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ON AFFINE HYPERSURFACES WITH EVERYWHERE NONDEGENERATE SECOND QUADRATIC FORM

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To Victor Vassiliev on the occasion of his 50th birthday

Abstract. An Arnold conjecture claims that a real projective hypersurface with second quadratic form of constant signature (k, l) should separate two projective subspaces of dimension k and l correspondingly. We consider affine versions of the conjecture dealing with hypersurfaces approaching at infinity two shifted halves of a standard cone. We prove that if the halves intersect, then the hypersurface does separate two affine subspaces. In the case of non-intersecting half-cones we construct an example of a surface of negative curvature in $\mathbb{R}^3$ bounding a domain without a line inside.

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1. Introduction

In this paper we prove three results related to Arnold conjectures about (k, l)-hyperbolic projective hypersurfaces formulated in [1].

Definition 1. A smooth hypersurface in $\mathbb{RP}^{n+1}$ is called (k, l)-hyperbolic if at any its point its second quadratic form has signature (k, l). In other words, near each its point in some affine system of coordinates (x, y, z) up to higher order terms the local equation of the hypersurface is $z = \sum_{i=1}^{k} x_i^2 - \sum_{j=1}^{l} y_i^2$.

Quadrics are the simplest examples of such hypersurfaces. Namely, let $Q(x, y)$ be a non-degenerate symmetric bilinear form in $\mathbb{R}^{n+1}$ of signature $(k+1, l+1)$. Then a hypersurface S_Q in $\mathbb{R}P^n$ given by equation $Q(x, x) = 0$ is smooth and (k, l)-hyperbolic.

The hypersurface S_Q has the following remarkable property: one of two domains bounded by S_Q (namely the domain $\{Q(x, x) < 0\}$) contains a l-dimensional projective subspace L_+, and its complement (the domain $\{Q(x, x) > 0\}$) contains a k-dimensional projective subspace L_-.

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Arnold's conjecture claims existence of such pair of subspaces for any $(k,\, l)$-hyperbolic hypersurface. It is the first conjecture from a long list in [1].

Conjecture 1 (Arnold Conjecture)**.** 1. *One of two components of a complement to a connected smooth $(k,\, l)$-hyperbolic hypersurface $B \subset \mathbb{R}P^n$ contains a projective subspace of dimension k, and another component contains a projective subspace of dimension l.*

2. *Any projective line joining these two subspaces intersects B at exactly 2 points.*

A case of $k = 0$ is proved in [1], and served as a motivation of this conjecture. Namely, Arnold proved that a locally convex connected hypersurface in $\mathbb{R}P^n$ doesn't intersects some hyperplane and therefore bounds a convex domain in the complement (which is isomorphic to $\mathbb{R}^n$).

$(k,\, l)$-hyperbolic hypersurfaces appear in [2] under the name of "hypersurfaces of type $(k_+,\, k_-)$". In particular, Gromov shows that domains bounded by such surfaces can be homotopically retracted to l-dimensional polyhedra and proves a version of the Lefshets-type theorem for such domains. Consider the class of domains with the following property: "any point of the complement lies on an k-dimensional affine subspace not intersecting the domain". Gromov mentions that this global property is stronger than the k-convexity requirement (see [2]), and Theorem 3 illustrates how much stronger this requirement is.

For connected hypersurfaces the $(k,\, l)$-hyperbolicity is equivalent to the requirement of non-degeneracy of the Gauss map (plus appropriate signature of the second quadratic form at one point). Therefore properties of $(k,\, l)$-hyperbolic hypersurfaces are closely related to the properties of their Gauss map, as mentioned in [2]. The Lemma 2 and the way the Theorem 1 is deduced from it can be considered as an illustration of this fact.

We prove in [3] the first case $k = l = 1$ of the Arnold Conjecture in some additional assumptions. Namely, we consider a class of *L-convex-concave* subsets of $\mathbb{R}P^3$. This class is smaller than the class of $(1,\, 1)$-hyperbolic surfaces, and is a projective analogue of the class of convex-concave sets considered in Section 4.1. We prove that any L-convex-concave subset of $\mathbb{R}P^3$ contains a line.

In this paper we deal with affine versions of the Arnold's conjecture. Namely, we consider $(k,\, l)$-hyperbolic hypersurfaces in $\mathbb{R}^n$ $(k + l = n - 1)$ with two different asymptotic behavior at infinity. Our results can be roughly summarized as follows: if the asymptotic condition at infinity forces the closure of the hypersurface in $\mathbb{R}P^n$ to be $(k,\, l)$-hyperbolic, then the domain bounded by the hypersurface contains an affine subspace of required dimension. And if the closure is not $(k,\, l)$-hyperbolic at the points of $\mathbb{R}P^n \setminus \mathbb{R}^n$, then this is not necessarily true. In other words, "the "compact origins" of the hypersurface", as Gromov puts it in [2], seems to force some geometrical properties of the domain bounded by it.

Here is more exact description of the results. Consider a $(k,\, l)$-hyperbolic hypersurface M in $\mathbb{R}^n$, $k + l = n - 1$. We say that M approaches a hypersurface L at infinity if M and L are arbitrarily C^1-close outside a big enough ball (see Section 2.1 for an exact definition). For example, the quadric $\left\{\sum_{i=0}^k x_i^2 - \sum_{j=1}^l x_{k+j}^2\right\} = A$ approaches at infinity the cone $K = \left\{\sum_{i=0}^k x_i^2 = \sum_{j=1}^l x_{k+j}^2\right\} \subset \mathbb{R}^n$.

We prove the following theorem:

Theorem 1. *The first claim of the Arnold conjecture is true for any (k, l)-hyperbolic closed connected hypersurface M approaching at infinity the quadratic cone K.*

We also prove the following result, supporting the second part of the Arnold Conjecture.

Theorem 2. *Any $(1, 1)$-hyperbolic closed connected surface M in $\mathbb{R}^3$ approaching at infinity the quadratic cone $K = \{x^2 + y^2 = z^2\}$ intersects any line passing through the origin at most two point. More exact, the central projection $M \to \mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$ is embedding.*

Note that the hypersurface described in Theorem 1 has a C^1-closure in $\mathbb{R}P^n$. Consider the simplest case $k = l = 1$, and denote by $K_- = K \cap \{z \leqslant 0\}$ and $K_+ = K \cap \{z \geqslant 0\}$ two halves of the quadratic cone. Will the result remain true if we consider surface M approaching union of translated K_- and K_+? There are two different cases: the translates can be disjoint or have a nonempty intersection. In both cases the projective closure $\overline{M}$ of the surface is not smooth at the points of $\overline{M} \setminus M$. However, in the first case the $\overline{B}$ can be made $(1, 1)$-hyperbolic after an arbitrarily small perturbation, and in the second case it is impossible.

It turns out that if the translates intersect, then the domain bounded by the surface M still contains a line (this essentially follows from the proof of Theorem 1), and if the intersection is empty, then this is not necessarily true:

Theorem 3. *Let $K' = \{(x, y, z)\colon x^2 + y^2 = (|z| - 1)^2,\ |z| \geqslant 1\}$ be a union of non-intersecting translates of K_- and K_+. There exists a $(1, 1)$-hyperbolic surface M approaching K' at infinity and bounding a domain not containing lines.*

This coincides with what the Arnold Conjecture predicts, further strengthening it.

All these result can be considered in more general context of existence of a solution of some boundary problem. A natural boundary problem is to find a compact smooth $(1, 1)$-hyperbolic affine surface bounded by a given set of non-intersecting closed smooth curves and with prescribed tangent planes at the boundary. Theorem 1 follows from the fact (this is how it is proved in the paper) that solution of some boundary problem of this type cannot intersect some open domains (the interior of the cone K for Theorem 1).

