Solvability of equations by explicit formulae (Liouville's theory, differential Galois's theory and topological obstructions)

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This appendix is dedicated to the study of the solvability of differential equations by explicit formulae. This is a quite old problem: the fist idea of solving it dates back to Abel. Today one knows three approaches to solve this problem. The first one belongs to Liouville; the second approach considers the problem from the point of view of the Galois's theory: it is related to the names of Picard, Vessiot, Kolchin and others; the third approach, topological, was firstly introduced, in the case of functions of one variable, in my thesis. I am infinitely grateful to my research director V.I. Arnold, who aroused my interest in this subject.

I had always believed that the topological approach cannot be completely applied to the case of many variables. Only recently I discovered that this is not true and that in the multi-dimensional case one can obtain absolutely analogous results [25].

This appendix contains the subject of my lectures to the Mathematical Society of Moscow and to the students of the École Normale Supérieure at the Independent University of Moscow (october 1994).

The section, concerning the functions of many variables, was added for this appendix in autumn 2002.

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1 Explicit solvability of equations

Some differential equations possess "explicit solutions". If it is the case, the solution gives itself the answer to the problem of solvability. But in general all attempts to find explicit solutions of equations turn out to be vain. One trays thus to prove that for some class of equations explicit solutions do not exist. We must now define correctly what this means (otherwise, it will not clear what we wish really to demonstrate). We choose the following way: we distinguish some classes of functions, and we say that an equation is explicitly solvable, if its solution belong to one of these classes. To different classes of functions there correspond different notions of solvability.

To define a class of function we give a list of *basic functions* and a list of *allowed operations*.

The class of functions is thus defined as the set of all functions which are obtained from the basic functions by means of the allowed operations.

EXAMPLE 1. The class of functions *representable by radicals*.

List of basic functions: constants and the identity function (whose value is equal to that of the independent variable).

List of the allowed operations: the arithmetical operations (addition, subtraction, multiplication, division) and the root extractions $\sqrt[n]{f}$, $n = 2, 3, \dots$ of a given function f.

The function $g(x) = \sqrt[3]{5x + 2\sqrt[2]{x}} + \sqrt[7]{x^3 + 3}$ is an example of a function representable by radicals.

The famous problem of the solvability of the algebraic equations by radicals is related to this class. Consider the algebraic equation

$$y^n + r_1(x)y^{n-1} + \cdots + r_n(x) = 0, \tag{1}$$

in which $r_i(x)$ are rational functions of one variable. The complete answer to the problem of solvability of equation (1) by radicals consists in the Galois theory (see §8).

Note that already in the simplest class, that in example 1, we encounter some difficulties: the functions we deal with are multivalued.

Let we see exactly, for example, what is the sum of two multivalued analytical functions $f(x)$ and $g(x)$. Consider an arbitrary point a, one of the germs f_a of function $f(x)$ at point a and one of the germs g_a of function $g(x)$ at the same point a. We say that the function $\varphi(x)$, defined by the germ $f_a + g_a$, is representable as sum of functions $f(x)$ and $g(x)$. This definition is not univocal. For example, one sees easily that there are exactly two functions representable as the sum $\sqrt{x} + \sqrt{x}$, namely $f_1 = 2\sqrt{x}$ and $f_2 \equiv 0$. The closure of a class of multivalued functions with respect to the addition is a class which contains, together with any two functions, all functions representable by their sum.

One can say the same for all the operations on the multivalued functions that we shall encounter in this chapter.

EXAMPLE 2. *Elementary functions.* Basic elementary functions are those functions which one learns at school and which are usually represented on the keyboard of calculators. Their list is the following: the constant function, the identity function (associating to every value, x, of the argument the value x itself), the n-th roots $\sqrt[n]{x}$, the exponential $\exp x$, the logarithm $\ln x$, the trigonometrical functions: $\sin x$, $\cos x$, $\tan x$, $\arcsin x$, $\arccos x$, $\arctan x$. The allowed operations are: the arithmetical operations, the composition.

Elementary functions are expressed by formulae, for instance:

$$f(x) = \arctan(\exp(\sin x) + \cos x).$$

From the beginning of the study of analysis, we learn that the integration of elementary functions is very far from being an easy task. Liouville proved in fact that the indefinite integrals of elementary functions are not, in general, elementary functions.

EXAMPLE 3. *Functions representable by quadratures.* The basic functions in this class are the basic elementary functions. The allowed operations are the arithmetical operations, the composition and the integration. A class is said *closed with respect to the integration*, if it contains with every function f also a function g such that $g' = f$.

For example, the function

$$\exp\left(\int^x \frac{\mathrm{d}t}{\ln t}\right)$$

is representable by quadratures. But, as Liouville had proven, this function is not elementary.

Examples 2 and 3 can be modified. We shall say that a class of functions is closed with respect to the solutions of the algebraic equations, if together with every set of functions $f_1, \dots, f_n$ it contains also a function y, satisfying the equation

$$y^n + f_1 y^{n-1} + \cdots + f_n = 0.$$

EXAMPLE 4. If in the definition of the class of elementary functions we add the operation of solution of algebraic equations, we obtain the class of the *generalized elementary functions.*

EXAMPLE 5. The class of functions representable by *generalized quadratures* contains the functions obtained from the class of functions representable by quadratures by adding the operation of solution of algebraic equations.

2 Liouville's theory

The first exact demonstrations of the non solvability of some equations neither by quadratures nor by elementary functions were obtained by Liouville in the middle of the XIX century. Here we briefly expose his results.

The reader can find a wider exposition of the Liouville method and of the works on analogous subjects by Chebychev, Mordukai-Boltovski, Ostrovski and Ritt in book [1].

First of all, Liouville showed that the classes of functions in Examples 2–5 can be constructed in a very simple way. Indeed, the set of basic elementary functions seems to be very large. Moreover, in the definition of this class one encounters some algebraic difficulties due to the composition operation. Liouville at first proved that one can reduce a lot the lists of basic functions, in one half of the cases leaving in it only the constants, ad in the remaining cases leaving only the constants and the identity function. Secondly, he proved that in the list of the allowed operations the composition is superfluous. One can define all the necessary operations using only arithmetical operations and differentiation. This fact plays an essential role for the algebraization of the problem of the differential fields numerability.

Let us formulate the corresponding definitions in differential algebra.

A field of functions F is called a *differential field* if it is closed with respect to the differentiation, i.e., if $g \in F$, then $g' \in F$. One can also consider the abstract differential fields, i.e., the field in which there is defined a supplementary differentiation operation, satisfying the Leibniz identity: $(a \cdot b)' = a' \cdot b + a \cdot b'$.

Suppose that a differential field F contains another smaller differential field F_0, $F_0 \subseteq F$. An element $y \in F$ is said *algebraic* over field F_0, if y satisfies an algebraic equation of type:

$$y^n + a_1 y^{n-1} + \cdots + a_n = 0,$$

where the coefficients a_i's belong to field F_0. In particular, element y is called *radical* over field F_0, if $y^k \in F_0$. The element y is said *integral* over field F_0, if $y' \in F_0$. The element y is said logarithmic over field F_0, if $y' = a'/a$, where $a \in F_0$. The element y is said *exponential integral* over field F_0, if $y' = ay$, $a \in F_0$. The element y is said *exponential* over field F_0, if $y' = a'y$. *The extension of field* F_0 by means of element y, denoted by $F_0\{y\}$, is called the *minimal differential field, containing* F_0 *and* y. The field $F_0\{y\}$ consists of the rational functions in $y, y', \dots, y^{(k)}, \dots$ with coefficients in F_0.

- 1) Element y is said *representable by radicals over field* F_0, if there exists a sequence $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k$, such that every extension $F_i \subseteq F_{i+1}$ is obtained by adding one radical to field F_i, and field $F_0\{y\}$ is contained in F_k. A sequence of this type is called a *tower*.

By this method one defines also other types of representability of an element y over a field F_0. The towers in these definitions are built by means of other types of extensions $F_i \subseteq F_{i+1}$:

- 2) Element y is said *elementary over field* F_0 when one can add logarithmic and exponential elements.
- 3) Element y is said *representable by quadratures over field* F_0 when adding integrals and exponential integrals is allowed.
- 4) Element y is called *generalized elementary element over field* F_0 when one can add algebraic, exponential and logarithmic elements.
- 5) Element y is said *representable by generalized quadratures over field* F_0 when one can add algebraic, integral and exponential integral elements.

THEOREM 1. (Liouville) *A function is elementary (a generalized elementary function), if and only if it is an elementary (generalized elementary) element over the field of the rational functions $\mathcal{R}$. A function is representable by quadratures (representable by generalized quadratures), if and only if it is representable by quadratures (representable by generalized quadratures) over the field of the complex numbers $\mathbb{C}$.*

For example, it follows from Theorem 1 that the basic elementary function $f(x) = \arctan x$ is representable by quadratures over the field $F_0 = \mathbb{C}$. Indeed, this becomes clear from the equation $f' \equiv \dfrac{1}{1+x^2}$, $x' \equiv 1$.

To prove, for example, the part of theorem 1 which concerns functions representable by quadratures, it suffices to verify, firstly, that there exist analogous representations for all the basic elementary functions, and, furthermore, that the class of functions representable by quadratures over field $\mathbb{C}$ is closed with respect to the composition.

Liouville constructed a nice theory on the solvability of equations. Let us show two examples of his results.

THEOREM 2 (Liouville). *The indefinite integral $y(x)$ of the algebraic function $A(x)$ of one complex variable is representable by generalized elementary functions if and only if it is representable in the form*

$$y(x) = \int^x A(t)\mathrm{d}t = A_0(x) + \sum_{i=1}^{k} \lambda_i \ln A_i(x),$$

where the $A_i(x)$'s, for $i = 0, 1 \ldots, k$, are algebraic functions.

