highest degree in c and the coefficient in $R^{[1]}_{\Pi\Delta}$ in front of this monomial is ± 1. But according to Theorem 19 the coefficient in R_Δ in front of this monomial is also ± 1. So $R_\Delta = \pm R^{[1]}_{\Pi,\Delta}$. □

13.3. The Poisson formula. The inductive Poisson formula for Δ-resultant (see [PSt], [DS]) and the summation formula (see Section 12.3) allow to provide an algorithm computing the Δ-resultant for 1-developed systems. To state the formula let us define all terms appearing in it. Let Δ be $(\Delta_1, \ldots, \Delta_{n+1})$ and let $(f_1, \ldots, f_{n+1})$ be a point in Ω_Δ. The only term in the formula depending on the coefficients of f_1 is the term $\Pi^{[1]}_\Delta$. To present the other terms we need some notation.

Denote by $\tilde{\Delta}_1$ the sum $\Delta_2 + \cdots + \Delta_{n+1}$. For each facet Γ of $\tilde{\Delta}_1$ denote by v_Γ the irreducible integral covector such that the inner product of $x \in \tilde{\Delta}_1$ with v_Γ attains its maximum value on Γ. With v_Γ one associates the value $H_{\Delta_1}(v_\Gamma)$ of the support function of Δ_1 on v_Γ, the faces $\Delta_j^{v_\Gamma}$ of Δ_j on which the inner product of $x \in \Delta_j$ with v_Γ attains its maximal value. By $f_j^{v_\Gamma}$ we denote the sum $\sum c_m x^m$ over $m \in \Delta_j^{v_\Gamma} \cap \mathbb{Z}^n$, where c_m is the coefficient in front of x^m in f_j. For each v_Γ the collection of n polyhedra $\tilde{\Delta}_1^{v_\Gamma} = (\Delta_2^{v_\Gamma}, \ldots, \Delta_{n+1}^{v_\Gamma})$ satisfies the inequality

$\dim(\Delta_2^{v_\Gamma} + \cdots + \Delta_{n+1}^{v_\Gamma}) \leqslant (n-1)$. This is why the generalized $\tilde{\Delta}_1^{v_\Gamma}$ resultant $R_{\tilde{\Delta}_1^{v_\Gamma}}(f_2^{v_\Gamma}, \ldots, f_{n+1}^{v_\Gamma})$ is defined. Now we are ready to state the Poisson formula.

Theorem 21. *The following Poisson formula holds*:

$$R_\Delta(f_1, \ldots, f_{n+1}) = \pm\Pi_\Delta^{[1]}(f_1, \ldots, f_{n+1}) \prod R_{\tilde{\Delta}_1^{v_\Gamma}}^{H_1(v_\Gamma)}(f_2^{v_\Gamma}, \ldots, f_{n+1}^{v_\Gamma}),$$

where the product is taken over all facets Γ *of* $\tilde{\Delta}_1$.

In a subsequent paper we are going to give an elementary proof of Theorem 21 (in fact, we will generalize the Poisson formula from the toric case to a larger class of algebraic varieties).

Remark 2. Mixed volume and Δ-resultants have many similar properties. For example the non-symmetric formula for the mixed volume $\mathrm{Vol}(\Delta_1, \ldots, \Delta_n) = \frac{1}{n}\sum_v H_1(v)\,\mathrm{Vol}(\Delta_2^v, \ldots, \Delta_n^v)$ is analogues to the Poisson formula for the Δ-resultants.

For a 1-developed collection $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ of Newton polyhedra the Poisson formula becomes much simpler: in this case the resultants $R_{\tilde{\Delta}_1^{v_\Gamma}}(f_1^{v_\Gamma}, \ldots, f_{n+1}^{v_\Gamma})$ can be computed explicitly. Below we present such computation.

By definition of Δ being 1-developed collection, for any facet Γ of $\tilde{\Delta}_1$ in the set $\{\Delta_j^{v_\Gamma}\}$ with $j > 1$ at least one polyhedron $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ is a point (if more then one polyhedron is a point denote by $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ any of them). Denote by $\mathrm{Vol}(v_\Gamma)$ the integral $(n-1)$ dimensional mixed volume of collection $\{\Delta_j^{v_\Gamma}\}$ with $j > 1$ in which the polyhedron $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ is skipped. Denote by $a_{j(v_\Gamma)}$ the coefficient of the Laurent polynomial $f_{j(v_\Gamma)}$ at the vertex $\Delta_{j(v_\Gamma)}^{v_\Gamma}$. In the above notation Example 4 provides us the formula:

$$R_{\tilde{\Delta}_1}^{v_\Gamma}(f_2^{v_\Gamma}, \ldots, f_{n+1}^{v_\Gamma}) = \pm a_{j(v_\Gamma)}^{(n-1)!\,\mathrm{Vol}(v_\Gamma)}.$$

Corollary 7. *With the notation as above the* Δ*-resultant of the* 1*-developed collection* Δ *is given by*:

$$R_\Delta(f_1, \ldots, f_{n+1}) = \pm\Pi_\Delta^{[1]}(f_1, \ldots, f_{n+1}) \prod_\Gamma a_{j(v_\Gamma)}^{(n-1)!\,\mathrm{Vol}(v_\Gamma) H_1(v_\Gamma)}.$$

Corollary 8. *Using Corollary* 6 *one can produce an algorithm for computing the* Δ*-resultant of a* 1*-developed collection* Δ.

Indeed, the only implicit term in the formula from the Corollary 7 is the term $\Pi_\Delta^{[1]}$. Corollary 6 provides an algorithm for its computation.

13.4. A sign version of Poisson formula. Let Δ be a developed collection. According to Theorem 13 with such Δ, an $(n+1)$-tuple of monomials $M_1, \ldots, M_{n+1}$ and an $(n+1)$-tuple of signs are defined. According to the formula from Corollary 7 the Poisson formula in that case can be written as $R_\Delta = \pm\Pi_\Delta^{[1]} M_1$. By definition Δ is not only 1-developed, it is i-developed for any i. Thus one can write the formula from Corollary 7 putting instead of f_1 any f_i.

Corollary 9. *The following equalities hold*:

$$\pm\Pi_\Delta^{[1]}M_1 = \cdots = \pm\Pi_\Delta^{[n+1]}M_{n+1} = \pm R_\Delta.$$

Thus from the theory of Δ-resultants one can prove the product identities from Theorem 13 up to sign. It is impossible to reconstruct the signs in these identities using Δ-resultants: the Δ-resultant itself is defined up to sign only. Our Theorem 13 provided the sign version

$$\Pi_\Delta^{[1]}M_1 s_1 = \cdots = \Pi_\Delta^{[n+1]}M_{n+1}s_{n+1}$$

of Poisson type identities and Theorem 20 provides identities $\Pi_\Delta^{[i]}M_i = \pm R_\Delta$ (which unavoidably could be up to sign only).

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