1.1. Gauss map and Quadrics. The proof of Theorem 1 uses properties of a Gauss image of the hypersurface M. Recall the definition of the Gauss map. For a smooth cooriented hypersurface $M \subset \mathbb{R}^n$ the Gauss map $G\colon M \to \mathbb{S}^{n-1}$ maps a point $x \in M$ to the vector normal to M at x. If $M = \{P = 0\}$ and $dP \neq 0$ on M, then the Gauss image of a point $x \in M$ is a normalized gradient $\nabla f(x)$. A classical computation shows that the Jacobian of the Gauss map is exactly the Gaussian curvature of the hypersurface, and is therefore non-zero for (k, l)-hyperbolic hypersurfaces.

A useful variant of the Gauss map for projective hypersurfaces maps a point $x \in M$ of the hypersurface to the intersection of its tangent plane T_xM with some fixed hyperplane. This intersection can be considered as a point in a projective space dual to this fixed hyperplane.

The hypersurface M of Theorem 1 is approaching at infinity a cone given by a non-degenerate quadratic form. We will need some standard facts about non-singular quadrics given below.

Proposition 1. *Let* $Q(x)=\sum_{i=0}^{k}x_i^2-\sum_{j=1}^{l}x_{k+j}^2$, *and let* $Q_\varepsilon=\{f=\varepsilon\}$ *be its level hypersurfaces. Denote by* $\mathbb{S}^{n-1}$ *the standard sphere* $\{\sum x_i^2=1\}\subset\mathbb{R}^n$, $k+l=n-1$.

1. *Projectivization of the cone hypersurface* Q_0 *is a* $(k,\,l-1)$*-hyperbolic hypersurface in* $\mathbb{R}P^{n-1}$;
2. *the hypersurface* Q_ε *is* $(k+1,\,l-1)$*-hyperbolic if* $\varepsilon<0$ *and* $(k,\,l)$*-hyperbolic if* $\varepsilon>0$;
3. *The Gauss map provides diffeomorphisms between* $\{Q=\varepsilon>0\}$ *and* $\mathbb{S}^n\cap\{Q>0\}$, *between* $\{Q=\varepsilon<0\}$ *and* $\mathbb{S}^n\cap\{Q<0\}$, *and maps* $\mathbb{S}^n\cap\{Q=0\}$ *diffeomorphically onto itself*;
4. $\{Q=\varepsilon>0\}$ *is diffeomorphic to* $\mathbb{S}^k\times B^l$, $\{Q=-\varepsilon<0\}$ *is diffeomorphic to* $\mathbb{S}^{l-1}\times B^{k+1}$ *and* $\mathbb{S}^n\cap\{Q=0\}$ *is diffeomorphic to* $\mathbb{S}^k\times\mathbb{S}^{l-1}$.

2. Theorem 1: Hypersurface Doesn't Intersect the Cone

2.1. Definitions and plan of the proof. In this section we prove Theorem 1. We start with a definition of an affine version of $(k,\,l)$-hyperbolicity and state more precisely the asymptotic conditions on the hypersurface M.

Definition 2. A smooth connected closed hypersurface M lying in $\mathbb{R}^n$ equipped with a standard Euclidean metric, is called $(k,\,l)$-hyperbolic if its second quadratic form is everywhere nondegenerate and have constant signature $(k,\,l)$.

Denote by B_R a ball of radius R with center at the origin, $B_R=\{\|x\|\leqslant R\}\subset\mathbb{R}^n$, and by $\mathbb{S}_R^{n-1}$ its boundary (i. e. a sphere of radius R with center at the origin).

Definition 3. We say that a hypersurface M approaches a hypersurface L at infinity if for any $\epsilon>0$ there exist $R'>R>0$ such that

1. there exists a diffeomorphism $\phi: L\setminus B_R\to M\setminus B_R$, such that $\|\phi(x)-x\|<\epsilon$ for any $x\in L\setminus B_{R'}$ and
2. there is a diffeomorphism ψ of the Gauss images of $L\cap\mathbb{S}_{R'}^{n-1}$ onto the Gauss image of $\phi(M\cap\mathbb{S}_{R'}^{n-1})\subset L$ such that $\operatorname{dist}(\psi(x),\,x)<\epsilon$ in standard metric on $\mathbb{S}^{n-1}$.

The proof of Theorem 1 goes as follows. First, using topological arguments, we prove, see Lemma 2 below, that the Gauss image of the hypersurface M does not intersect the Gauss image of any quadric $\{Q=-\varepsilon<0\}$.

Second, we note that the interior $U=\{Q<0\}$ of the cone K is a disjoint union of quadrics Q_ε. If $M\cap U$ is nonempty and bounded, then the level surface $\{Q=m=\min_{x\in M}Q(x)\}\subset U$ exists and is tangent to M at the point of the minimum of $Q|_M$. This contradicts to the fact that the Gauss images of M and $\{Q=m<0\}$ do not intersect. The assumption of boundedness can be relaxed to the asymptotic conditions above by a slight modification of these arguments.

2.2. The topological lemma. The proof of Theorem 1 starts from a simple topological lemma. This lemma will be applied later to the Gauss map of (k, l)-hyperbolic hypersurfaces.

Lemma 1. *Let M be a compact connected manifold with boundary ∂M and let $f\colon M \to N$ be a local diffeomorphism to a compact simply-connected manifold N, $\pi_1(N) = 0$. Suppose that the restriction $f|_{\partial M}$ of f to each connected component of the boundary of M is in addition an embedding. Then f is a diffeomorphism of M to $f(M)$.*

Proof. An easy case is $\partial M = \varnothing$. In this case f is a covering, so should be a trivial one (since $\pi_1(N)$, being trivial, has no nontrivial subgroups).

The general case will be reduced to this case by gluing "hats" to M, thus eliminating the boundary components one-by-one.

Namely, consider a connected component of ∂M (denote it by B). Its image $f(B)$ is a cooriented hypersurface in N. Indeed, $f(B)$ divides any sufficiently small neighborhood of any its point $f(b)$ into two parts, and one can choose in a canonical way one of them, namely an image of a small neighborhood of b in M. Therefore $N \setminus f(B)$ consists of two open parts, consisting of points having even and odd number of preimages under the mapping f correspondingly. Call the part which doesn't intersect the image of a sufficiently small neighborhood of B by "hat". We can glue the "hat" to M along $B \cong f(B)$: the resulting new manifold is a union of B and the "hat" with the neighborhood of $b \in B$ being defined as union of the connected component of $f^{-1}(U)$ containing b and the intersection of U and the "hat" (where $U \subset N$ is a sufficiently small open ball containing $f(b)$).

Repeating this operation with all components of the boundary, we get a new manifold $\widetilde{M}$ without a boundary and $M \subset \widetilde{M}$. The map f extends to a map $\widetilde{f}\colon \widetilde{M} \to N$ by an identity on "hats". The map $\widetilde{f}$ satisfies conditions of Lemma 1, so it is a global diffeomorphism by the first part of the proof. So f is also a diffeomorphism, since $f = \widetilde{f}|_M$. □

Corollary 1. *Let M be a compact connected oriented $(n-1)$-dimensional submanifold with boundary of $\mathbb{R}^n$, and suppose that its second fundamental form is everywhere nondegenerate, including the boundary. Assume that the restriction of the Gauss map of M to each connected component of the boundary ∂M is one-to-one. Then the Gauss map of M is one-to-one.*

2.3. Gauss image of (k, l)-hyperbolic hypersurface. Let, as before, $Q(x) = \sum_{i=0}^{k} x_i^2 - \sum_{j=1}^{l} x_{k+j}^2$ be a quadratic form on $\mathbb{R}^n$, $k + l = n - 1$. Consider a (k, l)-hyperbolic connected closed hypersurface $M \subset \mathbb{R}^n$ approaching the quadratic cone $K = \{Q = 0\}$ at infinity. We prove that the Gauss image of M coincides with the Gauss image of a (k, l)-hyperbolic level hypersurface of Q, thus ending the first part of the proof of Theorem 1.