A priori the integral of an algebraic function could be given by a very complicated formula. It could have the form

$$y = \exp(\exp(\exp(\exp(\exp(x))))).$$

Theorem 2 says that this does not happen. Either the integral of an algebraic function can be written in a simple way, or in general it is not a generalized elementary function.

THEOREM 3 (LIOUVILLE). *The differential linear equation*

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

where $p(x)$ and $q(x)$ are rational functions, is solvable by generalized quadratures, if and only if its solution can be written in the form

$$y = \exp(\int^x R(t)\mathrm{d}t),$$

where $R(x)$ is an algebraic function.

A priori the solution of equation (2) could be expressed by very complicated formulae. Theorem 3 says that this is nowhere the case. Either the equation has sufficiently simple roots, or in general it cannot be solved by generalized quadratures.

Liouville found a series of results of this type. The common idea is the following: simple equations have either simple solutions, or in general have no solutions in a given class (by quadratures, by elementary functions etc.)

The strategy of the proof in Liouville's theory is the following: prove that if a simple equation has a solution which is represented by a complicate formula, then this formula can be always simplified.

Liouville, undoubtedly, was inspired by the results by Lagrange, Abel and Galois on the non solvability by radicals of algebraic equations. Differently from the Galois theory, Liuoville's theory does not involve the notion of group of automorphisms. Liouville however uses, in order to simplify his formulae, " infinitely small automorphisms".

Let us come back to Theorem 2 on the integrability of algebraic functions. The following corollary follows from this theorem.

COROLLARY. *If the integral of an algebraic function A is a generalized elementary function, then the differential form $A(x)dx$ has some unavoidable singularities on the Riemann surface of the algebraic function A.*

It is well known that on every algebraic curve with positive genus there exist non singular differential forms (the so-called abelian differentials of first type). It follows that algebraic functions whose Riemann surfaces have positive genus, are not, in general, integrable by generalized elementary functions.

This was already known to Abel, who discovered it as he was proving the non solvability by radicals of the fifth-degree generic equation. Observe also that the Abel demonstration of the non solvability by radicals id based on topological arguments. I do not know whether the topological properties of the Riemann surfaces of functions representable by generalized quadratures are different from those of the Riemann surfaces of generalized elementary functions. Indeed, I am unable to prove through topological arguments that the integral of an algebraic function is not an elementary function: each one of such integrals is by definition a function representable by generalized quadratures. However, if an algebraic function depends on a parameter, its integral may depend on the parameter in an arbitrarily complicate manner. One can prove that *the integral of an algebraic function, as function of one parameter, can be not representable by generalized quadratures* and, consequently, *can be not a generalized elementary function of the parameter* (cf. example in §9).

3 Picard-Vessiot's theory

Consider the linear differential equation

$$y^{(n)} + r_1(x)y^{(n-1)} + \cdots + r_n(x)y = 0, \tag{3}$$

in which the $r_i(x)$'s are rational functions of complex argument.

Near a non singular point x_0 there exist n linearly independent solutions $y_1, \dots, y_n$ of equation (3). In this neighbourhood one can consider the functions field $\mathcal{R}\{y_1, \dots, y_n\}$, obtained by adding to the field of rational functions $\mathcal{R}$ all solutions y_i and all their derivatives $y_i^{(p)}$ until order $(n-1)$. (The derivatives of higher order are obtained from equation (3).

The field of functions $\mathcal{R}\{y_1, \dots, y_n\}$ is a differential field, i.e., it is closed with respect to the differentiation, as well as the field of rational functions $\mathcal{R}$. One calls *automorphism of the differential field F* an automorphism σ of field F, which preserves also the differentiation, i.e., $\sigma(g') = [\sigma(g)]'$. Consider an automorphism σ of the differential field $\mathcal{R}\{y_1, \dots, y_n\}$, which fixes all elements of field $\mathcal{R}$. The set of all automorphisms of this type forms a group, which is called the *Galois group* of equation (3). Every automorphism σ of the Galois group sends a solution of the equation to a solution of the equation. Hence to each one of such automorphisms there correspond a linear transform M_σ of the space of solutions, V^n. The automorphism σ is completely defined by transform M_σ, because field $\mathcal{R}\{y_1, \dots, y_n\}$ is generated by functions y_i's. In general, not every linear transform of space V^n is an automorphism σ of the Galois group. The raison is that the automorphism σ preserves all differential relations holding among the solutions. The Galois group can be considered as a special group of linear transforms of solutions. It turns out that this group is algebraic.

So, the Galois group of an equation is the algebraic group of linear transforms of the space of solutions preserving all differential relations among the solutions.

Picard began to translate systematically the Galois theory in the case of linear differential equations. As in the original Galois theory, also here one finds a one-to-one correspondence (the *Galois correspondence*) between the intermediate differential field and the algebraic subgroups of the Galois group.

Picard and Vessiot proved in 1910 that the sole responsible of the solvability of an equation by quadratures and by generalized quadratures is its Galois group.

PICARD-VESSIOT'S THEOREM. *A differential equation is solvable by quadratures if and only if its Galois group is soluble. A differential equation is solvable by generalized quadratures if and only if the connected component of unity in its Galois group is soluble.*

The reader can find the basic results of the differential Galois theory in book [2]. In [3] he will find a brief exposition of the actual state of this theory together with a rich bibliography.

Observe that from the Picard-Vessiot theorem it is not difficult to deduce that, if equation (3) is solvable by generalized quadratures, then it has a solution of the form $y_1 = \exp(\int^x A_1(t), \mathrm{d}t)$, where $A_1(x)$ is an algebraic function. If the equation has an explicit solution y_1, then one can decrease its order, taking as new unknown function $z = (y/y_1)'$. Function z satisfies a differential equation having an explicit form and a lower order. If the initial equation was solvable, also the new equation for function z is solvable. It must have therefore, according to the Picard-Vessiot theorem, a solution of the type $z_1 = \exp \int A_2(x)\mathrm{d}x$, where A_2 is an algebraic function etc. We see in this way that if a linear equation is solvable by generalized quadratures, formulae expressing the solutions are not exceedingly complicate. Here the Picard-Vessiot approach coincides with the Liouville approach. Moreover, the criterion of solvability by generalized quadratures can be formulated without mentioning the Galois group. Indeed, equation (3) of order n is solvable by generalized quadratures if and only if has a solution of the form $y_1 = \exp \int^x A(x)\mathrm{d}x$ and the equation of order $(n-1)$ for function z is solvable by generalized quadratures.

This theorem was enounced and proved by Murdakai-Boltovskii exactly in this form. Murdakai and Boltovskii obtained at the same time this result in 1910 using the Liouville method, independently of the Picard and Vessiot works. The Mordukai-Boltovskii theorem is a generalization of the Liouville theorem (cf. Theorem 3 in the preceding section) to linear differential equations of any order.

4 Topological obstructions to the representation of functions by quadratures

There exist a third approach to the problem of the representability of a function by quadratures. (cf. [4]–[10]). Consider the functions representable by quadratures as multivalued analytical functions of a complex variable. It turns out that there are some topological restrictions on the kind of disposition on the complex plane of the Riemann surface of a function representable by quadratures. If the function does not satisfy these conditions, it cannot be represented by quadratures.

This approach, besides the geometrical evidence, possesses the following advantage. The topological obstructions are related to the character of the multivalued function. They hold not only for functions representable by quadratures, but also for a more wide class of functions. This class is obtained adding to the functions representable by quadratures all the meromorphic functions and allowing the presence of such functions in all formulae. Hence the topological results on the non representability by quadratures are stronger that those of algebraic nature. The raison of this is that the composition of two functions is not an algebraic operation. In differential algebra instead of the composition of two functions, one considers the differential equation that they satisfy. But, for instance, the Euler function Γ does not satisfy any algebraic differential equation; therefore it is useless to search out an equation satisfied, for example, from the function $\Gamma(\exp x)$. The unique known results on the non representability of functions by quadratures and, for instance, by the Euler functions Γ are those obtained by our method.

On the other hand, by this method one cannot prove the non representability by quadratures of an arbitrary meromorphic single-valued function.

Using the Galois differential theory (and precisely its linear-algebraic part, related with the matrix algebraic groups and their differential invariants), one can prove that the sole reason of the non solvability by quadratures of the linear differential equations of Fuchs type (cf. §11) is of topological nature. In other words, when there are no topological obstructions to the solvability by quadratures for a differential equation of Fuchs type, this equation is solvable by quadratures.

The topological obstructions to the representation of a function by quadratures and by generalized quadratures are the following.

Firstly, functions representable by generalized quadratures and, as special case, by quadratures, can have at most a numerable set of singular points in the complex plane. (cf. §5) (whereas already for the simplest functions, representable by quadratures, the set of singular points may be everywhere dense!).

Secondly, the monodromy group of a function representable by quadratures is necessarily soluble (cf. §7) (whereas already for the simplest functions, representable by quadratures, the monodromy group may contain a continuum of elements!).

There exist also analogous topological restrictions on the disposition of the Riemann surface for functions representable by generalized quadratures. However, these restrictions cannot be simply formulated: in this case the monodromy group is not considered as an abstract group, but as the group of permutations of the sheets of the Riemann surface. In other words, in the formulation of such restrictions not only the monodromy group intervenes, but also the *monodromy pair* of the function. The monodromy pair of a function consists in its monodromy group and in a stationary subgroup for some germ (cf. §9). We shall see this geometrical approach to the problem of solvability more precisely.

