Izvestiya: *Mathematics* **70**:3 153–169 ©2006 RAS(DoM) and LMS
Izvestiya RAN: *Ser. Mat.* **70**:3 163–182 DOI 10.1070/IM2006v170n01ABEH002307

The Hilbert polynomial for systems of linear partial differential equations with analytic coefficients

A. G. Khovanskii, S. P. Chulkov

Abstract. We consider systems of linear partial differential equations with analytic coefficients and discuss existence and uniqueness theorems for their formal and analytic solutions. Using elementary methods, we define and describe an analogue of the Hilbert polynomial for such systems.

§ 1. Introduction

In this paper we consider systems of linear partial differential equations with analytic coefficients for one unknown function z in a domain U of the space $\mathbb{C}^n$. We study the spaces of germs of formal and analytic solutions at a point u of U. The following questions are discussed.

1) How to prescribe the initial data for formal and analytic solutions of such systems? More precisely, which sets of derivatives of z at u must be fixed in order to guarantee that there is a unique formal (analytic) solution with these data?

2) How does the dimension of the space formed by the k-jets of germs of formal and analytic solutions behave for various positive integers k and points u of U?

The following results are obtained. It is shown that there is a "bad" hypersurface Σ such that the space of germs of formal and analytic solutions has the same structure (in some sense) at every point of the complement $U \setminus \Sigma$. Namely, there is a set of partial derivatives (independent of the point u in the complement) that can be taken as the initial data for the formal solution at u (Theorem 1). The convergence of the formal solution is equivalent to that of the part of its Taylor series determined by the fixed derivatives (Theorem 3).

For every point u in $U \setminus \Sigma$ and every positive integer k we denote by $F_u(k)$ (resp. $A_u(k)$) the space of k-jets at u of the germs of the formal (resp. analytic) solutions at this point. For each k, the dimensions of $A_u(k)$ and $F_u(k)$ coincide and are independent of the point u (Corollary 4). For sufficiently large k, the function $H(k) = \dim A_u(k) = \dim F_u(k)$ is a polynomial in k (Corollary 3). Moreover, the algebraic meaning of the function H is clarified. Using the system of differential equations, one constructs a family of affine algebraic varieties which depend analytically on a parameter: the point u of U. When the parameter lies in the complement $U \setminus \Sigma$ of the hypersurface Σ, the Hilbert functions of these varieties coincide with H (see § 6.4).

The second author's work was carried out with the partial support of the Russian Foundation for Basic Research (grant no. 01-01-00739), the programme "Leading Scientific Schools of the Russian Federation" (grant no. NSh-1972.2003.1) and the NWO-RFBR 047.011.2004.026 (RFBR 05-02-89000-NWO).

AMS 2000 *Mathematics Subject Classification.* 13D40, 20M14, 35C10.

These questions are classical, and the results discussed are not absolutely new. The most important results in this field were obtained by Riquier [1] (an exposition of Riquier's theory in Russian is given in [2]) and Palamodov [3]. The question of the correct initial data for formal and analytic solutions of non-linear systems is considered in [1]. This remarkable work introduces a total ordering on the set of partial derivatives of a function of many variables. In the case of linear systems with constant coefficients, Riquier's method contains what was later given the name of Gröbner bases and caused a revolution in computational commutative algebra. However, since the problems considered in [1] are very general, the solutions obtained there are not definitive.

The simpler case of linear systems was considered by Palamodov, who was able to prove the existence and uniqueness of formal and analytic solutions in a general situation not covered by Riquier's approach. It is proved in [3] that the "bad" set for linear systems is smaller than the "bad" hypersurface Σ arising in Riquier's method. The work of Palamodov is technically much more involved and is based on *ad hoc* methods developed by him.

In this paper we use Riquier's approach. However, we apply it only to linear systems, where it gives rather good results.

The second author of this paper has recently realized that the more general theorems of Palamodov can also be proved by a modification of Riquier's method, thus avoiding the delicate techniques developed by Palamodov. This result will be published in a separate paper.

§ 2. Properties of the semigroup $\mathbb{Z}^n_{\geqslant 0}$

We consider the semigroup $\mathbb{Z}^n_{\geqslant 0} = \{(\alpha_1, \dots, \alpha_n) \mid \alpha_i \in \mathbb{Z},\ \alpha_i \geqslant 0\}$. Given an element α of the semigroup, we define its modulus $|\alpha|$ as the non-negative integer $\sum \alpha_i$.

This section contains the necessary information on the semigroup $\mathbb{Z}^n_{\geqslant 0}$.

2.1. The ordered semigroup $\mathbb{Z}^n_{\geqslant 0}$. We fix an ordering $\prec$ on the semigroup $\mathbb{Z}^n_{\geqslant 0}$ such that the following conditions hold.

a) If α, β are any elements of the semigroup whose moduli satisfy $|\alpha| < |\beta|$, then $\alpha \prec \beta$.

b) The ordering $\prec$ is compatible with addition in $\mathbb{Z}^n_{\geqslant 0}$. In other words, if α, β, γ belong to the semigroup and $\alpha \prec \beta$, then $\alpha + \gamma \prec \beta + \gamma$.

It turns out that the restriction of any such order $\prec$ to any finite subset of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ is determined by one linear functional. More precisely, we have the following lemma.

Lemma 1. *Let A be a finite subset of the semigroup $\mathbb{Z}^n_{\geqslant 0}$. Then there is a linear functional*

$$\Pi_A \colon \mathbb{Z}^n_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}, \qquad \Pi_A(\alpha) = \sum_{i=1}^{n} \pi_i \alpha_i,$$

with positive real π_i and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}$ such that the following condition holds. If $\alpha, \beta \in A$ are any elements with $\alpha \prec \beta$, then

$$\Pi_A(\alpha) < \Pi_A(\beta).$$

Proof. We consider the chain of natural inclusions $\mathbb{Z}^n_{\geqslant 0} \subset \mathbb{Z}^n \subset \mathbb{R}^n$. Let B be the following finite subset of the group $\mathbb{Z}^n$:

$$B = \{\delta \in \mathbb{Z}^n \mid \exists \alpha, \beta \in A\colon \alpha \prec \beta,\ \delta = \beta - \alpha\}.$$

Let $\operatorname{conv}(B)$ be the convex hull of B in the space $\mathbb{R}^n$. Since $\prec$ is compatible with addition in $\mathbb{Z}^n_{\geqslant 0}$, the set $\operatorname{conv}(B)$ does not contain 0. Indeed, assume the opposite. Write

$$\sum_{i=1}^{N} p_i \delta^i = 0, \tag{1}$$

where $\delta^i \in B \subset \mathbb{Z}^n$, $\delta^i = \beta^i - \alpha^i$, $\beta^i, \alpha^i \in A$ and $p_i \in \mathbb{R}$, $p_i > 0$. We can regard (1) as a system of homogeneous linear equations with integer coefficients, where the unknowns are the coordinates of the vector $p = (p_1, \dots, p_N) \in \mathbb{R}^N$. The existence of a non-trivial solution implies that this system has a non-trivial vector subspace of solutions. Since the coefficients of the equations in (1) are integers, the rational vectors are dense in the space of solutions. Therefore one can find rational (and hence also integral) positive numbers $\tilde{p}_i$ such that

$$\sum_{i=1}^{N} \tilde{p}_i \delta^i = 0.$$

Then we have

$$\sum_{i=1}^{N} \tilde{p}_i \beta^i = \sum_{i=1}^{N} \tilde{p}_i \alpha^i.$$

On the other hand, $\alpha_i \prec \beta_i$, whence $\sum_{i=1}^{N} \tilde{p}_i \alpha_i \prec \sum_{i=1}^{N} \tilde{p}_i \beta_i$. This is a contradiction.