Lemma 2. *Let M be a (k, l)-hyperbolic closed connected hypersurface approaching K at infinity. Then Gauss map of M is a diffeomorphism, and its image coincides with the Gauss image of a (k, l)-hyperbolic level hypersurface of Q.*

Remark 1. We will consider later a surface $M \subset \mathbb{R}^3$ approaching at infinity a modified cone $K' = \{(|z|-1)^2 = x^2+y^2\} \subset \mathbb{R}^3$ (i.e. $k-1=l=1$). Lemma 2 holds for this case as well: the Gauss image of such surface coincides with the Gauss image of the standard quadric $\{z^2+1 = x^2+y^2\}$.

Proof. The Jacobian of the Gauss map is equal to the Gaussian curvature, i.e. is non-vanishing. Therefore the Gauss map is a local diffeomorphism. We have to prove that, first, it is a global diffeomorphism onto one of the parts into which the Gauss image $G(K)$ of the cone K divides $\mathbb{S}^{n-1}$, and, second, that this part is the Gauss image of a $(k,\, l)$-hyperbolic level surface of Q. The first claim almost follows from Corollary 1, except that the M is not a compact manifold with boundary. The second claim follows from the topological type of M.

Here is a proof of the first claim. Consider the compact $M_R = M \cap B_R$, where $B_R \subset \mathbb{R}^n$ is a closed ball of the radius R with center at the origin, and denote by ∂M_R the boundary of M_R. As $R \to \infty$, its Gauss image $G(\partial M_R)$ C^1-converges to the Gauss image of $K \cap \partial B_R$, which coincides with the Gauss image of the cone K.

Lemma 3. *$G(M_R) \cap G(\partial M_{R'}) = \varnothing$ for any R and any $R' > R$.*

For big enough R' this follows immediately from Lemma 1 applied to $M_{R'}$ and its Gauss map. Lemma 1 is applicable since $M_{R'}$ is compact and the restriction of the Gauss map to its boundary is a diffeomorphism, due to the condition "approaching at infinity". Therefore this is true for any R' and R. □

Corollary 2. *The Gauss image of M_R doesn't intersect the $G(K)$.*

Indeed, if the intersection is non-empty, then there is a point $s \in G(K)$ which lies in the interior of $G(M_{R'})$, where R' is any number greater than R. Therefore m cannot be a limit point of $G(\partial M_R)$ as $R \to \infty$. This contradicts to the condition that M approaches K at infinity. □

Therefore the connected set $G(M_R)$ should lie entirely in one of the connected components into which the $G(K)$ divides the sphere $\mathbb{S}^{n-1}$. Since $\partial G(M_R)$ converge uniformly to $G(K)$, we conclude that $G(M)$ is exactly one of the connected components and the Gauss mapping is a diffeomorphism — the first statement of Lemma 2 is proved.

The second claim of Lemma 2 is that the $G(M)$ falls into the right connected component. If $k+1=l$, then all nonsingular level hypersurfaces of Q are $(k,\, l)$-hyperbolic (since $(k,\, l)$-hyperbolicity and $(k+1,\, l-1)$-hyperbolicity are then the same), and there is nothing to prove. So we suppose that $k+1 \neq l$. Then the Gauss image of K divides the sphere $\mathbb{S}^{n-1}$ into two topologically different domains: one is the Gauss image of a $(k,\, l)$-hyperbolic level surface and is diffeomorphic to $\mathbb{S}^k \times B^l$, and another is the Gauss image of a $(k+1,\, l-1)$-hyperbolic level hypersurface and is diffeomorphic to $\mathbb{S}^{l-1} \times B^{k+1}$. Denote these domains by D_+ and D_- correspondingly, so that $D_+ = G(\{Q=+1\})$ and $D_- = G(\{Q=-1\})$. Since the Gauss mapping of M is a diffeomorphism, we get that M is diffeomorphic to D_+ or D_-, and our goal is to exclude the last possibility.

We will prove that M is topologically different from D_- applying the Morse theory to the restriction $f = x_n|_M$ of the linear functional x_n to M. It turns out that the type of critical points of f dictated by (k, l)-hyperbolicity condition is incompatible with the assumption that M is diffeomorphic to D_-.

So suppose that $G(M) = D_-$. Since $\nabla(x_n) = e_n = (0, 0, \dots, 0, 1) \in D_-$, the function f has exactly two nondegenerate critical points on M, namely the preimages of $-e_n$ and e_n under the Gauss map of M (which is a diffeomorphism). The condition of (k, l)-hyperbolicity means that both these critical points are nondegenerate and their Morse indices are equal to k or l. Take $R \gg 1$ and let $\widetilde{M} = M \cap \{x_n < -R\}$. The main result of the Morse theory implies that the dimension of the group of relative homologies $H_i(M, \widetilde{M})$ is less than the number of critical points of index i of the function f. The pair $(M, \widetilde{M})$ is diffeomorphic to the pair $(Q_{-1}, \widetilde{Q}_{-1})$ by their respective Gauss maps, where $Q_{-1} = \{Q = -1\}$, and $\widetilde{Q}_{-1} = Q_{-1} \cap \{x_n < -R\}$, so $H_i(M, \widetilde{M}) = H_i(Q_{-1}, \widetilde{Q}_{-1})$. The latter can be easily computed to be equal to 1 for $i = k+1$ (since $Q_{-1}/\widetilde{Q}_{-1} \cong \mathbb{S}^{l-1} \vee \mathbb{S}^k$), and this contradicts to the fact that f has no critical points of index $k+1$ (recall that $k + 1 \neq l$). □

2.4. Rolle Lemma. Let $Q(x) = x_0^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_{k+l}^2$ be a quadratic form in $\mathbb{R}^n$, $n - 1 = k + l$. The cone $K = \{Q = 0\}$ divides $\mathbb{R}^n$ into two parts, $\{Q > 0\}$ and $\{Q < 0\}$. We prove in this paragraph that M does not intersect one of these domains.

Theorem 4. *Let M be a smooth connected hypersurface M such that* $\operatorname{dist}(x, K) \to 0$ *as $M \ni x \to \infty$. Suppose that the Gauss image of M is disjoint from the Gauss image of $\{Q = -1\}$. Then M does not intersect the whole domain $\{Q < 0\}$.*

The proof is a specialization of the following general lemma.

Lemma 4. *Let $M \subset \mathbb{R}^n$ be a smooth closed embedded hypersurface and suppose that its image under the Gauss map $G_M : M \to \mathbb{S}^{n-1}$ does not intersect a symmetric with respect to the antipodal map $x \to -x$ domain $U \subset \mathbb{S}^{n-1}$.*

Suppose that on $\mathbb{R}^n$ we are given a function f with nonnegative only critical values, and that $\frac{\nabla f(x)}{\|\nabla f(x)\|} \in U$ for $f(x) < 0$. Suppose that $\liminf f(x) \geqslant 0$ *as $M \ni x \to \infty$. Then f is nonnegative on M.*

Proof. Suppose that f is negative somewhere on M. Let $x_0 \in M$ the point of minimum of the restriction of f to M. It exists since $M \cap \{f \leqslant \varepsilon < 0\}$ is compact and nonempty for some ε. The point x_0 is a critical point of the restriction of f to M. Therefore the normal vector to M at x_0 is proportional to the nonzero vector $\nabla f(x_0)$, which means that $G_M(x_0) \in U$ — a contradiction. □

This lemma implies Theorem 4. The first candidate for the function f is the Q itself: Gauss images of $\{Q = t < 0\}$ are all equal and do not intersect the Gauss image of M. However, Q itself does not necessarily satisfy the last condition of Lemma 4: one should slightly adjust Q to ensure the last condition of the lemma.