5 $\mathcal{S}$-functions

We define a class of functions which will be the object of this section.

DEFINITION. One calls $\mathcal{S}$-*function* an analytical multivalued function of a complex variable, if the set of its singular points is at most numerable.

Let us make this definition more precise. Two regular germs f_a and g_b, defined at points a and b on the Riemann sphere S^2, are said equivalent, if germ g_b is obtained from the germ f_a by a regular continuation along some curve. Every germ g_b, equivalent to germ f_a, is called regular germ of the analytical multivalued function f, generated by germ f_a.

A point $b \in S^2$ is said *singular for the germ* f_a, if there exists a curve $\gamma\ [0, 1] \to S^2,\ \gamma(0) = a,\ \gamma(1) = b$, such that germ f_a cannot be regularly continued along this curve, but for every $t,\ 0 \leq t < 1$, this germ can be continued along the shortened curve $\gamma\ [0, t] \to S^2$. It is easy to see that the sets of singular points for equivalent germs do coincide.

A regular germ is called $\mathcal{S}$-*germ*, if the set of its singular points is at most numerable. An analytical multivalued function is called $\mathcal{S}$-function, if each one of its regular germs is a $\mathcal{S}$-germ.

We proved the the following theorem.

THEOREM ON THE CLOSURE OF THE CLASS OF $\mathcal{S}$-FUNCTIONS ([6],[8],1 0]). *The class* $\mathcal{S}$ *of all the* $\mathcal{S}$*-functions is closed with respect to the following operations:*

- *1) differentiation, i.e., if $f \in S$, then $f' \in S$;*
- *2) integration, i.e., if $f \in S$, then $\int f(x)dx \in S$;*
- *3) composition, i.e., if $g,\ f \in S$, then $g \circ f \in S$;*

- *4) meromorphic operation, i.e., if $f_i \in S$, $i = 1, \dots, n$, $F(x_1, \dots, x_n)$ is a meromorphic function of n variables and $f = F(f_1, \dots, f_n)$, then $f \in S$*[1]*;*

- *5) solution of algebraic equations, i.e., if $f_i \in S$, $i = 1, \dots, n$, and $f^n + f_1 f^{n-1} + \dots + f_n = 0$, then $f \in S$;*

- *6) solution of linear differential equations, i.e., if $f_i \in S$, $i = 1, \dots, n$, and $f^{(n)} + f_1 f^{(n-1)} + \dots + f_n = 0$, then $f \in S$.*

COROLLARY. *If the multivalued function f can be obtained from single-valued $\mathcal{S}$-functions by the operations of integration, differentiation, meromorphic operations, compositions, solutions of algebraic and linear differential equations, then function f has at most a numerable set of singular points. In particular, a function having a non numerable set of singular points is not representable by generalized quadratures.*

6 Monodromy group

The *monodromy group* of a $\mathcal{S}$-function f with a set A of singular points is the group of all permutations of the sheets of the Riemann surface of f which are visited when one moves round the points of set A.

More precisely, let F_a be the set of all germs of the $\mathcal{S}$-function f at point a, non belonging to set A of singular points. Consider a closed curve γ in $S^2 \setminus A$ beginning at point a. The continuation of every germ of set F_a along curve γ leads to a germ of set F_a.

Consequently, to every curve γ there corresponds a mapping of set F_a into itself, and to homotopic curves in $S^2 \setminus A$ there corresponds the same mapping. To the composition of curves there corresponds the mapping composition. One has thus defined an homomorphism τ of the fundamental group of set $S^2 \setminus A$ in the group $S(F_a)$ of the one-to-one mappings of set F_a into itself. One calls *monodromy group* of the $\mathcal{S}$-function f the image of the fundamental group $\pi_1(S^2 \setminus A, a)$ in group $S(F_a)$ under homomorphism τ.

We show some results, which are useful in the study of functions representable by quadratures as functions of one complex variable.

EXAMPLE. Consider the function $w(z) = \ln(1 - z^\alpha)$, where $\alpha > 0$ is an irrational number. Function w is an elementary function, given by a very simple formula. However its Riemann surface is very complicate. The set A of its singular points

[1] more precisely, *the meromorphic operation*, defined by the meromorphic function $F(x_1, \dots, x_n)$, puts in correspondence of functions $f_1, \dots, f_n$ a new function $F(f_1, \dots, f_n)$. The arithmetical operations and the exponential are examples of meromorphic operations, corresponding to functions $F_1(x, y) = x + y$, $F_2(x, y) = x \cdot y$, $F_3(x, y) = x/y$ and $F_4(x) = \exp x$.

consists of points $0, \infty$ and of points $a_k = e^{\frac{1}{\alpha}2k\pi i}$, where k is any integer. Since α is irrational, points a_k are densely distributed on the unitary circle. It is not difficult to prove that the fundamental group $\pi_1(S^2 \setminus A)$ and the monodromy group of function w are continuous. One can also prove that the image under homomorphism τ of the fundamental group $\pi_1(S^2 \setminus \{A \cup b\})$ of the complement of $A \cup b$, where $b \neq a_k$ is an arbitrary point on the unitary circle, is a proper subgroup of the monodromy group of function w. (The fact that the elimination of a sole point can produce a radical change of the monodromy group makes all demonstrations essentially difficult).

7 Obstructions to the representability of functions by quadratures

We proved the following theorem.

THEOREM ([6],[8],[10]). *The class of all $\mathcal{S}$-functions, having a soluble monodromy group, is closed with respect to the composition, the meromorphic operations, the integration and the differentiation.*

We thus obtain the following corollary.

RESULT ON QUADRATURES. *The monodromy group of a function f, representable by quadratures, is soluble. Moreover, also the monodromy group of every function f, which is obtained from single-valued $\mathcal{S}$-functions by means of compositions, meromorphic operations, integration and differentiation is soluble.*

We see now the application of this result to algebraic equations.

8 Solvability of algebraic equations

Consider the algebraic equation

$$y^n + r_1 y^{n-1} + \cdots + r_n = 0, \tag{4}$$

where the r_i's are rational functions of complex variable x.

Near to a non singular point x_0 there are all solutions $y_1, \ldots, y_n$ of equation (4). In this neighbourhood one can consider the field of all functions $\mathcal{R}\{y_1, \ldots, y_n\}$, obtained by adding to field $\mathcal{R}$ all solutions y_i.

Consider the automorphism σ of the field $\mathcal{R}\{y_1, \ldots, y_n\}$, which fixes every element of field $\mathcal{R}$. The totality of these automorphisms forms a group, which is called *Galois group of equation* (4). Every automorphism σ of the Galois group transforms a solution of the equation into a solution of the equation; consequently, to every automorphism σ there corresponds a permutation, S_σ, of the solutions. Automorphism σ is completely defined by permutation S_σ, because the field $\mathcal{R}\{y_1, \dots, y_n\}$ is generated by functions y_i. In general, not all permutations of the solutions can be continued to an automorphism σ of the Galois group: the reason is that automorphisms σ preserve all relations existing among the solutions.

The Galois group of an equation is thus the permutation group of the solutions which preserves all relations among the solutions.

Every permutation S_γ of the set of solutions can be continued, as automorphism of the monodromy group, to an automorphism of the entire field $\mathcal{R}\{y_1, \dots, y_n\}$. Indeed, with functions $y_1, \dots, y_n$, along curve γ, every element of field $\mathcal{R}\{y_1, \dots, y_n\}$ is continued meromorphically. This continuation gives the required automorphism, because during the continuation the arithmetical operations are preserved and every rational function comes back to its preceding values because of the univocity.

In this way, the monodromy group of the equation is contained in the Galois group: in fact, the Galois group coincides with the monodromy group. Indeed, the functions of the field $\mathcal{R}\{y_1, \dots, y_n\}$ which are fixed under the action of the monodromy group are the single-valued functions. These functions are algebraic, but every algebraic single-valued function is a rational function. Therefore the monodromy group and the Galois group have the same field of invariants, and thus, by the Galois theory, they coincide.

According to the Galois theory, equation (4) is solvable by radicals over the field of rational functions if and only if its Galois group is soluble over this field. In other words, the Galois theory proves the following facts:

1) *An algebraic function y, whose monodromy group is soluble, is representable by radicals.*

2) *An algebraic function y, whose monodromy group is not soluble, is not representable by radicals.*

Our theorem makes stronger result (2).

An algebraic function y, whose monodromy group is not soluble, cannot be represented though single-valued $\mathcal{S}$-functions by means of meromorphic operations, compositions, integrations and differentiations.

If an algebraic equation is not solvable by radicals, then it remains non solvable using the logarithms, the exponentials and the other meromorphic functions on the complex plane. A stronger version of this statement in given in §15.

9 The monodromy pair

The monodromy group of a function is not only an abstract group but is the group of transitive permutations of the sheets of its Riemann surface. Algebraically this object is given by a pair of groups: the permutations group and a subgroup of it, the stationary group of a certain element.

One calls the *monodromy pair of a $\mathcal{S}$-function* a pair of groups, consisting of the monodromy group of this function and the stationary subgroup of a sheet of the Riemann surface. The monodromy pair is defined correctly, i.e., this pair of groups, up to isomorphisms, does not depend on the choice of the sheet.

Definition. The pair of groups $[\Gamma, \Gamma_0]$ is called an *almost soluble pair of groups* if there exists a sequence of subgroups

$$\Gamma = \Gamma_1 \supseteq \cdots \supseteq \Gamma_m, \quad \Gamma_m \subset \Gamma_0,$$

such that for every $i, 1 \leq i \leq m-1$ group Γ_{i+1} is a normal divisor of group Γ_i and the quotient group Γ_i/Γ_{i+1} is either commutative, or finite.

