JOURNAL OF DYNAMICAL AND CONTROL SYSTEMS, Vol. 1, No. 1, 1995, 91-123

TOPOLOGICAL OBSTRUCTIONS TO THE REPRESENTABILITY OF FUNCTIONS BY QUADRATURES

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ABSTRACT. A topological variant of Galois theory, in which the monodromy group plays the role of the Galois group, is described. It turns out that there are topological restrictions on the way the Riemann surface of a function represented by quadratures covers the complex plane.

1. INTRODUCTION

1.1. Attempts to solve explicitly differential equations usually fail. The first rigorous proofs that some differential equations are not solvable by quadrature were obtained in the 1830's by Liouville. Liouville was undoubtedly inspired by the results of Lagrange, Abel, and Galois on the nonsolvability of algebraic equations by radicals. Unlike in Galois theory, automorphism groups do not play a central role in Liouville's method, even though Liouville uses "infinitely small automorphisms." His results have the following character: Liouville shows that "simple" equations cannot have solutions written by complicated formulas. "Simple" equations either have solutions of a sufficiently simple kind, or cannot be solved by quadrature. One can find an exposition of the Liouville method, as well as related work of Chebyshev, Mordukhai-Boltovskii, Ostrovskii, and Ritt, in [1].

Another approach to the problem of solvability of linear differential equations by quadrature was developed by Picard. Picard generalized Galois theory to the case of linear differential equations. Vessiot finished in 1910 the work started by Picard, and proved that a linear differential equation is solvable by quadrature if and only if its Galois group has a solvable normal subgroup of finite index. This theorem of Picard-Vessiot is analogous to the

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1991 *Mathematics Subject Classification.* 34A20.

This work was partially supported by Grant MBF000 from the International Science Foundation.

1079-2724/95/0100-0091\$07.50/0

Galois theorem on the solvability of algebraic equations by radicals. The main results of the differential Galois theory are contained in [2].

It is interesting that the Picard-Vessiot approach is close to the Liouville approach. Namely, the Galois group of a linear differential equation has a solvable normal subgroup of finite index if and only if the equation has solutions of a very specific simple kind. One can therefore state the Picard-Vessiot theorem without mentioning Galois groups. In that form, for second-order equations, it was discovered and proved by Liouville, and for nth order equations, by Mordukhai-Boltovskii. Mordukhai-Boltovskii obtained this result by Liouville's method in 1910, independently of and simultaneously with Vessiot.

There is a brief survey of the current state of the subject and a rather extensive bibliography in [3].

1.2. In this paper we describe a third approach to the problem of representing functions by quadratures. We consider functions that are representable by quadratures as multi-valued functions of one complex variable. It turns out that there are topological restrictions on the way the Riemann surface of a function representable by quadratures covers the complex plane. If the function does not satisfy these restrictions, then it is not representable by quadratures.

This approach has the following advantage, beside its geometric clarity. The topological prohibitions concern the character of the multivaluedness of the function. They are valid not only for functions that are representable by quadratures, but also for a much wider class of functions. One obtains this class if one adds the meromorphic functions to the class of functions representable by quadratures, as well as all functions representable by formulas containing the above. Because of this, the topological results on nonrepresentability of functions by quadratures are stronger than the algebraic results. This is because composition of functions is not an algebraic operation. In differential algebra, instead of composition of functions, one considers differential equations satisfied by the composition. But, for example, Euler's Γ-function does not satisfy any polynomial differential equation. Therefore, it is hopeless to look for an equation satisfied by, say, $\Gamma(\exp x)$. The only known results on the nonrepresentability of functions by quadratures and, say, Euler's Γ-functions have been obtained by our method.

On the other hand, it is impossible to prove that a single-valued meromorphic function is not representable by quadratures by using this method.

The approach we describe was announced in [4], [5] and developed in my Ph.D. Thesis [6], but was never published in detail. Here, we partially fill this gap (in the past almost quarter of a century, these results have not been rediscovered; solvability by quadrature, though classical, is not a fashionable theme).

By using differential Galois theory (more precisely, its linear-algebraic part dealing with algebraic matrix groups and their differential invariants), one can show that there are only topological reasons for a Fuchs-type linear differential equation not to be solvable by quadrature. In other words, if there are no topological obstructions to the solvability by quadrature of a Fuchs-type differential equation, then this equation is solvable by quadrature.

This and other facts pertaining to the topological Galois theory of Fuchs-type equations will be included in a forthcoming paper.

1.3. A rigorous definition of the representability of a function by quadratures was given by Liouville. Here we give variants of this definition that we will need.

A function f is said to be *representable via the functions* $\{\varphi_\alpha\}$ *by quadratures* if f can be represented by applying the arithmetic operations, differentiation, exponentiation, and integration on the φ_α's.

One can add solution of algebraic equations to the allowable operations.

A function f is said to be representable via the functions $\{\varphi_\alpha\}$ *by generalized quadratures* if f is representable via $\{\varphi_\alpha\}$ by quadratures and solutions of algebraic equations.

A function f is said to be representable via the functions $\{\varphi_\alpha\}$ *by n-quadratures* if f is representable via $\{\varphi_\alpha\}$ by quadratures and solutions of algebraic equations of degree at most n.

A function is said to be *representable by quadratures* (*generalized quadratures, n-quadratures*) if it is representable via constants $\varphi_\alpha \equiv C$ by quadratures (generalized quadratures, n-quadratures).

Each elementary function is representable by quadratures, as was discovered by Liouville. Thus, for example, $f(x) = \arctan x$ can be represented by

$$f' = \frac{1}{1+x^2}, \qquad x' \equiv 1.$$

In general, the class of all functions representable via $\{\varphi_\alpha\}$ by quadratures is not closed with respect to composition. But, as was shown by Liouville, the classes of all functions representable by quadratures, by generalized quadratures, and by n-quadratures are closed with respect to composition.

As we consider multi-valued functions, the definitions above need to be made more precise. Let us make more precise, for example, what we mean by the composition of two multi-valued functions $f(x)$ and $g(x)$. Take an arbitrary point a, one of the germs f_a of $f(x)$ at a, and one of the germs g_b of $g(x)$ at $b = f(a)$. We shall say that the function $\varphi(x)$ generated by the germ $g_b \circ f_a$ is representable as a composition of the functions $f(x)$ and $g(x)$. This definition does not produce a unique function. For example, there are exactly two functions that can be represented as $\sqrt{x^2}$, namely $f_1 = x$ and

$f_2 = -x$. To say that a class of multi-valued functions is closed with respect to composition means that if two functions belong to this class, then any function that can be represented as a composition of these two functions also belongs to this class.

The same has to be said about all the other operations on multi-valued functions that were mentioned in the above definitions and that will be encountered throughout this paper.

1.4. In this paper, we prove the existence of the following topological obstruction to the representability of a function by quadratures, generalized quadratures, and n-quadratures.

First, a function that is representable by generalized quadratures, and in particular a function that is representable by quadratures or n-quadratures, can have a no more than countable number of singular points on the complex plane. (Note that even for the simplest functions that are representable by quadratures, the set of singular points can be everywhere dense.)

This result is proved in Sec. 2.2, where we show that the class of functions with a no more than countable number of singular points is closed with respect to composition, the arithmetic operations, integration, and solution of algebraic equations.

Second, the monodromy group of a function representable by quadratures is always solvable. (Note that even for the simplest functions representable by quadratures the monodromy group can be uncountable.)

There are analogous restrictions on the placement of the Riemann surface of a function representable by generalized quadratures or by n-quadratures. But these restrictions are more complicated to state. There the monodromy group appears not as an abstract group, but as a group of permutations of the sheets of the function. In other words, these restrictions involve not just the monodromy group but the *monodromy pair* consisting of the monodromy group and the isotropy subgroup of some germ.

The definitions concerning monodromy pairs are given in Sec. 3.5. The main theorem (cf. Sec. 4) describes how the monodromy pair of a function changes under composition, integration, arithmetic operations, etc. In Sec. 5 we compute the classes of pairs of groups that appear in the main theorem. In Sec. 5.2 we gather all the results obtained.

In Sec. 6 we consider functions mapping the half-plane into polygons bounded by arcs of circles. We give an explicit classification of those polygons whose corresponding function is representable by quadratures. The main case of interest here turns out to be the well-known Christoffel-Schwarz case, in which the polygon has straight sides. There are two other interesting cases. The first reduces to the Christoffel-Schwarz case by taking logarithms, the second occurs when the function is algebraic. *There are no other integrable cases.* This result from [4], [6] is first published in detail here. It is, of course, closely connected with a whole series of classical research, including the work of Klein [7].

