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Elimination theory and Newton polytopes

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Abstract We study elimination theory in the context of Newton polytopes and develop its convex-geometry counterpart.

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1 Introduction

Let $N \subset \mathbb{C}^n$ be an affine algebraic variety and $\pi : \mathbb{C}^n \to \mathbb{C}^m$ be a projection. The goal of elimination theory is to describe the defining equations of $\pi(N)$ in terms of the defining equations of N. We study the defining equations of projections in the context of Newton polytopes: we assume that the variety $N \subset (\mathbb{C} \setminus 0)^n$ is defined by equations $f_1 = \cdots = f_k = 0$ with given Newton polytopes and generic coefficients and that the projection $\pi(N) \subset (\mathbb{C} \setminus 0)^m$ is given by one equation $g = 0$. Under this assumption, we describe the Newton polytope and the leading coefficients of the Laurent polynomial g in terms of the Newton polytope and the leading coefficients of the Laurent polynomials $f_1, \dots, f_k$ (by the leading coefficients, we mean the coefficients of monomials from the boundary of the Newton polytope).

In this section, we define the equation g of a projection of a complete intersection $f_1 = \cdots = f_k = 0$ (Definition 2) and describe its Newton polytope in terms of the Newton polytopes $\Delta_1, \dots, \Delta_k$ of the equations $f_1, \dots, f_k$ (Theorem 2). Section 2 contains certain

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facts about the geometry of this polytope. In particular, this polytope is an increasing function of the polytopes $\Delta_1, \dots, \Delta_k$ (Theorem 4) and coincides with the mixed fiber polytope of $\Delta_1, \dots, \Delta_k$ up to a shift and dilatation (Theorem 7). The existence and other basic properties of mixed fiber polytopes (Definition 7) are proved in Sect. 3. Sections 4 and 5 are concerned with computing the leading coefficients of g. For example, Theorems 16 and 19 can be used to explicitly compute the coefficients of the monomials corresponding to the vertices of the Newton polytope of g if the polytopes $\Delta_1, \dots, \Delta_k$ satisfy a condition of general position (Definition 20). In Sect. 6, we present some other versions of elimination theory in the context of Newton polytopes, such as the elimination theory for rational and analytic functions.

1.1 Elimination theory and Newton polytopes

Many important problems related to Newton polytopes and tropical geometry turn out to be special cases of this version of elimination theory. We give some examples of such problems:

1. To compute the number of common roots of polynomial equations with given Newton polytopes and generic coefficients: the answer is given by the Kushnirenko–Bernstein formula (see [1] or Theorem 1 below).
2. To compute the product of common roots of polynomial equations with given Newton polytopes and generic coefficients: if the Newton polytopes satisfy some conditions of general position, then the answer is given by Khovanskii's product formula (see [2] or Theorems 18 and 19 below).
3. To compute the sum of values of a polynomial over the common roots of polynomial equations with given Newton polytopes and generic coefficients: if the Newton polytopes satisfy some conditions of general position, then the answer is given by the Gelfond–Khovanskii formula (see [3] or Theorem 17 below).
4. To compute the Newton polytope and leading coefficients of the defining equation of a hypersurface parameterized by a polynomial map $(\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{n+1}$ with given Newton polytopes and generic coefficients of the components (implicitization theory): the Newton polytope was described by Sturmfels, Tevelev, and Yu (see [4]).
5. To describe the Newton polytope and the leading coefficients of a multidimensional resultant: the Newton polytope and the absolute values of leading coefficients were computed by Sturmfels (see [5]).
6. To prove the existence of mixed fiber polytopes (Definition 7): the existence of mixed fiber polytopes was predicted in [6] and proved in [7].

1.2 Problems 1–3 in the context of elimination theory

To put problems 1–3 in the context of elimination theory, we regard a Laurent monomial as a projection $\pi : (\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)$. Then the defining equation of the projection of a zero-dimensional complete intersection $\{f_1 = \dots = f_n = 0\} = \{z_1, \dots, z_N\}$ is a polynomial $g(t) = \prod_i \big(t - \pi(z_i)\big)$ in one variable.

Lemma 1 1. *The length of the* (*one-dimensional*) *Newton polytope of g is equal to the number of common roots of $f_1, \dots, f_n$.*

2. *The constant term of* g (*which is a leading coefficient in our terminology*) *is equal to the product of the values of the monomial* $-\pi$ *over all common roots of* $f_1, \dots, f_n$.

3. *Let* S_m *be a polynomial of* m *variables such that*

$$S_m\left(\sum_i x_i, \sum_i x_i^2, \dots, \sum_i x_i^m\right)$$

is equal to the m*th elementary symmetric function of the independent variables* x_i. *Then the coefficient of the monomial* t^m *in the polynomial* g *is equal to*

$$(-1)^{N-m} S_m(p_1, \dots, p_m),$$

where p_m *is the sum of the values of the monomial* π^m *over all common roots of* $f_1, \dots, f_n$.

All these facts are obvious, and we omit the proof. We generalize this lemma to projections of complete intersections of an arbitrary dimension (see Theorem 2 and Sect. 4).

Lemma 1 implies that the Kushnirenko–Bernstein formula (Theorem 1), Khovanskii's product formula (Theorems 18 and 19), and the Gelfond–Khovanskii formula (Theorem 17) can be seen as the respective explicit formulas for the Newton polytope of g, the leading coefficients of g, and all coefficients of g if the Newton polytopes of $f_1, \dots, f_n$ satisfy a certain condition of general position. We generalize these observations to projections of complete intersections of an arbitrary dimension (see Sect. 5).

1.3 Problems 4–6 in the context of elimination theory

Implicitization theory can be regarded as a special case of elimination theory. Indeed, we consider a map $g = (g_0, \dots, g_k)\colon (\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{k+1}$ and a k-dimensional complete intersection $F = \{f_1 = \dots = f_{n-k} = 0\} \subset (\mathbb{C} \setminus 0)^n$, where $g_0, \dots, g_k, f_1, \dots, f_{n-k}$ are Laurent polynomials on $(\mathbb{C} \setminus 0)^n$. Let π be the standard projection $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^{k+1} \to (\mathbb{C} \setminus 0)^{k+1}$ and $y_0, \dots, y_k$ be the standard coordinates on $(\mathbb{C} \setminus 0)^{k+1}$. Then the defining equation of the image $g(F) \subset (\mathbb{C} \setminus 0)^{k+1}$ coincides with the defining equation of the projection $\pi(\{g_0 - y_0 = \dots = g_k - y_k = f_1 = \dots = f_{n-k} = 0\})$.

A multidimensional resultant is the "universal" special case of elimination theory, which is clear from the following version of the definition of a resultant. We regard the polynomials

$$g_i(x_1, \dots, x_k) = \sum_{b \in B_i} c_{b,i} x^b, \quad i = 0, \dots, k, \quad B_i \subset \mathbb{Z}^k,$$

as polynomials f_i in the variables $c_{b,i}$ and x_j with all coefficients equal to 1. Let π be the projection of the domain of the polynomials $(f_0, \dots, f_k)$ along the domain of the polynomials $(g_0, \dots, g_k)$. Then the defining equation of the projection $\pi(\{f_0 = \dots = f_k = 0\})$ is called the $(B_0, \dots, B_k)$-*resultant*. This definition of the multidimensional resultant is somewhat different from the classical definition if we understand the defining equation of a projection in the sense of Definition 2 because it is not always square-free. We consider the square-free version of Definition 2 in Sect. 6 (see Theorem 20).

Elimination theory, implicitization theory, and the theory of multidimensional resultants are equivalent in the sense that they can be formulated in terms of each other. The contents of this paper could therefore be written in terms of resultants or implicitization theory. When written in these terms, Theorem 2 gives the descriptions of Newton polytopes in [4] and [5], while the facts in Sects. 4 and 5 give some new information about the leading coefficients.

For example, Theorems 16 and 19 can be used to compare the signs of the leading coefficients of a multidimensional resultant (Esterov AI, Khovanskii AG On the vertex coefficients of multidimensional resultants and discriminants. In preparation).

The theory of mixed fiber polytopes turns out to be the Newton-polyhedral counterpart of elimination theory in the following sense. We define the *composite polytope* of the polytopes $\Delta_1, \dots, \Delta_k$ as the Newton polytope of a projection of a complete intersection $f_1 = \dots = f_k = 0$ if the Newton polytope of f_i is Δ_i and the coefficients of $f_1, \dots, f_k$ are in general position. Then Theorems 2 and 7 imply that the composite polytope satisfies the definition of the mixed fiber polytope up to a shift and dilatation, which proves the existence of mixed fiber polytopes. We omit the details and prefer to give an independent elementary proof of the existence of mixed fiber polytopes in Sect. 3 to make our paper self-contained (the proof in [7] is based on work in preparation). We note that composite polytopes are more convenient than mixed fiber polytopes in some sense; for example, they are monotonic (Theorem 4).

1.4 The composite polynomial

Let $\overline{M}$ denote the Zariski closure of a set M. For an algebraic map $f\colon M \to (\mathbb{C}\setminus 0)^n$ of an irreducible algebraic variety M, let $m(f)$ denote the number of points in the preimage $f^{(-1)}(x)$ of a generic point $x \in f(M)$ if this number is finite, and let $m(f)$ be zero otherwise. We define a *cycle* $N = \sum_i a_i N_i$ in $(\mathbb{C}\setminus 0)^n$ as a formal linear combination of irreducible algebraic varieties $N_i \subset (\mathbb{C}\setminus 0)^n$ of the same dimension with integer coefficients a_i.

Definition 1 Let $\pi\colon (\mathbb{C}\setminus 0)^n \mapsto (\mathbb{C}\setminus 0)^{n-k}$ be an epimorphism of complex tori. For a cycle $N = \sum_i a_i N_i$ in $(\mathbb{C}\setminus 0)^n$, the cycle $\sum_i m(\pi|_{N_i}) a_i \overline{\pi(N_i)}$ is called the *projection* $\pi_* N$ of the cycle N.

Let $f_1, \dots, f_m$ be Laurent polynomials on $(\mathbb{C}\setminus 0)^n$ such that $\operatorname{codim}\{f_1 = \dots = f_m = 0\} = m$. Let $[f_1 = \dots = f_m = 0]$ denote the intersection cycle of the divisors of the polynomials $f_1, \dots, f_m$.

Definition 2 Let $f_0, \dots, f_k$ be Laurent polynomials on $(\mathbb{C}\setminus 0)^n$ such that $\operatorname{codim}\{f_0 = \dots = f_k = 0\} = k+1$. The Laurent polynomial $\pi_{f_0,\dots,f_k}$ on $(\mathbb{C}\setminus 0)^{n-k}$ such that $[\pi_{f_0,\dots,f_k} = 0] = \pi_*[f_0 = \dots = f_k = 0]$ is called the *composite polynomial* of the polynomials $f_0, \dots, f_k$ with respect to the projection π.

The composite polynomial $\pi_{f_0,\dots,f_k}$ is defined up to a monomial factor. To describe its Newton polytope, we need the Kushnirenko–Bernstein formula for the number of roots of a system of polynomial equations.

1.5 The Kushnirenko–Bernstein formula

The set of all convex bodies in $\mathbb{R}^m$ is a semigroup with respect to the operation of *Minkowski summation* $A + B = \{a + b \mid a \in A,\ b \in B\}$.

Definition 3 The *mixed volume* MV_μ induced by a volume form μ on $\mathbb{R}^m$ is the symmetric Minkowski-multilinear function of m convex bodies in $\mathbb{R}^m$ such that

$$\mathrm{MV}_\mu(\Delta, \dots, \Delta) = \int_\Delta \mu$$

for every convex body $\Delta \subset \mathbb{R}^m$. The mixed volume induced by the standard volume form is denoted by MV.

The *restriction* $f|_B$ of a Laurent polynomial $f(x) = \sum_{a \in \mathbb{Z}^n} c_a x^a$ onto a set $B \subset \mathbb{Z}^n$ is the polynomial $\sum_{a \in B} c_a x^a$. The *Newton polytope* Δ_f of a Laurent polynomial f is the convex hull of the set A such that $f(x) = \sum_{a \in A} c_a x^a$ and $c_a \neq 0$.

Definition 4 Laurent polynomials $f_0, \dots, f_k$ on $(\mathbb{C} \setminus 0)^n$ are said to be *Newton-nondegenerate* if for any collection of faces $A_0 \subset \Delta_{f_0}, \dots, A_k \subset \Delta_{f_k}$ such that the sum $A_0 + \dots + A_k$ is at most a k-dimensional face of the sum $\Delta_{f_0} + \dots + \Delta_{f_k}$, the restrictions $f_0|_{A_0}, \dots, f_k|_{A_k}$ have no common zeros in $(\mathbb{C} \setminus 0)^n$.