Any group Γ can be considered as the pair of groups $[\Gamma, e]$, where e is the unit subgroup (the group containing only the unit element). We say that group Γ is *almost soluble* if pair $[\Gamma, e]$ is almost soluble.

Theorem ([6],[8],[10]). *The class of all $\mathcal{S}$-functions, having a monodromy pair almost soluble, is closed with respect to the composition, the meromorphic operations, the integration, the differentiation and the solutions of algebraic equations.*

We thus obtain the following corollary.

Result on generalized quadratures. *The monodromy pair of a the function f, representable by generalized quadratures, is almost soluble. Moreover, also the monodromy pair of every function f, which is obtained from single-valued $\mathcal{S}$-functions by means of the composition, the meromorphic operations, the integration, the differentiation and the solutions of algebraic equations is almost soluble.*

Let us consider now some examples of functions non representable by generalized quadratures. Suppose the Riemann surface of a function f be a universal covering of $S^2 \setminus A$, where S^2 is the Riemann sphere and A is a finite set, containing at least three points. Thus *function f cannot be expressed in terms od $\mathcal{S}$-functions by means of generalized quadratures, compositions and meromorphic operations.* Indeed, the monodromy pair of this function consists in a free non commutative group, and its unit subgroup. One sees easily that such a pair of groups is not almost soluble.

Example. Consider the function z, which realizes the conformal transformation of the upper semi-plane into the triangle with vanishing angles, bounded by three arcs of circle. Function z is the inverse of the modular Picard function. The Riemann surface of function z is a universal covering of the sphere without three points;

consequently function z cannot be expressed in terms od single-valued $\mathcal{S}$-functions by means of generalized quadratures, compositions and meromorphic operations.

Observe that function z is strictly related to the elliptic integrals

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \text{ and } K'(k) = \int_0^{\frac{1}{k}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Every one of functions $K(k)$, $K'(k)$ and $z(w)$ can be obtained from the others by quadratures. It follows that no one of the integrals $K(k)$ and $K'(k)$ can be expressed in terms of single-valued $\mathcal{S}$-functions by means of generalized quadratures, compositions and meromorphic operations.

In the following section we will generalize the above example, finding all polygons, bounded by arcs of circle, to which the upper semi-plane can be sent by functions representable by generalized quadratures.

10 Mapping of the semi-plane to a polygon, bounded by arcs of circle

10.1 Application of the symmetry principle.

Consider in the complex plane a polygon G, bounded by arcs of circle. According to the Riemann theorem, there exists a function f_G, sending the upper semi-plane to polygon G. This mapping was studied by Riemann, Schwarz, Christoffel, Klein and others (cf, for example, [11]). Let we recall some classical results that shall be useful.

Denote by $B = \{b_j\}$ the preimage of the set of the vertices of polygon G under mapping f_G, by $H(G)$ the group of conformal transformations of the sphere generated by the inversions with respect to the sides of the polygon and by $L(G)$ the subgroup of homographic mappings (quotient of two linear functions). $L(G)$ is a subgroup of index 2 of group $H(G)$. From the Riemann-Schwarz symmetry principle one obtains the following results.

PROPOSITION.

- *1) Function f_G can be meromorphically continued along any curve avoiding set B.*
- *2) All germs of the multivalued functions f_G in a non singular point $a \notin B$ are obtained applying to a fixed germ f_a the group of homographic mappings $L(G)$.*

- *3) The monodromy group of function f_G is isomorphic to group $L(G)$.*
- *4) At points b_j, the singularities of function f_G are of the following types. If in vertex a_j of polygon G, corresponding to point b_j, the angle is equal to $\alpha_j \neq 0$, then function f_G, through an homographic transformation, is put in the form $f_G(z) = (z - b_j)^{\beta_j}\varphi(z)$, where $\beta_j = \alpha_j/2\pi$, and function φ is holomorphic in a neighbourhood of point b_j. If angle α_j is equal to zero, then function f_G by an homographic transformation is put in the form $f_G(z) = \ln(z) + \varphi(z)$, where function φ is holomorphic in a neighbourhood of b_j.*

From our results it follows that if function f_G is representable by generalized quadratures, then group $L(G)$) and group $H(G)$ are almost soluble.

10.2 Almost soluble groups of homographic and conformal mappings

Let π be the epimorphism of group $SL(2)$ of the matrices of order 2 with unit determinant onto the group of the homographic mappings L,

$$\pi \ : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{az+b}{cz+d}.$$

Since $\ker \pi = \mathbb{Z}_2$, group $\tilde{L} \subseteq L$ and group $\pi^{-1}(\tilde{L}) = \Gamma \subseteq SL(2)$ are both almost soluble. Group Γ is a group of matrices: therefore Γ *is almost soluble if and only if it has a normal subgroup Γ_0 of finite index which admits a triangular form.* (This version of the Lie theorem is true also in higher dimensions and plays an important role in differential Galois theory). Since group Γ_0 consists of matrices of order 2, group Γ_0 can be put in triangular form in one of the three following cases:

- 1) group Γ_0 has only one monodimensional eigenspace;
- 2) group Γ_0 has two monodimensional eigenspaces;
- 3) group Γ_0 has a two-dimensional eigenspace.

Consider now the group of homographic mappings $\tilde{L} = \pi(\Gamma)$. Group $\tilde{L}$ of homographic mappings is almost soluble if and only it has a normal subgroup $L_0 = \pi(\Gamma_0)$ of finite index, and the set of the invariant points consists of either a unique point, or two points, or the whole Riemann sphere.

The group of conformal mappings $\tilde{H}$ contains the group $\tilde{L}$ of index 2 (or of index 1), consisting of the homographic mappings. Hence for the almost soluble group $\tilde{H}$ of conformal mappings an analogous proposition holds.

Lemma. *A group of conformal mappings of the sphere is almost soluble if and only if it satisfies almost one of these conditions:*

- 1) *the group has only an invariant point;*
- *2) the group has an invariant set consisting of two points;*
- *3) the group is finite.*

This lemma follows from the preceding propositions, because the set of invariant points for a normal divisor is invariant under the action of the group. It is well known that *a finite group $\tilde{L}$ of homographic mappings of the sphere is sent by a homographic transformation of coordinates to a group of rotations.*

It is not difficult to prove that if the product of two inversions with respect to two different circles corresponds, under the stereographic projection, to a rotation of the sphere, then these circles correspond to great circles. Hence *every finite group $\tilde{H}$ of conformal mappings generated by the inversions with respect to some circles, is sent by a homographic transformation of coordinates to a group of motions of the sphere, generated by reflections.*

All the finite groups of the motions of the sphere generated by reflections are well known. They are exactly the symmetry groups of the following objects:

- 1) the regular pyramid with a regular n-gone as basis;
- 2) the n-dihedron, i.e., the solid made from two regular pyramids joining their bases.
- 3) the tetrahedron;
- 4) the cube or the octahedron;
- 5) the dodecahedron or the icosahedron.

All these groups of symmetries, except the group of the dodecahedron-icosahedron, are soluble. The sphere, whose centre coincides with the centre of gravity of the solid, is cut by the symmetry planes of the solid along a net of great circles. Lattices, corresponding to the mentioned solids, are called finite nets of great circles. The stereographic projections of these finite nets are shown in Figures 1–5.