1.5. Thus, in this paper we describe a topological variant of Galois theory, in which the monodromy group plays the role of Galois group. The results were obtained in 1969–1971 while I was a student of V. I. Arnold. I take this opportunity to thank my teacher.

I am grateful to A. A. Bolibruch who suggested that I reconsider and publish these old results, to my wife, T. V. Belokrinitskaya, who helped me in this endeavor, and to Smilka Zdravkovska who kindly agreed to translate the paper into English.

2. Function with a no more than countable set of singular points

2.1. Prohibited sets. We first define the class of functions that will be considered in what follows. A multi-valued analytic function of one complex variable is called an *S-function* if its set of singular points is no more than countable. Let us make this definition more precise.

Two regular germs f_a and g_b given at the points a and b of the Riemann sphere S^2 are said to be equivalent if g_b can be obtained from f_a by analytic continuation along some curve. Each germ g_b equivalent to a germ f_a is said to be a regular germ of the multi-valued function f generated by the germ f_a.

A point $b \in S^2$ is said to be singular for the germ f_a if there is a curve $\gamma\,[0,1] \to S^2$, $\gamma(0) = a$, $\gamma(1) = b$, such that there is no analytic continuation of the germ along this curve, but for every t, $0 \le t < 1$, there is an analytic continuation of the germ along the shorter curve $\gamma\,[0,t] \to S^2$. It is easy to see that the sets of singular points of two equivalent germs coincide.

A regular germ is called an S-germ if its set of singular points is no more than countable. A multi-valued analytic function is called an *S-function* if each of its regular germs is an S-germ.

In what follows we shall need a lemma according to which a curve can be "taken off" a countable set after a small perturbation.

Lemma on taking a curve off a countable set. *Let A be a no more than countable set in the complex plane $\mathbb{C}$, let $\gamma\,[0,1] \to \mathbb{C}$ be a curve, and let $\varphi(t)$ be a continuous positive function for $0 < t < 1$. Then there is a curve $\widehat{\gamma}\,[0,1] \to \mathbb{C}$ such that $\widehat{\gamma}(t) \notin A$ and $|\gamma(t) - \widehat{\gamma}(t)| < \varphi(t)$ for $0 < t < 1$.*

The "scientific" proof of this lemma consists of the following. In the function space of curves $\widetilde{\gamma}$ close to γ,

$$|\gamma(t) - \widetilde{\gamma}(t)| < \varphi(t),$$

the curves that do not intersect a given point of A form an open dense set. The intersection of a countable number of open dense sets in such function spaces is not empty.

We give here an elementary proof of this lemma. (It can be transferred almost verbatim to the case when A is uncountable but of Hausdorff length zero, cf. Sec. 7.) We first construct a curve $\overline{\gamma}$ which is the union of an infinite number of segments whose vertices do not belong to A and such that

$$|\gamma(t) - \overline{\gamma}(t)| < \frac{1}{2}\varphi(t).$$

Such a curve exists since the complement of A is everywhere dense. We now show how to change each segment $[p, q]$ of $\overline{\gamma}$ so as to obtain a curve that does not intersect A. Take the segment $[p, q]$ and let m be the normal to this segment passing through the midpoint of $[p, q]$. Consider the set of two-segment curves $[p, b] \bigcup [b, q]$, where $b \in m$ and b is sufficiently close to $[p, q]$. There is a continuum of such curves, and the intersection of any two such curves is $\{p, q\}$. Hence there is a two-segment curve in this set that does not intersect A. By replacing each segment of $\overline{\gamma}$ by such a two-segment curve we obtain the desired curve.

We shall also consider other sets outside of which a function has an analytic continuation. A no more than countable set A is called a *prohibited set* for the regular germ f_a, if f_a has a regular continuation along any curve $\gamma(t)$, $\gamma(0) = a$, that intersects A in at most $\gamma(0)$.

Theorem on prohibited sets. *A no more than countable set A is a prohibited set for a germ f_a if and only if A contains the set of singular points of f_a. In particular, a germ has a prohibited set if and only if it is a germ of an S-function.*

Proof. Suppose there is a singular point b of the germ f_a that does not belong to some prohibited set A of f_a. By definition, there exists a curve

$$\gamma\,[0, 1] \to S^2, \quad \gamma(0) = a, \quad \gamma(1) = b,$$

along which there is no regular continuation of f_a, but such a continuation exists through any $t < 1$. Without loss of generality, we can assume that a, b, and $\gamma(t)$ all lie in the finite part of the Riemann sphere, i.e., that $\gamma(t) \neq \infty$ for $0 \leq t \leq 1$. Let $R(t)$ denote the radius of convergence of the series $f_{\gamma(t)}$ which is obtained by continuation of f_a along $\gamma(t)$. Then $R(t)$ is a continuous function on $[0, 1)$. By the previous lemma, there is a curve $\widehat{\gamma}(t)$, $\widehat{\gamma}(0) = a$, $\widehat{\gamma}(1) = b$, such that

$$|\gamma(t) - \widehat{\gamma}(t)| < \frac{1}{3} R(t)$$

and $\widehat{\gamma}(t) \notin A$ for $t > 0$. The germ f_a has a continuation along $\widehat{\gamma}$ up to the point 1. But this clearly implies that f_a has a continuation along γ. This contradiction shows that the set of singular points of f_a is contained in each prohibited set for f_a. The converse statement (a countable set containing the set of singular points of f_a is a prohibited set for f_a) is obvious. □

2.2. The class of S-functions is closed.

Theorem that the class of S-functions is closed. *The class S of all S-functions is closed with respect to the following operations:*

(1) *differentiation; i.e., if $f \in S$, then $f' \in S$;*
(2) *integration; i.e., if $f \in S$, then $\int f(x)dx \in S$;*
(3) *composition; i.e., if $g,\ f \in S$, then $g \circ f \in S$;*
(4) *meromorphic operations; i.e., if $f_i \in S$, $i = 1, \dots, n$, and if $F(x_1, \dots, x_n)$ is a meromorphic function of n variables, then $f = F(f_1, \dots, f_n) \in S$;*
(5) *solutions of algebraic equations; i.e., if $f_i \in S$, $i = 1, \dots, n$, and if*

$$f^n + f_1 f^{n-1} + \cdots + f_n = 0,$$

then $f \in S$; and

(6) *solutions of linear differential equations; i.e., if $f_i \in S$, $i = 1, \dots, n$, and if*

$$f^{(n)} + f_1 f^{(n-1)} + \cdots + f_n = 0,$$

then $f \in S$.

Proof. (1) and (2). Let f_a, $a \neq \infty$, be the germ of an S-function with A as set of singular points. If there is a regular continuation of f_a along some curve γ lying in the finite part of the Riemann sphere, then the integral and the derivative of this germ have a regular continuation along γ. Therefore, one can take $A \bigcup \{\infty\}$ as prohibited set for the integral and for the derivative of f_a.

(3) Let f_a and g_b be germs of S-functions with A and B as sets of singular points, respectively, and let $f(a) = b$. Denote by $f^{-1}(B)$ the full inverse image of B under the multi-valued correspondence generated by f_a. In other words, $x \in f^{-1}(B)$ if and only if there is a germ ψ_x equivalent to f_a such that $\psi(x) \in B$. The set $f^{-1}(B)$ is no more than countable. One can take $A \bigcup f^{-1}(B)$ as prohibited set for $g_b \circ f_a$.

(4) Let f_{ia} be germs of S-functions, let A_i be their sets of singular points, and let F be a meromorphic function of n variables. We assume that f_{ia} and F are such that the germ

$$f_a = F(f_{1a}, \dots, f_{na})$$

is a well-defined meromorphic germ. By replacing a by a nearby point if necessary, we can assume that f_a is regular. If $\gamma(t)$ is a curve that does not intersect

$$A = \bigcup A_i$$

for $t > 0$, then f_a has a meromorphic continuation along γ. Let B denote the projection on the Riemann sphere of the set of poles of the function f generated by f_a. One can take $A \bigcup B$ as prohibited set in this case.

(5) Let f_{ia} be germs of S-functions, let A_i denote their sets of singular points, and let f_a be a regular germ such that

$$f_a^n + f_{1a} f_a^{n-1} + \cdots + f_{na} = 0.$$

If $\gamma(t)$ is a curve that does not intersect

$$A = \bigcup A_i$$

for $t > 0$, then there is a continuation of f_a along γ that contains, in general, meromorphic and algebraic elements. Let B denote the projection onto the Riemann sphere of the poles of the function f and the ramification points of its Riemann surface. One can take $A \bigcup B$ as prohibited set for f_a.