Since the closed bounded convex set $\operatorname{conv}(B)$ does not contain 0, there is a functional

$$L\colon \mathbb{R}^n \to \mathbb{R}, \qquad L(x) = \sum_{i=1}^{n} l_i x_i,$$

such that $L(x)$ is positive for all $x \in \operatorname{conv}(B)$. To get the desired functional Π_A, we put $\pi_i = S + l_i$, where S is a sufficiently large positive integer.

Remark 1. By generalizing this argument, one can easily prove the following well-known fact (see, for example, [4] or [5]).

Proposition 1. *Suppose that $\prec$ is an ordering on the semigroup $\mathbb{Z}^n_{\geqslant 0}$ that is compatible with addition. Then there are linear functionals $\Pi^1, \dots, \Pi^j$ with $j \leqslant n$ and*

$$\Pi^i\colon \mathbb{Z}^n_{\geqslant 0} \to \mathbb{R}$$

such that $\prec$ is lexicographic with respect to this set of functionals, that is, the assertion $\alpha \prec \beta$ holds if and only if we have

$$\Pi^1(\alpha) = \Pi^1(\beta), \quad \dots, \quad \Pi^i(\alpha) = \Pi^i(\beta), \qquad \Pi^{i+1}(\alpha) < \Pi^{i+1}(\beta)$$

for some i belonging to the set $\{0, \dots, j-1\}$.

Given a positive integer k, we denote by Π_k the functional satisfying the conditions of Lemma 1 for the set $A_k = \{\alpha \in \mathbb{Z}^n_{\geqslant 0} \mid |\alpha| \leqslant k\}$. Put $\mu_k = \min_{\{\alpha, \beta \in A_k,\, \alpha \neq \beta\}} |\Pi_k(\beta) - \Pi_k(\alpha)|$ and note that $\mu_k > 0$.

2.2. Properties of $\mathbb{Z}^n_{\geqslant 0}$-ideals. We define the *octant* $O^n(a) \subset \mathbb{Z}^n_{\geqslant 0}$ *with vertex* $a \in \mathbb{Z}^n_{\geqslant 0}$ to be the set of integer points $b \in \mathbb{Z}^n_{\geqslant 0}$ such that $a \preceq b$.

A subset of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ is called an *ideal in* $\mathbb{Z}^n_{\geqslant 0}$ (or a $\mathbb{Z}^n_{\geqslant 0}$-*ideal*) if, for each of its points, it contains the octant with vertex at this point. Clearly, every octant is an ideal in $\mathbb{Z}^n_{\geqslant 0}$.

We have the following two assertions (see, for example, [6]) about ideals in the semigroup $\mathbb{Z}^n_{\geqslant 0}$.

Proposition 2 ($\mathbb{Z}^n_{\geqslant 0}$ is Noetherian). *Every $\mathbb{Z}^n_{\geqslant 0}$-ideal is a union of finitely many octants.* (*In other words, every union of infinitely many octants is also a union of finitely many.*)

The semigroup $\mathbb{Z}^n_{\geqslant 0}$ contains 2^n coordinate semigroups: for every subset I of the set $\{1,\dots,n\}$ we have a subsemigroup $\mathbb{Z}_{\geqslant 0}(I)$ consisting of all integer points $a=(a_1,\dots,a_n)$ such that $a_i = 0$ for $i \in I$ and $a_i \geqslant 0$ for $i \notin I$. Among the semigroups $\mathbb{Z}_{\geqslant 0}(I)$, we have the zero semigroup (with $I = \{1,\dots,n\}$) and the semigroup $\mathbb{Z}^n_{\geqslant 0}$ (with $I = \varnothing$).

A subset of $\mathbb{Z}^n_{\geqslant 0}$ is called a *shifted coordinate subsemigroup* if it has the form $a + \mathbb{Z}_{\geqslant 0}(I)$ for some element $a \in \mathbb{Z}^n_{\geqslant 0}$.

Proposition 3. *The complement of every ideal in $\mathbb{Z}^n_{\geqslant 0}$ consists of finitely many disjoint shifted coordinate semigroups.*

§ 3. The Gröbner map and bases of differential ideals

In this section we define the Gröbner map for the ring of linear differential operators and use it to study ideals in this ring.

Consider an arbitrary domain U in the space $\mathbb{C}^n$ with coordinates $x_1,\dots,x_n$. Let B be a subring of the ring $\mathcal{O}(U)$ of all holomorphic functions on U such that B contains 1 and is closed under differentiation.

We denote by Dif_B the ring of linear differential operators on U whose coefficients lie in B. Take $d \in \mathrm{Dif}_B$. Then

$$d = \sum_{\alpha \in \operatorname{supp} d} b_\alpha \partial_\alpha,$$

where $b_\alpha (\not\equiv 0) \in B$ and $\operatorname{supp} d$ is a finite subset of $\mathbb{Z}^n_{\geqslant 0}$. Here ∂_α is the differentiation operator $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$. The finite set $\operatorname{supp} d$ is called the *support* of d.

Definition 1. The *Gröbner map* is given by

$$\mathrm{Grb}\colon \mathrm{Dif}_B \setminus \{0\} \to \mathbb{Z}^n_{\geqslant 0}, \qquad \mathrm{Grb}(d) = \max_{\alpha \in \operatorname{supp} d} \alpha,$$

where the maximum is taken with respect to $\prec$.

The following remarkable property of the Gröbner map is crucial for our constructions.

Lemma 2. *For any non-zero elements D, d of the ring Dif_B we have*

$$\mathrm{Grb}(D \circ d) = \mathrm{Grb}(d \circ D) = \mathrm{Grb}(D) + \mathrm{Grb}(d). \tag{2}$$

The coefficients of the leading derivatives (*with respect to the ordering introduced*) *in the decompositions of* $D \circ d$ *and* $d \circ D$ *are equal to the products of the coefficients of the leading derivatives in* d *and* D.

Proof. We denote the leading homogeneous parts of D and d by $\sum_{|\alpha|=N(D)} D_\alpha \partial_\alpha$ and $\sum_{|\alpha|=N(d)} d_\alpha \partial_\alpha$ respectively, where $D_\alpha, d_\alpha \in A$. The leading homogeneous parts of the operators $D \circ d$ and $d \circ D$ are equal to

$$\sum_{|\alpha|=N(D)+N(d)} \ \sum_{\{|\beta|=N(D),\, |\gamma|=N(d),\, \beta+\gamma=\alpha\}} D_\beta d_\gamma \partial_\alpha. \tag{3}$$

However, by condition a) on the ordering $\prec$, the image of any operator under the Gröbner map coincides with the image of its leading homogeneous part.

Corollary 1. *The image of any ideal of the ring* Dif_B *under the Gröbner map is an ideal of the semigroup* $\mathbb{Z}^n_{\geqslant 0}$.

Let $\mathcal{I}$ be a left ideal of the ring Dif_B. By Corollary 1, its image $\mathrm{Grb}(\mathcal{I})$ is an ideal in the semigroup $\mathbb{Z}^n_{\geqslant 0}$. Hence, by Proposition 2, it is a union of finitely many octants. We write

$$\mathrm{Grb}(\mathcal{I}) = \bigcup_{i=1}^{N} O(\gamma_i). \tag{4}$$

Take elements $l_1, \dots, l_N$ of $\mathcal{I}$ such that $\mathrm{Grb}(l_i) = \gamma_i$. For every i with $1 \leqslant i \leqslant N$ we denote by $a_{\gamma_i} \in B$ the coefficient of ∂_{γ_i} in the decomposition of the operator l_i.