Denote $\sqrt{x_0^2 + \dots + x_k^2}$ by a and $\sqrt{x_{k+1}^2 + \dots + x_{k+l}^2}$ by b, so that $Q(x) = a^2 - b^2$. Suppose that M intersects a domain $\{Q < -\varepsilon < 0\}$. Consider the function $f_1 = \sqrt{a^2 + \varepsilon} - b$, and denote by f its smoothing: the f_1 is not smooth at $b = 0$, but one can smoothen f_1 without changing it on $\{f_1 < 0\} = \{Q < -\varepsilon\}$, the only domain of interest. Up to a sign, $f(x)$ is equal to a distance between a point $x \in \{Q < -\varepsilon\}$ and the closest to it point $y \in \{Q = -\varepsilon\}$ with the same coordinates $x_0, \dots, x_k$.

One can show that f satisfies both conditions of Lemma 4. First, $\nabla f(x)$ is proportional to $\nabla Q(y)$ in $\{f < 0\}$, so the Gauss images of $\{f = t < 0\}$ and $\{Q = -\varepsilon\}$ coincide for any $t < 0$. Second, negative level sets of f are on positive distance from the cone K, so $\liminf f(x) \geqslant 0$ as $M \ni x \to \infty$.

Therefore, by Lemma 4, $M \cap \{f < 0\} = M \cap \{Q < -\varepsilon\} = \varnothing$, a contradiction. □

2.5. End of the proof of Theorem 1. The rest of the proof of Theorem 1 is a combination of Lemma 2 and Theorem 4.

First, let prove existence of an l-dimensional subspace in one of the domains into which M divides $\mathbb{R}^n$. Suppose first that $k + 1 \neq l$. In this case the quadrics $Q_1 = \{Q = 1\}$ and $Q_{-1} = \{Q = -1\}$ have different signatures of the second quadratic forms: the first one is (k, l)-hyperbolic, and the second is $(k + 1, l - 1)$-hyperbolic. By Lemma 2 the Gauss image of the (k, l)-hyperbolic hypersurface M coincide with the Gauss image of Q_1, and is therefore disjoint from the Gauss image of Q_{-1}. So, by Theorem 4, M does not intersect the domain $\{Q < 0\}$, which contains the l-dimensional subspace $\{x_0 = \cdots = x_k = 0\}$.

If $k + 1 = l$, then both Q_1 and Q_{-1} are (k, l)-hyperbolic, and Lemma 2 claims that the Gauss image of M coincides with the Gauss image of one of them. Taking $-Q$ instead of Q if necessary, we can assume that $G(M)$ coincides with $G(Q_1)$, and proceed as above.

The existence of a k-dimensional affine subspace in the second part of $\mathbb{R}^n \setminus M$ is evident. Since M approaches the cone K at infinity, the distance between $M \setminus B_R$ and $K \setminus B_R$ is less than distance between $K \setminus B_R$ and $L = \{x_{k+1} = \cdots = x_n = 0\}$ for big enough ball B_R. So any k-dimensional affine subspace of L lying outside B_R will not intersect M.

3. Projection from the Origin

Starting from this moment we deal with (1, 1)-hyperbolic surfaces in $\mathbb{R}^3$ only. So we will omit the (1, 1) and will call (1, 1)-hyperbolic surfaces hyberbolic surfaces.

Theorem 1 ensures that a hyperbolic surface approaching the cone $K = \{x^2 + y^2 = z^2\}$ at infinity do not intersect any line passing through the origin and lying in the domain $\{x^2 + y^2 < z^2\}$. We prove here that any ray emanating from the origin intersects M at at most one point.

Theorem 5. *Let $M \subset \mathbb{R}^3$ be a hyperbolic surface approaching the standard cone K at infinity.*

Then the restriction to M of the projection $\pi\colon \mathbb{R}^3 \to \mathbb{S}^2 = \{\|x\| = 1\}$ is embedding.

3.1. Arnold's formula. Theorem 5 follows from a remarkable formula due to Arnold, see [1]. Consider a smooth surface $M \subset \mathbb{R}P^3$. Denote by $\#\{M \cap \ell\}$ number of its points of intersections with a line ℓ and by $\operatorname{sign}(M, \ell)$ the number of point $x \in M$ containing the generic line ℓ in their tangent planes counted with multiplicities. The multiplicity is equal to "+1" if the Gaussian curvature of M at

x is positive and to "-1" if it is negative (if the curvature at x is zero then the formula for multiplicity is more complicated).

Lemma 5 (Arnold, 88)**.** *For any smooth hypersurface $M \subset \mathbb{R}P^3$ and for a generic line ℓ the sum $\#\{M \cap \ell\} + \operatorname{sign}(M, \ell)$ is equal to the Euler characteristic of M.*

Sketch of the proof for a semialgebraic M (due to O. Viro). Take out from M its points of intersection with ℓ and compute the Euler characteristic of the result using Fubini theorem for Euler characteristic. Namely, the Euler characteristic of $M \setminus \ell$ is equal to the integral over the space of all planes L_t containing ℓ, $t \in \mathbb{R}P^1$, of the Euler characteristic of the intersections $M_t = \{M \setminus \ell\} \cap L_t$. For simplicity, suppose that each section M_t has at most one singular point (if not, perturb ℓ slightly). Each nonsingular section M_t is a one-dimensional manifold, so is a union of circles and open intervals with ends at removed points. Therefore its Euler characteristic is equal to $-\#\{M \cap \ell\}$ (since Euler characteristic of a circle is equal to zero). Euler characteristics of singular sections differ from this number by $+1$ or by -1, depending on the sign of the curvature of M at the singular point lying on this section. Indeed, if the curvature at the singular point is negative, then the section has a self-intersection, so the Euler characteristic drops by 1. If the curvature at the singular point is positive, then the section has an isolated point, and Euler characteristic increases by 1.

Since the Euler characteristic of $\mathbb{R}P^1$ is equal to zero, the integration of $-\#\{M \cap \ell\}$ over $\mathbb{R}P^1$ gives zero. So the Euler characteristic of $M \setminus \ell$, being equal to the integral of the Euler characteristic of M_t over $\mathbb{R}P^1$, is equal to $\operatorname{sign}(M, \ell)$, and the result follows. □

3.2. Compactification of M and end of the proof of Theorem 2. We apply Lemma 5 to the closure of M in $\mathbb{R}P^3$. First, we have to show that the closure of M in $\mathbb{R}P^3$ is a smooth surface.