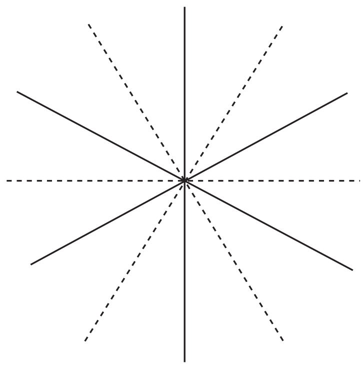

Figure 1: Pyramid

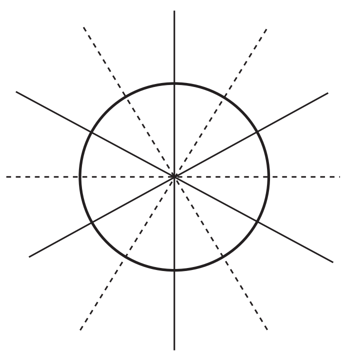

Figure 2: 6-dihedron

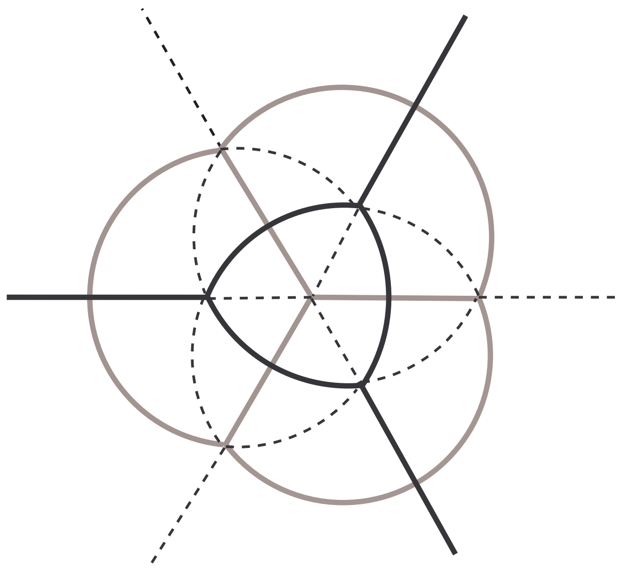

Figure 3: Tetrahedron–Tetrahedron

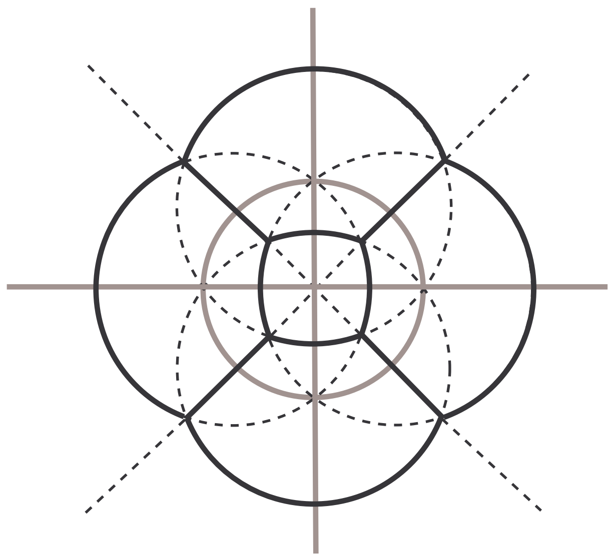

Figure 4: Cube–Octahedron

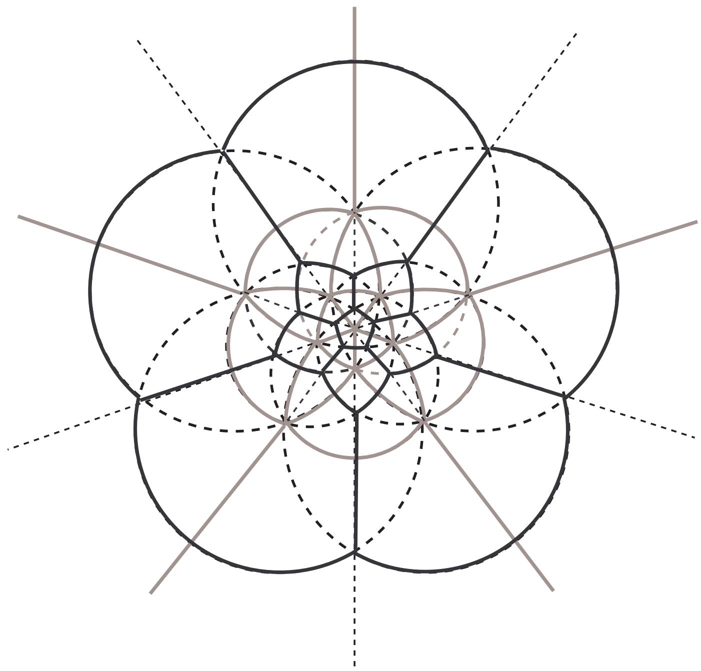

Figure 5: Dodecahedron–Icosahedron

10.3 The integrable case

Let us come back to the problem of the representability of the function f_G by generalized quadratures.

We consider now the different possible cases and we prove that the condition we have found is not only necessary but also sufficient for the representability of function f_G by generalized quadratures.

First integrability case. The group $H(G)$ has an invariant point. This means that the continuations of the edges of polygon G intersect in a point. Sending this point to infinity by a homographic transformation, we obtain the polygon $\overline{G}$, bounded by segments of straight lines (cf. Fig. 6).

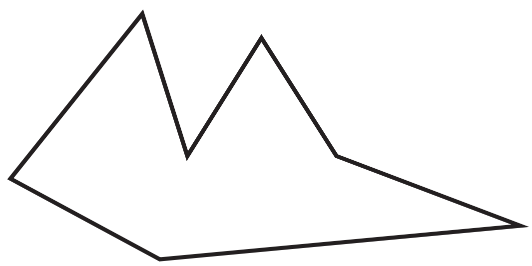

Figure 6: The first case of integrability.

All mappings of $L(\overline{G})$ have the form $z \to az + b$. All germs of function $\overline{f} = f_{\overline{G}}$ in a non singular point c are obtained applying to a fixed germ $\overline{f}_c$ the group $L(\overline{G})$, $\overline{f}_c \to a\overline{f}_c + b$. The germ $R_c = \overline{f}''_c/\overline{f}'_c$ is invariant under the action of group $L(\overline{G})$. This means that germ R_c is the germ of a single-valued function. A singular point b_j of function R_c can be only a pole (cf. proposition in §10.1). Thus function R_c is rational. The equation $\overline{f}''/\overline{f}' = R$ is integrable by quadratures. This case of integrability is well known. Function $\overline{f}$ in this case is called *Christoffel-Schwarz integral*.

Second integrability case. The invariant set of group $H(G)$ consists of two points. This means that there are two points with the following properties: for every side of polygon G, these points either are obtained by an inversion with respect to this side or belong to the continuation of this side. Sending one of these points to the origin and the other one to infinity by a homographic transformation, we obtain the polygon $\overline{G}$, bounded by arcs of circles with centre at point 0 and by segments of rays coming from point 0 (cf. Fig. 7).

All transformations of group $L(\overline{G})$ are of the form $z \to az$, $z \to \frac{b}{z}$. All germs of the function $\overline{f} = f_{\overline{G}}$ at a non singular point c are obtained applying to a fixed germ

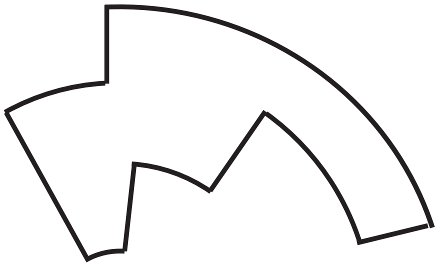

Figure 7: The second case of integrability.

$\overline{f}_c$ the transformations of the group $L(\overline{G})$

$$\overline{f}_c \to a\overline{f}_c,\ \overline{f}_c \to b/\overline{f}_c.$$

The germ $R_c = (\overline{f}'_c/\overline{f}_c)^2$ is invariant under the action of group $L(\overline{G})$ and it is the germ of the single-valued function R. The only singularities of function R are poles (cf. proposition in §10.1). Thus function R_c is rational. The equation $R = (\overline{f}''/\overline{f}')^2$ is integrable by quadratures.

Third integrability case. The group $H(G)$ is finite. This means that polygon G is sent by a homographic transformation to a polygon $\overline{G}$, whose sides lie on a finite net of great circles (see Figures 116-120). Group $L(G)$ is finite, and, as a consequence, function f_G has a finite number of values. Since all singularities of function f_G are of 'jump' type ((cf. proposition in §10.1), function f_G is an algebraic function.

Let us analyze the case when group $H(G)$ is finite and soluble. This happens if and only if polygon G is sent by a homographic transformation to a polygon $\overline{G}$, whose sides lie on a net of great circles different from that of the dodecahedron-icosahedron. In this case group $L(G)$ is soluble, and function f_G in expressed in terms od rational functions by means of arithmetical operations and of radicals (cf. §8).

From our results a theorem follows:

Theorem on the polygons bounded by of arcs of circles ([6],[8],[10]). *For an arbitrary polygon G, non belonging to the three above cases of integrability, function f_G not only is not representable by generalized quadratures, but it cannot be expressed in terms od single-valued $\mathcal{S}$-functions by means of generalized quadratures, compositions and meromorphic operations.*

11 Topological obstructions to the solvability of differential equations

11.1 The monodromy group of a linear differential equation and its relation with the Galois group

Consider the linear differential equation

$$y^{(n)} + r_1 y^{(n-1)} + \cdots + r_n y = 0, \tag{5}$$

where the r_i's are rational functions of the complex variable x. The poles of functions r_i and ∞ are called the singular points of equation (5).

Near a non singular point x_0 the solutions of the equations form a space V^n of dimension n. Consider now an arbitrary curve $\gamma(t)$ on the complex plane, beginning at x_0 and ending at point x_1 avoiding the singular points a_i. The solutions of the equation can be analytically continued along the curve, remaining solutions of the equation. Hence to every curve γ there corresponds a linear mapping M_γ of the space $V^n_{x_0}$ of the solutions at point x_0 in the space $V^n_{x_1}$ of the solutions at point x_1.

If one changes curve γ, avoiding the singular points and leaving fixed its ends, mapping M_γ does not vary. To a closed curve there corresponds therefore a linear transform of the space V^n into itself. The totality of these linear transforms of space V^n forms a group, which is called *monodromy group of equation* (5). So, the monodromy group of an equation is the group of the linear transforms of the solutions, which correspond to different turns round the singular points. The monodromy group of an equation characterizes the multivocity of its solutions.

Near a non singular point x_0 there are n linearly independent solutions, $y_1, \ldots, y_n$, of equation (5). In this neighbourhood one can consider the field of functions $\mathcal{R}\{y_1, \ldots, y_n\}$, obtained adding to field of rational functions $\mathcal{R}$ all solutions y_i and all their derivatives.

Every transformation M_γ of the monodromy group of the space of solutions can be continued to an automorphism of the entire field $\mathcal{R}\{y_1, \ldots, y_n\}$. Indeed, with functions $y_1, \ldots, y_n$, along curve γ every element of field $\mathcal{R}\{y_1, \ldots, y_n\}$ can be analytically continued. This continuation gives the required automorphism, because during the continuation the arithmetical operations and the differentiation are preserved, and the rational functions come back to their initial values because of their univocity.

In this way, *the monodromy group of an equation is contained in its Galois group.*

The field of the invariants of the monodromy group is a subfield of $\mathcal{R}\{y_1, \ldots, y_n\}$, consisting of the single-valued functions. Differently from the algebraic case, for differential equations the field of invariants under the action of the monodromy group can be bigger than the field of rational functions.