Newton-nondegenerate collections of polynomials form a dense subset in the space of all collections of polynomials with given Newton polytopes.

Theorem 1 (Kushnirenko–Bernstein [1]) 1. *The number of common roots of Newton-nondegenerate Laurent polynomials* $f_1, \dots, f_n$ *in* $(\mathbb{C} \setminus 0)^n$ *with multiplicities taken into account is equal to* $n!\mathrm{MV}(\Delta_{f_1}, \dots, \Delta_{f_n})$.

2. *Without the assumption of Newton-nondegeneracy, the number of isolated common roots of* $f_1, \dots, f_n$ *in* $(\mathbb{C} \setminus 0)^n$ *with multiplicities taken into account is not greater than* $n!\mathrm{MV}(\Delta_{f_1}, \dots, \Delta_{f_n})$.

1.6 The Newton polytope of the composite polynomial

The Newton polytope of the polynomial $\pi_{f_0,\dots,f_k}$ is uniquely determined up to a shift by equality $(*)$ below. This equality is a corollary of the Kushnirenko–Bernstein formula and can be seen as its generalization (see Lemma 1.1).

Theorem 2 1. *Let* $\pi^\times : \mathbb{Z}^{n-k} \hookrightarrow \mathbb{Z}^n$ *be the inclusion of character lattices defined by the epimorphism* $\pi : (\mathbb{C} \setminus 0)^n \mapsto (\mathbb{C} \setminus 0)^{n-k}$. *Let* $A_0, \dots, A_k \subset \mathbb{Z}^n$ *and* $A \subset \mathbb{Z}^{n-k}$ *be the Newton polytopes of the polynomials* $f_0, \dots, f_k$ *and* $\pi_{f_0,\dots,f_k}$. *Then for any convex bodies* $B_1, \dots, B_{n-k-1} \subset \mathbb{Z}^{n-k}$,

$$\begin{aligned}(n-k)!\mathrm{MV}(A, B_1, \dots, B_{n-k-1}) \\ = n!\mathrm{MV}(A_0, \dots, A_k, \pi^\times B_1, \dots, \pi^\times B_{n-k-1})\end{aligned} \tag{$*$}$$

if the polynomials $f_0, \dots, f_k$ *are Newton-nondegenerate.*

2. *Without the assumption of Newton-nondegeneracy*,

$$\begin{aligned}(n-k)!\mathrm{MV}(A, B_1, \dots, B_{n-k-1}) \\ \le n!\mathrm{MV}(A_0, \dots, A_k, \pi^\times B_1, \dots, \pi^\times B_{n-k-1}).\end{aligned} \tag{$**$}$$

This theorem yields the "elimination theory for convex bodies," which describes the polytope A in terms of $A_0, \dots, A_k$ proceeding from equality $(*)$ and estimates it proceeding from inequality $(**)$ (see Sect. 2 for the details).

Proof By the continuity and linearity of the mixed volume, it suffices to prove this theorem under the assumption that $B_1, \dots, B_{n-k-1}$ are polytopes with integer vertices. Under this assumption, we consider generic Laurent polynomials $g_1, \dots, g_{n-k-1}$ on $(\mathbb{C} \setminus 0)^{n-k}$ with the Newton polytopes $B_1, \dots, B_{n-k-1}$. Because $\pi_{f_0,\dots,f_k}$ is not identically zero, the collection $\pi_{f_0,\dots,f_k}, g_1, \dots, g_{n-k-1}$ is Newton-nondegenerate. If the collection $f_0, \dots, f_k$ is Newton-nondegenerate, then the collection $f_0, \dots, f_k, g_1 \circ \pi, \dots, g_{n-k-1} \circ \pi$ is also Newton-nondegenerate.

By the Kushnirenko–Bernstein formula, the numbers of solutions of the systems $f_0 = \dots = f_k = g_1 \circ \pi = \dots = g_{n-k-1} \circ \pi = 0$ and $\pi_{f_0,\dots,f_k} = g_1 = \dots = g_{n-k-1} = 0$ are respectively equal to $n!V(A_0, \dots, A_k, B_1, \dots, B_{n-k-1})$ and $(n-k)!V(A, B_1, \dots, B_{n-k-1})$. On the other hand, the solutions of the second system are the projections of the solutions of the first system. □

2 Elimination theory for convex bodies

Theorem 2 motivates the following definition, which yields the "elimination theory for convex bodies." A convex body B in an $(n-k)$-dimensional subspace $L \subset \mathbb{R}^n$ is called a *composite body* of convex bodies $\Delta_0, \dots, \Delta_k \subset \mathbb{R}^n$ if the mixed volume $n!\mathrm{MV}(\Delta_0, \dots, \Delta_k, B_1, \dots, B_{n-k-1})$ in $\mathbb{R}^n$ is equal to the mixed volume $(n-k)!\mathrm{MV}(B, B_1, \dots, B_{n-k-1})$ in L for every collection of convex bodies $B_1, \dots, B_{n-k-1} \subset L$ (see Definition 5 for the details).

For every collection of convex bodies $\Delta_0, \dots, \Delta_k$, there exists a unique composite body up to a shift (Theorem 3). The existence of composite bodies follows because the mixed fiber body of the bodies $\Delta_0, \dots, \Delta_k$ satisfies the definition of a composite body up to a shift and dilatation (Definition 7 and Theorem 7). Hence, the theory of composite bodies is a version of the theory of mixed fiber polytopes, conjectured in [6] and constructed in [7]. Because [7] is based on work in preparation, we prefer to present another approach to mixed fiber polytopes in Sect. 3 to make our paper self-contained. At the same time, we prove some basic facts about composite bodies:

- A composite body of polytopes is a polytope (Theorem 6).
- A composite body of integer polytopes (i.e., polytopes such that all their vertices are integer lattice points) is a shifted integer polytope (Theorem 13).
- Composite bodies are monotonic (Theorem 4).
- The linear span of a composite body depends on the linear spans of its arguments (Theorem 5).
- Codimension-m faces of a composite polytope depend on codimension-m faces of its arguments (Theorem 11).
- In particular, vertices of the composite polytope of the polytopes $\Delta_0, \dots, \Delta_k$ can be expressed in terms of moments of their k-dimensional faces (Theorem 12).

2.1 Composite bodies

Definition 5 Let $L \subset \mathbb{R}^n$ be a vector subspace of codimension k, μ be a volume form on $\mathbb{R}^n/L$, and $\Delta_0, \dots, \Delta_k$ be convex bodies in $\mathbb{R}^n$. A convex body $B \subset L$ is called a *composite body* of $\Delta_0, \dots, \Delta_k$ in L and is denoted by $\mathrm{CB}_\mu(\Delta_0, \dots, \Delta_k)$ if for every collection of convex bodies $B_1, \dots, B_{n-k-1} \subset L$,

$$n!\mathrm{MV}_{\mu' \wedge \mu}(\Delta_0, \dots, \Delta_k, B_1, \dots, B_{n-k-1}) = (n-k)!\mathrm{MV}_{\mu'}(B, B_1, \dots, B_{n-k-1}),$$

where μ' is a volume form on L.

Theorem 3 1. *For any collection of convex bodies $\Delta_0, \dots, \Delta_k \subset \mathbb{R}^n$, there exists a composite body $\mathrm{CB}_\mu(\Delta_0, \dots, \Delta_k)$.*

2. *A composite body $\mathrm{CB}_\mu(\Delta_0, \dots, \Delta_k)$ is unique up to a shift.*

Proof Part 1 of the theorem follows from an explicit formula for composite bodies (see Theorem 7). Part 2 follows from the monotonicity (see Theorem 4). □

The proof of uniqueness implies that in Definition 5, it suffices to consider collections $B_1, \dots, B_{n-k-1}$ such that $B_1, \dots, B_{n-k-1}$ are simplices.

2.2 Monotonicity of a composite body

Theorem 4 *If $\Delta_i \subset \Delta_i'$ for $i = 0, \dots, k$, then*

$$\mathrm{CB}_\mu(\Delta_0, \dots, \Delta_k) \subset \mathrm{CB}_\mu(\Delta_0', \dots, \Delta_k').$$

This is a corollary of the monotonicity of the mixed volume and the following fact.

Lemma 2 *Let Δ and Δ' be convex bodies in $\mathbb{R}^m$. If*

$$\mathrm{MV}(\Delta, B, \dots, B) \le \mathrm{MV}(\Delta', B, \dots, B)$$

for every simplex B, then for a certain shift $a \in \mathbb{R}^m$, the shifted body $\Delta + a$ is contained in Δ'.

Proof We choose a such that the minimax distance

$$\mathrm{dist}(\Delta + a, \Delta') = \max_{x \in \Delta + a} \min_{y \in \Delta'} |x - y|$$

is minimal. We suppose that $\mathrm{dist}(\Delta + a, \Delta') > 0$. Then the set of all covectors $\gamma \in (\mathbb{R}^m)^*$ such that

$$\max_{x \in \Delta + a} \langle \gamma, x \rangle > \max_{y \in \Delta'} \langle \gamma, y \rangle$$

is not contained in a half-space. In particular, it contains covectors $\gamma_0, \dots, \gamma_m$ such that none of them is a linear combination of the others with nonnegative coefficients. Let B denote an m-dimensional simplex with the external normal covectors $\gamma_0, \dots, \gamma_m$. Then $\mathrm{MV}(\Delta, B, \dots, B) > \mathrm{MV}(\Delta', B, \dots, B)$ because of the following formula for mixed volumes. □

Lemma 3 *Let Δ be a convex body, $B_1, \dots, B_{m-1}$ be polytopes, and μ be a volume form in $\mathbb{R}^m$. Let $\Gamma \subset (\mathbb{R}^n)^*$ be a set that contains one external normal covector for each $(m-1)$-dimensional face of the sum $B_1 + \cdots + B_{m-1}$. Then*

$$\mathrm{MV}_\mu(\Delta, B_1, \dots, B_{m-1}) = \frac{1}{m} \sum_{\gamma \in \Gamma} \max_{x \in \Delta} \langle \gamma, x \rangle \mathrm{MV}_{\mu/\gamma}(B_1^\gamma, \dots, B_{m-1}^\gamma),$$

where B_i^γ is the maximal face of B_i on which γ attains its maximum as a function on B_i.

The mixed volume in the right-hand side makes sense because its arguments are all parallel to the same $(m-1)$-dimensional subspace $\ker\gamma$.

Proof If $\Delta = B_1 = \cdots = B_{m-1}$ contains the origin, then this formula states that the volume of Δ is equal to the sum of volumes of the convex hulls $\mathrm{conv}(\{0\} \cup F)$, where F ranges all $(m-1)$-dimensional faces of Δ. In general, the formula follows from this special case by the additivity and continuity of the mixed volume. □

2.3 Linear span of a composite body

We need one more fact about composite bodies, which, in the context of Newton polytopes, reflects the fact that elimination of variables preserves the homogeneity of equations. Namely, the following theorem expresses the linear span of a composite body in terms of linear spans of its arguments.

For a set $\Delta \subset \mathbb{R}^n$, let $\langle\Delta\rangle$ denote the linear span of all vectors of the form $a-b$, where $a \in \Delta$ and $b \in \Delta$. For a subspace $L \subset \mathbb{R}^n$, let p denote the projection $\mathbb{R}^n \mapsto \mathbb{R}^n/L$. We recall that μ is a volume form on $\mathbb{R}^n/L$.

Theorem 5 1. *If* $\dim p(\Delta_{i_1} + \cdots + \Delta_{i_q}) < q-1$ *for some numbers* $0 \le i_1 < \cdots < i_q \le k$, *then* $\mathrm{CB}_\mu(\Delta_0, \ldots, \Delta_k)$ *consists of one point.*

2. *Otherwise, there exists a unique minimal nonempty set* $\{i_1, \ldots, i_q\} \subset \{1, \ldots, n\}$ *such that* $\dim p(\Delta_{i_1} + \cdots + \Delta_{i_q}) = q-1$. *In this case*

$$\langle \mathrm{CB}_\mu(\Delta_0, \ldots, \Delta_k)\rangle = \langle \Delta_{i_1} + \cdots + \Delta_{i_q}\rangle \cap L.$$

Proof By the definition of a composite body, this theorem follows from a similar fact about mixed volumes, namely, from Bernstein's criterion for the vanishing of the mixed volume (see below). The uniqueness of a minimal nonempty set $\{i_1, \ldots, i_q\} \subset \{1, \ldots, n\}$ such that $\dim p(\Delta_{i_1} + \cdots + \Delta_{i_q}) = q-1$ follows because the family of all such sets is closed under the operation of intersection (see Theorem 1.1 in [5] for the details). □

Lemma 4 (Bernstein's criterion [8]) *The mixed volume of convex bodies* $B_1, \ldots, B_n$ *in* $\mathbb{R}^n$ *is equal to zero iff* $\dim\langle B_{i_1} + \cdots + B_{i_q}\rangle < q$ *for some numbers* $1 \le i_1 < \cdots < i_q \le n$.