Lemma 6. *The closure $\overline{M}$ of M in $\mathbb{R}P^3$ is smooth.*

Proof. Take affine coordinates $\tilde{x} = \frac{x}{z}$, $\tilde{y} = \frac{y}{z}$, $\tilde{w} = \frac{1}{z}$. We are interested in the points of $\overline{M} \cap \{\tilde{w} = 0\}$. The first part of the condition "M approaches K at infinity" implies that $\overline{M}$ approaches $\{\tilde{x}^2 + \tilde{y}^2 = 1\}$ faster than $|\tilde{w}|$, so $\overline{M}$ is smooth at these points. The second part means that as $x \in M$ tends to $x_0 \in \overline{M} \cap \{\tilde{w} = 0\}$ the limit of tangent planes $T_x M$ exists and is equal to the tangent plane at x_0. This means C^1-smoothness of $\overline{M}$. □

For the hyperbolic surface M the curvature is always negative. Therefore the sign in Lemma 5 is always "$-$". By Lemma 2 the Euler characteristic of $\overline{M}$ is equal to the Euler characteristic of $\{x^2 + y^2 = z^2 + w^2\}$, i. e. is equal to zero. So Lemma 5 claims in this case that for generic ℓ

$$\#\{M \cap \ell\} = \#\{x \in M \colon \ell \subset T_x M\}. \tag{3.1}$$

We want to prove first that projection of M to $\mathbb{S}^2$ is a local diffeomorphism. In other words, we have to show that tangent plane to M cannot contain the vertex O of the cone. Suppose otherwise, and take a plane tangent to M and passing through O. The normal to this plane lies in the Gauss image of M, i. e. in

$\mathbb{S}^2 \cap \{x^2+y^2-z^2>0\}$. Equivalently, this plane intersects the domain $\{x^2+y^2-z^2<0\}$. Therefore this plane, which passes through the vertex of the cone, contains a line $\ell \subset \{x^2+y^2<z^2\} \cup \{0\} \subset K$. By Lemma 4 $\ell \cap \overline{M} = \varnothing$. Therefore, by compactness of $\overline{M}$, it is true for all lines close enough to ℓ. Moreover, if ℓ is contained in a plane T_xM, then, due to the nonzero curvature of M at x, any line close enough to ℓ is also contained in some tangent plane to M. So for a generic line sufficiently close to ℓ the left hand side of (3.1) is equal to zero, while the right hand side is at least one, a contradiction.

Theorem 5 now follows from Lemma 1 applied to the restriction of the projection to $M_R = M \cap B_R$ — intersection of M with a big enough ball B_R. Indeed, we just proved that the projection is a local diffeomorphism. Also, the restriction of the projection to the boundary of M_R is diffeomorphism since the boundary of M_R is C^1-close to $K \cap \{x^2+y^2+z^2=R^2\}$, so is embedded by projection. □

4. Example

In this section we construct an example of a domain in $\mathbb{R}^3$ not containing a line and bounded by a hyperbolic closed connected surface without boundary. This surface has an asymptotic behavior similar to considered above (namely it approaches the pseudo-cone K' at infinity), but its closure in $\mathbb{R}P^3$ is not everywhere hyperbolic (even after smoothing).

Construction starts by a definition of an affine convex-concave set. Consider a hyperbolic surface bounding some domain in $\mathbb{R}^3$. At each point it has a direction of positive sectional curvature and an orthogonal direction of negative sectional curvature. The affine convex-concave sets come from a requirement that these directions should not be far from a vertical (= parallel to z-axis) and horizontal (= perpendicular to z-axis) respectively. More exact, we want the horizontal sectional curvature to be always negative, and boundary of every horizontal projection to be locally convex. The first requirement implies that the horizontal sections of a domain bounded by the hyperbolic surface are convex, and the second one implies a concave dependence of these sections on the plane of the section. We introduce the affine convex-concave sets as sets satisfying these two last properties, i. e. using only the notion of convexity. This class is an affine version of the class of L-convex-concave subsets of $\mathbb{R}P^n$ defined in [4], and is similarly closed under surgeries considered there. Analogues of the first part of the Arnold conjecture can be formulated for both L-convex-concave sets and affine convex-concave sets. We prove this analogue to be true for the first nontrivial case of L-convex-concave subsets of $\mathbb{R}P^3$, see [3].

The first step of constructions of this section is a construction of a counterexample to the analogue of Arnold conjecture for convex-concave subsets of $\mathbb{R}^3$. This counterexample is a so-called strip — a piece of a two-dimensional surface which is at the same time a convex-concave set. A strip is not a domain bounded by a hyperbolic surface, but belongs to a compactification of the class of convex-concave domains bounded by hyperbolic surfaces. Absence of lines inside is an open condition in the class of convex-concave sets, so any convex-concave body close enough to the strip also does not contains a line inside.

The second step consists of a small perturbation of a set E — a union of the cone K' with the strip — resulting in a convex-concave domain bounded by a hyperbolic surface. Namely, the class of convex-concave sets is closed under taking the section-wise addition, or, more general, integration by Minkowsky. A suitable averaging of a two-parametric family of convex-concave set obtained from the set E by translations and rotations around z-axis gives the required perturbation of the set E.

4.1. Affine convex-concave sets. We will call *horizontal* any object parallel to the coordinate (x, y)-plane in $\mathbb{R}^3$, e. g. planes $\{z = c\}$, directions $(a, b, 0)$ etc.

Definition 4. We say that a set $A \subset \mathbb{R}^3$ is convex-concave if the following two conditions are satisfied:

(1) its horizontal sections $S_t = A \cap \{z = t\}$ are nonempty, convex and compact.
(2) S_t depend concavely on t.

The second condition means that for any $t_1 < t_2 < t_3$ the section S_{t_2} is contained inside a three-dimensional convex hull of the union $S_{t_1} \cup S_{t_3}$. If we denote the projections of S_{t_i} along the z-axis by $\hat{S}_{t_i}$, then (2) is equivalent to the condition that $\hat{S}_{t_2}$ lies inside the linear (in Minkowski sense) combination $\frac{t_2-t_1}{t_3-t_1}\hat{S}_{t_3} + \frac{t_3-t_2}{t_3-t_1}\hat{S}_{t_1}$.

Condition (2) can be reformulated in several equivalent ways. The first equivalent formulation is

(2′) for any $t_1 < t_2 < t_3$ any point of the section S_{t_2} lies on a line intersecting both S_{t_1} and S_{t_3}.

Another equivalent reformulation is the condition that the boundary of any horizontal projection of A consists of two convex curves.

(2″) For any projection π along any horizontal direction the boundary of the projection $\pi(A) = \{-\phi_1(z) \leqslant w \leqslant \phi_2(z)\}$ is defined by two convex functions $\phi_1(z)$ and $\phi_2(z)$ of the coordinate z.

Here "f is a convex function" means that $\frac{f(x+a)-f(x)}{a} - \frac{f(x)-f(x-b)}{b} \geqslant 0$ for all $a, b > 0$. For C^2-smooth functions this is equivalent to "$f''(z) \geqslant 0$ for all $z \in \mathbb{R}$". More general, a continuous function $f(z)$ is convex if and only if $\int f g'' dz \geqslant 0$ for any smooth nonnegative function g with compact support.

4.1.1. *Support function and another reformulation of the second condition.* To any compact affine convex set $S \subset \mathbb{R}^n$ corresponds its support function $F_S(\ell) = \max_{x \in S} \ell(x)$ defined on the dual space $(\mathbb{R}^n)^*$. This function is $\mathbb{R}_+$-homogeneous of degree 1, and satisfies $F_S(\alpha\ell_1 + \beta\ell_2) \leqslant \alpha F_S(\ell_1) + \beta F_S(\ell_2))$ for any positive α and β.

Conversely, for any $F\colon (\mathbb{R}^n)^* \to \mathbb{R}$ satisfying these two conditions one can construct a convex compact figure $S = \bigcap_{\ell \in (\mathbb{R}^n)^*} \{\ell(x) \leqslant F(\ell)\}$. Here is another description of S for smooth F. The gradient ∇F is $\mathbb{R}_+$-homogeneous of degree 0. In other words, ∇F is a composition of the radial projection $\mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ and a mapping of $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$. One can check that the image $F(\mathbb{S}^{n-1})$ bounds a convex domain S, and $F_S = F$. So, the correspondence $S \leftrightarrow F_S$ is a bijection.

This correspondence preserves the semigroup structure of the class of convex sets: if we define the Minkowski sum of two sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ as $A + B = \{a + b\colon a \in A,\, b \in B\}$, then $F_{A+B} = F_A + F_B$.