For example, for differential equation (5), in which all coefficients $r_i(x)$'s are polynomials, all solutions are single-valued. But, of course, the solutions of such equations are not always polynomials. The reason is that here the solutions of differential equations may grow exponentially in approaching the singular points. One knows an extension of the class of linear differential equations, for which there are no similar complications, i.e., for which the solutions, while approaching the singular points, grow at most as some power. Differential equations which possess this property are called equations of Fuchs type.

For differential equations of Fuchs type the Frobenius theorem holds.

THEOREM 1. *For the differential equations of Fuchs type, the subfield of the differential field $\mathcal{R}\{y_1, \dots, y_n\}$, consisting of single-valued functions, coincides with the field of rational functions.*

According to the differential Galois theory, from the Frobenius theorem it follows that the algebraic closure of the monodromy group, M, (i.e., the smallest algebraic group containing M) coincides with the Galois group.

The differential Galois theory gives thus the following criterion of solvability of differential equations of Fuchs type.

THEOREM 2. *A differential equation of Fuchs type is solvable by quadratures or by generalized quadratures if its monodromy group is, respectively, soluble or almost soluble.*

The differential Galois theory provides at the same time two results:

- *1) if the monodromy group of a differential equation of Fuchs type is soluble (almost soluble), then this equation is solvable by quadratures (by generalized quadratures).*

- *2) if the monodromy group of a differential equation of Fuchs type is not soluble (almost soluble), then this equation is not solvable by quadratures (by generalized quadratures).*

Our theorem makes stronger result 2. Indeed, it is easy to see that for almost every solution of differential equation (5) the monodromy pair is $[M, e]$, where M is the monodromy group of the equation, and e its trivial subgroup. We thus have the following:

THEOREM 3 ([6],[8]). *If the monodromy group of differential equation (5) is not soluble (almost soluble), then almost every solution of this equation is not representable in terms of single-valued $\mathcal{S}$-functions by means of compositions, meromorphic operations, integrations, differentiations and solutions of algebraic equations.*

Is the monodromy group of a given linear differential equation soluble (almost soluble)? This question turns out to be quite difficult. However, there exists an interesting example, where the answer to this question is very simple.

11.2 Systems of differential equations of Fuchs type with small coefficients

Consider a system of linear differential equations of Fuchs type, i.e., a system of type

$$y' = Ax \tag{6}$$

where $y = y_1, \ldots, y_n$ is the unknown vectorial function and A is a $n \times n$ matrix, consisting of rational functions of the complex variable x, having the following form:

$$A(x) = \sum_{i=1}^{k} \frac{A_i}{x - a_i},$$

where the A_i's are constant matrices.

If matrices A_i's are put at the same time in triangular form, then system (6), as every triangular system, is solvable by quadratures. There are undoubtedly non triangular systems which are solvable. However, if matrices A_i's are sufficiently small, then such systems do not exist. More precisely, we obtained the following results:

THEOREM 4([9]). *The non triangular system (6), with matrices A_i's sufficiently small, $\|A_i\| < \varepsilon(a_1, \ldots, a_k, n)$, is strictly non solvable, i.e., it is not solvable even using all single-valued $\mathcal{S}$-functions, compositions, meromorphic operations, integrations, differentiations and solutions of algebraic equations.*

The demonstration of this theorem uses the Lappo-Danilevskij theory [12].

12 Algebraic functions of several variables

Up to now we have considered only single-valued functions. We are ready to make two observations concerning functions of several variables, whose demonstrations do not require new notions and are obtained by the same method we used for single-valued functions.

Consider the algebraic equation

$$y^n + r_1 y^{n-1} + \cdots + r_n = 0, \tag{7}$$

where the r_i's are rational functions of k complex variables $x_1, \ldots, x_k$.

1) According to the Galois theory, equation (7), having a soluble monodromy group, is solvable by radicals. *But if the monodromy group of equation (7) is not soluble, then not only the equation is not solvable by radicals, but cannot be solved even using radicals of entire functions of several variables, arithmetical operations and compositions.* This statement can be considered as a variation of the Abel theorem on the non solvability of algebraic equations of degree higher than four. (A stronger result is presented in §15).

2) Equation (7) defines an algebraic function of k variables. Which are the conditions for representing a function of k variables by algebraic functions of a smaller number of variables, using compositions and arithmetical operations? The 13-th Hilbert problem consists in this question.[2]. If one excludes the remarkable

[2]The problem on the composition was formulated by Hilbert for classes of continuous functions, not for algebraic functions. A.G. Vitushkin considered this problem for smooth functions and proved the non representability of functions of n variables with continuous derivatives up to order p by functions of k variables with continuous derivatives up to order q having a lower "complexity", i.e., for $k/q < n/p$ [15]. Afterwards he applied his method to the study of the complexity in the problem of tabularizations [16].

Vitushkin's results was also proven by Kolmogorov, developing his own theory of the ϵ-entropy for classes of functions, measuring as well their complexity: this entropy, expressed by the logarithm of the number of ϵ-different functions, grows, for ϵ decreasing, as $(1/\epsilon)^{n/p}$ [17].

Finally, the solution of the problem in the Hilbert initial formulation turned out to be opposed to that conjectured by Hilbert himself: Kolmogorov [18] was able to represent continuous functions of n variables by means of continuous functions of 3 variables, Arnold [19] represented the functions of three variables by means of functions of two, and finally Kolmogorov [20] represented functions of two variables as composition of functions of a sole variable with the help of the sole addition. (*Note of the translator*) results [13],[14] on this subject[3], up to now there is no proof that there exist algebraic functions of several variables which are not representable by algebraic functions of a sole variable.

We know however the following result:

Theorem ([4],[5]). *An entire function y of two variables (a, b), defined by the equation*

$$y^5 + ay + b = 0,$$

cannot be expressed in terms of entire functions of a sole variable by means of compositions, additions and subtractions.

The raison is the following. To every singular point p of an algebraic function one can associate a *local monodromy group*, i.e. the group of permutations of the sheets of the Riemann surface which is obtained going round the singularities of the function along curves lying in an arbitrarily small neighbourhood of point p. For algebraic functions of one variable this local group is commutative; by consequence the local monodromy group of an algebraic function which is expressed by means of sums and differences of integer functions *must be soluble.* But the local monodromy

[3]V. I. Arnol'd [13] invented a completely new approach to the demonstration of the non representability of an algebraic entire function of several variables as composition of algebraic entire functions of fewer variables. This approach is based on the study of the cohomology of the complement of the set of the branches of the function, which leads to the study of the cohomology of the braid groups.

We must remark that here the definition of representability of an algebraic function differs from the classical definition. Classical formulae for the solutions by radicals of equations of degree 3 and 4 cannot be completely considered mere compositions: these multivalued expressions by radicals contain, with the required roots, also "parasite" values. The new methods show that these parasite values are unavoidable: even equations of degree 3 and 4 are not *strictly* (i.e., without parasite values) solvable by radicals.

In particular, Arnold proved [13] that if $n = 2^r$ $(r \geq 2)$ the algebraic function $\lambda(z)$ of n complex variables $z = (z_1, \ldots, z_n)$ defined by the equation

$$\lambda^n + z_1\lambda^{n-1} + \cdots + z_n = 0$$

is not strictly representable in any neighbourhood of the origin as composition of algebraic entire functions (division is not allowed) with fewer than $n-1$ variables and of single-valued holomorphic functions of any number of variables. V. Ya. Lin [14] proved the same proposition for any $n \geq 3$.

The Arnol'd work had a great resonance: the successive calculations of the cohomologies with other coefficients of the generalized braid groups allowed to find results more and more extended.

The methods of the theory of the cohomologies of the braid groups, elaborated in the study of compositions of algebraic functions, have been afterward applied by Vassiliev and Smale [21],[22],[23] to the problem of finding the topologically necessary number of ramifications in the numeric algorithms for the approximate calculation of roots of polynomials. (The number of ramifications is of order of n for a polynomial of degree n).(*Note of the translator*) group of the function

$$y^5 + ay + b = 0$$

, near the point $(0, 0)$, is the group $S(5)$ of all permutations of five elements, which is not soluble. This explains the statement of the theorem.

Observe that if the operation of division is allowed, then the above argument no longer holds. Indeed, the division is an operation killing the continuity and its application destroys the locality. In fact, the function y satisfying

$$y^5 + ay + b = 0$$

can be expressed by means of the division in terms of a function, $g(x)$, of one variable, defined by the equation

$$g^5 + g + x = 0,$$

and of the function of one variable $f(a) = \sqrt[4]{a}$. It is not difficult to see that

$$y(a, b) = g(b/\sqrt[4]{a^5})\sqrt[4]{a}.$$

13 Functions of several complex variables representable by quadratures and generalized quadratures

The multi-dimensional case is more complicate than the monodimensional one. We have to reformulate the basic definitions and, in particular, to slightly change the definition of representability of functions by quadratures and by generalized quadratures. In this section we give a new formulation of the problem.