(6) If the coefficients of the equation

$$f_a^{(n)} + f_{1a} f_a^{(n-1)} + \cdots + f_{na} = 0$$

have a regular continuation along some curve γ lying in the finite part of the Riemann sphere, then each solution f_a of this equation also has a regular continuation along γ. Therefore, one can take

$$A = \bigcup A_i \bigcup \{\infty\},$$

where A_i is the set of singular points of f_{ai}, as prohibited set for f_a. □

Remark. The arithmetic operations and exponentiation are examples of meromorphic operations. Hence, the class of S-functions is closed under arithmetic operations and exponentiation.

Corollary. *If a multi-valued function $f(x)$ can be obtained from single-valued S-functions by integration, differentiation, meromorphic operations, compositions, and solutions of algebraic equations and of linear differential equations, then $f(x)$ has a no more than countable set of singular points. In particular, a function with uncountable number of singular points cannot be represented by generalized quadratures.*

3. The monodromy group

3.1. The monodromy group with prohibited set. The monodromy group of an S-function f with prohibited set A is the group of permutations of the sheets of f as a result of going around the points of A. We now make this sentence more precise.

Let F_a be the set of all germs of an S-function f at a point a that does not belong to some prohibited set A. Take a closed curve γ in $S^2 \setminus A$ starting at a. Given a germ f_a in F_a, after analytic continuation of f_a along γ we get another germ in F_a.

Thus, each curve γ in $S^2 \setminus A$ determines a map from F_a to itself, and homotopic curves determine the same map. The product of curves determines the product (=composition) of the corresponding maps. We thus get a homomorphism τ of the fundamental group of $S^2 \setminus A$ into the group $S(F_a)$ of permutations of F_a. We shall call this the *A-monodromy homomorphism.* The monodromy group of an S-function f with prohibited set A (or the *A-monodromy group*) is the image of the fundamental group $\pi_1(S^2 \setminus A, a)$ in $S(F_a)$ under τ.

Proposition.

(1) *The A-monodromy group of an S-function does not depend on the choice of the base point.*
(2) *The A-monodromy group of an S-function f acts transitively on the sheets of f.*

Both statements can easily be proved by using the lemma from Sec. 2.1. Let us prove, for example, the second.

Proof. Let f_{1a} and f_{2a} be germs of f at a. As f_{1a} and f_{2a} are equivalent germs, there is a curve γ such that continuation of f_{1a} along γ produces f_{2a}. By the lemma in Sec. 2.1, there is an arbitrarily close to γ curve $\widehat{\gamma}$ that does not intersect A. If $\widehat{\gamma}$ is sufficiently close to γ, then the permutation of F_a corresponding to $\widehat{\gamma}$ will map f_{1a} to f_{2a}. □

We now show some effects that one should take into account when studying quadrature functions as functions of one complex variable.

Example. Consider the function

$$w(z) = \ln(1 - z^{\alpha}),$$

where $\alpha > 0$ is an irrational number. Then $w(z)$ is an elementary function, given by a very simple formula. Nevertheless, the Riemann surface of $w(z)$ covers the complex plane in a very complicated way. The set A of singular points of $w(z)$ consists of the points 0, ∞ and the logarithm ramification points

$$a_k = e^{2k\pi i/\alpha}, \qquad k \in \mathbb{Z}.$$

Since α is irrational, the points a_k form a dense set on the unit circle. It is not difficult to prove that the fundamental group $\pi_1(S^2 \setminus A)$ and the A-monodromy group of $w(z)$ have the cardinality of the continuum. One can also show that the B-monodromy group of $w(z)$, where

$$B = A \bigcup \{a\}$$

and $a \neq a_k$ is an arbitrary point on the unit circle, is a proper subgroup of the A-monodromy group of $w(z)$.

3.2. The closed monodromy group. Because of the dependence of the A-monodromy group on the choice of A, we are led to consider the group of permutations of the sheets of the function with a topology, the Tikhonov topology. It turns out that the closure of the A-monodromy group does not depend on A.

Given a set M, we consider the following topology on the group $S(M)$ of permutations of M. For each finite set $L \subset M$ define a neighborhood U_L of the identity permutation as the set of permutations p such that $p(l) = l$ for $l \in L$. Take as basis of neighborhoods of the identity permutation all the sets U_L, where L runs over all finite subsets of M.

Lemma on the closure of the monodromy group. *The closure of the monodromy group of an S-function f with prohibited set A in the group $S(F)$ of all permutations of the leaves of f does not depend on the choice of prohibited set A.*

Proof. Let A_1 and A_2 be two prohibited sets for f and let F_a be the set of leaves of f at a,

$$a \notin A_1 \bigcup A_2.$$

Let

$$\Gamma_1,\ \Gamma_2 \subseteq S(F_a)$$

be the monodromy groups of f corresponding to these two prohibited sets. It is sufficient to prove that, for each permutation $\mu_1 \in \Gamma_1$ and for each finite set $L \subseteq F_a$, there is a permutation $\mu_2 \in \Gamma_2$ such that

$$\mu_1 \big|_L = \mu_2 \big|_L .$$

Let

$$\gamma \in \pi_1(S^2 \setminus A_1, a)$$

be a curve corresponding to μ_1. Since L is finite, any curve

$$\widehat{\gamma} \in \pi_1(S^2 \setminus A_1, a)$$

that is sufficiently close to γ will induce a permutation $\widehat{\mu}_1$ which coincides with μ_1 on L. By the lemma in Sec. 2.1 such a curve $\widehat{\gamma}$ can be chosen so that $\widehat{\gamma}$ does not intersect A_2. Hence $\widehat{\mu}_1$ will be an element of Γ_2. □

Because of this lemma the notion of *closed monodromy group* of an S-function f is well defined: it is the closure in $S(F)$ of the monodromy group of f with some prohibited set A.

3.3. Transitive actions of groups on sets and the monodromy pair of an S-function. The monodromy group of a function f is not just an abstract group, but also a transitive group of permutations of the sheets of f. In this section we recall the algebraic description of transitive actions of groups on sets.

An action of a group Γ on a set M is a homomorphism τ from Γ to the group $S(M)$. Two actions

$$\tau_1\, \Gamma \to S(M_1) \quad \text{and} \quad \tau_2\, \Gamma \to S(M_2)$$

are said to be equivalent if there exists a one-to-one map

$$q\, M_1 \to M_2$$

such that

$$\overline{q} \circ \tau_1 = \tau_2,$$

where $\overline{q}\, S(M_1) \to S(M_2)$ is the isomorphism induced by q.

The *isotropy subgroup* Γ_a of an element $a \in M$ under τ is the subgroup of all $\mu \in \Gamma$ such that

$$\tau\mu(a) = a.$$

The action τ is said to be *transitive* if for any two elements $a,\ b \in M$ there is a $\mu \in \Gamma$ such that

$$\tau\mu(a) = b.$$

The following is obvious.

Proposition.

(1) *An action τ of Γ is transitive if and only if the isotropy groups of any two elements $a,\ b \in M$ are conjugate. The image of Γ under a transitive action τ is isomorphic to the quotient*

$$\Gamma / \bigcap_{\mu \in \Gamma} \mu \Gamma_a \mu^{-1}.$$

(2) *There is a unique up to equivalence transitive action of Γ with a given subgroup as the isotropy group of some element.*

Thus, transitive actions of Γ can be described by a pair of groups. We say that a pair of groups $[\Gamma, \Gamma_a]$, where Γ_a is the isotropy subgroup of some element a under a transitive action τ of Γ, is the *monodromy pair* of a with respect to τ. The group

$$\tau(\Gamma) \sim \Gamma / \bigcap_{\mu \in \Gamma} \mu \Gamma_a \mu^{-1}$$

will be called the monodromy group of the pair $[\Gamma, \Gamma_a]$.

The A-monodromy homomorphism τ determines a transitive action of the fundamental group $\pi_1(S^2 \setminus A)$ on the set F_a of sheets of f over a.

The monodromy pair of a germ f_a under the action τ will be called the *monodromy pair of f_a with prohibited set A*. The monodromy pair of a germ f_a under the action of the closed monodromy group will be called the *closed monodromy pair of f_a*. Any two germs of an S-function f have isomorphic monodromy pairs with prohibited set A, so one can talk of the monodromy pair of an S-function f with prohibited set A, and of the closed monodromy pair of f. We shall denote by $[f]$ the closed monodromy pair of an S-function f.