The functions defining the boundaries of projection in Definition 4, version (2″), are the values of the support function of S_t. Namely, if the projection is defined by $\pi\colon (x,\, y,\, z) \to (\ell(x,\, y),\, z)$, $\ell \in (\mathbb{R}^2)^*$, then $\phi_1(z) = \max_{(x,y,z)\in S_z}[-\ell(x,\, y)]$ and $\phi_2(z) = \max_{(x,y,z)\in S_z} \ell(x,\, y)$. The second condition of Definition 4 means that the support functions of $S_t(\ell)$ depend concavely on t for any fixed $\ell \in (\mathbb{R}^2)^*$.

4.2. Strips. First we construct a convex-concave set all horizontal sections of which are segments.

Definition 5. A *strip* is a surface with boundary defined parametrically as

$$S = \{(x,\, y,\, z) \in \mathbb{R}^3\colon\ x = u_1(z) + tf_1(z),\ \ y = u_2(z) + tf_2(z),\ \ \|t\| \leqslant 1,\ \ z \in \mathbb{R}\}$$

which is also a convex-concave set.

Horizontal sections of the strip are just segments, so the strip can be defined using two curves, one formed by the middle-points of the segments $M = (u_1(z),\, u_2(z),\, z)$ and another formed by the ends of the segments, $M_1 = (u_1(z) + f_1(z),\, u_2(z) + f_2(z),\, z)$.

Example. A *degenerate* strip corresponds to linear f_1, f_2 and $u_i \equiv 0$. The degenerate strip is a piece of a quadric which contains two one-parametric families of lines: a family of horizontal lines and another including two lines bounding the degenerate strip. Any point of the degenerate strip lies on a line intersecting all sections and convex-concavity follows by Definition 4, version (2′).

It turns out that there exist non-degenerate strips, and they survive suitably chosen small perturbations. The property to contain lines is however lost after this perturbation, and thus we get a strip not containing lines.

4.2.1. *An unperturbed strip with exactly one line inside.* A strip is called unperturbed if $u_i(z) \equiv u_2(z) \equiv 0$. Equivalently, centers of all horizontal sections of an unperturbed strip lie on the z-axis.

Remark. We consider further only the case of $f_i(z)$ being two linearly independent solutions of a linear differential equation of second order $y'' = g(z)y$. This is not very restrictive. Indeed, any two functions are solutions of a differential equation of second order as soon as their Wronskian is nonzero. But if the Wronskian $W(f_1,\, f_2)$ changes sign at some point z_0 then, using arguments similar to those in the proof of Lemma 8 below, one can see that the projection along the direction $(f_1(z_0),\, f_2(z_0),\, 0)$ doesn't satisfy condition (2″) of Definition 4.

First, we prove that almost all unperturbed strips contain only one line — the z-axis.

Lemma 7. *Let f_1, f_2 be two linearly independent solutions of a second order linear differential equation $y'' = g(z)y$. Then the set $\{x = tf_1(z),\, y = tf_2(z),\, \|t\| \leqslant 1\}$ contains a line ℓ different from the z-axis if and only if $g(z) \equiv 0$.*

Proof. The "if" part is evident and correspond to the degenerate strip above.

Let prove the "only if" part. If the line ℓ and the z-axis lie in one plane, then the strip lies in a plane $Ax + By = 0$, which means that f_1, f_2 are linearly dependent.

So these two lines (ℓ and the z-axis) are not in the same plane. After a rotation around z-axis we can assume that $\ell \in \{x = A\}$, so the line ℓ is defined by equations $x = A$, $y = az + b$ (equivalently, we replace f_1, f_2 by their linear combinations). Consider the quotient $k(z) = \frac{f_2(z)}{f_1(z)}$. Then $k'(z) = \frac{W(f_1, f_2)}{f_1^2} = \frac{\text{const}}{f_1^2}$. Indeed, $W(f_1, f_2) \equiv \text{const}$ since the equation $y'' = g(z)y$ has no term with y'. From the other side, $k(z) = \frac{az+b}{A}$ is a linear function, so its derivative is a constant. So $f_1(z) \equiv \text{const}$ and therefore $g(z) \equiv 0$. □

Lemma 8. *Let f_i be two linearly independent solutions of $y'' = g(z)y$. If $g(z) \geqslant 0$ for all $z \in \mathbb{R}$ then the set $S = \{x = tf_1(z),\ y = tf_2(z),\ \|t\| \leqslant 1\}$ is a strip (i. e. is convex-concave).*

Proof. Projections of S along the $(-b, a, 0)$ direction can be described as $\pi(A) = \{-|\phi(z)| \leqslant w \leqslant |\phi(z)|\}$, where $\phi(z) = af_1(z) + bf_2(z)$ is again a solution of the same equation $y'' = g(z)y$. We have to prove that $|\phi(z)|$ is convex, or, equivalently, that $\phi(z)'' \geqslant 0$ when $\phi(z) > 0$ and that $\phi(z)'' \leqslant 0$ when $\phi(z) < 0$. In other words, $\phi(z)$ and $\phi''(z)$ should have the same sign for all z. But this follows immediately from the equation and positivity of $g(z)$. □

4.2.2. *Perturbation of a strip.* Here we perturb the strip of Lemma 8 in such a way that the perturbed strip does not contain lines. The perturbation is local (i. e. between two levels), and there is only one line passing through the unperturbed part. So we only have to ensure that

(1) the perturbed part does not contain this line, and
(2) that the perturbed strip remains convex-concave.

Here is the outline of the construction below. Take any strip S described in Lemma 8. Take any function $\rho(t)$ such that, first, $|\rho(z)| \leqslant g(z)$ and, second, $\rho(z) \neq \text{const} \cdot g(z)$. Take $u_i(z)$ such that $u_i''(z) = \rho(z)f_i(z)$, $i = 1, 2$. Consider the strip

$$\widetilde{S} = \{(x, y, z) \in \mathbb{R}^3 \colon\ x = u_1(z) + tf_1(z),\ \ y = u_2(z) + tf_2(z),\ \ \|t\| \leqslant 1\}.$$

In other words, we shift the segments S_t — the horizontal sections of S — by the vector $(u_1(t), u_2(t), 0)$. We prove below that the first condition on $\rho(z)$ implies concave-convexity of $\widetilde{S}$, and the second condition implies that z-axis is not in $\widetilde{S}$.

Lemma 9. *$\widetilde{S}$ is convex-concave.*

Proof. The horizontal sections of S are segments, so the first condition of Definition 4 is satisfied. We check the second condition of Definition 4 in the form $(2'')$. A horizontal projection of S along a direction $(-b, a, 0)$ is given by $\pi(A) = \{\psi(z) - |\phi(z)| \leqslant w \leqslant \psi(z) + |\phi(z)|\}$, where $\psi(z) = au_1(z) + bu_2(z)$ and $\phi(z) = af_1(z) + bf_2(z)$.