Suppose to have fixed a class of basic functions and a set of allowed operations. Is a given function (being, for instance, solution of a given algebraic or differential equation, or the result of one of the other allowed operation) representable in terms od the basic functions by means of the allowed operations? First of all, we are interested exactly in this problem but we give to it a bit different meaning. We consider the distinct *single-valued branches* of a multivalued function as single-valued functions on different domains: we consider also every multivalued function as the set of its single-valued branches. We apply the allowed operations (as the arithmetical operations or the composition) only to the single-valued branches on different domains. Since our functions are analytical, it suffices to consider as domains only small neighbourhoods of points. The problem now is the following: *is it possible to* *express a given germ of a function at a given point in terms od the germs of the basic functions by means of the allowed operations?* Of course, here the answer depends on the choice of the single-valued germ of the multivalued function at that point. However, it happens that (for the class of basic functions we are interested in) either the searched representation does not exist for any germ of the single-valued function at any point, or, on the contrary, all germs of the given multivalued function are expressed by the same representation at almost all points. In the former case we say that *no branches of the given multivalued function can be expressed in terms od the branches of the basic functions by means of the allowed operations*; in the latter case that this representation *exists.*

First of all, observe the difference between this formulation of the problem and that of the problem exposed in §1. *For analytical functions of a sole variable among the allowed operations there exists, in fact, the operation of analytical continuation.*

Consider the following example. Let f_1 be an analytical function, defined in a domain U of the plane $\mathbb{C}^1$, which cannot be continued beyond the boundaries of domain U, and let f_2 be the analytical function in domain U, defined by the equation $f_2 = -f_1$. According to the definition in §1, the zero function is representable in the form $f_1 + f_2$ *for all the values of the argument.* From this new point of view, equation $f_1 + f_2 = 0$ is fulfilled only inside domain U, not outside. Previously we was not interested in the existence of a *unique domain*, in which all required properties should hold on the single-valued branches of the multivalued function: a result of an operation could hold on a domain, another result in another domain on the analytical continuations of the obtained functions. For the $\mathcal{S}$-functions of a sole variable one can obtain the needed topological limitations even with this extended notion of operation on analytical multivalued functions. For functions of several variables we can no longer use this extended notion and we must adopt a new formulation with some restrictions, which can seem less (but which is perhaps more) natural.

Let us start by giving the exact definitions. Fixe the standard space $\mathbb{C}^n$ with coordinates system $x_1, \ldots, x_n$.

Definition. 1) *The germ of a function φ at point $a \in \mathbb{C}^n$ can be expressed in terms od the germs of the functions $f_1, \ldots, f_n$ at point a by means of the integration* if it fulfils the equation $d\varphi = \alpha$, where $\alpha = f_1 dx_1 + \cdots + f_n dx_n$. For the germs of the given functions $f_1, \ldots, f_n$, germ φ exists if and only if the 1-form α is closed. Germ φ is thus defined up to additive constants.

2) *The germ of a function φ at point $a \in \mathbb{C}^n$ can be expressed in terms od the germs of the functions $f_1, \ldots, f_n$ at point a by means of the exponential and of integrations*, if it fulfils the equation $d\varphi = \alpha\phi$, where $\alpha = f_1 dx_1 + \cdots + f_n dx_n$). For the germs of the given functions $f_1, \ldots, f_n$, germ φ exists if and only if the 1-form

α is closed. The germ φ is thus defined up to multiplicative constants.

3) *The germ of a function y at point $a \in \mathbb{C}^n$ can be expressed in terms od the germs of the functions $f_0, \dots, f_k$ at point a by means of a solution of an algebraic equation*, if the germ f_0 does not vanish and fulfils the the equation

$$f_0 y^k + f_1 y^{k-1} + \cdots + f_k = 0.$$

Definition.

1) *The class of the germs of functions in $\mathbb{C}^n$ representable by quadratures* (over the field of the constants) is defined by the following choice: the germs of the basic functions are the germs of the constant functions (at every point of the space $\mathbb{C}^n$); the allowed operations are the arithmetical operations, the integration, and raising to the power of the integral.

2) *The class of the germs of functions in $\mathbb{C}^n$ representable by generalized quadratures* (over the field of the constants) is defined by the following choice: the germs of the basic functions are the germs of the constant functions (at every point of the space $\mathbb{C}^n$); the allowed operations are the arithmetical operations, the integration, the raising to the power of the integral and the solution of algebraic equations.

Remark that the above definitions can be translated almost literally in the case of abstract differential fields, provided with n commutative differentiation operations $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. In such generalized form these definitions are due to Kolchin.

Consider now the class of the germs of functions, representable by quadratures and by generalized quadratures in the spaces $\mathbb{C}^n$ of whatever dimension $n \geq 1$. Repeating the Liouville argument (cf. theorem 1 in §2), it is not difficult to prove that *the class of the germs of functions of several variables representable by quadratures and by generalized quadratures contains the germs of the rational functions of several variables and the germs of all elementary basic functions; these classes of germs are closed with respect to the composition.* (The closure with respect to the composition of a class of germs of functions representable par quadratures means the following: if $f_1, \dots, f_m$ are germs of functions representable by quadratures at point $a \in \mathbb{C}^n$ and g is a germ of a function representable by quadratures at point $b \in \mathbb{C}^m$, where $b = (f_1(a), \dots, f_m(a))$, then the germ $g(f_1, \dots, f_m)$ at point $a \in \mathbb{C}^n$ is the germ of a function representable by quadratures).

14 $\mathcal{SC}$-germs

Does it exists a class of germs of functions of several variables sufficiently wide (containing the germs of functions representable by generalized quadratures, the germs of entire functions of several variables and closed with respect to the natural operations as the composition) for which the monodromy group is defined? In this section we define the class of $\mathcal{SC}$-germs and we enounce the theorem on the closure of this class with respect to the natural operations: this gives an affirmative answer to the posed question. I discovered the class of $\mathcal{SC}$-germs relatively recently: up to that time I believed the answer were negative.

In the case of functions of a sole variable it was useful to introduce the class of the $\mathcal{S}$-functions. Let us start by a direct generalization of the class of $\mathcal{S}$-functions to the multidimensional case.

A subspace $A \subset M$ in a connected k-dimensional analytical manyfold M is said *thin*, if there exists a numerable set of open subsets $U_i \subset M$ and a numerable set of analytical subspaces $A_i \subset U_i$ in these open subsets such that $A \subseteq \bigcup A_i$. An analytical multivalued function on manyfold M is called a $\mathcal{S}$-*function*, if the set of its singular points is thin. Let us make more precise this definition.

Two regular germs f_a and g_b, given at points a and b of manyfold M, are said *equivalent* if germ g_b is obtained by a regular continuation of germ f_a along some curve. Every germ g_b, equivalent to germ f_a, is also called a *regular* germ of the analytical multivalued function f, generated by germ f_a.

A point $b \in M$ is said *singular for germ* f_a, if there exists a curve $\gamma\ [0,1] \to M,\ \gamma(0) = a,\ \gamma(1) = b$, such that germ f_a cannot be regularly continued along this curve, but for every $t,\ 0 \leq t < 1$, this germ can be continued along the shortened curve $\gamma\ [0,t] \to M$. It is easy to see that the sets of singular points for equivalent germs do coincide.

A regular germ is said $\mathcal{S}$-*germ*, if the set of its singular points is thin. An analytical multivalued function is called a $\mathcal{S}$-function, if everyone of its regular germs is a $\mathcal{S}$-germ.

Remark. For functions of one complex variable we had given two definitions of $\mathcal{S}$-functions. The first one is the above definition, the second one is given by the theorem in §5. These definitions, evidently, coincide.

For $\mathcal{S}$-functions of several variables the notions of *monodromy group* and of *monodromy pair* are automatically translated.

Let us clarify way the multidimensional case is more complicate than the monodimensional one.

Imagine the following situation. Let $f(x, y)$ be a multivalued analytical function of two variables with a set A of ramification points, where $A \subset \mathbb{C}^2$ is an analytical curve on the complex plane. It can happen that at one of the points $a \in A$ there exists an analytical germ f_a of the multivalued analytical function f (by the definition of the set A of ramification points, at point a there exists not every germs of the function f, but some germ may exist). Let now $g_1(t)$ and $g_2(t)$ be two analytical functions of the complex variable t, given by the mapping of the complex line $\mathbb{C}$ in the complex plane $\mathbb{C}^2$, such that the image of line $\mathbb{C}$ is contained in A, i.e., $(g_1(t), g_2(t)) \in S$ for every $t \in \mathbb{C}$. Let b be the preimage of point a under this mapping, i.e., $a = (g_1(b), g_2(b))$. What can we say on the multivalued analytical function on the complex line, generated by the germ of $f(g_1, g_2)$ at point b, obtained as result of the composition of the germs of rational function g_1, g_2 at point b and of the germ of function f at point a? It is clear that the analytical properties of this function depend essentially on the continuation of germ f_a along singular curve A.

Nothing like this may happen under composition of functions of a sole variable. Indeed, the set of singularities of a $\mathcal{S}$-function of one variable consists of isolated points. If the image of the complex space under an analytical mapping g is entirely contained in the set of the singular points of a function f, then function g is a constant. It is evident that if function g is a constant, after having defined f on its set of singular points, function $f(g)$, too, results to be constant.

In the monodimensional case for our purpose it suffices to study the character of multivocity of the analytical function only in the complement of its singular points. In the multidimensional case, we have to study the possibility of continuing those germs of functions which meet along their set of singularities (if, of course, the germ of the function is defined in an arbitrary point of the set of singularities). It happens that the germs of multivalued functions sometimes are automatically continued along their set of singularities [24]: this thus allows us to pass all difficulties.

An important role is played by the following definition:

Definition. The germ f_a of an analytical function at point a of space $\mathbb{C}^n$ is called $\mathcal{SC}$-*germ* if the following condition is satisfied. For every connected complex analytical manyfold M, every analytical mapping $G\ M \to \mathbb{C}^n$ and every preimage c of point a, $G(c) = a$, there exists a thin subset $A \subset M$ such that for every curve $\gamma\ [0, 1] \to M$, beginning at point c, $\gamma(0) = c$, and having no intersection with the set A, except, at most, at the initial point, i.e., $\gamma(t) \notin A$ for $t > 0$, the germ f_a can be analytically continued along the curve $G \circ \gamma\ [0, 1] \to \mathbb{C}^n$.

