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COMPUTER SCIENCE

Estimation of the Distance between Images under Translation

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Description of an effect. Suppose given a finite set of standard images (images of symbols) in the plane, each of which lies inside some standard convex figure (e.g., inside a given rectangle). There is one more image to be tested, which we must recognize, i.e., identify with one of the standard images. We superpose one of the standard images on that being tested.

Practical problem. Find a translation of a template image which ensures the best coincidence of this image the image being tested.

This problem is indeed encountered in practice. Many experiments have revealed the following effect.

Observed effect. If a random superposition is good, i.e., the region where the template image does not coincide with the image to be tested occupies a small part of the template image, then the translation maximizing the coincidence of images is small (i.e., its length is a small part of the diameter of the template image).

In what follows, we give a mathematical model explaining this effect on a qualitative level. Instead of images, the model considers a pair consisting of a set A (the colored part of an image) and a density $\phi : A \to R$ (the intensity of coloring). The degree of coincidence between images with densities ϕ_1 and ϕ_2 is modeled by the integral

$$\int |\phi_1 - \phi_2| d\mu.$$

In Sections 1 and 2, we estimate from below the measure of noncoincidence between a template image and its image under a measure-preserving mapping. The specifics of the problem for translations is discussed in Section 3. In Section 4, less accurate but more comprehensible estimates are given. The effect being discussed is explained in Section 5.

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1. ABSTRACT PROBLEM

Let Ω be a space with measure $d\mu$, and let X be a set of pairs $x = (A, \phi)$, where $A \subset \Omega$ is a measurable subset and $\phi : A \to R$ is an integrable nonnegative function, called density. The set X is endowed with the distance $p(x_1, x_2) = \int_\Omega |\phi_1 - \phi_2| d\mu$, where

$$x_1 = (A_1, \phi_1), \quad x_2 = (A_2, \phi_2).$$

Suppose that $F : \Omega \to \Omega$ is a one-to-one measure-preserving mapping, i.e., $F^* d\mu = d\mu$. The mapping F induces an isometry $\Phi : X \to X$, which takes each point $x = (A, \phi)$ to the point $\Phi(x) = (F(A), (F^{-1})^* \phi)$. From a subset $\Theta \subset \Omega$ and a number K we define the subset $Y \subset X$ consisting of points $((A, \phi))$ such that (i) $A \subset \Theta$ and (ii) $\int_\Omega \phi d\mu = K$.

Problem 1. Find a (the best) lower bound for the function $S(\Theta, K, F) = \min_{x \in Y} p(x, \Phi(x))$.

One of the possible solutions of Problem 1 is presented in the next section.

Consider the following modification of Problem 1. Let us expand the set X to the set Z consisting of pairs $(z = (A, \psi))$ in which the density ψ is allowed to concentrate at points. To be more precise, in Z, densities are generalized densities ψ, which are sums of integrable functions and finite linear combinations of δ-functions at points $q \in \Omega$ (we denote them by δ_q); i.e., $\psi = \phi + \sum a_k \delta_{q_k}$. To the space Z the definitions of the distance p and the mapping Φ are transferred automatically. Suppose given a mapping $F : \Omega \to \Omega$, a set $\Theta \subset \Omega$, and a number K. Let $W \subset Z$ be the set of points (A, ψ), where $\psi = \phi + \sum a_k \delta_{q_k}$, such that (i) $A \subset \Theta$, (ii) $\phi \geq 0$, $a_k \geq 0$, the support of the function ϕ is contained in A, and $q_k \in A$, and (iii) $\int_\Omega \phi d\mu + \sum a_k = K$.

Problem 1'. Find a (the best) lower bound for the function $G(\Theta, K, F) = \min_{z \in W} p(z, \Phi(z))$.

2. PARTIAL SOLUTION OF PROBLEMS 1 AND 1'

The following theorem is valid.

Theorem 1. *If there exists an integer n for which the set $F^n(\Theta)$ does not intersect Θ, then $S(\Theta, K, F) \geq \frac{2K}{n}$ and $G(\Theta, K, F) \geq \frac{2K}{n}$.*

Under additional assumptions, the bound of Theorem 1 is unimprovable. The main of these assumptions is as follows.

Condition I_n. There exists a point $a \in \Omega$ such that the points $a, F(a), \ldots, F^n(a)$ are different, and all of these points, except $F^n(a)$, belong to the set Θ.