2.4 Mixed fiber bodies and the existence of composite bodies

The notion of a composite body turns out to be a version of the notion of a mixed fiber body. We use this relation to prove the existence and some basic properties of composite bodies.

Let $L \subset \mathbb{R}^n$ be a vector subspace of codimension k, μ be a volume form on $\mathbb{R}^n/L$, and p denote the projection $\mathbb{R}^n \to \mathbb{R}^n/L$.

Definition 6 [9] For a convex body $\Delta \subset \mathbb{R}^n$, the set of all points of the form

$$\int_{p(\Delta)} s\mu \in \mathbb{R}^n,$$

where $s\colon p(\Delta) \to \Delta$ is a continuous section of the projection p, is called the *Minkowski integral* of Δ and is denoted by $\int p|_\Delta \mu$.

The following fact explains the relation between composite bodies and the Minkowski integrals.

Lemma 5 *The convex body $(k+1)!\int p|_\Delta \mu$ is contained in a fiber of the projection p and satisfies the definition of the composite body* $\mathrm{CB}_\mu(\Delta, \dots, \Delta)$ *up to a shift.*

Proof We first consider a special case. If $\Delta = A + B$, where $B \subset L$ and the restriction $p|_A$ is injective, then the statement follows from the additivity of the mixed volume. Indeed, for arbitrary convex bodies $B_1, \dots, B_{n-k-1} \subset L$ and a volume form μ' on L,

$$
\begin{aligned}
n!\mathrm{MV}_{\mu\wedge\mu'}(A+B, \dots, A+B, B_1, \dots, B_{n-k-1}) \\
&= (k+1)\cdot n!\mathrm{MV}_{\mu\wedge\mu'}(A, \dots, A, B, B_1, \dots, B_{n-k-1}) \\
&= (k+1)\cdot k!\Big(\int_{p(A)}\mu\Big)\cdot (n-k)!\mathrm{MV}_{\mu'}(B, B_1, \dots, B_{n-k-1}) \\
&= (n-k)!\mathrm{MV}_{\mu'}\Big((k+1)!\Big(\int_{p(A)}\mu\Big)\cdot B, B_1, \dots, B_{n-k-1}\Big) \\
&= (n-k)!\mathrm{MV}_{\mu'}\Big((k+1)!\cdot\int p|_\Delta\mu, B_1, \dots, B_{n-k-1}\Big).
\end{aligned}
$$

In general, the projection $p(\Delta)$ can be subdivided into small pieces and Δ can be subdivided into the inverse images Δ_i of these pieces. Representing the mixed volume $\mathrm{MV}_{\mu\wedge\mu'}(\Delta, \dots, \Delta, B_1, \dots, B_{n-k-1})$ as the sum of the mixed volumes

$$\sum_i \mathrm{MV}_{\mu\wedge\mu'}(\Delta_i, \dots, \Delta_i, B_1, \dots, B_{n-k-1})$$

for arbitrary convex bodies $B_1, \dots, B_{n-k-1}$ in L and approximating each Δ_i by a sum $A_i + B_i$ such that $B_i \subset L$ and the restriction $p|_{A_i}$ is injective, we reduce the general case to the special case. □

The following theorem provides a way to generalize Lemma 5 to composite bodies of arbitrary collections of convex bodies.

Theorem 6 *Let $u\colon \mathbb{R}^n \to L$ be a linear projection.*

1. *There exists a unique symmetric multilinear map* $\mathrm{MF}_{\mu,u}$ *from collections of* $k+1$ *convex bodies in $\mathbb{R}^n$ to convex bodies in L such that* $\mathrm{MF}_{\mu,u}(\Delta, \dots, \Delta) = u\int p|_\Delta\mu$ *for each convex body $\Delta \subset \mathbb{R}^n$.*
2. *This map assigns polytopes to polytopes.*

This theorem is proved below.

Definition 7 The convex body $\mathrm{MF}_{\mu,u}(\Delta_0, \dots, \Delta_k)$ is called the *mixed fiber body* of the bodies $\Delta_0, \dots, \Delta_k$.

Theorem 7 *The convex body* $(k+1)!\mathrm{MF}_{\mu,u}(\Delta_0, \dots, \Delta_k)$ *is contained in a fiber of the projection p and satisfies the definition of the composite body* $\mathrm{CB}_\mu(\Delta_0, \dots, \Delta_k)$ *up to a shift.*

Proof By the additivity of mixed fiber bodies and mixed volumes, the statement can be reduced to the special case $\Delta_0 = \dots = \Delta_k$ considered in Lemma 5 above. □

2.5 Virtual bodies

It is more convenient to prove Theorem 6 in the context of virtual bodies instead of convex bodies because an explicit formula for mixed fiber bodies (see Lemma 7) involves subtraction of convex bodies.

We recall that the *Grothendieck group* K_G of a commutative semigroup K is the group of formal differences of elements from K. In more detail, it is the quotient of the set $K \times K$ by the equivalence relation $(a, b) \sim (c, d) \Leftrightarrow \exists k : a + d + k = b + c + k$ with the operations $(a, b) + (c, d) = (a + c, b + d)$ and $-(a, b) = (b, a)$. For each semigroup K with the cancellation law $a + c = b + c \Rightarrow a = b$, the map $a \to (a + a, a)$ induces the inclusion $K \hookrightarrow K_G$. An element of the form $(a + a, a) \in K_G$ is said to be *proper* and is usually identified with $a \in K$. Under this convention, we can write $(a, b) = a - b$.

Definition 8 The *group of virtual bodies* in $\mathbb{R}^n$ is the Grothendieck group of the semigroup of convex bodies in $\mathbb{R}^n$ with the operation of Minkowski summation. It contains the *group of virtual polytopes* in $\mathbb{R}^n$, i.e., the Grothendieck group of the semigroup of convex polytopes in $\mathbb{R}^n$.

These commutative groups are real vector spaces with the operation of scalar multiplication defined as dilatation.

Definition 9 For a virtual body Δ in $\mathbb{R}^n$, its *support function* $\Delta(\cdot) \colon (\mathbb{R}^n)^* \to \mathbb{R}$ is defined as

$$\Delta(\gamma) = \max_{x \in \Delta_1} \langle \gamma, x \rangle - \max_{x \in \Delta_2} \langle \gamma, x \rangle,$$

where Δ_1 and Δ_2 are convex bodies such that $\Delta = \Delta_1 - \Delta_2$.

The following statement describes the group of virtual bodies more explicitly. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *positively homogeneous* if $f(tx) = tf(x)$ for each $t \geq 0$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a *d.c. function* if it can be represented as the difference of two convex functions.

Lemma 6 1. *The map $\Delta \to \Delta(\cdot)$ induces an isomorphism between the group of virtual bodies in $\mathbb{R}^n$ and the group of positively homogeneous d.c. functions on $(\mathbb{R}^n)^*$.*

2. *This isomorphism induces an isomorphism between the group of virtual polytopes and the group of continuous piecewise linear positively homogeneous functions.*

Proof The map $\Delta \to \Delta(\cdot)$ is surjective by the definition of a d.c. function. It is injective because a convex body is uniquely determined by its support function. Part 2 of the lemma follows because each continuous piecewise linear function can be represented as a difference of two convex piecewise linear functions. $\square$

The operations of taking the mixed volume, the composite body, and the mixed fiber body can be extended to virtual bodies by linearity. This extension is unique, but its properties are quite different. For example, the mixed volume of virtual polytopes is not monotonic (e.g., $\mathrm{MV}(-A, A) > \mathrm{MV}(-A, 2A)$ for a convex polygon A) and is not nondegenerate in the sense of Lemma 4 (e.g., $\mathrm{MV}(B - C, 2B + 2C) = 0$ for nonparallel segments B and C in the plane). As a result, Theorems 4 and 5 are not applicable to virtual composite bodies.

2.6 Proof of Theorem 6

The uniqueness and part 2 of the theorem are corollaries of the following formula for mixed fiber bodies.

Lemma 7 *For any convex bodies* $\Delta_0, \dots, \Delta_k \subset \mathbb{R}^n$,

$$\begin{aligned}&\mathrm{MF}_{\mu,u}(\Delta_0, \dots, \Delta_k)\\ &\quad = \frac{1}{(k+1)!} \sum_{0 \le i_1 < \cdots < i_q \le k} (-1)^{k+1-q} u \textstyle\int p|_{(\Delta_{i_1}+\cdots+\Delta_{i_q})}\, \mu.\end{aligned}$$

Proof Let $m\colon A \times \cdots \times A \to B$ be a symmetric multilinear map, where A and B are semi-groups. Then

$$\begin{aligned}&m(a_1, \dots, a_k)\\ &\quad = \sum_{0 \le i_1 < \cdots < i_q \le k} (-1)^{k+1-q} m(a_{i_1} + \cdots + a_{i_q}, \dots, a_{i_1} + \cdots + a_{i_q}).\end{aligned}$$

To prove this formula, we open the parentheses in the right-hand side by the linearity of m and cancel like terms. □

Lemma 8 (see Sect. 3 or [7]) *Let* $u\colon \mathbb{R}^n \to L$ *be a linear projection and* μ *be a volume form on* $\mathbb{R}^n/L$.

1. *There exists a symmetric multilinear map* $\mathrm{MF}_{\mu,u}$ *from collections of* $k+1$ *virtual polytopes in* $\mathbb{R}^n$ *to virtual polytopes in* L *such that* $\mathrm{MF}_{\mu,u}(\Delta, \dots, \Delta) = u \int p|_\Delta \mu$ *for each convex polytope* $\Delta \subset \mathbb{R}^n$.
2. *The map* $\mathrm{MF}_{\mu,u}$ *sends convex polytopes to convex polytopes.*

The existence of mixed fiber bodies can be reduced to this special case as follows. For arbitrary convex bodies $\Delta_0, \dots, \Delta_k$ in $\mathbb{R}^n$, we define the virtual body $\mathrm{MF}_{\mu,u}(\Delta_0, \dots, \Delta_k)$ as in Lemma 7. It follows from the definition that

1. $\mathrm{MF}_{\mu,u}$ is symmetric,
2. $\mathrm{MF}_{\mu,u}(\Delta, \dots, \Delta) = u \int p|_\Delta \mu$ for each convex body $\Delta \subset \mathbb{R}^n$, and
3. $\mathrm{MF}_{\mu,u}$ is continuous in the sense of the norm $|\Delta| = \max_{\gamma \in B} |\Delta(\gamma)|$, where $B \in (\mathbb{R}^n)^*$ is a compact neighborhood of the origin, because the Minkowski integral is continuous in this sense.

Lemma 8 implies that $\mathrm{MF}_{\mu,u}$ is multilinear and preserves convexity under the assumption that the arguments are polytopes. Namely, for any virtual polytopes $\Delta_0, \Delta_0', \Delta_1, \dots, \Delta_k$,

4. $\mathrm{MF}_{\mu,u}(\Delta_0 + \Delta_0', \Delta_1, \dots, \Delta_k) = \mathrm{MF}_{\mu,u}(\Delta_0, \dots, \Delta_k) + \mathrm{MF}_{\mu,u}(\Delta_0', \dots, \Delta_k)$,
5. $\mathrm{MF}_{\mu,u}(t \cdot \Delta_0, \dots, \Delta_k) = t \cdot \mathrm{MF}_{\mu,u}(\Delta_0, \dots, \Delta_k)$, and
6. $\mathrm{MF}_{\mu,u}(\Delta_0, \dots, \Delta_k)$ is convex if $\Delta_0, \dots, \Delta_k$ are convex.