Let $\mathcal{M}$ be the multiplicative system generated by the functions $1, a_{\gamma_1}, \dots, a_{\gamma_N}$ in the ring B, that is, $\mathcal{M}$ is the minimal subset of B which is closed under multiplication and contains $1, a_{\gamma_1}, \dots, a_{\gamma_N}$. We consider the localization $\mathcal{M}^{-1}B$ of the ring B with respect to the multiplicative system $\mathcal{M}$. The elements of the ring $\mathcal{M}^{-1}B$ are the equivalence classes of formal quotients $\frac{b}{m}$, where b is any element of B and m is an element of the multiplicative system $\mathcal{M}$. It is natural to regard $\mathcal{M}^{-1}B$ as a subring of the ring $\mathcal{O}_{U\setminus M}$ of holomorphic functions on the domain $U \setminus M$, where $M = \{a_{\gamma_1} \dots a_{\gamma_N} = 0\}$. The ring $\mathcal{M}^{-1}B$ is obviously closed under differentiation.

Consider the ring $\mathrm{Dif}_{\mathcal{M}^{-1}B}$. We regard the elements of the ring Dif_B as elements of $\mathrm{Dif}_{\mathcal{M}^{-1}B}$, having in mind their images in $\mathrm{Dif}_{\mathcal{M}^{-1}B}$ under the natural embedding

$$\begin{gathered} \pi\colon \ \mathrm{Dif}_B \to \mathrm{Dif}_{\mathcal{M}^{-1}B}, \\ \sum_\alpha b_\alpha \partial_\alpha \mapsto \sum_\alpha \frac{b_\alpha}{1}\, \partial_\alpha. \end{gathered} \tag{5}$$

Proposition 4. *The elements* $l_1, \dots, l_N$ *form a basis of the ideal* $\mathrm{Dif}_{\mathcal{M}^{-1}B} \cdot \mathcal{I}$ *in the ring* $\mathrm{Dif}_{\mathcal{M}^{-1}B}$.

Here $\mathrm{Dif}_{\mathcal{M}^{-1}B} \cdot \mathcal{I}$ is the minimal left ideal of the ring $\mathrm{Dif}_{\mathcal{M}^{-1}B}$ containing the image of $\mathcal{I}$ under the embedding (5).

Remark 2. Geometrically, Proposition 4 means that the elements $l_1, \dots, l_N$ form a basis of the ideal $\mathcal{I}$ if we restrict the coefficients of the differential operators to $U \setminus M$.

Proof of Proposition 4. The Gröbner map Grb is defined for the ring $\mathrm{Dif}_{\mathcal{M}^{-1}B}$ as well as for Dif_B, and we have $\mathrm{Grb}(\pi(d)) = \mathrm{Grb}(d)$ for every element d of Dif_B. Therefore we have $\mathrm{Grb}(\mathrm{Dif}_{\mathcal{M}^{-1}B} \cdot \mathcal{I}) = \mathrm{Grb}(\mathcal{I})$.

Consider any non-zero element u of the ideal $\mathrm{Dif}_{\mathcal{M}^{-1}B} \cdot \mathcal{I}$ and write $u = f_u \partial_{\mathrm{Grb}(u)} + r$, where r denotes lower terms. The image $\mathrm{Grb}(u)$ belongs to the octant $O^n(\gamma_k)$ for some $k = 1, \dots, N$. Therefore $\mathrm{Grb}(u) = \gamma_k + \alpha$, where $\alpha \in \mathbb{Z}^n_{\geqslant 0}$. We put

$$u_1 = u - \frac{f_u}{a_{\gamma_k}} \partial_\alpha \circ l_k.$$

We have either $u_1 = 0$ or $\mathrm{Grb}(u_1) \prec \mathrm{Grb}(u)$ by Lemma 2. If the element u_1 of the ideal $\mathrm{Dif}_{\mathcal{M}^{-1}B} \cdot \mathcal{I}$ is non-zero, the process can be repeated. However there is no infinite chain $u, u_1, u_2, \ldots \in \mathrm{Dif}_{\mathcal{M}^{-1}B} \cdot \mathcal{I}$ with $\mathrm{Grb}(u) \succ \mathrm{Grb}(u_1) \succ \mathrm{Grb}(u_2) \succ \cdots$. Indeed, condition a) on $\prec$ guarantees that there are only finitely many elements of $\mathbb{Z}^n_{\geqslant 0}$ smaller than a given one, and $(\mathbb{Z}^n_{\geqslant 0}, \preceq)$ is a totally ordered set. Hence there is an l such that $u_l = 0$. Replacing the element u_l by its expression in terms of u and l_i, we get the desired result.

Remark 3. The decomposition $u = \sum_{i=1}^k p_i \circ l_i$ (constructed in the proof of Proposition 4) of elements of the ideal is such that the inequality $\mathrm{Grb}(p_i \circ l_i) \preceq \mathrm{Grb}(u)$ holds for each i.

In the proof of the last assertion, we constructed a system of generators of the ideal. This system is induced by the Gröbner map and is called a *Gröbner basis of the ideal.*

Consider the submodule $\mathrm{MI} \subset \mathrm{Dif}_{\mathcal{M}^{-1}B}$ which is generated over $\mathcal{M}^{-1}B$ by the generators $\{\partial_\alpha\}$, $\alpha \in \mathbb{Z}^n_{\geqslant 0} \setminus \mathrm{Grb}(\mathcal{I})$.

Proposition 5. *We have a direct sum decomposition*

$$\mathrm{Dif}_{\mathcal{M}^{-1}B} = \mathrm{MI} \oplus \mathrm{Dif}_{\mathcal{M}^{-1}B} \cdot \mathcal{I}. \tag{6}$$

Proof. We must prove that every element d of the ring $\mathrm{Dif}_{\mathcal{M}^{-1}B}$ can be uniquely written as a sum

$$d = m(d) + i(d), \tag{7}$$

where $i(d) \in \mathrm{Dif}_{\mathcal{M}^{-1}B} \cdot \mathcal{I}$ and $m(d) \in \mathrm{MI}$, that is, $\operatorname{supp} m(d) \subset \mathbb{Z}^n_{\geqslant 0} \setminus \mathrm{Grb}(\mathcal{I})$. To prove the uniqueness, we assume that there are two decompositions

$$d = i_1 + m_1 = i_2 + m_2,$$

where $i_1, i_2 \in \mathcal{I}$ and $m_1, m_2 \in \mathrm{MI}$. Then $0 \neq i_1 - i_2 = m_2 - m_1$, whence $\mathrm{Grb}(i_1 - i_2) = \mathrm{Grb}(m_2 - m_1)$. However, $(i_1 - i_2) \in \mathcal{I}$ and, therefore, $\mathrm{Grb}(i_1 - i_2) \in \mathrm{Grb}(\mathcal{I})$. On the other hand, $\operatorname{supp}(m_2 - m_1) \subset \mathbb{Z}^n_{\geqslant 0} \setminus \mathrm{Grb}(\mathcal{I})$, whence $\mathrm{Grb}(m_2 - m_1) \notin \mathrm{Grb}(\mathcal{I})$, a contradiction.