3.4. Almost normal functions. A pair of groups

$$[\Gamma, \Gamma_0], \quad \Gamma_0 \subseteq \Gamma,$$

is called an *almost normal pair* if there exists a finite set $P \subset \Gamma$ such that

$$\bigcap_{\mu \in \Gamma} \mu \Gamma_0 \mu^{-1} = \bigcap_{\mu \in P} \mu \Gamma_0 \mu^{-1}.$$

Lemma on discrete actions. *The image $\tau(\Gamma)$ of Γ under a transitive action $\tau\,\Gamma \to S(M)$ is a discrete subgroup of $S(M)$ if and only if the monodromy pair $[\Gamma, \Gamma_0]$ of some element $x_0 \in M$ is almost normal.*

Proof. Let the group $\tau(\Gamma)$ be discrete. Denote by $\overline{P}$ a finite subset of M such that the neighborhood $U_{\overline{P}}$ of the identity permutation does not contain elements of $\tau(\Gamma)$ other than the identity. This means that the intersection

$$\bigcap_{x \in \overline{P}} \Gamma_x$$

of the isotropy subgroups of the elements $x \in \overline{P}$ acts trivially on M, i.e.,

$$\bigcap_{x \in \overline{P}} \Gamma_x \subseteq \bigcap_{\mu \in \Gamma} \mu \Gamma_0 \mu^{-1}.$$

The groups Γ_x are conjugate to Γ_0, so we can choose a finite set $P \subset \Gamma$ such that

$$\bigcap_{\mu \in P} \mu\Gamma_0\mu^{-1} = \bigcap_{\mu \in \Gamma} \mu\Gamma_0\mu^{-1}.$$

The converse is proved analogously. □

We shall say that an S-function f is almost normal if the monodromy group of f is discrete. It follows from the lemma above that f is almost normal if and only if the closed monodromy pair $[f]$ is almost normal.

A differential rational function of several functions f_i is a rational function of the f_i's and the derivatives of the f_i's.

Lemma on finitely generated functions. *Let f be an S-function such that each germ of f at a is a differential rational function of a finite number of given germs of f at a. Then f is almost normal.*

Indeed, if the given germs do not change after continuation along a curve, then neither do differential rational functions of these germs.

It follows from this lemma that any solution of a linear differential equation with rational coefficients is an almost normal function. The same holds for many other functions that one encounters in differential algebra.

3.5. Classes of pairs of groups. In this section we describe how the closed monodromy pairs of functions transform under composition, integration, differentiation, etc. We need some notions related to pairs of groups, which we introduce now.

By a pair of groups we shall always mean a pair consisting of a group Γ and a subgroup of Γ. We shall identify Γ with the pair consisting of Γ and the trivial subgroup of Γ.

Definition. A class $\mathcal{L}$ of pairs of groups will be called an *almost complete class of pairs of groups* if

(1) for each pair of groups $[\Gamma, \Gamma_0] \in \mathcal{L}$, $\Gamma_0 \subseteq \Gamma$, and for each homomorphism

$$\tau\, \Gamma \to G,$$

where G is some group, the pair of groups $[\tau\Gamma, \tau\Gamma_0]$ also belongs to $\mathcal{L}$;

(2) for each pair of groups $[\Gamma, \Gamma_0] \in \mathcal{L}$, $\Gamma_0 \subseteq \Gamma$, and for each homomorphism

$$\tau\, G \to \Gamma,$$

where G is some group, the pair of groups $[\tau^{-1}(\Gamma), \tau^{-1}(\Gamma_0)]$ also belongs to $\mathcal{L}$; and

(3) for each pair of groups $[\Gamma, \Gamma_0] \in \mathcal{L}$, $\Gamma_0 \subseteq \Gamma$, and for each group G with a T_2-topology and such that $\Gamma \subseteq G$, the pair of groups $[\overline{\Gamma}, \overline{\Gamma}_0]$ also belongs to $\mathcal{L}$, where $\overline{\Gamma}$ and $\overline{\Gamma}_0$ denote the closures of Γ and Γ_0 in G, respectively.

Definition. An almost complete class $\mathcal{M}$ of pairs of groups will be called a *complete class of pairs of groups* if

(1) for each pair of groups $[\Gamma, \Gamma_0] \in \mathcal{M}$, and for each group Γ_1 such that
$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma,$$
the pair of groups $[\Gamma, \Gamma_1]$ also belongs to $\mathcal{M}$; and

(2) for any two pairs of groups
$$[\Gamma, \Gamma_1],\ [\Gamma_1, \Gamma_2] \in \mathcal{M},$$
the pair of groups $[\Gamma, \Gamma_2]$ also belongs to $\mathcal{M}$.

The minimal almost complete and the minimal complete class of pairs of groups containing a given class $\mathcal{B}$ of pairs of groups will be denoted by $\mathcal{L}\langle\mathcal{B}\rangle$ and $\mathcal{M}\langle\mathcal{B}\rangle$, respectively.

Lemma.

(1) *If the monodromy group of a pair of groups $[\Gamma, \Gamma_0]$ is contained in some complete class $\mathcal{M}$ of pairs of groups, then $[\Gamma, \Gamma_0]$ is also contained in $\mathcal{M}$.*

(2) *If an almost normal pair $[\Gamma, \Gamma_0]$ is contained in some complete class $\mathcal{M}$ of pairs of groups, then the monodromy group of $[\Gamma, \Gamma_0]$ is also contained in $\mathcal{M}$.*

Let us prove the second claim. Let Γ_i, $i = 1, \dots, n$, be a finite number of subgroups conjugate to Γ_0 and such that
$$\bigcap_{i=1}^{n} \Gamma_i = \bigcap_{\mu \in \Gamma} \mu \Gamma_0 \mu^{-1}.$$
The pairs $[\Gamma, \Gamma_i]$ are isomorphic to the pair $[\Gamma, \Gamma_0]$, so
$$[\Gamma, \Gamma_i] \in \mathcal{M}.$$
Let $\tau\, \Gamma_2 \to \Gamma$ denote the inclusion homomorphism. Then
$$\tau^{-1}(\Gamma_1) = \Gamma_2 \bigcap \Gamma_1,$$
hence
$$[\Gamma_2, \Gamma_2 \bigcap \Gamma_1] \in \mathcal{M}.$$

Since $\mathcal{M}$ contains $[\Gamma, \Gamma_2]$ and $[\Gamma_2, \Gamma_2 \bigcap \Gamma_1]$, we have

$$[\Gamma, \Gamma_1 \bigcap \Gamma_2] \in \mathcal{M}.$$

Continuing this argument, we obtain that $\mathcal{M}$ contains the pair $[\Gamma, \bigcap_{i=1}^{n} \Gamma_i]$ and hence $\mathcal{M}$ also contains the group

$$\Gamma / \bigcap_{\mu \in \Gamma} \mu \Gamma_0 \mu^{-1}.$$

Proposition on the class $\mathcal{L}\langle [f] \rangle$. *An almost complete class of pairs $\mathcal{L}$ contains the closed monodromy pair $[f]$ of an S-function f if and only if $\mathcal{L}$ contains the monodromy pair of f with prohibited set A.*

Proof. Let $[\Gamma, \Gamma_0]$ denote the monodromy pair of f with prohibited set A. Then

$$[f] = [\overline{\Gamma}, \overline{\Gamma}_0].$$

Therefore, any almost complete class $\mathcal{L}$ containing $[\Gamma, \Gamma_0]$ also contains $[f]$. Conversely, if $[\overline{\Gamma}, \overline{\Gamma}_0]$ belong to $\mathcal{L}$, then

$$[\Gamma, \Gamma_0] \in \mathcal{L}.$$

Indeed, the topology in the permutation group is such that

$$\Gamma_0 = \Gamma \bigcap \overline{\Gamma}_0.$$

Therefore the pair $[\Gamma, \Gamma_0]$ is the inverse image of $[\overline{\Gamma}, \overline{\Gamma}_0]$ under the inclusion of Γ in $\overline{\Gamma}$. □

4. The main theorem

Main Theorem. *The class $\widehat{\mathcal{M}}$ of S-functions whose closed monodromy pairs belong to some complete class $\mathcal{M}$ of pairs is closed with respect to differentiation, composition, and meromorphic operations. If, moreover, $\mathcal{M}$ contains*

1. *the group $\mathbb{C}$ of complex numbers with respect to addition, then $\widehat{\mathcal{M}}$ is closed with respect to integration,*
2. *the group $S(n)$ of permutations on n elements, then $\widehat{\mathcal{M}}$ is closed with respect to solving algebraic equations of degree at most n.*

The proof of this theorem consists of the following lemmas.