Approximating arbitrary convex bodies with convex polytopes and using the continuity of $\mathrm{MF}_{\mu,u}$ (property 3), we can extend properties 4, 5 and 6 to arbitrary convex bodies. □

3 Mixed fiber polytopes

In this section, we prove the existence of mixed fiber polytopes (Lemma 8). Namely, let $L \subset \mathbb{R}^n$ be a vector subspace of codimension k, $u: \mathbb{R}^n \to L$ be a linear projection, and μ be a volume form on $\mathbb{R}^n/L$. Let p denote the projection $\mathbb{R}^n \to \mathbb{R}^n/L$. Then (1) there exists a symmetric multilinear map $\mathrm{MF}_{\mu,u}$ from collections of $k+1$ virtual polytopes in $\mathbb{R}^n$ to virtual polytopes in L such that $\mathrm{MF}_{\mu,u}(\Delta, \dots, \Delta) = u \int p|_\Delta \mu$ for each convex polytope $\Delta \subset \mathbb{R}^n$, and (2) $\mathrm{MF}_{\mu,u}$ maps convex polytopes to convex polytopes. The proof of these statements follows because the Minkowski integral is a polynomial map from the space of virtual polytopes in $\mathbb{R}^n$ to the space of virtual polytopes in L. Each polynomial map of vector spaces yields a certain symmetric multilinear function, which is called the polarization of the polynomial. In more detail, statement 1 follows from Theorems 8, 9, and 10, and statement 2 follows from the corollary in Sect. 3.3.

3.1 Polarizations of polynomials on Zariski dense sets

The existence of mixed fiber polytopes is a corollary of the following general construction.

Definition 10 A set A in a vector space W is said to be *Zariski dense* if each finite-dimensional subspace $U \subset W$ is contained in a finite-dimensional subspace $V \subset W$ such that $A \cap V$ is Zariski dense in V (i.e., $A \cap V$ is not contained in a proper algebraic subset of V).

Definition 11 A map $f: A \to V$ from a subset A of a vector space W to a vector space V is called a (homogeneous) *polynomial of degree* k if for each finite-dimensional subspace $U \subset W$ and for each linear function $l: V \to \mathbb{R}$, the composition $l \circ f|_U: A \cap U \to \mathbb{R}$ is a restriction of a (homogeneous) polynomial of degree at most k on U.

Theorem 8 1. *A (homogeneous) polynomial map of degree k on a Zariski dense subset of a vector space W has a unique extension to a (homogeneous) polynomial map of degree k on W.*

2. *For a homogeneous polynomial map $f: W \to V$ of degree k, there exists a unique symmetric multilinear function*

$$Mf: \underbrace{W \oplus \cdots \oplus W}_{k} \to V \quad \textit{such that} \quad Mf(w, \dots, w) = f(w)$$

for every $w \in W$.

Definition 12 The function Mf is called the *polarization* of the polynomial f.

Proof of Theorem 8 1. Let A be a Zariski dense subset in W and $f: A \to V$ be a (homogeneous) polynomial map of degree k. For a subspace U such that $A \cap U$ is Zariski dense in U, there exists a unique (homogeneous) polynomial map $f_U: U \to V$ of degree k such that $f_U = f$ on $U \cap A$. For any two such finite-dimensional subspaces U and U', the sum $U + U'$ is contained in a finite-dimensional subspace U'' such that $U'' \cap A$ is Zariski dense. Hence, $f_{U''} = f_{U'}$ on U' and $f_{U''} = f_U$ on U. In particular, $f_U = f_{U'}$ on the intersection $U \cap U'$. This implies that polynomials f_U glue together into a map $\tilde{f}: W \to V$ such that $\tilde{f} = f$ on A.

2. For numbers $t_1, \ldots, t_k$ and vectors $w_1, \ldots, w_k \in W$, the expression $f(t_1 w_1 + \cdots + t_k w_k)/k!$ is a homogeneous polynomial as a function of $t_1, \ldots, f_k$. The coefficient of the monomial $t_1 \cdots t_k$ in this polynomial satisfies the definition of the polarization Mf. □

We apply polarizations in the following context. Let $V(K)$ be the space of virtual polytopes in a k-dimensional vector space K. Let $A(K) \subset V(K)$ be the set of convex polytopes.

Theorem 9 *The subset $A(K) \subset V(k)$ is Zariski dense in $V(K)$.*

Definition 13 A polytope $\Delta' \in V(K)$ is said to be *compatible* with a polytope $\Delta \in V(K)$ if the support function $\Delta'(\cdot)$ is linear on every domain of linearity of $\Delta(\cdot)$.

Proof of Theorem 9 Let $V(\Delta) \subset V(K)$ be the space of all virtual polytopes compatible with $\Delta \in V(K)$. Theorem 9 is a corollary of the following facts:

1. For every polytope $\Delta \in V(K)$, the space $V(\Delta)$ is finite dimensional. Indeed, the space of piecewise-linear functions with the prescribed domains of linearity is finite dimensional.
2. For every convex polytope $\Delta \in A(K)$, the intersection $V(\Delta) \cap A(K)$ is Zariski dense in $V(K)$.
3. Every finite-dimensional vector subspace $U \subset V(K)$ is contained in the space $V(\Delta)$ for some convex polytope $\Delta \in A(K)$. Indeed, if U is generated by the differences $A_i - B_i$ of the convex polytopes A_i and B_i, then we can choose $\Delta = \sum_i A_i + B_i$.

The theorem is proved. □

3.2 The Minkowski integral is a polynomial

Let $u\colon \mathbb{R}^n \to L$ be a linear projection, μ be a volume form on the k-dimensional vector space $\mathbb{R}^n/L$, and p be the projection $\mathbb{R}^n \to \mathbb{R}^n/L$.

Theorem 10 *The Minkowski integral $\mathcal{M}(\Delta) = u \int p|_\Delta \mu$ is a homogeneous polynomial map $A(\mathbb{R}^n) \to A(L)$ of degree $k+1$.*

For a convex polytope $\Delta \in \mathbb{R}^n$, we define $A(\Delta)$ as the set of all convex polytopes compatible with Δ. For a convex k-dimensional polytope Δ, the restriction of $\mathcal{M}$ to $A(\Delta)$ is a homogeneous polynomial map of degree $k+1$ because of the following two facts (the first follows from the definition of the Minkowski integral, and the second is well known).

Lemma 9 *The Minkowski integral $\mathcal{M}(\Delta)$ of a convex k-dimensional polytope Δ consists of one point, and this point coincides with the projection u of the first moment $\int_\Delta x p^*(\mu)$ of Δ, where x ranges Δ and $p^*(\mu)$ is the volume form μ on $\mathbb{R}^n/L$ lifted to Δ.*

Lemma 10 *The first moment is a homogeneous polynomial of degree $k+1$ on the space $A(\Delta)$ if Δ is a convex k-dimensional polytope.*

Theorem 10 can be reduced to k-dimensional polytopes as follows. For a covector $\gamma \in (\mathbb{R}^n)^*$ and a convex polytope $\Delta \subset \mathbb{R}^n$, let Δ^γ be the maximal face where γ attains its maximum as a function on Δ.

Lemma 11 *For every covector $\gamma \in L^*$,*

$$(u \textstyle\int p|_\Delta \mu)^\gamma = \sum_{\delta \in (\mathbb{R}^n)^*,\, \delta|_L = \gamma} u \int p|_{\Delta^\delta} \mu.$$

This equality easily follows from the definition of the Minkowski integral, and we omit the proof. The sum in the right-hand side makes sense because it contains finitely many nonzero summands. We note that Lemmas 9 and 11 are respectively similar to Propositions 5.1 and 5.2 in [7].

Lemma 12 *The map $\mathcal{M}$ preserves the compatibility of convex polytopes*:

$$\mathcal{M}\big(A(\Delta)\big) \subset A\big(\mathcal{M}(\Delta)\big).$$

Proof The integral of a continuous family of convex functions is a linear function if every function in the family is linear. We apply this to the following description of the support function of $\mathcal{M}(\Delta)$. □

For a convex body $\Delta \subset \mathbb{R}^n$ and a point $a \in \mathbb{R}^n/L$, let Δ_a denote the convex body $u\big(\Delta \cap p^{(-1)}(a)\big) \subset L$; roughly speaking, Δ_a is a fiber of Δ over the point a.

Lemma 13 *The support function of the body $u\big(\mathcal{M}(\Delta)\big)$ is equal to the integral of the support functions of the bodies Δ_a over $a \in p(\Delta)$.*

This equality easily follows from the definition of the Minkowski integral, and we omit the proof.

Proof of Theorem 10 For a face B of a polytope Δ, let $\widetilde{B}\colon A(\Delta) \to A(B)$ be the map that sends each $\Delta' \in A(\Delta)$ to its face $B' \in A(B)$ such that $B + B'$ is a face of $\Delta + \Delta'$. For an n-dimensional convex polytope $\Delta \in A(\mathbb{R}^n)$, let $a_1, \dots, a_I$ denote the vertices of $\mathcal{M}(\Delta)$ and $B_1, \dots, B_J$ denote the k-dimensional faces of Δ. By Lemma 12, the points $\tilde{a}_1\big(\mathcal{M}(\Delta')\big), \dots, \tilde{a}_I\big(\mathcal{M}(\Delta')\big)$ are the vertices of the polytope $\mathcal{M}(\Delta')$ for each convex polytope $\Delta' \in A(\Delta)$. By Lemma 11, each vertex $\tilde{a}_i\big(\mathcal{M}(\Delta')\big)$ is equal to a finite sum of the Minkowski integrals of k-dimensional faces $\widetilde{B}^j(\Delta')$. By Lemmas 9 and 10, the Minkowski integral $\mathcal{M}$ is a homogeneous polynomial of degree $k+1$ on the image of each linear map $\widetilde{B}_j$. □

3.3 Faces and convexity of mixed fiber polytopes

By Theorems 9 and 10, there exists a unique polarization of the Minkowski integral of a polytope in $\mathbb{R}^n$ with respect to a volume form μ on $\mathbb{R}^n/L$. It is denoted by $\mathrm{MF}_{\mu,u}(\Delta_0, \dots, \Delta_k)$ and is called the *mixed fiber polytope*. To prove that it preserves convexity, we extend Lemma 11 to mixed fiber polytopes as follows.

For a virtual polytope Δ equal to the difference of convex polytopes A and B in $\mathbb{R}^n$ and for a covector $\gamma \in (\mathbb{R}^n)^*$, the *support face* Δ^γ is defined as $A^\gamma - B^\gamma$.

Theorem 11 *For virtual polytopes $\Delta_0, \dots, \Delta_k \subset \mathbb{R}^n$ and a covector $\gamma \in L^*$, the face $\big(\mathrm{MF}_{\mu,u}(\Delta_0, \dots, \Delta_k)\big)^\gamma$ coincides with the Minkowski sum*

$$\sum_{\delta \in (\mathbb{R}^n)^*,\, \delta|_L = \gamma} \mathrm{MF}_{\mu,u}(\Delta_0^\delta, \dots, \Delta_k^\delta).$$

This theorem follows from Lemma 11 by the linearity of mixed fiber polytopes.

The length of a one-dimensional Minkowski integral of a convex polytope Δ is by definition equal to the volume of Δ. This fact extends by linearity as follows.

Lemma 14 *If the convex polytopes $\Delta_0, \ldots, \Delta_k$ are all parallel to a $(k+1)$-dimensional subspace $K \subset \mathbb{R}^n$ and t is a coordinate on the line $K \cap L$, then a mixed fiber body $\mathrm{MF}_{\mu,u}(\Delta_0, \ldots, \Delta_k)$ is a segment, parallel to the line $K \cap L$, and its length (in the sense of the coordinate t) is equal to $\mathrm{MV}_{dt \wedge p^*\mu}(\Delta_0, \ldots, \Delta_k)$.*

This mixed volume makes sense because its arguments are all parallel to the same $(k+1)$-dimensional subspace K. The volume form $dt \wedge p^*\mu$ makes sense on K because $\ker(p|_K) = K \cap L$.

In particular, a one-dimensional mixed fiber polytope of convex polytopes is convex. Because each edge of a mixed fiber polytope is a sum of one-dimensional mixed fiber polytopes by Theorem 11, each edge of a mixed fiber polytope of convex polytopes is convex. A polytope with all convex edges is convex.

Corollary *A mixed fiber polytope of convex polytopes is convex.*

3.4 Vertices and integrality of mixed fiber polytopes

The proof of Theorem 10 is based on the fact that vertices of the Minkowski integral of Δ can be expressed in terms of the first moments of faces of Δ. We extend this fact to mixed fiber polytopes in order to prove their integrality. To formulate this, we need the polarization of the first moment, which exists by Lemma 10. For the virtual polytopes $\Delta_0, \ldots, \Delta_k$ in $\mathbb{R}^n$, the subspace $\langle \Delta_0, \ldots, \Delta_k \rangle \subset \mathbb{R}^n$ is defined as the minimal subspace containing convex polytopes B_i^j such that $\Delta_i = B_i^0 - B_i^1$ up to a shift for $i = 0, \ldots, k$.