Suppose, in addition, that the set Ω is endowed with a topology, in which the set Θ is open and the mapping F is continuous.

Theorem 2. *If the conditions listed above hold and $F^n(\Theta) \cap \Theta = \phi$, then $S(\Theta, K, F) = G(\Theta, K, F) = \frac{2K}{n}$.*

For Problem 1', Theorem 2 can be strengthened: the requirement that the set Ω is endowed with a topology in which the set Θ is open and the mapping F is continuous is not necessary.

Theorem 2'. *Suppose that condition I_n holds and $F^n(\Theta) \cap \Theta = 0$. Then, $G(\Theta, K, F) = \frac{2K}{n}$.*

3. THE CASE OF A TRANSLATION

Suppose that a space Ω with a measure $d\mu$ is the space R^n with Euclidean measure, a mapping F is the translation by a fixed vector $\mathbf{a} \in R^n$ (i.e., $F(x) = x + \mathbf{a}$), and a set Θ is any n-dimensional convex subset in R^n. Consider the set of nonnegative integrable functions ϕ on R^n identically vanishing outside the convex set Θ and satisfying the relation

$$\int_{R^n} \phi(x) d\mu = K. \quad (1)$$

Problem 1 for a translation. Find a (the best) lower bound for the function $S(\Theta, K, \mathbf{a})$ defined by

$$S = \min_{\phi} \int_{R^n} |\phi(x) - \phi(x - \mathbf{a})| d\mu.$$

The statement of Problem 1' in this case is similar.

We write a vector $\mathbf{a} \in R^n$ in the form $\mathbf{a} = |\mathbf{a}|\mathbf{v}$, where $\mathbf{v}$ is a vector of unit length and $|\mathbf{a}|$ is the length of the vector $\mathbf{a}$. For a vector $\mathbf{v}$ of unit length, we define the $\mathbf{v}$-diameter $D_v(\Theta)$ of the convex figure Θ as the supremum of the lengths of intervals being the sections of the domain Θ by straight lines parallel to the vector $\mathbf{v}$. For a nonzero vector $\mathbf{a} = |\mathbf{a}|\mathbf{v}$, we define the a-diameter $D_{\mathbf{a}}(\Theta)$ of the convex figure Θ as the ratio $\frac{D_{\mathbf{v}}(\Theta)}{|\mathbf{a}|}$ of its v-diameter to the length $|\mathbf{a}|$ of the vector $\mathbf{a}$.

A solution of Problems 1 and 1' for a vector $\mathbf{a} = |\mathbf{a}|\mathbf{v}$ in the case where the a-diameter of the figure Θ is not a positive integer is given by the following theorem.

Theorem 3. *If*

$$n|\mathbf{a}| < D_{\mathbf{v}}(\Theta) < (n+1)|\mathbf{a}|,$$

for some integer $n \geq 0$, then $S = G = \frac{2K}{n+1}$.

A solution of Problem 1' for a vector $\mathbf{a} = |\mathbf{a}|\mathbf{v}$ in the case where the a-diameter of the figure Θ is a positive integer is given by the following theorem.

Theorem 4. *If*

$$n|\mathbf{a}| = D_{\mathbf{v}}(\Theta),$$

for some integer $n \geq 0$, then $G = \frac{2K}{n+1}$.

If $n|\mathbf{a}| = D_{\mathbf{v}}(\Theta)$, then the minimum in the problem is not attained by usual functions. However, this minimum is attained by a linear combination of δ-functions, and this minimum equals $\frac{2K}{n+1}$.

4. ESTIMATION OF THE DISTANCE BETWEEN IMAGES

In this section, for brevity, we discuss only Problem 1. The formula for the function $S(\Theta, K, \mathbf{a})$ is rather cumbersome. Corollary 1 gives a nonsharp but fairly good bound for this function, which is quite comprehensible. Corollary 2 gives a bound which is linear for a fixed vector $\mathbf{v}$ and small values of the length $|\mathbf{a}|$ with respect to the length $|\mathbf{a}|$, i.e., equals $C|\mathbf{a}|$, where C is an explicitly written constant.