We claim that the boundary of any horizontal projection is given by convex functions, i. e. that

- $(\psi(z)+\phi(z))'' \geqslant 0$ and $(\psi(z)-\phi(z))'' \leqslant 0$ when $\phi(z) \geqslant 0$ and that
- $(\psi(z)-\phi(z))'' \geqslant 0$ and $(\psi(z)+\phi(z))'' \leqslant 0$ when $\phi(z) \leqslant 0$

In other words, we have to prove that $(\phi(z) \pm \psi(z))''$ has the same sign as $\phi(z)$. This is evident since their ratio is equal to $g \pm \rho$ which is nonnegative. □

Lemma 10. *If $\rho \neq \mathrm{const} \cdot g$ for a $|c| < 1$ then the z-axis does not lie in $\widetilde{S}$.*

Proof. Suppose opposite, i.e. that $(u_1(z), u_2(z)) = \lambda(z)(f_1(z), f_2(z))$ for all z. Then $\rho(z)(f_1(z), f_2(z)) = (u_1''(z), u_2''(z)) = (\lambda''+\lambda g)(f_1(z), f_2(z)) + 2\lambda'(f_1'(z), f_2'(z))$. Therefore the vector $2\lambda'(f_1', f_2')$ is proportional to the vector (f_1, f_2). Therefore either $\lambda' \equiv 0$ or the vector (f_1', f_2') is proportional to the vector (f_1, f_2) for all z. The second possibility contradicts to the linear independence of f_1 and f_2. The first one means that $\lambda \equiv \mathrm{const}$, i.e. that ρ and g are proportional. □

The perturbation can be made local:

Lemma 11. *We can find an even $\rho(z)$ satisfying all previous conditions and such that $u_i(z) \equiv 0$ for $|z| \geqslant 1/2$.*

Proof. Indeed, consider the space L of even C^2-smooth functions $\rho(z)$ vanishing identically for $|z| \geqslant 1/2$. The functions u_i solving $u_i'' = \rho(z)f_i$ with $\rho \in L$ and initial conditions $u_i(-1) = u_i'(-1) = 0$ are identical zero on $z \leqslant -1/2$ and are linear on $z \geqslant 1/2$, i.e. $u_i(z) = a_{i1}z + a_{i0}$ for $z > 1/2$. Since a_{ij} depend linearly on ρ, the infinite-dimensional space L contains a subspace L', $\operatorname{codim} L' \leqslant 4$, of functions corresponding to $a_{ij} = 0$. □

4.2.3. *Specification of the strip.* For greater convenience in construction of the counterexample we impose some additional restrictions on $g(z)$ and $\rho(z)$, f_i and u_i.

Corollary 3 (of the constructions above)**.** *Let $g(z)$ be any even smooth function identically equal to 0 for $|z| \geqslant 1$ and strictly positive otherwise. One can find an even nonzero function $\rho(z)$ vanishing identically for $|z| \geqslant 1/2$, and functions $u_1(z)$, $u_2(z)$, $f_1(z)$, $f_2(z)$ such that*

1. *$f_1(z)$, $f_2(z)$ are two linearly independent solutions of $f''(z) = g(z)f(z)$, and $u_i''(z) = \rho(z)f_i(z)$;*
2. *the perturbed strip $S = \{(x, y, z) \in \mathbb{R}^3 \colon x = u_1(z) + tf_1(z),\ y = u_2(z) + tf_2(z),\ \|t\| \leqslant 1\}$ does not contain lines and is symmetric with respect to the rotation $(x, y, z) \to (x, -y, -z)$ of $\mathbb{R}^3$;*
3. *the part of the strip S — a piece of a quadric — lying in $\{z \geqslant 1\}$ is bounded by rays whose directions lie inside the cone $z^2 > x^2 + y^2$;*
4. *$f_1^2(2) + f_2^2(2) < 1$.*

Proof. Take a $\rho(z)$ as in Lemma 11. Let $f_1(z)$ and $f_2(z)$ be a pair of an even and an odd solutions of $f''(z) = g(z)f(z)$, and choose $u_i(z)$ to be solutions of equations $u_i'' = \rho(z)f_i(z)$ of the same oddity as f_i. Together this means that the strip S is symmetric with respect to the rotation $(x, y, z) \to (x, -y, -z)$ of $\mathbb{R}^3$.

Since $\rho(z)$ and $g(z)$ are not proportional, the strip S does not contain lines.

Since $f_i''(z) = u_i(z) = 0$ for $|z| > 1$, the boundary of the part of S lying in $\{|z| > 2\}$ is just four rays $\{(x, y, z) \colon x = \pm(a_0 + a_1|z|),\ y = \pm b_1|z|,\ |z| > 1\}$.

Multiplying $f_i(z)$ and $u_i(z)$ by a small number (i. e. after a dilatation of (x, y)-plane), we can assume that $a_1^2 + b_1^2 < 1$ and $f_1^2(\pm 2) + f_2^2(\pm 2) < 1$, as required. $\square$

4.3. Gluing to the quasi-cone. Here we glue the strip S of Corollary 3 to the quasi-cone $K' = \{(x, y, z)\colon x^2 + y^2 = (|z| - 1)^2, |z| \geqslant 1\}$. More exact, we construct a convex-concave set E with horizontal sections equal to those of K' for $|z| \geqslant 2$ and coinciding with sections of S for $|z| \leqslant 1$.

Here is outline of the construction of E. Take the union E_1 of S and K'. Horizontal sections of E_1 are sometimes segments, sometimes closed discs and sometimes their unions. Define the set E as a set whose horizontal sections are the convex hulls of the corresponding horizontal sections of E_1, i. e. we take convex hull level-wise. E coincides with S in $\{|z| \leqslant 1\}$, so in particular doesn't contain a line. The last two conditions of Corollary 3 taken together guarantee that the part of S lying in $\{|z| \geqslant 2\}$ lies inside K', i. e. E coincide with K' outside $\{|z| \leqslant 2\}$.

The set E is convex-concave since its support function is convex: it is equal to the maximum of the support function of S and a linear function — a support function of K' — overtaking it as $z \to \infty$. Here are the details.

Lemma 12. *E is a convex-concave set.*

Proof. All horizontal sections of E are nonempty and convex by definition of the set E, so the first condition of Definition 4 holds.

Let a projection of E be given by $\pi(E) = \{-\phi_2(z) \leqslant w \leqslant \phi_1(z)\}$. The second condition of Definition 4 is that both $\phi_1(z)$ and $\phi_2(z)$ are convex.

We prove it for $\phi_1(z)$ (for $\phi_2(z)$ the proof is the same). Taking convex hull of sections doesn't change horizontal projections, so $\pi(E) = \pi(E_1)$. Let projections of S and K' be defined by $\pi(S) = \{-\phi_2^S(z) \leqslant w \leqslant \phi_1^S(z)\}$ and $\pi(K') = \{-\phi_2^{K'}(z) \leqslant w \leqslant \phi_1^{K'}(z), |z| \geqslant 1\}$. Then $\phi_1(z) = \max(\phi_1^S(z), \phi_1^{K'}(z))$ for $|z| \geqslant 1$ and $\phi_1(z) = \phi_1^S(z)$ for $|z| < 1$.

Let $\widetilde{\phi}_1^{K'}(z)$ be a convex piecewise linear function equal to $\phi_1^{K'}(z)$ for $|z| \geqslant 1$ and equal to 0 for $|z| \leqslant 1$. Note that by choice of $\rho(z)$ in Corollary 3 the middle point of the intervals $S \cap \{z = t\}$ lie on the z-axis for $|t| \geqslant 1/2$, so $\phi_1^S(z) = \phi_2^S(z) \geqslant 0$ for $|z| \geqslant 1/2$. Therefore $\phi_1(z) = \max(\widetilde{\phi}_1^S(z), \phi_1^{K'}(z))$ for $z \in [1/2, \infty)$, so is convex on this interval as a maximum of two convex functions. Similarly $\phi_1(z)$ is convex on $(-\infty, -1/2]$. By definition $\phi_1(z)$ is a convex function on $[-1, 1]$. Therefore $\phi_1(z)$ is convex on the whole real line. $\square$

4.4. Smoothing. The convex-concave body E built in the previous section doesn't contain a line but still is not a domain bounded by a hyperbolic surface. The last step of the construction of the counterexample is a smoothing of E. As a result we get a convex-concave domain D bounded by a smooth hyperbolic surface.

First we prove that any domain sufficiently close to E doesn't contain a line.