Lemma 15 *There exists a unique symmetric multilinear function MM_μ of $k+1$ convex bodies such that*

1. *the domain of MM_μ consists of all collections of virtual polytopes*

$$\Delta_0, \ldots, \Delta_k \subset \mathbb{R}^n$$

such that $\dim\langle \Delta_0, \ldots, \Delta_k \rangle \leq k$ and

2. *we have*

$$\mathrm{MM}_\mu(\Delta, \ldots, \Delta) = \int_\Delta x p^*(\mu)$$

for each k-dimensional convex polytope $\Delta \subset \mathbb{R}^n$, where x ranges Δ and $p^*(\mu)$ is the volume form μ on $\mathbb{R}^n/L$ lifted to Δ.

Definition 14 The point $\mathrm{MM}_\mu(\Delta_0, \ldots, \Delta_k) \in \mathbb{R}^n$ is called the *mixed moment* of $\Delta_0, \ldots, \Delta_k$.

By linearity, Lemma 9 extends to mixed fiber polytopes as follows.

Lemma 16 *If* $\dim\langle\Delta_0,\dots,\Delta_k\rangle \le k$, *then the mixed fiber polytope*

$$\mathrm{MF}_{\mu,u}(\Delta_0,\dots,\Delta_k)\in\mathbb{R}^n$$

consists of one point $u\mathrm{MM}_\mu(\Delta_0,\dots,\Delta_k)\in L$.

Lemma 16 and Theorem 11 give the following expression for the vertices of a mixed fiber polytope.

Theorem 12 *In the notation in Theorem* 11,

1. *if* $\dim\langle\Delta_0^\delta,\dots,\Delta_k^\delta\rangle \le k$ *for each covector* $\delta\in(\mathbb{R}^n)^*$ *such that* $\delta|_L=\gamma$, *then the face* $\big(\mathrm{MF}_{\mu,u}(\Delta_0,\dots,\Delta_k)\big)^\gamma$ *is a vertex of* $\mathrm{MF}_{\mu,u}(\Delta_0,\dots,\Delta_k)$ *and is equal to* $\sum_{\delta|_L=\gamma} u\mathrm{MM}_\mu(\Delta_0^\delta,\dots,\Delta_k^\delta)$;
2. *almost all covectors* $\gamma\in L^*$ *satisfy the condition in part* 1; *and*
3. *the set of all points of the form* $\sum_{\delta|_L=\gamma} u\mathrm{MM}_\mu(\Delta_0^\delta,\dots,\Delta_k^\delta)$, *where* $\gamma\in L^*$ *satisfies the condition in part* 1, *coincides with the set of all vertices of* $\mathrm{MF}_{\mu,u}(\Delta_0,\dots,\Delta_k)$.

In part 2 of the theorem, "almost all (co)vectors in a space V" means all covectors from the complement of a finite union of proper vector subspaces of V.

In particular, because the mixed moment of integer polytopes is a rational number with the denominator $(k+1)!$, the same is true for mixed fiber polytopes.

Theorem 13 *If* $\Delta_0,\dots,\Delta_k$ *are integer polytopes* (*i.e., their vertices are integer lattice points*), $L\subset\mathbb{R}^n$ *is a* k*-dimensional rational subspace*, $u(\mathbb{Z}^n)=L\cap\mathbb{Z}^n$, *and* μ *is the integer volume form on* $\mathbb{R}^n/L$ (*i.e.,* $\int_{\mathbb{R}^n/(L+\mathbb{Z}^n)}\mu=1$), *then* $(k+1)!\mathrm{MF}_{\mu,u}(\Delta_0,\dots,\Delta_k)$ *is an integer polytope.*

4 Leading coefficients of a composite polynomial in terms of composite polynomials of fewer variables

In this section, we present some technical facts about how to compute the leading coefficients of a composite polynomial in terms of composite polynomials of fewer variables. In the next section, we use these facts to compute the leading coefficients of a composite polynomial $\pi_{f_0,\dots,f_k}$ explicitly under the assumption that the Newton polytopes of the polynomials $f_0,\dots,f_k$ satisfy a certain condition of general position.

We recall that for a covector $\gamma\in(\mathbb{R}^n)^*$ and a convex polytope $A\subset\mathbb{R}^n$, the polytope A^γ is defined as the maximal face of A where γ attains its maximum as a function on A.

Definition 15 For a covector $\gamma\in(\mathbb{R}^n)^*$ and a Laurent polynomial $f(x)=\sum_{a\in A}c_ax^a$ on $(\mathbb{C}\setminus 0)^n$, the polynomial $\sum_{a\in A^\gamma}c_ax^a$ is called the *truncation* of f in the direction γ and is denoted by f^γ.

Theorem 14 expresses a truncation of a composite polynomial in terms of composite polynomials of truncations. Theorem 15 represents a homogeneous composite polynomial as a composite polynomial of fewer variables. Because truncations of polynomials are homogeneous, Theorem 15 can be used to simplify the answer in the formulation of Theorem 14. As a result, a truncation of a composite polynomial can be expressed in terms of composite polynomials of fewer variables.

Definition 16 The *vertex coefficients* of a polynomial f are the coefficients of its monomials corresponding to the vertices of the Newton polytope Δ_f.

Because a composite polynomial is unique up to a monomial factor, we are interested in ratios of its vertex coefficients rather than in individual vertex coefficients. Theorem 16 expresses the ratio of two vertex coefficients of a composite polynomial as the product of values of some monomial over the roots of some system of polynomial equations. By Lemma 1.2, this product over roots can be seen as a vertex coefficient of a corresponding composite polynomial of one variable.

4.1 Truncation and dehomogenization

The operations of truncating and taking the composite polynomial commute in the following sense.

Theorem 14 *Let $\pi : (\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{n-k}$ and $\pi^\times : \mathbb{Z}^{n-k} \hookrightarrow \mathbb{Z}^n$ be an epimorphism of complex tori and the corresponding embedding of their character lattices, and let $f_0, \dots, f_k$ be Newton-nondegenerate Laurent polynomials on $(\mathbb{C} \setminus 0)^n$. Then for every $\gamma \in (\mathbb{Z}^{n-k})^*$, the truncation $\pi^\gamma_{f_0,\dots,f_k}$ is equal to the product $\prod_\delta \pi_{f_0^\delta,\dots,f_k^\delta}$ over all $\delta \in (\mathbb{Z}^n)^*$ such that $\delta|_{\pi^\times\mathbb{Z}^{n-k}} = \gamma$.*

Because composite polynomials are defined up to a monomial multiplier, we can assume that whenever $\pi_{f_0^\delta,\dots,f_k^\delta}$ is a monomial, it is equal to 1. Under this assumption, the product $\prod_{\delta\in\mathbb{Z}^n,\,\delta|_{\pi^\times\mathbb{Z}^{n-k}}=\gamma} \pi_{f_0^\delta,\dots,f_k^\delta}$ contains a finite number of factors different from 1. The proof of this theorem is given at the end of this section. Theorem 11 is the geometric counterpart of this theorem.

A Laurent polynomial $f : (\mathbb{C} \setminus 0)^n \to \mathbb{C}$ is said to be *homogeneous* if there exist an epimorphism of complex tori $(\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{n'}$ and a Laurent polynomial $g : (\mathbb{C} \setminus 0)^{n'} \to \mathbb{C}$ such that $n' < n$ and $f = g \circ h$ up to a monomial factor. The polynomial g is called a *dehomogenization* of f. Theorem 15 below implies that the operations of dehomogenization and taking the composite polynomial commute in the following sense: if polynomials $f_0, \dots, f_k$ are "sufficiently homogeneous," then their composite polynomial is also homogeneous, and its dehomogenization is equal to the composite polynomial of dehomogenizations of $f_0, \dots, f_k$ raised to some power.

Every pair of tori epimorphisms $(\mathbb{C} \setminus 0)^{n-k} \xleftarrow{\pi} (\mathbb{C} \setminus 0)^n \xrightarrow{h} (\mathbb{C} \setminus 0)^{n'}$ and corresponding character lattice embeddings $\mathbb{Z}^{n-k} \overset{\pi^\times}{\hookrightarrow} \mathbb{Z}^n \overset{h^\times}{\hookleftarrow} \mathbb{Z}^{n'}$ can be included in the commutative squares

$$\begin{array}{ccc} (\mathbb{C} \setminus 0)^n & \overset{h}{\mapsto} & (\mathbb{C} \setminus 0)^{n'} \\ \downarrow \pi & & \downarrow \pi' \\ (\mathbb{C} \setminus 0)^{n-k} & \overset{h'}{\mapsto} & (\mathbb{C} \setminus 0)^{n'-k} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{Z}^n & \overset{h^\times}{\hookleftarrow} & \mathbb{Z}^{n'} \\ \uparrow \pi^\times & & \uparrow \pi'^\times \\ \mathbb{Z}^{n-k} & \overset{h'^\times}{\hookleftarrow} & \mathbb{Z}^{n'-k} \end{array}$$

such that the image of $\mathbb{Z}^{n'-k}$ in $\mathbb{Z}^n$ is equal to the intersection $\pi^\times\mathbb{Z}^{n-k} \cap h^\times\mathbb{Z}^{n'}$.

Theorem 15 *In this notation, if Laurent polynomials $f_0, \dots, f_k$ on $(\mathbb{C} \setminus 0)^n$ are homogeneous in the sense that $f_i = g_i \circ h$ up to a monomial factor for some Laurent polynomials $g_0, \dots, g_k$ on $(\mathbb{C} \setminus 0)^{n'}$, then their composite polynomial $\pi_{f_0,\dots,f_k}$ is homogeneous in the sense that it is equal to $g \circ h'$, where $g = (\pi'_{g_0,\dots,g_k})^{\left|\mathbb{Z}^n / (\pi^\times\mathbb{Z}^{n-k} + h^\times\mathbb{Z}^{n'})\right|}$ is a Laurent polynomial on $(\mathbb{C} \setminus 0)^{n'-k}$.*

The proof is given at the end of this section.

4.2 Vertex coefficients

Definition 17 The *product over roots* $R_{A_1,\dots,A_m}(g_0; g_1,\dots,g_m)$ is a rational function on the space of collections of Laurent polynomials $(g_1,\dots,g_m)$ such that the Newton polytope of g_i is $A_i \subset \mathbb{Z}^m$. By definition, this function is equal to the product of values of a polynomial g_0 over the roots of the system $g_1 = \cdots = g_m = 0$ for Newton-nondegenerate polynomials $g_1,\dots,g_m$.

Lemma 1.2 is a formula for the vertex coefficient of a composite polynomial of one variable in terms of products over roots. The following theorem extends this formula to composite polynomials of several variables. Let $\pi\colon \mathbb{C}^k \times \mathbb{C}^{n-k} \to \mathbb{C}^{n-k}$ be the standard projection and $u_1,\dots,u_k$ be the standard coordinates on $\mathbb{C}^k$. We suppose that $f_0,\dots,f_k$ are polynomials on $\mathbb{C}^k \times \mathbb{C}^{n-k}$ and that their Newton polytopes $A_0,\dots,A_k \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ intersect all coordinate hyperplanes. We let $A \subset \mathbb{R}^{n-k}$ denote the Newton polytope of the composite polynomial $\pi_{f_0,\dots,f_k}$ and consider covectors γ_1 and γ_2 in $(\mathbb{Z}^{n-k})^*$ with positive integer coordinates. Let $\tilde{f}_i(u,t)$ be a Laurent polynomial $f_i(u, t^{\gamma_2} + t^{-\gamma_1})$ of $k{+}1$ variables $u_1,\dots,u_k,t$ and $\tilde{A}_i$ be its Newton polytope.