Corollary 1. *The bound*

$$S(\Theta, K, \mathbf{a}) \geq \frac{2K}{D_{\mathbf{v}}(\Theta) + |\mathbf{a}|}|\mathbf{a}|.$$

holds. This bound is sharp for $|\mathbf{a}| = D_{\mathbf{v}}(\Theta), \frac{D_{\mathbf{v}}(\Theta)}{2}, \frac{D_{\mathbf{v}}(\Theta)}{3}$ and as $|\mathbf{a}| \to \infty$ (i.e., figuratively, for $|\mathbf{a}| = \frac{D_{\mathbf{v}}(\Theta)}{0}$).

Corollary 2. *For any $0 \leq y_0$ and $0\ 0 \leq |\mathbf{a}| \leq y_0$,*

$$S(\Theta, K, \mathbf{a}) \geq \frac{2K}{D_{\mathbf{v}}(\Theta) + y_0}|\mathbf{a}|,$$

and for $|\mathbf{a}| > y_0$,

$$S(\Theta, K, \mathbf{a}) \geq \frac{2Ky_0}{D_{\mathbf{v}}(\Theta) + y_0}.$$

In particular, for $y_0 = D_v(\Theta)$, *the following assertion holds: if* 0 $0 \le |\mathbf{a}| \le D_v(\Theta)$, *then*

$$S(\Theta, K, \mathbf{a}) \ge \frac{K}{D_v(\Theta)}|\mathbf{a}|,$$

and if $|\mathbf{a}| > D_v(\Theta)$, *then* $S(\Theta, K, \mathbf{a}) \ge K$ (*note that, in fact,* $S(\Theta, K, \mathbf{a}) = 2K$ *for* $|\mathbf{a}| > D_v(\Theta)$).

5. QUALITATIVE EXPLANATION OF THE EFFECT

Let ϕ be a nonnegative integrable function on R^n vanishing outside a convex set Θ and satisfying relation (1), and let $g(x)$ be another nonnegative integrable function on R^n. Suppose that

$$\int_{R^n} |\phi(x) - g(x)| dx = I.$$

and the integral I is much less than K.

Problem 2. Varying the parameter a, minimize the integral

$$\int_{R^n} |\phi(x-a) - g(x)| dx = M(a)$$

Our purpose is to show that if $K \gg I$, then, from the point of view of minimization of $M(a)$, only vectors $\mathbf{a} = |\mathbf{a}|\mathbf{v}$ that are significantly shorter that the v-diameter $D_v(\Theta)$ of the domain Θ are of interest.

Let $S(\mathbf{a}) = \min \int_{R^n} |\phi(x) - \phi(x - \mathbf{a})| d\mu$, where the minimum is over all nonnegative functions ϕ satisfying relation (1) and supported on the domain Θ. We can apply the bounds for the function $S(\mathbf{a})$ obtained above.

Theorem 5. *Let* $\mathbf{v}$ *be a fixed vector of unit length, and let* $d(\mathbf{v})$ *be the least number such that any vector* $\mathbf{a} = |\mathbf{a}|\mathbf{v}$ *with* $|\mathbf{a}| > d(\mathbf{v})$ *satisfies the inequality* $S(\mathbf{a}) > 2I$. *Then, for* $|\mathbf{a}| > d(\mathbf{v})$,

$$\int_{R^n} |\phi(x - \mathbf{a}) - g(x)| d\mu > I.$$

In particular, to minimize $M(a)$, *it is sufficient to consider only vectors* $a = |a| v$ *for which* $|\mathbf{a}| < d(\mathbf{v})$.

Corollary 3. *Suppose that, for a fixed unit vector* v, *the inequality* $I < \frac{D_v(\Theta)}{2}$ *holds. Then, to minimize the integral* $M(\mathbf{a})$ *on the straight line spanned by the vector* $\mathbf{v}$, *it is sufficient to consider vectors a satisfying the inequality* $|\mathbf{a}| < \frac{2I}{K} D_v(\Theta)$. *For* $|\mathbf{a}| > \frac{2ID_v(\Theta)}{K}$, *the inequality* $\int_{R^n} |\phi(x - \mathbf{a}) - g(x)| d\mu > I$ *holds.*

Corollary 3 gives a qualitative explanation of the effect discussed above. Indeed, if $I \ll K$, then the number $\frac{2I}{K}$ is small. Therefore, $M(\mathbf{a})$ attains a minimum at a vector $\mathbf{a}$ of length significantly smaller than the diameter $D_v(\Theta)$ of the figure Θ in the direction of the vector $\mathbf{v}$.

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