Let us now prove the existence of such a decomposition. We use the following algorithm. Take $d \in \mathrm{Dif}_{\mathcal{M}^{-1}B}$. If $\operatorname{supp} d \cap \mathrm{Grb}(\mathcal{I}) = \varnothing$, then we put $m(d) = d$ and $i(d) = 0$. Otherwise we consider $\mu(d) = \max_{\alpha \in \operatorname{supp} d \cap \mathrm{Grb}(\mathcal{I})} \alpha$. Write $\mu(d) = \gamma_k + \alpha$ for some $1 \leqslant k \leqslant N$ and $\alpha \in \mathbb{Z}^n_{\geqslant 0}$. We consider the difference $d - a\partial_\alpha\bigl(\frac{l_k}{a_{\alpha_k}}\bigr)$, where the coefficient $a \in B$ is chosen in such a way that either

$$\operatorname{supp}\biggl(d - a\partial_\alpha\biggl(\frac{l_k}{a_{\alpha_k}}\biggr)\biggr) \cap \mathrm{Grb}(\mathcal{I}) = \varnothing,$$

and then we put $m(d) = d - a\partial_\alpha\bigl(\frac{l_k}{a_{\alpha_k}}\bigr)$ and $i(d) = a\partial_\alpha\bigl(\frac{l_k}{a_{\alpha_k}}\bigr)$, or

$$\operatorname{supp}\biggl(d - a\partial_\alpha\biggl(\frac{l_k}{a_{\alpha_k}}\biggr)\biggr) \cap \operatorname{Grb}(\mathcal{I}) \neq \varnothing,$$

and then we repeat the process. Note that the following inequality holds:

$$\mu\biggl(d - a\partial_\alpha\biggl(\frac{l_k}{a_{\alpha_k}}\biggr)\biggr) \prec \mu(d).$$

Since $(\mathbb{Z}^n_{\geqslant 0}, \preceq)$ is a totally ordered set, we get the desired decomposition after finitely many steps.

§ 4. Formal solutions of systems of linear partial differential equations

Let us introduce some notation. We fix a domain U in the space $\mathbb{C}^n$ of independent variables and fix a subring A of the ring $\mathcal{O}(U)$ of holomorphic functions on U such that A contains 1 and is closed under differentiation.

Consider a system of homogeneous linear differential equations in U:

$$\begin{aligned} D_1 z &= 0, \\ &\dots\dots\dots \\ D_k z &= 0, \\ &\dots\dots\dots \end{aligned} \tag{8}$$

where $D_i \in \operatorname{Dif}_A$, $i = 1, 2, \dots$.

The system (8) may contain infinitely many equations. In this section we describe the space of formal solutions of (8) in a neighbourhood of u, where u is any point of some open dense subset of U.

4.1. Formal solutions of the system as functionals on the ring of differential operators. Let u be a point of U. We consider a subring B of the ring $\mathcal{O}_u$ of germs of functions holomorphic at u.

Definition 2. A map $\varphi\colon M \mapsto \mathbb{C}$ of a B-module M is said to be *u-linear* if

$$\varphi\biggl(\sum_{j=1}^{N} f_j L_j\biggr) = \sum_{j=1}^{N} f_j(u)\varphi(L_j) \tag{9}$$

for any $L_i \in M$ and $f_i \in B$.

Let $L_u(M)$ be the space of u-linear maps of the module M.

Lemma 3. *For every point u in U there is a natural isomorphism of vector spaces*:

$$L_u(\operatorname{Dif}_A) \cong \mathbb{C}[[x-u]]. \tag{10}$$

Proof. It is easy to verify that the following map determines an isomorphism:

$$\mathbb{C}[[x-u]] \to L_u(\operatorname{Dif}_A), \tag{11}$$

$$f(d) = d(f)|_{x=u}, \tag{12}$$

where $d \in \operatorname{Dif}_A$, $f, d(f) \in \mathbb{C}[[x-u]]$ and $d(f)|_{x=u}$ denotes the constant term of the series $d(f)$.

We denote by $\mathcal{I}(S)$ the left ideal in the ring Dif_A generated by the operators on the left-hand sides of the equations in (8).

Proposition 6. *Consider the u-linear map of* Dif_A *determined by a formal series* $f \in \mathbb{C}[[x-u]]$. *This map vanishes identically on the ideal* $\mathcal{I}(S)$ *if and only if we have* $d(f) = 0$ *for every operator* $d \in \mathcal{I}(S)$, *that is, if and only if* f *is a formal solution of* (8).

Proof. Take any operator $d \in \operatorname{Dif}_A$. Then the equation $d(f) = 0$ holds if and only if $\partial_\alpha(d(f))|_{x=u} = (\partial_\alpha \circ d)(f)|_{x=u} = 0$ for all $\alpha \in \mathbb{Z}^n_{\geqslant 0}$. This proves the desired assertion.

We denote by $F_u(S)$ the space of formal solutions of the system (8) at the point u.

Corollary 2. *We have a natural isomorphism*

$$F_u(S) = L_u(\operatorname{Dif}_A / \mathcal{I}(S)). \tag{13}$$

We use the following lemma to describe spaces of u-linear maps. Let $\mathcal{M}$ be a multiplicative system in A, that is, a subset of A which contains 1 and is closed under multiplication (see § 3). As mentioned above, the ring of quotients $\mathcal{M}^{-1}A$ is closed under differentiation.

We consider an arbitrary left ideal $\mathcal{I}$ in the ring Dif_A and denote by $\mathcal{M}^{-1}\mathcal{I}$ the minimal left ideal in the ring $\operatorname{Dif}_{\mathcal{M}^{-1}A}$ containing the image $\pi_*(\mathcal{I})$ of $\mathcal{I}$ under the embedding

$$\pi\colon \operatorname{Dif}_A \to \operatorname{Dif}_{\mathcal{M}^{-1}A}. \tag{14}$$

Lemma 4. *Let* u *be a point of* U. *If every function of the multiplicative system* $\mathcal{M}$ *takes a non-zero value at* u, *then the vector spaces* $L_u(\operatorname{Dif}_A / \mathcal{I})$ *and* $L_u(\operatorname{Dif}_{\mathcal{M}^{-1}A} / \mathcal{M}^{-1}\mathcal{I})$ *of u-linear maps are naturally isomorphic.*

Proof. The embedding π induces a map

$$\pi^*\colon L_u(\operatorname{Dif}_{\mathcal{M}^{-1}A} / \mathcal{M}^{-1}\mathcal{I}) \to L_u(\operatorname{Dif}_A / \mathcal{I}). \tag{15}$$

The following formula shows that π^* is an isomorphism:

$$(\pi^*)^{-1}(l)\biggl(\biggl[\sum_{j=1}^{N} \frac{d_\alpha}{x_\alpha}\,\partial^\alpha\biggr]\biggr) = \sum_{j=1}^{N} \frac{d_\alpha(u)}{x_\alpha(u)}\, l([\partial^\alpha]), \tag{16}$$

where $d_{\alpha_j} \in A$, $x_{\alpha_j} \in \mathcal{M}$. For every element d of the ring Dif_A, we denote by $[d]$ the element of $\operatorname{Dif}_{\mathcal{M}^{-1}A} / \mathcal{M}^{-1}\mathcal{I}$ whose representative is d.