Theorem 16 *If the polynomials $f_0,\dots,f_k$ are Newton-nondegenerate and the covectors γ_1 and γ_2 are generic in the sense that the face $\big((\{a\}\times\mathbb{R}^{n-k}) \cap \sum_i A_i\big)^{\gamma_j}$ is a vertex for each $a \in \mathbb{R}^k$, then*

1. *the face A^{γ_j} of the polytope A is a vertex (let B_j denote it); the difference $B_1 - B_2$ is equal to*

$$(k+1)! \sum_{\delta\in(\mathbb{Z}^k)^*} \mathrm{MM}_\mu(A_0^{\gamma_1+\delta},\dots,A_k^{\gamma_1+\delta}) - \mathrm{MM}_\mu(A_0^{\gamma_2+\delta},\dots,A_k^{\gamma_2+\delta}),$$

   where μ is the unit volume form on $\mathbb{Z}^k$ and the mixed moment MM *is defined in Lemma 15; and*

2. *the ratio of the coefficients of the composite polynomial $\pi_{f_0,\dots,f_k}$ at the vertices B_1 and B_2 is equal to*

$$(-1)^{\gamma_1\cdot B_1+\gamma_2\cdot B_2} R_{\tilde{A}_0,\dots,\tilde{A}_k}(t; \tilde{f}_0,\dots,\tilde{f}_k).$$

After an appropriate monomial change of coordinates and multiplication of the polynomials f_i by appropriate monomials, this theorem can be used to find the ratio of the coefficients of the composite polynomial $\pi_{f_0,\dots,f_k}$ at two arbitrary vertices B_1 and B_2 of its Newton polytope. If $\pi_{f_0,\dots,f_k}$ is homogeneous, then a monomial change of coordinates is unnecessary. If the Newton polytopes of the polynomials $f_0,\dots,f_k$ satisfy a certain condition of general position (see Definition 20), then Theorem 19 can be used to compute $R(t; \tilde{f}_0,\dots,\tilde{f}_k)$ explicitly.

Proof Part 1 of the theorem follows from Theorem 12. To prove part 2, we apply the following lemma to the composite polynomial $\pi_{f_0,\dots,f_k}$ multiplied by a monomial such that its Newton polytope belongs to the positive octant and intersects all coordinate hyperplanes. □

Lemma 17 *If the Newton polytope A of a polynomial g intersects all coordinate hyperplanes and if γ_1 and γ_2 are covectors with positive integer components, then the ratio of the coefficients of g at the vertices A^{γ_1} and A^{γ_2} is equal to $(-1)^{\gamma_1\cdot B_1+\gamma_2\cdot B_2}$ times the product of roots of the Laurent polynomial in one variable $g(t^{\gamma_2}+t^{-\gamma_1})$.*

This lemma is a corollary of the Vieta theorem.

4.3 Proof of Theorem 15

We extend the commutative square

$$\begin{array}{ccc} (\mathbb{C}\setminus 0)^n & \overset{h}{\mapsto} & (\mathbb{C}\setminus 0)^{n'} \\ \downarrow \pi & & \downarrow \pi' \\ (\mathbb{C}\setminus 0)^{n-k} & \overset{h'}{\mapsto} & (\mathbb{C}\setminus 0)^{n'-k} \end{array} \quad \text{to} \quad \begin{array}{ccccc} (\mathbb{C}\setminus 0)^n & \overset{p}{\mapsto} & T & \overset{p_2}{\mapsto} & (\mathbb{C}\setminus 0)^{n'} \\ & & \downarrow p_1 & & \downarrow \pi' \\ & & (\mathbb{C}\setminus 0)^{n-k} & \overset{h'}{\mapsto} & (\mathbb{C}\setminus 0)^{n'-k}, \end{array}$$

where p_1 and p_2 are the projections of $(\mathbb{C}\setminus 0)^{n-k}\times(\mathbb{C}\setminus 0)^{n'}$ to the multipliers, T is the kernel of the epimorphism $h'\circ p_1-\pi'\circ p_2\colon (\mathbb{C}\setminus 0)^{n-k}\times(\mathbb{C}\setminus 0)^{n'}\to(\mathbb{C}\setminus 0)^{n'-k}$, and $p=(\pi,h)\colon(\mathbb{C}\setminus 0)^n\to(\mathbb{C}\setminus 0)^{n-k}\times(\mathbb{C}\setminus 0)^{n'}$. The corresponding commutative diagram of embeddings of character lattices implies that the image of $p^\times$ is a sublattice of index $q=\left|\mathbb{Z}^n/(\pi^\times\mathbb{Z}^{n-k}+h^\times\mathbb{Z}^{n'})\right|$ in $\mathbb{Z}^n$. Hence, a fiber of the epimorphism p consists of q points.

For a cycle $N=\sum_i a_iN_i$ in a complex torus $(\mathbb{C}\setminus 0)^m$ and an epimorphism $p\colon(\mathbb{C}\setminus 0)^n\to(\mathbb{C}\setminus 0)^m$, let $p^{(-1)}(N)$ denote the cycle $\sum_i a_i p^{(-1)}(N_i)$. If $m=n$ and a fiber of p consists of q points, then $p_*\circ p^{(-1)}(N)=q\cdot N$. Hence, $\pi_*\circ h^{(-1)}=(p_1)_*\circ p_*\circ p^{(-1)}\circ p_2^{(-1)}=q\cdot(p_1)_*\circ p_2^{(-1)}=q\cdot h^{(-1)}\circ\pi'_*$. To prove the statement of the theorem, we apply both sides of this equality to the cycle $[g_0=\cdots=g_k=0]$.

4.4 Truncations of varieties

The proof of Theorem 14 is based on the following definition of a truncation of a variety (just a more geometric reformulation of the usual definition; see [10]). By varieties, we mean formal sums of irreducible algebraic varieties of the same dimension with positive coefficients. By the intersection of varieties, we mean the intersection counting multiplicities, which makes sense only for proper intersections (the intersection of the varieties V_i is said to be *proper* if its codimension is equal to the sum of the codimensions of V_i). For an algebraic curve $C\subset(\mathbb{C}\setminus 0)^n$, there exists a unique compactification $\widetilde{C}=C\sqcup\{p_1,\dots,p_I\}$ that is smooth near all infinite points p_i. A variety $N\subset(\mathbb{C}\setminus 0)^n$ is said to be *γ-homogeneous* for a linear function γ on the character lattice of the torus $(\mathbb{C}\setminus 0)^n$ if N is invariant under the action of the corresponding one-parameter subgroup $\{t^\gamma\mid t\in(\mathbb{C}\setminus 0)\}\subset(\mathbb{C}\setminus 0)^n$.

Definition 18 1. The truncation of an irreducible curve $C\subset(\mathbb{C}\setminus 0)^n$ in the direction $\gamma\in\mathbb{Z}^n$ is a curve $C^\gamma=\sum A_i$, where the summation is over all infinite points p_i of its compactification $\widetilde{C}$, and a curve A_i is given by a parameterization c_it^γ if C is given by a parameterization $c_it^\gamma+\dots$ near p_i.

2. The truncation of an arbitrary curve $C=\sum m_iC_i$ in the direction $\gamma\in\mathbb{Z}^n$ is a curve $C^\gamma=\sum m_iC_i^\gamma$.

3. The truncation of an m-dimensional variety $M\subset(\mathbb{C}\setminus 0)^n$ in the direction $\gamma\in\mathbb{Z}^n$ is an m-dimensional γ-homogeneous variety M^γ such that for any γ-homogeneous variety N of dimension $\operatorname{codim} M+1$,

a. if $M^\gamma \cap N$ is a curve, then $M \cap N$ is a curve, and

b. under this assumption, $M^\gamma \cap N = (M \cap N)^\gamma$.

Lemma 18 1. *There exists a unique truncation of a given variety in a given direction.*

2. *Let $f_1 = \cdots = f_k = 0$ be a Newton-nondegenerate complete intersection. Then its truncation in a direction γ is the complete intersection $f_1^\gamma = \cdots = f_k^\gamma = 0$.*

3. *There is a finite number of different truncations of a given variety.*

Proof The uniqueness follows from the definition. The existence is a corollary of the following explicit construction for the truncation of $M \subset (\mathbb{C} \setminus 0)^n$ in the direction $\gamma \in \mathbb{Z}^n$. Without loss of generality, we can assume that $\gamma = (k, 0, \dots, 0)$ and define M^γ as $p_1^{-1}(\overline{M} \cap \{x_1 = 0\})$, where $x_1, \dots, x_n$ are the standard coordinates in $\mathbb{C}^n$, $p_1 : (\mathbb{C} \setminus 0)^n \to \{x_1 = 0\}$ is the standard projection, and $\overline{M} \subset \mathbb{C} \times (\mathbb{C} \setminus 0)^{n-1}$ is the closure of the variety $M \subset (\mathbb{C} \setminus 0)^n \subset \mathbb{C} \times (\mathbb{C} \setminus 0)^{n-1}$ counting multiplicities.

Part 2 of the lemma also follows from this construction. Indeed, the variety

$$p_1^{-1}(\overline{\{f_1^\gamma = \cdots = f_k^\gamma = 0\}} \cap \{x_1 = 0\})$$

is given by the ideal I generated by the γ-truncations of all the elements of the ideal $\langle f_1, \dots, f_k \rangle$. The ideal I is equal to $\langle f_1^\gamma, \dots, f_k^\gamma \rangle$ because for any relation $\sum g_i f_i^\gamma = 0$, the polynomials g_i are contained in the ideal $\langle f_1^\gamma, \dots, f_k^\gamma \rangle$ (the last fact is equivalent to the vanishing of the first homology group of the Koszul complex for a regular sequence $f_1^\gamma, \dots, f_k^\gamma$).

In general, part 3 of the lemma follows from the existence of the c-fan or Gröbner fan of the ideal of a variety M (see [12]). If M is a Newton-nondegenerate complete intersection, which is the only important case for the proof of Theorem 14, then part 3 follows from part 2. Indeed, γ_1- and γ_2-truncations of a Newton-nondegenerate complete intersection $f_1 = \cdots = f_k = 0$ coincide if $A^{\gamma_1} = A^{\gamma_2}$, where A is the sum of the Newton polytopes Δ_{f_i}. □

4.5 Proof of Theorem 14

Theorem 14 is a special case of the following theorem.

Theorem 17 *Let $\pi : (\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{n-k}$ and $\pi^\times : \mathbb{Z}^{n-k} \hookrightarrow \mathbb{Z}^n$ be an epimorphism of complex tori and the corresponding embedding of their character lattices, and let $M \subset (\mathbb{C} \setminus 0)^n$ be a variety. Then the truncation $\big(\pi_*(M)\big)^\gamma$ is equal to the sum $\sum_\delta \pi_*(M^\delta)$ over all $\delta \in (\mathbb{Z}^n)^*$ such that $\delta|_{\pi^\times \mathbb{Z}^{n-k}} = \gamma$ (in particular, there is a finite number of nonempty summands).*

Proof If M is one-dimensional, then this theorem follows from the definition of the truncation of a curve. If the dimension is arbitrary, then the number of nonempty summands is finite by Lemma 18.3 because $M^{\delta_1} = M^{\delta_2}$, $\delta_1 \neq \delta_2$, and $\delta_1|_{\pi^\times \mathbb{Z}^{n-k}} = \delta_2|_{\pi^\times \mathbb{Z}^{n-k}}$ implies $\pi_* M^{\delta_1} = \pi_* M^{\delta_2} = \varnothing$. If a $(\operatorname{codim} M - k + 1)$-dimensional γ-homogeneous variety $N \subset (\mathbb{C} \setminus 0)^{n-k}$ intersects all summands $\pi_*(M^\delta)$ properly, then

$$N \cap \sum_{\substack{\delta \in (\mathbb{Z}^n)^*, \\ \delta|_{\pi^\times \mathbb{Z}^{n-k}} = \gamma}} \pi(M^\delta) = \big(N \cap \pi(M)\big)^\gamma \quad \overset{(1)}{\Leftrightarrow}$$

$$\pi_*\Big(\sum_{\substack{\delta\in(\mathbb{Z}^n)^*,\\ \delta|_{\pi\times\mathbb{Z}^{n-k}}=\gamma}} \pi^{(-1)}(N)\cap M^\delta\Big) = \Big(\pi_*\big(\pi^{(-1)}(N)\cap M\big)\Big)^\gamma \quad \overset{(2)}{\Leftrightarrow}$$

$$\pi_*\Big(\sum_{\substack{\delta\in(\mathbb{Z}^n)^*,\\ \delta|_{\pi\times\mathbb{Z}^{n-k}}=\gamma}} \pi^{(-1)}(N)\cap M^\delta\Big) = \pi_*\Big(\sum_{\substack{\delta\in(\mathbb{Z}^n)^*,\\ \delta|_{\pi\times\mathbb{Z}^{n-k}}=\gamma}} \big(\pi^{(-1)}(N)\cap M\big)^\delta\Big).$$

Here the last equation follows from the definition of a truncation of the variety M. Equivalence (2) is the statement of the theorem for a curve $\pi^{(-1)}(N)\cap M$. Equivalence (1) is a corollary of the fact that $\pi_*(A\cap\pi^{(-1)}B) = (\pi_*A)\cap B$ for any varieties $A\subset(\mathbb{C}\setminus 0)^n$ and $B\subset(\mathbb{C}\setminus 0)^{n-k}$. □

5 Leading coefficients of a composite polynomial: Explicit answers for generic Newton polytopes

Definition 19 The *edge coefficients* of a polynomial f are the coefficients of its monomials that correspond to the integer lattice points on the edges of the Newton polytope Δ_f.