Lemma on the derivative. *For each S-function f we have*

$$[f'] \in \mathcal{M}\langle [f] \rangle.$$

Proof. Let A be the set of singular points of the S-function f and let f_a be a germ of f at a nonsingular point a. Denote by the Γ the fundamental group $\pi_1(S^2 \setminus A, a)$ and by Γ_1 and Γ_2 the isotropy subgroups of f_a and f'_a, respectively. Then Γ_2 contains Γ_1. Indeed, f_a does not change after continuation along a curve $\gamma \in \Gamma_1$, so the derivative f'_a does not change either. It follows from the definition of a complete class that

$$[\Gamma, \Gamma_2] \in \mathcal{M}\langle[\Gamma, \Gamma_1]\rangle.$$

Using the proposition in Sec. 3.5 we obtain

$$[f'] \in \mathcal{M}\langle[f]\rangle. \quad \square$$

Lemma on composition. *For any two S-functions f and g, we have*

$$[g \circ f] \in \mathcal{M}\langle[f], [g]\rangle.$$

Proof. Let A and B denote the sets of singular points of f and g, respectively. Let $f^{-1}(B)$ be the inverse image of B under the multi-valued correspondence generated by the multi-valued function f. Set

$$Q = A \bigcup f^{-1}(B).$$

Let f_a be some germ of f at $a \notin Q$, and let g_b be some germ of g at $b = f(a)$. Then Q is a prohibited set for the germ $g_b \circ g_a$. Denote by Γ the fundamental group $\pi_1(S^2 \setminus Q, a)$ and by Γ_1 and Γ_2 the isotropy subgroups of the germs f_a and $g_b \circ f_a$, respectively. Denote by G the fundamental group $\pi_1(S^2 \setminus B, b)$ and by G_0 the isotropy subgroup of g_b. Define a homomorphism

$$\tau \, \Gamma_1 \to G$$

as follows. To each curve $\gamma \in \Gamma_1$ associate the curve

$$\tau \circ \gamma(t) = f(\gamma(t)),$$

where $f_{\gamma(t)}$ is the germ obtained from f_a after continuation along γ up to time t. The curves $\tau \circ \gamma$ are closed because f_a does not change after continuation along curves belonging to Γ_1. A homotopy of γ within $S^2 \setminus Q$ determines a homotopy of $\tau \circ \gamma$ within $S^2 \setminus B$ because

$$f^{-1}(B) \subseteq Q.$$

Hence the homomorphism is well defined. The germ $g_b \circ f_a$ does not change after continuation along curves belonging to the group $\tau^{-1}(G_0)$; in other words,

$$\tau^{-1}(G_0) \subseteq \Gamma_2.$$

This implies the lemma. Indeed, we have

$$\Gamma \supseteq \Gamma_2 \supseteq \tau^{-1}(G_0) \subseteq \tau^{-1}(G) = \Gamma_1 \subseteq \Gamma,$$

and hence

$$[\Gamma, \Gamma_2] \in \mathcal{M}\langle [G, G_0], [\Gamma, \Gamma_1] \rangle.$$

Using the proposition from Sec. 3.5, we get

$$[g \circ f] \in \mathcal{M}\langle [f], [g] \rangle. \quad \square$$

Lemma on the integral. *For each S-function f, the following inclusion holds:*

$$\left[\int f(x)\, dx\right] \in \mathcal{M}\langle [f], \mathbb{C} \rangle,$$

where $\mathbb{C}$ denotes the group of complex numbers with respect to addition.

Proof. Let A be the set of singular points of f and let $Q = A \bigcup \{\infty\}$. Let f_a be some germ of f at a point $a \notin Q$ and let g_a be a germ of $\int f(x)\, dx$ at this point, $g'_a = f_a$. We can take Q as prohibited set for the germs f_a and g_a. Denote by Γ the fundamental group $\pi_1(S^2 \setminus Q, a)$, and by Γ_1 and Γ_2 the isotropy subgroups of f_a and g_a, respectively.

Define a homomorphism

$$\tau\, \Gamma_1 \to \mathbb{C}$$

as follows. To each curve $\gamma \in \Gamma_1$ associate the number

$$\int_\gamma f(\gamma(t))\, dx,$$

where $f_{\gamma(t)}$ is the germ obtained by continuation of f_a along γ up to the point t, and $x = \gamma(t)$. The isotropy subgroup Γ_2 of g_a coincides with the kernel of τ, and hence

$$[\Gamma, \Gamma_2] \in \mathcal{M}\langle [\Gamma, \Gamma_1], \mathbb{C} \rangle.$$

By using the proposition from Sec. 3.5, we obtain

$$\left[\int f(x)\, dx\right] \in \mathcal{M}\langle [f], \mathbb{C} \rangle. \quad \square$$

In what follows it will be convenient to use vector-valued functions. The definitions of prohibited set, *S-function*, and monodromy group extend in a straightforward way to vector-valued functions.

Lemma on vector-valued functions. *For each vector-valued S-function*

$$\mathbf{f} = (f_1, \dots, f_n)$$

the following equality holds:

$$\mathcal{M}\langle [\mathbf{f}] \rangle = \mathcal{M}\langle [f_1], \dots, [f_n] \rangle.$$

Proof. Let A_i denote the set of singular points of f_i. The set of singular points of $\mathbf{f}$ is

$$Q = \bigcup A_i.$$

Let $\mathbf{f}_a = (f_{1a}, \dots, f_{na})$ be some germ of $\mathbf{f}$ at a point $a \notin Q$. Denote by Γ the fundamental group $\pi_1(S^2 \setminus Q, a)$, by Γ_i the isotropy subgroups of the germs f_{ia}, and by Γ_0 the isotropy subgroup of $\mathbf{f}_a$. Then

$$\Gamma_0 = \bigcap_{i=1}^{n} \Gamma_i,$$

so

$$\mathcal{M}\langle[\Gamma, \Gamma_0]\rangle = \mathcal{M}\langle[\Gamma, \Gamma_1], \dots, [\Gamma, \Gamma_n]\rangle.$$

By using the proposition in Sec. 3.5, we obtain

$$\mathcal{M}\langle[\mathbf{f}]\rangle = \mathcal{M}\langle[f_1], \dots, [f_n]\rangle. \quad \square$$

Lemma on meromorphic operations. *For each vector-valued S-function*

$$\mathbf{f} = (f_1, \dots, f_n)$$

and meromorphic function $F(x_1, \dots, x_n)$ such that $F \circ \mathbf{f}$ is defined, the following holds:

$$[F \circ \mathbf{f}] \in \mathcal{M}\langle[\mathbf{f}]\rangle.$$

Proof. Let A be the set of singular points of $\mathbf{f}$ and B the projection of the set of poles of $F \circ \mathbf{f}$ into the Riemann sphere. As prohibited set for $F \circ \mathbf{f}$ we can take $Q = A \bigcup B$. Let $\mathbf{f}_a$ be some germ of $\mathbf{f}$ at a point $a \notin Q$. Denote by Γ the fundamental group $\pi_1(S^2 \setminus Q, a)$ and by Γ_1 and Γ_2 the isotropy subgroups of $\mathbf{f}_a$ and $F \circ \mathbf{f}_a$, respectively. Then Γ_2 is a subgroup of Γ_1: indeed, the vector-valued function $\mathbf{f}$ does not change after continuation along a curve $\gamma \in \Gamma_1$, hence a meromorphic function of $\mathbf{f}$ does not change either. It follows from $\Gamma_2 \subseteq \Gamma_1$ that

$$[\Gamma, \Gamma_2] \in \mathcal{M}\langle[\Gamma, \Gamma_1]\rangle.$$

By using the proposition in Sec. 3.5, we obtain

$$[F \circ \mathbf{f}] \in \mathcal{M}\langle[\mathbf{f}]\rangle. \quad \square$$

Lemma on algebraic functions. *Let $\mathbf{f} = (f_1, \dots, f_n)$ be a vector-valued function and y an algebraic function of $\mathbf{f}$ defined by the equation*

$$y^n + f_1 f^{n-1} + \dots + f_n = 0. \tag{1}$$

Then

$$[y] \in \mathcal{M}\langle[f], S(n)\rangle,$$

where $S(n)$ is the symmetric group on n elements.

Proof. Let A denote the set of singular points of $\mathbf{f}$ and B the projection of the set of algebraic ramification points of y into the Riemann sphere. As prohibited sets for y and $\mathbf{f}$ we can take $Q = A \bigcup B$. Let y_a and $\mathbf{f}_a$ be some germs of y and $\mathbf{f}$ at a point $a \notin Q$ related by the formula

$$y_a^n + f_{1a}y_a^{n-1} + \cdots + f_{na} = 0.$$

Denote by Γ the fundamental group $\pi_1(S^2 \setminus Q, a)$ and by Γ_1 and Γ_2 the isotropy subgroups of $\mathbf{f}_a$ and y_a, respectively. The coefficients of (1) do not change after analytic continuation along a curve $\gamma \in \Gamma_1$, so the roots of (1) will only be permuted after such a continuation. We get a homomorphism

$$\tau \, \Gamma_1 \to S(n).$$

The group Γ_2 is contained in the kernel of τ, and hence

$$[\Gamma, \Gamma_2] \in \mathcal{M}\langle [\Gamma, \Gamma_1], S(n)\rangle.$$

By using the proposition in Sec. 3.5, we obtain

$$[y] \in \mathcal{M}\langle [f], S(n)\rangle. \quad \square$$

This finishes the proof of the main theorem.