Lemma 13. *Suppose that horizontal sections $E'_t = E' \cap \{z = t\}$ of a set $E' \subset \mathbb{R}^3$ are compact and lie in an ε-neighborhood of $E \cap \{z = t\}$ for all $-10 \leqslant t \leqslant 10$. If ε is sufficiently small then E' doesn't contain lines.*

Proof. It is enough to check non-horizontal lines, i. e. the lines given by $\ell = \{(a + bz,\, c + dz,\, z) \colon z \in \mathbb{R}\}$. The continuous function

$$\operatorname{maxdist}(\ell,\, E) = \max_{x \in \ell \cap \{|z| \leqslant 10\}} \operatorname{dist}(x,\, E)$$

achieves its nonzero minimum c on the set of all non-horizontal lines. Indeed, $\operatorname{maxdist}(\ell,\, E)$ tends to infinity as $(a,\, b,\, c,\, d) \to \infty$, so there is a global minimum of $\operatorname{maxdist}(\ell,\, E)$. Moreover, this minimum is positive since $\operatorname{maxdist}(\ell,\, E) = 0$ would imply that $\ell \cap \{|z| \leqslant 1\} \subset E \cap \{|z| \leqslant 1\} = S \cap \{|z| \leqslant 1\}$, which is impossible.

If $\varepsilon < c$, then $\operatorname{maxdist}(\ell,\, E') > \operatorname{maxdist}(\ell,\, E) - \varepsilon \geqslant 0$ for any line ℓ. This means that $\ell \not\subset E' \cap \{|z| \leqslant 1\}$, so $\ell \not\subset E'$. □

4.4.1. *Convolution.* The procedure of smoothing of E described below is a general method of smoothing of convex-concave sets with moderate growth of support function as $|z| \to \infty$. The procedure is a generalization of the well-known fact that convolution of an integrable convex function with a $\mathcal{C}^\infty$-smooth positive function is a convex $\mathcal{C}^\infty$-smooth function.

An affine Minkowski sum $\lambda A + \mu B$, $\lambda + \mu = 1$, of two convex sets A and B with non-negative coefficients λ, μ, can be described in two ways:

(1) $\lambda A + \mu B = \{\lambda a + \mu b \colon a \in A,\, b \in B\}$;
(2) $F_{\lambda A + \mu B} = \lambda F_A + \mu F_B$, where F_S denotes the support function of a convex set S.

This operation can be applied section-wise to convex-concave sets.

Lemma 14. *Let A and B be two convex-concave sets, and let λ_1 and λ_2 be two non-negative numbers, $\lambda_1 + \lambda_2 = 1$. Define $C = \lambda A + \mu B$ as a set whose horizontal sections C_z are equal to the $\lambda_1 A_z + \lambda_2 B_z$. Then C is convex-concave.*

Indeed, the sections C_z are convex by definition, and support function of C_z is convex in z as a convex sum of two convex in z functions — the support functions of A_z and B_z. □

We will apply a generalization of this operation to the set E. Consider a group of affine transformations of $\mathbb{R}^3$ generated by translations in vertical directions and rotations around z-axis. This group Γ is isomorphic to a cylinder $\Gamma \cong \mathbb{R} \times \mathbb{S}^1$: to $(z,\, \phi) \in \Gamma$ corresponds a composition $g_{z,\phi}$ of a shift by $(0,\, 0,\, z)$ and rotation by angle ϕ around z-axis. Fix a standard Lebesgue measure $\mu = \mathrm{d}z\mathrm{d}\phi$ on Γ.

The group Γ preserves the class of convex-concave sets.

Take a δ-like function $K_\varepsilon(z,\, \phi) \colon \Gamma \to \mathbb{R}$ with a following properties (where $\varepsilon > 0$ is sufficiently small):

(1) $\int_\Gamma K_\varepsilon(z,\, \phi)\,\mathrm{d}\mu = 1$, and $\int_{|z|,|\phi| \leqslant \varepsilon} K_\varepsilon(x,\, \phi)\,\mathrm{d}\mu \geqslant 1 - \varepsilon$,
(2) $K_\varepsilon(z,\, \phi)$ is $\mathcal{C}^\infty$-smooth and strictly positive,
(3) $K_\varepsilon(z,\, \phi)$ and all its partial derivatives decrease exponentially as $|z| \to \infty$,
(4) $K_\varepsilon(z,\, \phi)$ is an even function of z.

For example, one can take the $K_\varepsilon(z,\, \phi) = C(\varepsilon) \exp((\cos\phi - z^2)/\varepsilon)$ with a suitable choice of the constant $C(\varepsilon)$.

We define D as an affine Minkowski combination with weight K_ε of shifts and rotations of E: $D = \int_\Gamma K_\varepsilon(z,\, \phi) g_{z,\phi}(E)\,\mathrm{d}\mu$. Alternatively, the support function of the set D is defined by the convolution $F_{D_z}(\ell) = \int_\Gamma K_\varepsilon(t, \psi) F_{E_{z-t}}(R_{-\psi}(\ell))\, dt\, d\psi$, where $R_{-\psi}$ is a rotation of $(\mathbb{R}^2)^*$ by the angle $-\psi$.

These integrals converge since the support function of E grows as a linear function of $|z|$: as soon as $|z| \geqslant 2$, the sections of E are just circles of radius $|z| - 1$.

We claim that

Theorem 6. *The set D is the required counterexample.*

(1) *Sections of D are strictly convex with nonempty interior. Moreover, boundaries of sections of D_ε are smooth and have everywhere non-vanishing curvature.*
(2) *Boundaries of projections of D_ε are bounded by graphs of smooth strictly convex functions.*
(3) *D is close to E in the sense of Lemma 13. It means that for any $\delta > 0$ we can find an $\epsilon > 0$ such that the sections of domain D_ε are in the δ-neighborhoods of the corresponding sections of E. Therefore D_ε does not contain lines for a sufficiently small ε.*
(4) *Boundary of D is a smooth hyperbolic surface. Moreover, D_ε approaches K' at infinity.*

Proof. Evidently, the support function of D is infinitely smooth as a convolution with an infinitely smooth kernel. Moreover, since the support function of E is convex and non-linear in z, the support function of D is strictly convex in z.

The non-degeneracy of the curvature of the sections of D follows from the receipt of the reconstruction of a convex set from its smooth support function, see Section 4.1.1. Indeed, one can easily check that boundaries of sections D_t are smooth and have nonzero curvature if the kernel of the Hessian of the support function is one-dimensional, i. e. coincides with the line joining the origin and the point. But the Hessian of the support function of D_t is a convolution of K_ε with the Hessian $H(F_{E_s}) = \begin{pmatrix} F_{E_s,xx} & F_{E_s,xy} \\ F_{E_s,xy} & F_{E_s,yy} \end{pmatrix}$ of the support functions of sections E_s. Since the latter is somewhere nonzero and everywhere positively semi-definite as a quadratic form, the convolution will be everywhere nonzero and positive semi-definite. Thus the boundaries of sections of D_t are smooth with nonzero curvature.

Moreover, the gradient mapping $\nabla_{x,y} F_{D_t}(\ell) \colon \mathbb{R} \times \mathbb{S}^1 \to \partial D_t$ is a smooth parameterization of the boundary of D. Taken together, this means that D_ε is convex-concave and its boundary is smooth with everywhere nonzero curvature.

By standard arguments one can prove that as $\varepsilon \to 0$, the result of a convolution of a function F_E with K_ε converges to F_E itself. This implies that the set D_ε lies in a δ-neighborhood of E, as required.

The last claim is that D_ε approaches E at infinity. First, the sections E_z of E are circles of radius $|z| - 1$ for $|z| \geqslant 2$, and therefore $F_E(z, \ell) = (|z| - 1)\|\ell\|$ for $|Z| \geqslant 2$. Since K_ε is even as a function of z, the convolution with K_ε preserves linear functions of z. Therefore the difference $|F_{D_\varepsilon}(z, \ell) - F_E(z, \ell)|$ decreases exponentially together with all its derivatives as $|z| \to \infty$. Therefore the parameterizations of boundaries of E and D_ε by the gradient of their support functions as before are exponentially close, which proves the claim. □

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