We can compute the Newton polytope and the vertex and edge coefficients of a composite polynomial $\pi_{f_0,\dots,f_k}$ explicitly if the Newton polytopes of the polynomials $f_0,\dots,f_k$ satisfy the following condition of general position.

Definition 20 Polytopes $A_0,\dots,A_k$ in $\mathbb{R}^n$ are said to be *developed* if the following condition is satisfied: if the faces $B_0,\dots,B_k$ of the polytopes $A_0,\dots,A_k$ sum to a k-dimensional face of the Minkowski sum $A_0+\dots+A_k$, then B_i is a vertex of A_i for some i.

5.1 Elimination theory for polynomials with developed Newton polytopes

If the Newton polytopes of the polynomials $f_0,\dots,f_k$ are developed, then the explicit computation of the Newton polytope and the vertex and edge coefficients of the composite polynomial $\pi_{f_0,\dots,f_k}$ is based on the following facts:

- The polynomials $f_0,\dots,f_k$ are Newton-nondegenerate, and the assumption of Newton nondegeneracy in Theorems 2.1, 14, 15, and 16 is redundant.
- Theorems 14, 15, and 16 express the vertex and edge coefficients of a composite polynomial of several variables in terms of composite polynomials of one variable.
- Passing to the right-hand side in the formulation of Theorems 14, 15, and 16 preserves the property that the Newton polytopes are developed (see Lemmas 19 and 20 below).
- If $\pi_{f_0,\dots,f_k}$ is a composite polynomial of one variable, then Lemma 1 implies that Khovanskii's product formula (Theorems 18 and 19) and the Gelfond-Khovanskii formula (Theorem 17) can be seen as the respective explicit formulas for the vertex coefficient and the edge coefficients of $\pi_{f_0,\dots,f_k}$.

Lemma 19 1. *In the notation in Theorem 14, if the Newton polytopes of the polynomials $f_0,\dots,f_k$ are developed, then the Newton polytopes of the polynomials $f_0^\delta,\dots,f_k^\delta$ are also developed for every covector δ.*

2. *In the notation in Theorem 15, if the Newton polytopes of the polynomials $f_0,\dots,f_k$ are developed, then the Newton polytopes of the polynomials $g_0,\dots,g_k$ are developed.*

These facts follow from the definitions, and we omit the proof.

But in the notation in Theorem 16, the Newton polytopes of the polynomials $\tilde{f}_0, \dots, \tilde{f}_k$ are usually not developed (regardless of the Newton polytopes of the polynomials $f_0, \dots, f_k$), and we must consider the following (weaker) condition.

Definition 21 Polytopes $A_1, \dots, A_n$ in $\mathbb{R}^n$ are said to be *developed with respect to a point* $b \in \mathbb{R}^n$ if the following condition is satisfied: if the faces $B_1, \dots, B_n$ of the polytopes $A_1, \dots, A_n$ sum to a face of the Minkowski sum $A_1 + \dots + A_n$, then B_i is a vertex of A_i for some i unless $B_1 + \dots + B_n$ contains a segment parallel to the vector $b \in \mathbb{R}^n$.

Lemma 20 *In the notation in Theorem 16, if the Newton polytopes of the polynomials $f_0, \dots, f_k$ are developed, then the Newton polytopes of the polynomials $\tilde{f}_0, \dots, \tilde{f}_k$ are developed with respect to the degree of the monomial t.*

5.2 The Gelfond–Khovanskii formula and Khovanskii's product formula

Definition 22 For a collection of polytopes $A_1, \dots, A_n$ in $\mathbb{R}^n$, let ϕ_i be a nonnegative real-valued function on the boundary $\partial(A_1 + \dots + A_n)$ such that its zero set is the union of all faces of the form $B_1 + \dots + B_n$, where $B_1, \dots, B_n$ are the respective faces of $A_1, \dots, A_n$ and B_i is a vertex. The *combinatorial coefficient* C_a of a vertex $a \in (A_1 + \dots + A_n)$ is the local degree of the map $(\phi_1, \dots, \phi_n)\colon \partial(A_1 + \dots + A_n) \to \partial\mathbb{R}^n_+$ near a if $\phi_1 \cdot \dots \cdot \phi_n = 0$ near a.

In particular, the definition of the combinatorial coefficient makes sense for all vertices of the sum of developed polytopes.

Definition 23 Let $f_1, \dots, f_n, g$ be Laurent polynomials of the variables $x_1, \dots, x_n$, and let their Newton polytopes $A_1, \dots, A_n$ be developed. The *residue* $\operatorname{res}_a \omega_{f.,g}$ of a form $\omega_{f.,g} = g\, dx_1 \wedge \dots \wedge dx_n/(f_1 \cdots f_n x_1 \cdots x_n)$ at a vertex a of the polytope $\sum A_i$ is defined as the constant term of the series

$$g \frac{1}{p(a)} \frac{1}{p/p(a)},$$

where p is the product $f_1 \cdot \dots \cdot f_n$, $p(a)$ is its term of degree a of the polynomial p, and $1/(p/p(a))$ is the inverse of the polynomial $p/p(a)$ near the origin.

Theorem 18 [3] *Let $f_1, \dots, f_n$ be Laurent polynomials on $(\mathbb{C} \setminus 0)^n$, and let their Newton polytopes $A_1, \dots, A_n$ be developed. Then the sum of the values of a Laurent polynomial h over the roots of the system $f_1 = \dots = f_n = 0$ (with multiplicities of the roots taken into account) is equal to $(-1)^n \sum_a C_a \operatorname{res}_a \omega_{f.,h \det \frac{\partial f.}{\partial x.}}$, where a ranges all vertices of the polytope $\sum A_i$.*

Let $\mathbb{Z}_2^{n \times m}$ be the space of $\mathbb{Z}_2$ matrices with n rows and m columns.

Definition 24 There exists a unique nonzero function $\det_2\colon \mathbb{Z}_2^{n \times (n+1)} \to \mathbb{Z}_2$ that is linear and symmetric as a function of columns and vanishes at degenerate matrices. It is called the *2-determinant*.

Definition 25 Let $f_1,\dots,f_n$ be Laurent polynomials on $(\mathbb{C}\setminus 0)^n$, and let their Newton polytopes $A_1,\dots,A_n$ be developed. The *Parshin symbol* $[f_1,\dots,f_n,x^b]_a$ of the monomial x^b at a vertex a of the polytope $\sum A_i$ is the product

$$(-1)^{\det_2(a_1,\dots,a_n,b)} f_1(a_1)^{-\det(b,a_2,\dots,a_n)}\dots f_n(a_n)^{-\det(b,a_1,\dots,a_{n-1})},$$

where $f_i(a)$ is the term of degree a of the polynomial f_i.

Theorem 19 [2] *Let $f_1,\dots,f_n$ be Laurent polynomials on $(\mathbb{C}\setminus 0)^n$, and let their Newton polytopes $A_1,\dots,A_n$ be developed. Then the product of the values of the monomial x^{A_0} over the roots of the system $f_1=\dots=f_n=0$ (with multiplicities of the roots taken into account) is equal to $\prod_a [f_1,\dots,f_n,x^{A_0}]_a^{(-1)^n C_a}$, where a ranges all vertices of the polytope $\sum A_i$.*

In particular, this product is a monomial as a function of the vertex coefficients of the polynomials $f_1,\dots,f_n$, and this theorem can be seen as a multidimensional generalization of the fact that the constant term of a polynomial in one variable is equal to the product of the negatives of its roots.

5.3 Khovanskii's product formula for Newton polytopes developed with respect to a point

Lemma 20 implies that we must generalize Theorem 18 to polytopes developed with respect to a point for it to be applicable in the context of Theorem 16. For a polytope $A\subset\mathbb{R}^n$ and a concave piecewise-linear function $v\colon A\to\mathbb{R}$, let $N(v)$ denote the polyhedron $\{(a,t)\,|\,a\in A,\ t\le v(a)\}\subset\mathbb{R}^n\oplus\mathbb{R}^1$. Let $v_1,\dots,v_n$ be piecewise-linear functions on the polytopes $A_1,\dots,A_n\subset\mathbb{R}^n$. Let Γ denote the union of all bounded faces of the polytope $\sum_i N(v_i)\subset\mathbb{R}^n\oplus\mathbb{R}^1$; Γ is a topological disc. Let $\Gamma_j\subset\partial\Gamma$ be the union of all faces that can be represented as $\sum_i B_i$, where B_i are faces of $N(v_i)$, $i=1,\dots,n$, and B_j is a point. We consider a continuous map $(\phi_1,\dots,\phi_n)\colon\partial\Gamma\to\mathbb{R}^n_+$ such that the zero set of a function ϕ_j is Γ_j.

Definition 26 Functions $v_1,\dots,v_n$ are said to be *developed* if the image of the map $(\phi_1,\dots,\phi_n)$ is contained in the boundary of the positive octant $\mathbb{R}^n_+$. A point $a\in\partial(A_1+\dots+A_n)$ is called a *vertex of the sum $A_1+\dots+A_n$ with respect to the functions v_j* if it is equal to the projection of some vertex $b\subset\partial\Gamma$ of the sum $N(v_1)+\dots+N(v_n)$. In this case $b=b_1+\dots+b_n$, where b_j is a vertex of the polyhedron $N(v_j)$, and we let a_j denote the projection of b_j.

The *combinatorial coefficient* C_a of a vertex a is the local topological degree of the map $(\phi_1,\dots,\phi_n)\colon\Gamma\to\partial\mathbb{R}^n_+$ at the point b. The *Parshin symbol* $[f_1,\dots,f_n,x^k]_a$ of the monomial x^k at this vertex is the product

$$(-1)^{\det_2(a_1,\dots,a_n,k)} f_1(a_1)^{-\det(k,a_2,\dots,a_n)}\dots f_n(a_n)^{-\det(k,a_1,\dots,a_{n-1})},$$

where $f_i(a)$ is the term of degree a of the polynomial f_i.

Theorem 20 *If the polytopes $A_1,\dots,A_n\subset\mathbb{Z}^n$ are developed with respect to A_0, then the function $R_{A_1,\dots,A_n}(x^{A_0};f_1,\dots,f_n)$ (see Definition 17) is equal to the monomial in the vertex coefficients of polynomials $f_1,\dots,f_n$*

$$\prod_{\substack{a\text{ is a vertex of }A_1+\dots+A_n\\ \text{with respect to }v_1,\dots,v_n}} [f_1,\dots,f_n,x^{A_0}]_a^{(-1)^n C_a},$$

where $v_1, \dots, v_n$ are arbitrary developed functions on the polytopes $A_1, \dots, A_n$ such that all vertices of $A_1 + \dots + A_n$ with respect to $v_1, \dots, v_n$ are integer.

Theorem 18 is a special case of this theorem for developed polytopes $A_1, \dots, A_n$ and developed functions $v_i = 0$ on them. The statement in Theorem 19 holds for Newton-degenerate polynomials $f_1, \dots, f_n$, but $R_{A_1,\dots,A_n}(x^{A_0}; f_1, \dots, f_n)$ is not equal to the product of the values of the monomial x^{A_0} over the roots of the system $f_1 = \dots = f_n = 0$ in this case.

Proof The main point in the proof of Theorem 18 (see [2]) is the following fact: if the polynomials $f_1, \dots, f_n$ depend on a parameter $s \in (\mathbb{C} \setminus 0)$ and if their Newton polytopes are developed and are independent of s, then the product of the values of the monomial x^{A_0} over the roots of the system $f_1 = \dots = f_n = 0$ as a function of s is a monomial because it is a rational function of s that has no zeroes and no poles. We can easily verify that the same holds under the assumption that the Newton polytopes of $f_1, \dots, f_n$ are developed with respect to A_0 if we consider the function $R_{A_1,\dots,A_n}(x^{A_0}; f_1, \dots, f_n)$ instead of the product of x^{A_0} over the roots of the system $f_1 = \dots = f_n = 0$. $\square$

6 Other versions of elimination theory in the context of Newton polytopes

In this paper, we have discussed common zeros of Laurent polynomials with the multiplicities of zeros taken into account. Of course, the same theory can be developed in many other contexts. We give some examples.