4.2. Existence of formal solutions. We consider the $\mathbb{Z}^n_{\geqslant 0}$-ideal $\operatorname{Grb}(\mathcal{I}(S))$. By Proposition 2 it can be written as a finite union of octants:

$$\operatorname{Grb}(\mathcal{I}(S)) = \bigcup_{i=1}^{l} O(\gamma_i). \tag{17}$$

Choose elements $s_1, \dots, s_l$ of $\mathcal{I}(S)$ such that $\operatorname{Grb}(s_i) = \gamma_i$ for each i. Denote by Γ the multiplicative system in the ring A generated by 1 and the leading coefficients of the operators s_i (that is, the coefficients s_{γ_i} of the derivatives ∂_{γ_i}). We also denote by Σ the analytic hypersurface given by the equation $s_{\gamma_1} \dots s_{\gamma_l} = 0$.

Clearly, Proposition 4 can be restated in the following geometric form.

Proposition 7. *The elements* $s_1, \dots, s_l$ *generate the ideal* $\operatorname{Dif}_{\Gamma^{-1}A} \cdot \mathcal{I}(S)$. *In other words, the system* (8) *is equivalent in the domain* $U \backslash \Sigma$ *to the finite system consisting only of the equations* $s_i z = 0$ *where* i *runs from* 1 *to* l.

We define the *support* $\operatorname{supp} f$ *of a formal (convergent) series*

$$f = \sum_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} f_\alpha (x-u)^\alpha$$

to be the following subset of the semigroup:

$$\operatorname{supp} f = \{\alpha \in \mathbb{Z}^n_{\geqslant 0} \mid f_\alpha \neq 0\}. \tag{18}$$

Theorem 1. *Suppose that* $u \in U \setminus \Sigma$. *Then there is an isomorphism of vector spaces*

$$F_u(S) \cong \bigl\{f \in \mathbb{C}[[x-u]] \mid \operatorname{supp} f \subset \mathbb{Z}^n_{\geqslant 0} \setminus \operatorname{Grb}(\mathcal{I}(S))\bigr\}. \tag{19}$$

Proof. Consider the free $\Gamma^{-1}A$-module $\operatorname{MI}(S)$ with basis $\{\partial_\alpha\}$, $\alpha \in \mathbb{Z}^n_{\geqslant 0} \setminus \operatorname{Grb}(\mathcal{I}(S))$. Clearly,

$$L_u(\operatorname{MI}) \cong \bigl\{f \in \mathbb{C}[[x-u]] \mid \operatorname{supp} f \subset \mathbb{Z}^n_{\geqslant 0} \setminus \operatorname{Grb}(\mathcal{I}(S))\bigr\}. \tag{20}$$

By Proposition 5 there is an isomorphism of $\Gamma^{-1}A$-modules:

$$\operatorname{MI} \cong \operatorname{Dif}_{\Gamma^{-1}A} / \Gamma^{-1}\mathcal{I}(S). \tag{21}$$

Since $u \in U \setminus \Sigma$, all the functions of the multiplicative system $\Gamma \subset A$ are different from zero at u. Hence, by Lemma 4,

$$F_u(S) \cong L_u(\operatorname{Dif}_A / \mathcal{I}(S)) \cong L_u(\operatorname{Dif}_{\Gamma^{-1}A} / \Gamma^{-1}\mathcal{I}(S)). \tag{22}$$

Combining (20)–(22), we get a proof of the theorem.

For every non-negative integer i we consider the space

$$F_{u,i}(S) = F_u(S) / \bigl(f \sim g \overset{\text{def}}{\Leftrightarrow} f - g = o((x-u)^i)\bigr) \tag{23}$$

of i-jets of formal solutions at $u \in U$. The function $H(u,i) = \dim F_{u,i}(S)$ of the integer argument i is called the *Hilbert function* of the system (8) at the point u.

Corollary 3. *The Hilbert function* $H(u,i)$ *is the same for all* $u \in U \setminus \Sigma$ *and is a polynomial for all sufficiently large* i.

Proof. By (19) we have an equation

$$\dim F_{u,i}(S) = \bigl|\bigl\{\alpha \in \mathbb{Z}^n_{\geqslant 0} \setminus \operatorname{Grb}(\mathcal{I}(S)) \mid |\alpha| \leqslant i\bigr\}\bigr|, \tag{24}$$

which yields the first assertion. (Given a finite set A, we denote by $|A|$ the number of its elements.) By Proposition 3 we can represent $\mathbb{Z}^n_{\geqslant 0} \setminus \operatorname{Grb}(\mathcal{I}(S))$ as

$$\mathbb{Z}^n_{\geqslant 0} \setminus \operatorname{Grb}(\mathcal{I}(S)) = \bigcup_{k=1}^{l} \{a_k + \mathbb{Z}_{\geqslant 0}(I_k)\}. \tag{25}$$

However, for each shifted coordinate semigroup $a_k+\mathbb{Z}_{\geqslant 0}(I_k)$, the function

$$H_{(a_k,I_k)}(i)=\bigl|\{\alpha\in a_k+\mathbb{Z}_{\geqslant 0}(I_k)\mid |\alpha|\leqslant i\}\bigr|$$

is a polynomial for $i\geqslant |a_k|$. $\bigl($It is easy to verify that $H_{(a_k,I_k)}(i)=\binom{n-|I|+i-a_k}{i-a_k}$ for $i\geqslant |a_k|$.$\bigr)$ Hence the Hilbert function

$$H(u,i)=\sum_{j=1}^{l}H_{(a_j,I_j)}(i)$$

is a polynomial for $i\geqslant \max_k|a_k|$.

§ 5. The convergence theorem

5.1. The convergence theorem and its corollaries. To prove existence and uniqueness theorems for germs of analytic solutions we need the following assertion. Suppose that the formal series

$$z(x)=\sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}}a_\alpha x^\alpha$$

satisfies the following finite system of differential relations:

$$\begin{array}{ll}\partial_{\gamma_1}z=F_1(x,\partial_\alpha z), & \alpha\prec\gamma_1,\\ \dots\dots\dots\dots\dots\dots & \dots\dots\\ \partial_{\gamma_k}z=F_k(x,\partial_\alpha z), & \alpha\prec\gamma_k,\end{array}\tag{26}$$

where $F_1,\dots,F_k$ are holomorphic functions of the variables $x_1,\dots,x_n$ and the derivatives $\partial_\alpha z$ whose exponents α satisfy the inequalities in the right column of the system. We consider the subset $I=\bigcup_{i=1}^k O(\gamma_i)$ of the semigroup $\mathbb{Z}^n_{\geqslant 0}$.

Theorem 2. *Suppose that the truncation $\tilde z(x)=\sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}\setminus I}a_\alpha x^\alpha$ of the series z has non-zero radius of convergence. Then so does the formal solution z.*

A proof of this theorem is given in § 5.2 below.

Let $A_u(S)$ be the space of germs of analytic solutions of the system at the point u. Combining Theorems 2 and 1, we get the following theorem.

Theorem 3. *For every point $u\in U\setminus\Sigma$ there is an isomorphism of vector spaces*

$$A_u\cong\bigl\{f\in\mathbb{C}\{(x-u)\}\mid \operatorname{supp}f\subset\mathbb{Z}^n_{\geqslant 0}\setminus\operatorname{Grb}(\mathcal{I})\bigr\}.\tag{27}$$

Proof. Consider the system of linear differential equations consisting of the equations $s_1z=0,\dots,s_lz=0$, where the s_i are the elements of $\mathcal{I}(S)$ chosen above. (They satisfy $\operatorname{Grb}(s_i)=\gamma_i$ for each i.) By Proposition 7, this system is equivalent to the original system (8) in the domain $U\setminus\Sigma$. We resolve each equation $s_iz=0$ with respect to the leading (in our order) derivative γ_i. (This is possible by the choice of the hypersurface Σ.) We can now apply Theorem 2 to the resolved system.