Proposition. *If the set of singular points of a $\mathcal{S}$-function is an analytical set, then every germ of this function is a $\mathcal{SC}$-germ.*

This proposition follows directly from the results exposed in [24].

It is evident that every $\mathcal{SC}$-germ is the germ of a $\mathcal{S}$-function. For the $\mathcal{SC}$-germs the notions of monodromy group and of monodromy pair are thus well defined.

In the sequel we will need the notion of holonomic system of linear differential equations. A system of N linear differential equations $L_j(y) = 0$, $j = 1, \dots, N$,

$$L_j(y) = \sum a^j_{i_1,\dots,i_n} \frac{\partial^{i_1+\dots+i_n} y}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} = 0$$

for the unknown function y, whose coefficients $a^j_{i_1,\dots,i_n}$ are analytical functions of n complex variables $x_1, \dots, x_n$, is said *holonomic* if the space of its solutions has a finite dimension.

THEOREM ON THE CLOSURE OF THE CLASS OF $\mathcal{SC}$-GERMS. *The class of the $\mathcal{SC}$-germs in $\mathbb{C}^n$ is closed with respect to the following operations:*

- *1) differentiation, i.e., if f is a $\mathcal{SC}$-germ at point $a \in \mathbb{C}^n$, then for every $i = 1, \dots, n$ the germs of the partial derivatives $\frac{\partial f}{\partial x_i}$ are as well $\mathcal{SC}$-germs at point a;*

- *2) integration, i.e. , if $df = f_1 dx_1 + \dots + f_n dx_n$, where $f_1, \dots, f_n$ are $\mathcal{SC}$-germs at point $a \in \mathbb{C}^n$, then also f is a $\mathcal{SC}$-germ at point a;*

- *3) composition with the $\mathcal{SC}$-germs of m variables, i.e., if $f_1, \dots, f_m$ are $\mathcal{SC}$-germs at point $a \in \mathbb{C}^n$ and g is a $\mathcal{SC}$-germ at point $(f_1(a), \dots, f_m(a))$ in the space $\mathbb{C}^m$, then also $g(f_1, \dots, f_m)$ is a $\mathcal{SC}$-germ at point a;*

- *4) solutions of algebraic equations, i.e., if $f_0, \dots, f_k$ are $\mathcal{SC}$-germs at point $a \in \mathbb{C}^n$, the germ f_0 is not zero and the germ y fulfils the equation $f_0 y^k + f_1 y^{k-1} + \dots + f_k = 0$, then also germ y is a $\mathcal{SC}$-germ at point a;*

- *5) solutions of holonomic systems of linear differential equations, i.e., if the germ of function y at point $a \in \mathbb{C}^n$ satisfies the holonomic system of N linear differential equations*

  $$L_j(y) = \sum a^j_{i_1,\dots,i_n} \frac{\partial^{i_1+\dots+i_n} y}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} = 0,$$

  whose all coefficients $a^j_{i_1,\dots,i_n}$ are $\mathcal{SC}$-germs at point a, then also y is a $\mathcal{SC}$-germ at point a.

COROLLARY. *If the germ of a function f can be obtained from the germs of single-valued $\mathcal{S}$-functions having an analytical set of singular points by mans of integrations, of differentiations, meromorphic operations, compositions, solutions of algebraic equations and solutions of holonomic systems of linear differential equations, then the germ of f is a $\mathcal{SC}$-germ. In particular, a germ which is not a $\mathcal{SC}$-germ, cannot be represented by generalized quadratures.*

15 Topological obstructions to the representability by quadratures of functions of several variables

This section is dedicated to the topological obstructions to the representability by quadratures and by generalized quadratures of functions of several complex variables. These obstructions are analogous to those holding for functions of one variable considered in §§7-9.

THEOREM 1. *The class of all $\mathcal{SC}$-germs in $\mathbb{C}^n$, having a soluble monodromy group, is closed with respect to the operations of integration and of differentiation. Moreover, this class is closed with respect to the composition with the $\mathcal{SC}$-germs of m variables ($m \geq 1$) having soluble monodromy groups.*

RESULT ON QUADRATURES. *The monodromy group of any germ of a function f, representable by quadratures, is soluble. Moreover, also every germ of a function, representable by the germs of single-valued $\mathcal{S}$-functions having an analytical set of singular points, is soluble by means of integrations, of differentiations and compositions.*

COROLLARY. *If the monodromy group of the algebraic equation*

$$y^k + r_1 y^{k-1} + \cdots + r_k = 0,$$

in which the r_i's are rational functions of n variables, is not soluble, then any germ of its solutions not only is not representable by radicals, but cannot be represented in terms od the germs of single-valued $\mathcal{S}$-functions having an analytical set of singular points by means of integrations, of differentiations and compositions.

This corollary represents the strongest version of the Abel theorem.

THEOREM 2. *The class of all $\mathcal{SC}$-germs in $\mathbb{C}^n$, having a monodromy pair almost soluble, is closed with respect to the operations of integration, differentiation and solution of algebraic equations. Moreover, this class is closed with respect to the composition with the $\mathcal{SC}$-germs of m variables ($m \geq 1$) having a monodromy pair almost soluble.*

RESULT ON GENERALIZED QUADRATURES. *The monodromy pair of a germ of a function f, representable by generalized quadratures, is almost soluble. Moreover, also the monodromy pair of every germ of a function f, representable in terms od the germs of single-valued $\mathcal{S}$-functions having an analytical set of singular points by means of integrations, differentiations, compositions and solutions of algebraic equations is almost soluble.*

16 Topological obstruction to the solvability of the holonomic systems of linear differential equations

16.1 The monodromy group of a holonomic system of linear differential equations

Consider a holonomic system of N differential equations $L_j(y) = 0$, $j = 1, \dots, N$,

$$L_j(y) = \sum a^j_{i_1,\dots,i_n} \frac{\partial^{i_1+\dots+i_n} y}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} = 0,$$

where y is the unknown function, and the coefficients $a^j_{i_1,\dots,i_n}$ are rational functions of the n complex variables $x_1, \dots, x_n$.

One knows that,for any holonomic system, there exists a singular algebraic surface Σ in space $\mathbb{C}^n$, having the following properties. Every solution of the system can be analytically continued along whatever curve avoiding hypersurface Σ. Let V be the finite-dimensional space of the solutions of a holonomic system near a point x_0, which lies outside the hypersurface Σ. Consider an arbitrary curve $\gamma(t)$ in space $\mathbb{C}^n$ with initial point x_0, non crossing the hypersurface Σ. The solutions of the system can be analytically continued along curve γ, remaining solutions of the system. Consequently, to every curve γ of this type there corresponds a linear transformation M_γ of the space of solutions V in itself. The totality of the linear transformations M_γ, corresponding to all curves γ, forms a group, which is called *monodromy group of the holonomic system.*

Kolchin generalized the Picard-Vessiot theory to the case of holonomic systems of differential equations. From the Kolchin theory we obtain two corollaries concerning the solvability by quadratures of the holonomic systems of differential equations. As in the monodimensional case, a holonomic system is said *regular* if approaching the singular set Σ and the infinity its solutions grow at most as some power.

Theorem 1. *A regular holonomic system of linear differential equations is soluble by quadratures and by generalized quadrature if its monodromy group is, respectively, soluble and almost soluble.*

Kolchin theory proves at the same time two results.

- 1) *If the monodromy group of a regular holonomic system of linear differential equations is soluble (almost soluble), then this system is solvable by quadratures (by generalized quadratures).*

- 2) *If the monodromy group of a regular holonomic system of linear differential equations is not soluble (is not almost soluble), then this system is not solvable by quadratures (by generalized quadratures).*

Our theorem makes stronger result (2).

THEOREM 2. *If the monodromy group of a holonomic system of equations of linear differential equations is not soluble (is not almost soluble), then every germ of almost all solutions of this system cannot be expressed in terms od the germs of single-valued $\mathcal{S}$-functions having an analytical set of singular points by means of compositions, meromorphic operations, integrations and differentiations (by means of compositions, meromorphic operations, integrations, differentiations and solutions of algebraic equations).*

16.2 Holonomic systems of equations of linear differential equations with small coefficients

Consider a system of linear differential equations completely integrable of the following form

$$dy = Ay \tag{8}$$

where $y = y_1, \ldots, y_n$ is the unknown vector-function and A is a $(n \times n)$ matrix, consisting of differential 1-forms with rational coefficients in the space $\mathbb{C}^n$, satisfying the condition of complete integrability $dA + A \wedge A = 0$ and having the following form:

$$A = \sum_{i=1}^{k} A_i \frac{dl_i}{l_i},$$

where the A_i's are constant matrices and the l_i's are linear non homogeneous functions in $\mathbb{C}^n$.

If the matrices A_i can be put at the same time in triangular form, then system (8), as every completely integrable triangular system, is solvable by quadratures. There exist undoubtedly integrable non triangular systems. However, if the matrices A_i's are sufficiently small, such systems do not exist. More precisely, we proved the following theorem.

THEOREM 3.*A completely integrable non triangular system (8), with the modules of the matrices A_i's sufficiently small, is strictly not solvable, i.e., its solution cannot be represented even trough the germs of all single-valued $\mathcal{S}$-functions, having an analytical set of singular points, by means of compositions, meromorphic operations, integrations, differentiations and solutions of algebraic equations.*

The proof of this theorem uses a multidimensional variation of the Lappo-Danilevskij theorem [26].

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