5. Group obstructions to representability in quadratures

5.1. Computation of some classes of pairs of groups. The main theorem raises the problem of describing the minimal class of pairs of groups containing the group $\mathbb{C}$ of complex numbers with respect to addition, as well as the minimal classes of pairs of groups containing, respectively, $\mathbb{C}$ and the finite groups, and also $\mathbb{C}$ and the group $S(n)$. In this section we solve these problems.

Proposition 1. *The minimal complete class of pairs, $\mathcal{M}\langle \mathcal{L}_\alpha \rangle$, containing some given almost complete classes $\mathcal{L}_\alpha$ of pairs consists of pairs of groups $[\Gamma, \Gamma_0]$ for which there is a chain of subgroups*

$$\Gamma = \Gamma_1 \supseteq \cdots \supseteq \Gamma_m \subseteq \Gamma_0$$

such that for each i, $1 \le i \le m-1$, the pair $[\Gamma_i, \Gamma_{i+1}]$ is contained in some $\mathcal{L}_{\alpha(i)}$.

For the proof, it suffices to show that the pairs $[\Gamma, \Gamma_0]$ as described in the proposition (a) belong to the complete class $\mathcal{M}\langle \mathcal{L}_\alpha \rangle$ and (b) form a complete class of pairs. Both claims follow immediately from the definitions.

One can easily check the following propositions.

Proposition 2. *The class of pairs of groups $[\Gamma, \Gamma_0,]$ such that Γ_0 is a normal subgroup of Γ and Γ/Γ_0 is commutative forms the minimal almost complete class $\mathcal{L}\langle \mathcal{A}\rangle$ containing the class $\mathcal{A}$ of all abelian groups.*

Proposition 3. *The class of pairs of groups* $[\Gamma, \Gamma_0]$ *such that* Γ_0 *is a normal subgroup of* Γ *and* Γ/Γ_0 *is finite forms the minimal almost complete class* $\mathcal{L}\langle\mathcal{K}\rangle$ *containing the class* $\mathcal{K}$ *of all finite groups.*

Proposition 4. *The class of pairs* $[\Gamma, \Gamma_0]$ *such that*

$$\operatorname{ind}(\Gamma, \Gamma_0) \leq n$$

forms an almost complete class of groups.

We shall denote the class in Proposition 4 by $\mathcal{L}\langle\operatorname{ind} \leq n\rangle$. Proposition 4 is of interest to us because of the following characteristic property of subgroups of $S(n)$.

Lemma 1. *A group* Γ *is isomorphic to* $S(n)$ *if and only if* Γ *contains subgroups* Γ_i, $i = 1, \dots, k$, *such that*

(1) $\bigcap_{i=1}^{k} \Gamma_i$ *does not contain nontrivial normal subgroups of* Γ, *and*
(2) $\sum_{i=1}^{k} \operatorname{ind}(\Gamma, \Gamma_i) \leq n$.

Proof. Let Γ be a subgroup of $S(n)$. Consider the representation of Γ as a subgroup of the permutation group on a set M consisting of n elements. Let M decompose into k orbits under the action of Γ. Choose a point x_i in each of the orbits. The set of isotropy subgroups Γ_i of the x_i satisfy the conditions of the lemma. Conversely, assume Γ contains subgroups satisfying the conditions of the lemma. Denote by P the disjoint union of the sets $P_i = \{P_i^j\}$, where P_i^j are the conjugacy classes of Γ_i in Γ. There is a natural action of Γ on P. The corresponding representation of Γ in $S(P)$ is faithful since the kernel of this representation lies in

$$\bigcap_{i=1}^{k} \Gamma_i.$$

Finally, $S(P)$ is a subgroup of $S(n)$ since P contains

$$\sum_{i=1}^{k} \operatorname{ind}(\Gamma, \Gamma_i) \leq n$$

elements. □

A chain of groups

$$\Gamma_i, \quad i = 1, \dots m, \quad \Gamma = \Gamma_1 \supseteq \cdots \supseteq \Gamma_m \subseteq \Gamma_0,$$

is called a normal tower of the pair of groups $[\Gamma, \Gamma_0]$ if Γ_{i+1} is a normal subgroup of Γ_i for each $i = 1, \dots, m-1$. The quotient groups Γ_i/Γ_{i+1} are called the quotients of the normal tower.

Theorem on the classes $\mathcal{M}\langle\mathcal{A},\mathcal{K}\rangle, \mathcal{M}\langle\mathcal{A},S(n)\rangle$, and $\mathcal{M}\langle\mathcal{A}\rangle$.

(1) *A pair of groups $[\Gamma,\Gamma_0]$ belongs to the minimal complete class $\mathcal{M}\langle\mathcal{A},\mathcal{K}\rangle$ containing all finite and all abelian groups if and only if $[\Gamma,\Gamma_0]$ admits a normal tower with each quotient either finite or abelian.*

(2) *A pair of groups $[\Gamma,\Gamma_0]$ belongs to the minimal complete class $\mathcal{M}\langle\mathcal{A},S(n)\rangle$ containing $S(n)$ and all the abelian groups if and only if $[\Gamma,\Gamma_0]$ admits a normal tower with each quotient either a subgroup of $S(n)$ or an abelian group.*

(3) *A pair of groups $[\Gamma,\Gamma_0]$ belongs to the minimal complete class $\mathcal{M}\langle\mathcal{A}\rangle$ containing all abelian groups if and only if the monodromy group of $[\Gamma,\Gamma_0]$ is solvable.*

Proof. (1) follows from the description in Propositions 2 and 3 of the classes $\mathcal{L}\langle\mathcal{A}\rangle$ and $\mathcal{L}\langle\mathcal{K}\rangle$ and from Proposition 1.

In order to prove (2), consider the minimal complete class of pairs of groups containing $\mathcal{L}\langle\mathcal{A}\rangle$ and $\mathcal{L}\langle\text{ind}\le n\rangle$. This class consists of pairs $[\Gamma,\Gamma_0]$ for which there is a chain of subgroups

$$\Gamma=\Gamma_1\supseteq\cdots\supseteq\Gamma_m\subseteq\Gamma_0,$$

such that for each i, $1\le i\le m-1$, either Γ_i/Γ_{i+1} is abelian or $\text{ind}(\Gamma_i,\Gamma_{i+1})\le n$ (cf. Propositions 3 and 4, as well as Proposition 1). This class contains $S(n)$ (cf. Lemma 1) as well as all abelian groups, and it is clearly the minimal complete class of pairs that has this property. All that remains is to reformulate the answer. We shall transform successively the chain of subgroups

$$\Gamma=\Gamma_1\supseteq\cdots\supseteq\Gamma_m\subseteq\Gamma_0,$$

into a normal tower for $[\Gamma,\Gamma_0]$. Assume that for $j\le i$ the group Γ_{i+1} is a normal subgroup of Γ_i and that

$$\text{ind}(\Gamma_i,\Gamma_{i+1})\le n.$$

Denote by $\overline{\Gamma}_{i+1}$ the largest normal subgroup of Γ_i contained in Γ_{i+1}. Clearly, $\Gamma_i/\overline{\Gamma}_{i+1}$ is a subgroup of $S(n)$. Instead of the original chain of subgroups, consider the chain

$$\Gamma=G_1\supseteq\cdots\supseteq G_m=\Gamma_0,$$

where $G_j=\Gamma_j$ for $j\le i$ and

$$G_j=\Gamma_j\bigcap\overline{\Gamma}_{i+1}$$

for $j>i$. By continuing this process (no more than m times) we obtain a normal tower and the required description of $\mathcal{M}\langle\mathcal{A},S(n)\rangle$.