We fix a partition of the set $\mathbb{Z}^n_{\geqslant 0} \setminus \operatorname{Grb}(\mathcal{I})$ into shifted coordinate semigroups:

$$\mathbb{Z}^n_{\geqslant 0} \setminus \operatorname{Grb}(\mathcal{I}) = \bigcup_{k=1}^{l} \{a_k + \mathbb{Z}_{\geqslant 0}(I_k)\}. \tag{28}$$

The following theorem is a corollary of Theorem 3.

Theorem 4. *For every point $u \in U \setminus \Sigma$ there is a unique solution $z(x)$ of the system (8) such that $z(x)$ is holomorphic in a neighbourhood of u and satisfies the following l initial conditions*:

$$\partial_{a_k} z(x)|_{\{x_i = u_i,\, i \in I_k\}} = \psi_k(x_{i_1}, \dots, x_{i_m}), \qquad 1 \leqslant k \leqslant l,$$

where $\{i_1, \dots, i_m\} = \{1, \dots, n\} \setminus I_k$ and the ψ_i are arbitrary holomorphic functions of their arguments in a neighbourhood of u. (If $I_k = \{1, \dots, n\}$ for some k, then ψ_k is simply a complex number.)

Proof. To deduce this theorem from the previous one, it suffices to note that every convergent series consisting of monomials whose exponents belong to some coordinate semigroup $\mathbb{Z}_{\geqslant 0}(I)$, where $I = \{1, \dots, n\} \setminus \{i_1, \dots, i_m\}$, is a holomorphic function of the variables $x_{i_1}, \dots, x_{i_m}$.

Let $A_{u,i}(S)$ be the space of i-jets of germs of analytic solutions at the point u.

Corollary 4. *For every point $u \in U \setminus \Sigma$, the following dimensions coincide for every i*:

$$\dim F_{u,i}(S) = \dim A_{u,i}(S). \tag{29}$$

5.2. Proof of Theorem 2. The proof is based on the majorant method. We consider the ring $\mathbb{C}[[y_1, \dots, y_l]]$ of formal series in some variables $y_1, \dots, y_l$. Take $A, B \in \mathbb{C}[[y_1, \dots, y_l]]$.

Definition 3. The formal series $A(y) = \sum_{\alpha \in \mathbb{Z}^l_{\geqslant 0}} a_\alpha y^\alpha$ *majorizes* the series $B(y) = \sum_{\alpha \in \mathbb{Z}^l_{\geqslant 0}} b_\alpha y^\alpha$ if the following conditions hold for every element α of the semigroup $\mathbb{Z}^l_{\geqslant 0}$:

$$a_\alpha \in \mathbb{R}_{\geqslant 0} \quad \text{and} \quad |b_\alpha| \leqslant a_\alpha.$$

The idea of the proof is to construct a convergent series that majorizes the given formal solution. This majorizing series is constructed from a solution of some equation that majorizes every equation of the original system in some informal sense. The majorizing equation turns out to be an ordinary differential equation, and the existence of solutions follows from standard existence and uniqueness theorems.

In § 5.2.1 we prove lemmas on majorization. In §§ 5.2.2–5.2.5 we prove the theorem in the special case when the equations of the system are linear in the derivatives of highest order and the main derivatives have the same order. The conditions of this special case are stated in § 5.2.2. In § 5.2.3 we change the coordinates in such a way that the majorizing equation can easily be obtained for the transformed system. In § 5.2.4 we present the majorizing equation and prove that it has an analytic solution. In § 5.2.5 we construct a convergent series from the solution of the majorizing equation and use Lemma 5 to prove that this series majorizes the original formal solution of the system. In § 5.2.6 we reduce the general case to this special case and thus prove the theorem.

5.2.1. *The majorization lemmas.* In the first lemma we state the necessary version of the fact that the majorization property is preserved under composition.

Consider holomorphic functions f_1 and f_2 on a neighbourhood of zero in the space $\mathbb{C}^{n+m} = \{(x_1, \dots, x_n, \xi_1, \dots, \xi_m) \mid x_i, \xi_j \in \mathbb{C}\}$. Suppose that the series expansion of f_2 majorizes that of f_1.

We fix a set $\alpha_1 \prec \dots \prec \alpha_m \prec \alpha_0$ of $m+1$ elements of the semigroup $\mathbb{Z}^n_{\geqslant 0}$.

Let $w = \sum w_\alpha x^\alpha \in \mathbb{R}_{\geqslant 0}[[x]]$ and $z = \sum z_\alpha x^\alpha \in \mathbb{C}[[x]]$ be some series. We assume that the following conditions hold:

1) we have $|z_\alpha| \leqslant w_\alpha$ for all $\alpha \prec \alpha_0$,

2) we have $w_{\alpha_i} = z_{\alpha_i} = 0$ for every i with $1 \leqslant i \leqslant m$.

Let $W \in \mathbb{R}_{\geqslant 0}[[x]]$ be the well-defined series obtained by replacing the variables ξ_i in the decomposition of f_2 by the series $\partial_{\alpha_i} w$ $(1 \leqslant i \leqslant m)$, and let $Z \in \mathbb{C}[[x]]$ be obtained from the decomposition of f_1 by replacing the variables ξ_i by the series $\partial_{\alpha_i} z$ $(1 \leqslant i \leqslant m)$.

Lemma 5. *We have* $\bigl|\partial_\beta Z|_0\bigr| \leqslant \partial_\beta W|_0$ *for every* $\beta \prec \alpha_0$.

Proof. Write $Z = \sum_\alpha Z_\alpha x^\alpha$ and $W = \sum_\alpha W_\alpha x^\alpha$. The values of the derivatives $\partial_\beta Z|_0 = \beta!\, Z_\beta$ and $\partial_\beta W|_0 = \beta!\, W_\beta$ are sums (over the same set of indices) of products of the forms

$$\beta!\, f^1_{(\alpha,\delta)} \prod_{i=1}^m (\alpha_i!)^{\delta_i} z_{\theta_1+\alpha_i} \cdots z_{\theta_{\delta_i}+\alpha_i} \tag{30}$$

and

$$\beta!\, f^2_{(\alpha,\delta)} \prod_{i=1}^m (\alpha_i!)^{\delta_i} w_{\theta_1+\alpha_i} \cdots z_{\theta_{\delta_i}+\alpha_i} \tag{31}$$

respectively. In (30) and (31) we have $\alpha, \theta \in \mathbb{Z}^n_{\geqslant 0}$, $\delta = (\delta_1, \dots, \delta_m) \in \mathbb{Z}^m_{\geqslant 0}$, and the $f^j_{(\alpha,\delta)}$, $j = 1, 2$, are the coefficients of the series expansions of the f^j. We note that $\alpha + \sum \theta_i = \beta \prec \alpha_0$ in (30) and (31). Hence, by conditions 1) and 2), every product in (31) is greater than or equal to the modulus of the corresponding product (30). This proves the lemma.

The following easy lemma is proved, for example, in [2].