6.1 Square-free composite polynomials

The square-free standpoint is usual when discussing Newton polytopes of multidimensional resultants. For a finite set $A \in \mathbb{Z}^n$, let $\mathbb{C}[A]$ denote the set of all Laurent polynomials $\sum_{a\in A} c_a x^a$. We consider an epimorphism $\pi\colon (\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{n-k}$ and finite sets $A_0, \dots, A_k$ in the character lattice $\mathbb{Z}^n$ of the complex torus $(\mathbb{C} \setminus 0)^n$.

The composite polynomial $\pi_{f_0,\dots,f_k}$ is not square-free for a collection of polynomials $(f_0, \dots, f_k) \in \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$ if the sets $A_0, \dots, A_k \subset \mathbb{Z}^n$ are degenerate in some sense. Let $\pi^0_{f_0,\dots,f_k}$ be the square-free polynomial that has the same zeros as $\pi_{f_0,\dots,f_k}$. The theorem stated below expresses the *square-free composite polynomial* $\pi^0_{f_0,\dots,f_k}$ in terms of $\pi_{f_0,\dots,f_k}$.

Definition 27 Let $L \subset \mathbb{Z}^n$ be an $(n-k)$-dimensional lattice and $p\colon \mathbb{Z}^n \to \mathbb{Z}^k$ be the projection along L. The *multiplicity* $d(A_0, \dots, A_k, L)$ of the collection of finite sets $A_0, \dots, A_k \subset \mathbb{Z}^n$ with respect to L is defined as follows:

1. if $\dim p(A_{i_1} + \dots + A_{i_q}) < q - 1$ for some numbers $0 \le i_1 < \dots < i_q \le k$, then $d(A_0, \dots, A_k, L) = 0$;
2. otherwise, we choose the minimal nonempty set $\{i_1, \dots, i_q\} \subset \{0, \dots, k\}$ such that $\dim p(A_{i_1} + \dots + A_{i_q}) = q - 1$, choose the minimal sublattice $M \subset \mathbb{Z}^n$ that contains the sum $A_{i_1} + \dots + A_{i_q} + L$ up to a shift, and note that $\operatorname{codim} M = k + 1 - q$. Let r denote the projection $\mathbb{Z}^n \to \mathbb{Z}^{k+1-q}$ along M and $\{j_1, \dots, j_{k+1-q}\}$ denote the set $\{0, \dots, k\} \setminus \{i_1, \dots, i_q\}$. In this notation,

$$d(A_0, \dots, A_k, L) = (k + 1 - q)!\mathrm{MV}(rA_{j_1}, \dots, rA_{j_{k+1-q}}) \cdot |\ker r/M|.$$

For example, let $L \subset \mathbb{Z}^2$ be the horizontal coordinate axis. Then $d(A_1, A_2, L) = 0$ iff both A_1 and A_2 are contained in horizontal segments. If one of them is contained in a horizontal segment, then $d(A_1, A_2, L)$ is equal to the height of the other one. If neither A_1 nor A_2 is contained in a horizontal segment, then $d(A_1, A_2, L)$ is equal to the GCD of the lengths of vertical segments connecting points of the set $A_1 + A_2 + L$.

Theorem 21 *We consider an epimorphism of complex tori $\pi : (\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{n-k}$, the corresponding embedding of their character lattices $L \subset \mathbb{Z}^n$, and finite sets $A_0, \ldots, A_k \subset \mathbb{Z}^n$.*

1. *If $d(A_0, \ldots, A_k, L) = 0$, then $\pi^0_{f_0,\ldots,f_k} = \pi_{f_0,\ldots,f_k} = 1$ for all collections of polynomials $(f_0, \ldots, f_k) \in \mathbb{C}[A_0] \oplus \cdots \oplus \mathbb{C}[A_k]$.*
2. *Otherwise, $(\pi^0_{f_0,\ldots,f_k})^{d(A_0,\ldots,A_k,L)} = \pi_{f_0,\ldots,f_k}$ for all collections of polynomials $f_0, \ldots, f_k$ from some Zariski open subset of the space $\mathbb{C}[A_0] \oplus \cdots \oplus \mathbb{C}[A_k]$.*

We note that this Zariski open subset neither contains nor is contained in the set of all Newton-nondegenerate collections of polynomials. In particular, this theorem implies that the Newton polytope of $\pi^0_{f_0,\ldots,f_k}$ is $d(A_0, \ldots, A_k, L)$ times smaller than the Newton polytope of $\pi_{f_0,\ldots,f_k}$ for a generic collection of polynomials $(f_0, \ldots, f_k) \in \mathbb{C}[A_0] \oplus \cdots \oplus \mathbb{C}[A_k]$.

Proof Part 1 of the theorem is a corollary of Theorem 5.1. Applying Theorem 5.2 and Theorem 15, we can reduce part 2 to the following special case. □

Definition 28 Let $L \subset \mathbb{Z}^n$ be an $(n - k)$-dimensional lattice $L \subset \mathbb{Z}^n$ and $p: \mathbb{Z}^n \to \mathbb{Z}^k$ be the projection along L. A collection of finite sets $A_0, \ldots, A_k \subset \mathbb{Z}^n$ is said to be *essential with respect to $L \subset \mathbb{Z}^n$* if $\dim p(A_{i_1} + \cdots + A_{i_q}) > q - 1$ for every collection of numbers $0 \le i_1 < \cdots < i_q \le k$, $q \le k$, and the sum $A_0 + \cdots + A_k + L$ is not contained in a shifted proper sublattice of $\mathbb{Z}^n$.

We note that $d(A_0, \ldots, A_k, L) = 1$ if the collection $A_0, \ldots, A_k \subset \mathbb{Z}^n$ is essential with respect to L, but the converse is not true. For example, if $L \subset \mathbb{Z}^2$ is the horizontal coordinate axis, then A_1 and A_2 form an essential collection iff neither of them is contained in a horizontal segment and $d(A_1, A_2, L) = 1$.

Lemma 21 *If $A_0, \ldots, A_k$ are essential with respect to L, then $\pi^0_{f_0,\ldots,f_k} = \pi_{f_0,\ldots,f_k}$ for all collections of polynomials $f_0, \ldots, f_k$ from some Zariski open subset of the space $\mathbb{C}[A_0] \oplus \cdots \oplus \mathbb{C}[A_k]$.*

The proof is a straightforward generalization of a similar argument for multidimensional resultants (see Theorem 1.1 in [5], where the notion of essential sets was introduced for $k = n$).

6.2 Composite functions of rational functions

Definition 29 A *vertex* of a virtual polytope $A - B$ is a pair of vertices (a, b) of the polytopes A and B such that $a + b$ is a vertex of the sum $A + B$. The *Newton polytope of a rational function f/g* is the difference of the Newton polytopes of f and g. The *vertex coefficient of a rational function f/g* at the vertex (a, b) of its Newton polytope is the ratio of the vertex coefficients of the polynomials f and g at the respective vertices a and b.

Elimination theory can be readily generalized from Laurent polynomials and convex polytopes to rational functions and virtual polytopes.

6.3 Composite functions of germs of analytic functions

A *convex polyhedron* in $\mathbb{R}^n$ is an intersection of a finite number of half-spaces (which may be unbounded). Two convex polyhedra in $\mathbb{R}^n$ are said to be *parallel* if their support functions have the same domain. For a germ of an analytic function $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ of the variables $x_1,\dots,x_n$, the Newton polyhedron is defined as the minimal polyhedron parallel to the positive octant in the lattice of monomials in $x_1,\dots,x_n$ and containing all monomials of the Taylor expansion of f. Elimination theory can be readily generalized from Laurent polynomials and bounded polyhedra to germs of analytic functions and polyhedra parallel to the positive octant. It requires the following version of Bernstein's theorem.

Definition 30 [11, 12] Let P_C be the set of all pairs of polyhedra (A,B) such that A and B are both parallel to a cone C and the difference $A\triangle B$ is bounded. The notions of the Minkowski sum $(A,B)+(C,D)=(A+C,B+D)$ and volume $\mathrm{Vol}\big((A,B)\big)=\mathrm{Vol}(A\setminus B)-\mathrm{Vol}(B\setminus A)$ for such pairs yield the mixed volume $V_C\colon \underbrace{P_C\times\cdots\times P_C}_{n}\to\mathbb{R}$, which is the polarization of the volume with respect to Minkowski summation.

If the cone C consists of one point $*$, then P_C is the set of pairs of bounded polyhedra, $\mathrm{Vol}\big((A,B)\big)=\mathrm{Vol}(A)-\mathrm{Vol}(B)$, and hence

$$V_*((A_1,B_1),\dots,(A_n,B_n))=\mathrm{MV}(A_1,\dots,A_n)-\mathrm{MV}(B_1,\dots,B_n).$$

If $C\neq\{*\}$, then the mixed volumes in the right-hand side are infinite, but "their difference is well defined."

Lemma 22 [12] *We have*

$$V_C((A_1,B_1),\dots,(A_n,B_n))+V_C((B_1,C_1),\dots,(B_n,C_n))$$
$$=V_C((A_1,C_1),\dots,(A_n,C_n)).$$

Let μ be the unit volume form in $\mathbb{R}^n$, S be the positive octant in $(\mathbb{R}^n)^*$, and $S_0\subset S$ be a set of covectors that contains a unique multiple of each covector in S.

Lemma 23 [12] *We have*

$$V_C((A_1,B_1),\dots,(A_n,B_n))=\frac{1}{n}\sum_{\gamma\in S_0}\sum_{i=1}^{n}(\max\gamma(A_i)-\max\gamma(B_i))$$
$$\times\,\mathrm{MV}_{\mu/\gamma}(A_1^\gamma,\dots,A_{i-1}^\gamma,B_{i+1}^\gamma,\dots,B_n^\gamma).$$

We note that the right-hand side of this formula is not symmetric under permutations of pairs. The sum in the right-hand side makes sense because it contains finitely many nonzero summands (which correspond to normal covectors of bounded $(n-1)$-dimensional faces of the sum $A_1+B_1+\cdots+A_n+B_n$). The $(n-1)$-dimensional mixed volume in the right-hand side makes sense because all arguments are contained in the $(n-1)$-dimensional space $\ker\gamma$.

Definition 31 A polyhedron is called an *M-far stabilization* of a polyhedron $\Delta \subset \mathbb{R}^n_+$ parallel to the positive octant $\mathbb{R}^n_+$ if it can be represented as the convex hull of a union $\Delta \cup \Gamma$ for some polyhedron $\Gamma \subset \mathbb{R}^n_+$ such that the distance between Γ and the origin is greater than M and the difference $\mathbb{R}^n_+ \setminus \Gamma$ is bounded. The *mixed volume of (unbounded) polyhedra* $\Delta_1, \dots, \Delta_n \subset \mathbb{R}^n_+$ parallel to $\mathbb{R}^n_+$ is defined as the mixed volume of pairs $(\mathbb{R}^n_+, \tilde{\Delta}_1), \dots, (\mathbb{R}^n_+, \tilde{\Delta}_n)$, where $\tilde{\Delta}_i$ is an M-far stabilization of Δ_i if the mixed volume of these pairs is independent of the choice of M-far stabilizations for some M (in this case, we say that the mixed volume of $\Delta_1, \dots, \Delta_n$ is *well defined*).

Theorem 22 1. *The mixed volume of polyhedra $\Delta_1, \dots, \Delta_n \subset \mathbb{R}^n_+$ parallel to the positive octant $\mathbb{R}^n_+$ is well defined iff each k-dimensional coordinate plane intersects at least k of these polyhedra.*

2. *If the germs of the functions $f_1, \dots, f_n \colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ have an isolated common root of multiplicity μ, then the mixed volume V of their Newton polyhedra is well defined, and $\mu \geq n!V$.*

3. *If the mixed volume V of the integer polyhedra $\Delta_1, \dots, \Delta_n \subset \mathbb{R}^n_+$ parallel to the positive octant $\mathbb{R}^n_+$ is well defined, then the germs of the analytic functions $f_1, \dots, f_n$: $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ have an isolated common root of multiplicity $n!V$ if their Newton polyhedra are equal to $\Delta_1, \dots, \Delta_n$ and their leading coefficients are in general position in the sense that for any collection of bounded faces $A_1 \subset \Delta_1, \dots, A_n \subset \Delta_n$ such that the sum $A_1 + \cdots + A_n$ is a face of the sum $\Delta_1 + \cdots + \Delta_n$, the Laurent polynomials $f_1|_{A_1}, \dots, f_n|_{A_n}$ have no common zeros in $(\mathbb{C} \setminus 0)^n$.*

Proof Part 1 of the theorem follows from Lemma 22, and parts 2 and 3 follow from part 1 and a local version of Bernstein's formula (see Theorem 3 in [12]). □

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