We now prove (3). By Propositions 1 and 2 a pair of $[\Gamma, \Gamma_0]$ belongs to $\mathcal{M}\langle\mathcal{A}\rangle$ if and only if there is a chain

$$\Gamma = \Gamma_1 \supseteq \cdots \supseteq \Gamma_m \subseteq \Gamma_0,$$

such that Γ_i/Γ_{i+1} is abelian. Consider the chain

$$\Gamma = G^1 \supseteq \cdots \supseteq G^m,$$

where G^{i+1}, $i = 1, \dots, m-1$, is the derived subgroup (commutator subgroup) of G^i. Each automorphism of Γ maps G^i to itself, hence each G^i is a normal subgroup of Γ. Induction on i shows that $G^i \subseteq \Gamma_i$, and, in particular,

$$G^m \subseteq \Gamma_m \subseteq \Gamma_0.$$

As G^m is a normal subgroup of Γ and as $G^m \subseteq \Gamma_0$ we have

$$G^m \subseteq \bigcap_{\mu\in\Gamma} \mu\Gamma_0\mu^{-1}.$$

The definition of the chain of G^i's implies that Γ/G^m is solvable. The group

$$\Gamma/\bigcap_{\mu\in\Gamma} \mu\Gamma_0\mu^{-1}$$

is solvable since it is a subgroup of Γ/G^m. The converse assertion (a pair of groups with solvable monodromy group belongs to $\mathcal{M}\langle\mathcal{A}\rangle$) is obvious. □

Proposition 5. *Let Γ be an abelian group of cardinality at most $\mathfrak{c}$ (continuum). Then*

$$\Gamma \in \mathcal{L}\langle\mathcal{A}\rangle.$$

Proof. The complex numbers $\mathbb{C}$ form a vector space over the rational numbers whose dimension is $\mathfrak{c}$. Let $\{e_\alpha\}$ be some basis for this vector space. The subgroup $\widetilde{\mathbb{C}}$ of $\mathbb{C}$ generated by the numbers $\{e_\alpha\}$ is a free abelian group with $\mathfrak{c}$ generators. Each abelian group Γ of cardinality $\leq \mathfrak{c}$ is a quotient of $\widetilde{\mathbb{C}}$, and hence

$$\Gamma \in \mathcal{L}\langle\mathcal{A}\rangle. \quad \square$$

It follows from Proposition 5 and the results of the computation of

$$\mathcal{M}\langle\mathcal{A}, \mathcal{K}\rangle, \quad \mathcal{M}\langle S(n)\rangle, \quad \text{and} \quad \mathcal{M}\langle\mathcal{A}\rangle$$

that a pair of groups $[\Gamma, \Gamma_0]$ with Γ of cardinality $\leq \mathfrak{c}$ belongs to

$$\mathcal{M}\langle\mathbb{C}, \mathcal{K}\rangle, \ \mathcal{M}\langle\mathbb{C}, S(n)\rangle, \quad \text{and} \quad \mathcal{M}\langle\mathbb{C}\rangle,$$

if and only if $[\Gamma, \Gamma_0]$ belongs to

$$\mathcal{M}\langle\mathcal{A}, \mathcal{K}\rangle, \ \mathcal{M}\langle\mathcal{A}, S(n)\rangle, \quad \text{and} \quad \mathcal{M}\langle\mathcal{A}\rangle,$$

respectively.

We shall restrict ourselves to this result, as the group of permutations of the sheets of the function has cardinality $\leq \mathfrak{c}$.

Lemma 2. *A free nonabelian group* Λ *does not belong to* $\mathcal{M}\langle\mathcal{A},\mathcal{K}\rangle$.

Proof. Assume that $\Lambda \in \mathcal{M}\langle\mathcal{A},\mathcal{K}\rangle$, i.e., that Λ admits a normal tower

$$\Lambda = \Gamma_1 \supseteq \cdots \supseteq \Gamma_m = e$$

with each quotient either finite or abelian. Each Γ_i is free since it a subgroup of a free group (cf. [8], p. 179). The group $\Gamma_m = e$ is abelian. Let Γ_{i+1} be the first abelian group in this tower. For any two elements $a, b \in \Gamma_i$ there is a nontrivial relation: if Γ_i/Γ_{i+1} is abelian, then the elements

$$aba^{-1}b^{-1} \quad \text{and} \quad ab^2a^{-1}b^{-2}$$

commute; if Γ_i/Γ_{i+1} is finite, then some powers a^p and b^p of a and b commute. Therefore, Γ_i has no more than one generator, and hence is Γ_i abelian. This contradiction proves that $\Lambda \notin \mathcal{M}\langle\mathcal{A},\mathcal{K}\rangle$. □

Lemma 3. *For* $n > 4$ *the symmetric group* $S(n)$ *does not belong to* $\mathcal{M}\langle\mathbb{C}, S(n-1)\rangle$.

Proof. For $n > 4$, the alternating group $A(n)$ is simple and nonabelian. Hence $A(n) \notin \mathcal{M}\langle\mathbb{C}, S(n-1)\rangle$, and therefore, for $n > 4$,

$$S(n) \notin \mathcal{M}\langle\mathbb{C}, S(n-1)\rangle. \quad \square$$

Lemma 4. *The only transitive group of permutations on* n *elements that is generated by transpositions is the symmetric group* $S(n)$.

Proof. Let Γ be a group of permutations of a set M consisting of n elements, and let Γ be generated by transpositions. We shall say that a subset $M_0 \subseteq M$ is complete if each permutation of M_0 extends to some permutation of M that belongs to Γ. Complete subsets exist: for example, if an element of Γ transposes two elements a, $b \in M$, then $M_0 = \{a, b\}$ is complete. Let M_0 be a complete subset of M with maximal number of elements. Assume $M_0 \neq M$. Since Γ is transitive there is a transposition $\mu \in \Gamma$ among the generators of Γ which transposes an element $a \notin M_0$ with some element $b \in M_0$. The permutation group generated by μ and $S(M_0)$ is $S(M_0 \bigcup \{a\})$. So $M_0 \bigcup \{a\}$ is complete and contains M_0 as a proper subset. This contradiction proves that $\Gamma = S(M)$. □

5.2. Necessary conditions for the representability of functions by quadratures, n-quadratures, and generalized quadratures. If a function f is representable by quadratures, n-quadratures, or generalized quadratures, then f has at most countably many singular points (cf. Sec. 2).

We gather here the information we have obtained for the monodromy groups of such functions.

Result on generalized quadratures. Let f be a function representable by generalized quadratures. Then the closed monodromy pair $[f]$ of f has a normal tower with each quotient either finite or abelian. Moreover, this condition is also satisfied by the closed monodromy pair $[f]$ of a function f that is representable by generalized quadratures, compositions, and meromorphic operations via single-valued S-functions. If, moreover, f is almost normal, then this condition is also satisfied by the monodromy group $[f]$ of f.

Result on n-quadratures. Let f be a function representable by n-quadratures. Then the closed monodromy pair $[f]$ of f has a normal tower with each quotient either a subgroup of $S(n)$ or an abelian group. Moreover, this condition is satisfied by the closed monodromy pair $[f]$ of a function f that is representable by n-quadratures, compositions, and meromorphic functions via single-valued functions. If, moreover, f is almost normal, this condition is also satisfied by the monodromy group $[f]$ of f.

Result on quadratures. The closed monodromy group of a function f representable by quadratures is solvable. Moreover, if a function f is representable via single-valued S-functions by quadratures, compositions, and meromorphic operations, then the closed monodromy group of f is also solvable.

To prove these results, it suffices to apply the main theorem to the classes

$$\widehat{\mathcal{M}}\langle\mathbb{C},\mathcal{K}\rangle,\ \widehat{\mathcal{M}}\langle\mathbb{C},S(n)\rangle,\ \text{and } \widehat{\mathcal{M}}\langle\mathbb{C}\rangle$$

of S-functions, and use the computations for the classes

$$\mathcal{M}\langle\mathbb{C},\mathcal{K}\rangle,\ \mathcal{M}\langle\mathbb{C},S(n)\rangle,\ \text{and } \mathcal{M}\langle\mathbb{C}\rangle.$$

We now give examples of functions that are not representable by generalized quadratures. Let the Riemann surface of f be the universal cover of $S^2 \setminus A$, where S^2 is the Riemann sphere and A is a finite set with at least three points. Then *f is not representable via single-valued S-functions by generalized quadratures, compositions, and meromorphic operations.* Indeed, f is an almost normal function. The closed monodromy group of f is free and nonabelian since $\pi_1(S^2 \setminus A)$ has the same properties.

Example 1. Consider the function $z(w)$ which maps conformally the upper half-plane into a triangle bounded by arcs of circles with zero angles. This function $z(w)$ is inverse to the Picard modular function. The

Riemann surface of $z(w)$ is the universal covering of the sphere minus three points. Therefore $z(w)$ is not representable via single-valued S-functions by generalized quadratures, compositions, and meromorphic operations.

Note that the function $z(w)$ is closely related to the elliptic integrals

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

and

$$K'(k) = \int_0^{\frac{1}{k}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Each two of the functions $K(k)$, $K'(k)$, and $z(w)$ form a pair such that each function in the pair is representable via the other by quadratures (cf. [9], pp. 257–280 and pp. 333–340). Therefore, neither of the functions $K(k)$ and $K'(k)$ is representable via single-valued S-functions by generalized quadratures, compositions, and meromorphic operations.