Lemma 6. *Suppose that the series* $A(x) = \sum_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} a_\alpha x^\alpha$ *converges absolutely at the point* $x_1 = \dots = x_n = \rho > 0$. *Let* M *be a positive number which is larger than the absolute value of every term of the series* $A(\rho)$. *Then the power series expansions of the functions* $F_1(x) = \frac{M}{(1-x_1/\rho)\dots(1-x_n/\rho)}$ *and* $F_2 = \frac{M}{(1-(x_1+\dots+x_n)/\rho)}$ *in a neighbourhood of zero majorize the series* $A(x)$.

5.2.2. *Statement of conditions of the special case.* There is no loss of generality in assuming that the coefficients z_α of the series z are equal to 0 for $\alpha \in \mathbb{Z}^n_{\geqslant 0} \setminus I$ because the series naturally composed from them determines an analytic function in a neighbourhood of the origin.

We consider the following special case. Suppose that the system (26) is linear in the highest-order derivatives, and all the main derivatives (that is, those that appear on the left-hand sides of the equations) are of the same order. More precisely, the system is given by

$$\begin{aligned} \partial_{\gamma_1} z &= \sum_{|\alpha|=N,\, \alpha\prec\gamma_1} f^1_\alpha\, \partial_\alpha z + f^1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \partial_{\gamma_k} z &= \sum_{|\alpha|=N, \alpha\prec\gamma_k} f^k_\alpha\, \partial_\alpha z + f^k, \end{aligned} \tag{32}$$

where $|\gamma_1| = \dots = |\gamma_k| = N > 0$, and f^i_α, f^i are holomorphic functions of the variables $x_1, \dots, x_n$ and the derivatives $\partial_\beta z$ with $|\beta| < N$.

5.2.3. *A change of coordinates.* Here we make a change of coordinates such that the leading derivatives become "main". Under this coordinate change, the coefficients of the other derivatives are multiplied by small numbers.

By assumption, f^i_α and f^i are holomorphic functions of their arguments in a neighbourhood of zero. For all admissible values of i and α we expand the functions f^i_α, f^i and M^α_i into power series in the neighbourhood of zero. Suppose that all these series converge absolutely at the point $x_1 = \dots = x_n = \partial_\alpha z = \rho$, where the parameter α is an arbitrary element of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ with $|\alpha| < N$. Using Lemma 6, we choose a positive real constant C such that the expansion of the function

$$\frac{C}{1 - (x_1 + \dots + x_n + \sum_{|\alpha|<N} \partial_\alpha z)/\rho}$$

majorizes the corresponding expansions of f^i_α, f^i.

We denote by Π the functional Π_N (see § 2.1). Put $\mu = \mu_N$ and consider a positive real number $\theta < 1$ such that

$$\theta^\mu < \varepsilon = \frac{1}{2}\,\frac{1}{\Delta_N C},$$

where Δ_i $(i \in \mathbb{Z}_{\geqslant 0})$ is the number of elements α in $\mathbb{Z}^n_{\geqslant 0}$ such that $|\alpha| = i$.

We put $y_i = \theta^{-\pi_i} x_i$, $1 \leqslant i \leqslant n$. Then

$$\frac{\partial^{|\alpha|} z}{\partial x^\alpha}(x) = \theta^{-\Pi(\alpha)} \frac{\partial^{|\alpha|} z}{\partial y^\alpha}(y(x)).$$

Making the change of coordinates and dividing by the coefficient of the leading derivative, we rewrite equations (32) in the form

$$\begin{aligned} \partial_{\gamma_i} z = \sum_{|\alpha|=N} & f^i_\alpha\big(y, \theta^{-\Pi(\gamma)} \partial_\gamma z(y)\big) \theta^{\Pi(\gamma_i)-\Pi(\alpha)} \partial_\alpha z \\ & + f^i\big(y, \theta^{-\Pi(\gamma)} \partial_\gamma z(y)\big) \theta^{\Pi(\gamma_i)}, \end{aligned} \tag{33}$$

where i runs from 1 to k.

Here and in what follows we consider operators of partial differentiation in the new system of coordinates. The formal series $z(y) = z(x(y))$ satisfies the differential equations (33) in the new system of coordinates (with "identically vanishing initial conditions"). For all admissible α, i we put

$$\tilde{f}^i_\alpha(y, \partial_\gamma z) = f^i_\alpha\big(y, \theta^{-\Pi(\gamma)}\partial_\gamma z\big)\theta^{\Pi(\gamma_i)-\Pi(\alpha)},$$
$$\tilde{f}^i(y, \partial_\gamma z) = f^i\big(y, \theta^{-\Pi(\gamma)}\partial_\gamma z\big)\theta^{\Pi(\gamma_i)}.$$

Note that in (33) we have

$$\theta^{\Pi(\gamma_i)} \leqslant \theta^{\Pi(\gamma_i)-\Pi(\alpha)} \leqslant \theta^{\Pi(\gamma)} \leqslant \theta^\mu < \varepsilon$$

since $|\gamma| < |\alpha| = |\gamma_i| = N$. Hence the following lemma is proved.

Lemma 7. *For some $0 < \rho_1 \ll \rho$, the power series expansion of*

$$\frac{\varepsilon C}{1 - (y_1 + \dots + y_n + \sum_{|\alpha|<N} \partial_\alpha z)/\rho_1}$$

majorizes the corresponding expansions of the functions $\tilde{f}^i_\alpha$, $\tilde{f}^i$ that depend on the variables y_i and some partial derivatives with respect to these variables, for all admissible values of i and α.

5.2.4. *The construction of a majorizing equation.* Consider the ordinary differential equation

$$Y^{(N)}(t) = \frac{\varepsilon C}{1 - \big(t + \sum_{j=1}^{N-1} \Delta_j Y^{(j)}(t)\big)/\rho_1}\big(\Delta_N Y^{(N)}(t) + 1\big). \tag{34}$$

Resolving this equation with respect to the leading derivative, we get

$$Y^{(N)}(t) = \frac{2\varepsilon C}{1 - 2\big(t + \sum_{j=1}^{N-1} \Delta_j Y^{(j)}(t)\big)/\rho_1}\,. \tag{35}$$

In (35) we took into account the fact that $\varepsilon\Delta_N C = 1/2$. By the existence and uniqueness theorem for ordinary differential equations, the equation (35) ((34)) has a unique solution $Z(t)$ with initial conditions $Z^{(0)} = \dots = Z^{(N-1)} = 0$. It is clear that the power series expansion of $Z(t)$ in a neighbourhood of zero has strictly positive coefficients. We put

$$G(Y, t) = \frac{\varepsilon C}{1 - \big(t + \sum_{j=1}^{N-1} \Delta_j Y^{(j)}(t)\big)/\rho_1}\,.$$

5.2.5. *The construction of a majorizing series.*

Lemma 8. *The power series expansion of the function $Z\big(\sum_{i=1}^n y_i\big)$ in a neighbourhood of zero majorizes the formal solution $z(y)$. Hence the series $z(y)$ converges in a neighbourhood of zero.*

Proof. Write $Z\big(\sum_i y_i\big) = \sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}} Z_\alpha y^\alpha$. We use induction on $\alpha \in \mathbb{Z}^n_{\geqslant 0}$ to establish the inequality

$$|z_\alpha| \leqslant Z_\alpha. \tag{36}$$

Indeed, (36) holds for $\alpha \in \mathbb{Z}^n_{\geqslant 0} \setminus I$ and, therefore, for $\alpha = 0$. Suppose that (36) holds for all $\alpha \prec \alpha_0$ and let us prove it for α_0. We write $\alpha_0 = \beta + \gamma_i$ for some i, $1 \leqslant i \leqslant k$.

Then

$$\alpha_0!\,|z_{\alpha_0}| = \bigl|\partial_{\alpha_0} z|_0\bigr| = \left|\left(\partial_\beta\left(\sum_{|\alpha|=N,\,\alpha\prec\gamma_i} \tilde{f}^i_\alpha \partial_\alpha z + \tilde{f}^i\right)\Big|_0\right)\right|.$$

Using Lemma 5 and the equation

$$\partial_\alpha F\Bigl(\sum y_i\Bigr) = F^{(|\alpha|)}\Bigl(\sum y_i\Bigr), \tag{37}$$

where F is an arbitrary holomorphic function, we get

$$\left|\left(\partial_\beta\left[\sum_{|\alpha|=N,\alpha\prec\gamma_i} \tilde{f}^i_\alpha \partial_\alpha z + \tilde{f}^i\right]\right)\Big|_0\right)\right| \\ \leqslant \left(\partial_\beta\left[G\Bigl(Z, \sum_i y_i\Bigr)\Bigl(\Delta_N Z^{(N)}\Bigl(\sum_i y_i\Bigr) + 1\Bigr)\right]\right)\Big|_{y=0}.$$

Indeed, Lemma 7 and the induction hypothesis show that every term of the form

$$\tilde{f}^i_\alpha \partial_\alpha z \tag{38}$$

is majorized by

$$G\Bigl(Z, \sum_i y_i\Bigr) Z^{(N)}\Bigl(\sum_i y_i\Bigr), \tag{39}$$

and the number of terms of the form (38) does not exceed Δ_N. We get the necessary estimate for the other terms and then apply Lemma 5. By (34) and (37) we have

$$\left(\partial_\beta\left[G\Bigl(Z, \sum_i y_i\Bigr)\Bigl(\Delta_N Z^{(N)}\Bigl(\sum_i y_i\Bigr) + 1\Bigr)\right]\right)\Big|_{y=0} \\ = \partial_{\alpha_0} Z\Bigl(\sum_i y_i\Bigr)\Big|_0 = \alpha_0!\, Z_{\alpha_0},$$

which proves the lemma.

5.2.6. *Completion of the proof of the theorem.* It remains to note that the case of arbitrary differential equations reduces to the case studied above. Indeed, we can replace the original equations by the finite set of all their consequences of the form

$$\partial_\beta \partial_{\gamma_i} z = \partial_\beta f_i(x, \partial_\alpha z),$$

where $|\beta| + |\gamma_i| = N$ and N is sufficiently large (say, $N = \max_i |\gamma_i| + 1$). Using this transformation, we get a set of differential equations of the form (32). The formal series $z(x)$ satisfies the new system of equations. The set $\mathbb{Z}^n_{\geqslant 0} \setminus I$ is increased by finitely many elements. This does not influence the convergence of the series $\tilde{z}(x)$ that determines the truncation of $z(x)$.

§ 6. Examples and remarks

6.1. On the conditions imposed on the ordering $\prec$. It turns out that the following condition (which is weaker than condition a)) also enables one to construct formal solutions of the system.

a′) We have $0 \prec \alpha$ for every element α of the semigroup.

All the lemmas on the ordered semigroup and the properties of the Gröbner map remain valid if we replace condition a) by a′). In particular, if condition a′) holds, then the ordered semigroup $(\mathbb{Z}^n_{\geqslant 0}, \preceq)$ is a totally ordered set. The statement and proof of Theorem 1 do not change in this case.

The following example of Kowalevsky (see [7] or [2]) shows that condition a′) does not guarantee that an analogue of Theorem 3 holds. Consider the equation

$$\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial x^2}\,.$$

It is easy to construct an ordering $\prec$ such that conditions a′), b) hold and $\frac{\partial^2}{\partial x^2} \prec \frac{\partial}{\partial y}$. One easily verifies that the formal series constructed from the initial data

$$z|_{y=y_0} = \phi(x),$$

where $\phi(x)$ is an arbitrary holomorphic function, may have zero radius of convergence.

6.2. The case of several unknown functions. It is important to note that our theorems on the existence and uniqueness of formal and analytic solutions can easily be extended to the case of linear systems with several unknown functions $z_1, \dots, z_p$.

The set of derivatives of $z_1, \dots, z_p$ is parametrized by points of the product $Z = \mathbb{Z}^n_{\geqslant 0} \times \{1, \dots, p\}$. Let $\prec_{\mathbb{Z}^n_{\geqslant 0}}$ be an ordering on the semigroup $\mathbb{Z}^n_{\geqslant 0}$ satisfying conditions a), b) of § 2.1. Consider the following total ordering $\prec$ on Z. Given any elements (α, i), (β, j) of Z, we compare the elements α, β of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ with respect to the order $\prec_{\mathbb{Z}^n_{\geqslant 0}}$. If they coincide, then we compare the numbers i and j (as integers). In this case, one can carry out analogues of all the constructions in this paper and prove direct analogues of Theorems 1–4.

6.3. On spaces of solutions at points of the "bad" hypersurface Σ. The techniques described above enable one to study the spaces of formal and (germs of) analytic solutions of the system at points of the complement of an analytic hypersurface Σ (see § 4.2). The following example shows that the structure of spaces of formal and analytic solutions at some points of Σ may differ from their structure at points of the complement.

Consider an equation of the form

$$\sum_{i=1}^{n} a_i x_i \frac{\partial z}{\partial x_i} = 0,$$

where the a_i are integers. Then the "bad" hypersurface Σ is one of the hyperplanes $x_i = 0$, depending on the choice of the ordering $\prec$. It is easy to choose the a_i in such a way that Theorems 1–4 do not hold for the point 0 of Σ. In particular, the function $H(0, i)$ may fail to be a polynomial on a set of sufficiently large positive integers.

6.4. The algebraic meaning of the Hilbert function. Consider the space $\mathbb{C}^n$ with coordinates $\xi_1, \dots, \xi_n$. We recall that the principal symbol of the system is a family $M(u)$ of algebraic varieties (more precisely, ideals in the ring $\mathbb{C}[\xi_1, \dots, \xi_n]$ of polynomials), where the parameter u belongs to the domain U. The family $M(u)$ is defined as follows. For the operator

$$D_i = \sum_{\alpha \in \operatorname{supp} D_i} d_\alpha \partial_\alpha \tag{40}$$

on the left-hand side of the ith equation of (8), we consider the family of homogeneous polynomials

$$\widetilde{D}_i(u, \xi) = \sum_{\substack{\alpha \in \operatorname{supp} D_i \\ |\alpha| = r(D_i)}} d_\alpha(u) \xi^\alpha, \tag{41}$$

where ξ^α is the monomial $\xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ and $r(D_i)$ is the order of the operator D_i. Then $M(u)$ is the family of ideals in $\mathbb{C}[\xi_1, \dots, \xi_n]$ generated by all the polynomials $\widetilde{D}_i$.

Using the construction of Gröbner bases, one can easily prove the following assertion on the Hilbert function of the system (8).

Proposition 8. *For every point $u \in U \setminus \Sigma$ and every non-negative integer i we have*

$$H_{M(u)}(i) = H(u, i), \tag{42}$$

where $H_{M(u)}$ is the Hilbert function of the algebraic variety $M(u)$.

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A. G. Khovanskii
Institute of System Analysis of RAS
E-mail: askold@math.toronto.edu

Received 8/SEP/03
Translated by A. V. DOMRIN

S. P. Chulkov
Moscow Independent University
E-mail: chulkov@mccme.ru