In the following section we shall generalize Example 1 and enumerate all the polygons bounded by arcs of circles onto which the upper half-plane can be mapped by a function representable by generalized quadratures.

Example 2. Let f be an n-valued algebraic function with simple ramification points whose images on the Riemann sphere are distinct. If $n > 4$, then f is not representable via single-valued S-functions by $(n-1)$-quadratures, compositions, and meromorphic operations. In particular, f is not representable by $(n-1)$-quadratures.

Indeed, if one goes around a simple ramification point of f, two sheets of f get transposed. The monodromy group of f is a transitive permutation group generated by transpositions, i.e., is equal to $S(n)$. For $n > 4$, $S(n)$ does not belong to $\mathcal{M}\langle \mathbb{C}, S(n-1)\rangle$.

6. Mapping the half-plane onto a polygon bounded by arcs of circles

6.1. Application of the symmetry principle. Let G be a polygon bounded by arcs of circles in the complex plane. By the Riemann mapping theorem there is a function f_G mapping the upper half-plane onto G. This mapping has been studied by Riemann, Schwarz, Christoffel, Klein, and others. We shall recall the classical results that will be needed.

Denote by $B = \{b_j\}$ the inverse image of the set of vertices of G under f_G, by $H(G)$ the group of conformal transformations of the sphere generated by reflections about the sides of G, and by $L(G)$ the subgroup of $H(G)$ of index 2 consisting of fractional linear transformations. The following proposition follows from the Riemann-Schwarz symmetry principle.

Proposition.

(1) *The function* f_G *has a meromorphic continuation along curves that do not intersect* B.

(2) *All germs of a multi-valued function* f_G *at a nonsingular point* $a \notin B$ *can be obtained by applying the fractional linear group* $L(G)$ *to a fixed germ of* f_G *at* a.

(3) *The monodromy group of* f_G *is isomorphic to* $L(G)$.

(4) *The singularities of* f_G *around the points* b_j *are of the following kind. If the angle* α_j *of* G *at the vertex* a_j *corresponding to* b_j *is nonzero, then, after applying a fractional linear transformation,* f_G *can be reduced to the following:*

$$f_G(z) = (z - b_j)^{\beta_j} \varphi(z),$$

where

$$\beta_j = \alpha_j / 2\pi,$$

and $\varphi(z)$ *is holomorphic around* b_j. *If* $\alpha_j = 0$, *then, after applying a fractional linear transformation,* f_G *can be reduced to the following form:*

$$f_G(z) = \ln(z) + \varphi(z),$$

where $\varphi(z)$ *is holomorphic around* b_j.

It follows from our results that if f_G is representable by generalized quadratures, then $L(G)$, and hence $H(G)$, belongs to $\mathcal{M}\langle \mathbb{C}, \mathcal{K} \rangle$.

6.2. Groups of fractional linear and of conformal transformations belonging to $\mathcal{M}\langle \mathbb{C}, \mathcal{K} \rangle$. Let π be the epimorphism of the group $SL(2)$ of 2×2 matrices with determinant equal to 1 into the group L of fractional linear transformations defined by

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{az + b}{cz + d}.$$

Since $\ker \pi = \mathbb{Z}_2$, a group $\widetilde{L} \subseteq L$ belongs to $\mathcal{M}\langle \mathbb{C}, \mathcal{K} \rangle$ if and only if the group $\pi^{-1}(\widetilde{L}) = \Gamma \subseteq SL(2)$ belongs to $\mathcal{M}\langle \mathbb{C}, \mathcal{K} \rangle$. As Γ is a matrix group, $\Gamma \in \mathcal{M}\langle \mathbb{C}, \mathcal{K} \rangle$ if and only if Γ has a normal subgroup Γ_0 of finite index that can be reduced to triangular form. (This version of Lie's theorem applies also to higher dimensions and plays an important role in differential Galois theory. We shall discuss such theorems in detail in Part 2 of this paper, which will be devoted to solvability by quadrature of Fuchs-type differential equations.) The group Γ_0 consists of 2×2 matrices, hence Γ_0 can be reduced to triangular form in one of the following three cases:

(1) Γ_0 has a unique 1-dimensional eigenspace;
(2) Γ_0 has two 1-dimensional eigenspaces;
(3) Γ_0 has a 2-dimensional eigenspace.

We now consider the linear fractional group $\widetilde{L} = \pi(\Gamma)$. A group $\widetilde{L}$ of fractional linear transformations belongs to $\mathcal{M}\langle\mathbb{C},\mathcal{K}\rangle$ if and only if $\widetilde{L}$ has a normal subgroup $L_0 = \pi(\Gamma)$ of finite index such that the set of fixed points of L_0 consists of either one or two points or is the whole Riemann sphere.

A group $\widetilde{H}$ of conformal transformations has a subgroup $\widetilde{L}$ of index 2 (or of index 1) consisting of fractional linear transformations. Hence

$$\widetilde{H} \in \mathcal{M}\langle\mathbb{C},\mathcal{K}\rangle$$

is equivalent to a condition analogous to the above-mentioned one.

Lemma on conformal transformation groups in $\mathcal{M}\langle\mathbb{C},\mathcal{K}\rangle$.
A group $\widetilde{H}$ of conformal transformations belongs to $\mathcal{M}\langle\mathbb{C},\mathcal{K}\rangle$ if and only if one of the following three conditions holds:

(1) *$\widetilde{H}$ has a fixed point;*
(2) *$\widetilde{H}$ has an invariant set consisting of two points;*
(3) *$\widetilde{H}$ is finite.*

The lemma follows from the discussion above as the set of fixed points of a normal subgroup is invariant under the action of the group. It is well known that *a finite group $\widetilde{L}$ of fractional linear transformations of the sphere can be reduced to a group of rotations by a fractional linear transformation.*

If the product of two inversions with respect to two distinct circles corresponds under stereographic projection to a rotation of the sphere, then it is not difficult to show that these two circles are great circles. Hence *if $\widetilde{H}$ is a finite group of conformal transformations that is generated by inversions with respect to circles, then $\widetilde{H}$ can be reduced by a linear fractional transformation of the coordinates to a group of motions of the sphere that is generated by reflections.*

All finite groups of motions generated by reflections are well known. Each such group is the group of motions of one of the following solids:

(1) a pyramid with base a regular n-gon;
(2) an n-gonal dihedron, i.e., the solid obtained by gluing along the base two identical pyramids as in (1);
(3) a tetrahedron;
(4) a cube or octahedron;
(5) a dodecahedron or icosahedron.

All these groups of motions, except for the dodecahedral-icosahedral, are solvable. Take a sphere centered at the center of gravity of the solid. The planes of symmetry of this solid intersect this sphere in a network of great circles. The networks corresponding to the solids enumerated above shall be called finite networks of great circles. Figure 3 depicts the finite networks.

6.3. Integrable cases. We now return to the problem of representation of a function f_G by quadratures.

A case-by-case inspection will show that the conditions we found on the monodromy group are not just necessary but also sufficient for f_G to be representable by quadratures.

First integrable case. The group $H(G)$ has a fixed point. This means that the sides of G, after extension, intersect in one point. Map this point to infinity by a fractional linear transformation. Then G gets mapped to a polygon $\overline{G}$ bounded by straight segments.

All the transformations in $L(\overline{G})$ have the form

$$z \to az + b.$$

All germs of $\overline{f} = f_{\overline{G}}$ at a nonsingular point c can be obtained by applying the group $L(\overline{G})$ to a fixed germ $\overline{f}_c$:

$$\overline{f}_c \to a\overline{f}_c + b.$$

The germ

$$R_c = \overline{f}''_c / \overline{f}'_c$$

is invariant under the action of $L(\overline{G})$. This means that R_c is the germ of a single-valued function. The singular points of R_c can only be poles (cf. the proposition in Sec. 6.1.). Therefore, R_c is rational. The equation

$$\overline{f}''/\overline{f}' = R$$

is solvable by quadrature. This is a well-known case of integrability. In this case $\overline{f}$ is called a Christoffel-Schwarz integral.

Second integrable case. The group $H(G)$ has an invariant set consisting of two points. This means that for each side of G, either these two points are inverse with respect to this side, or they both lie on the extension of this side. Map these two points to 0 and ∞, respectively, by a fractional linear transformation. Then G is mapped to a polygon $\overline{G}$ bounded by arcs of circles centered at 0 and by rays emanating from 0 (cf. Figure 2). All transformations in $L(\overline{G})$ have the form

$$z \to az, \quad \text{or} \quad z \to b/z.$$

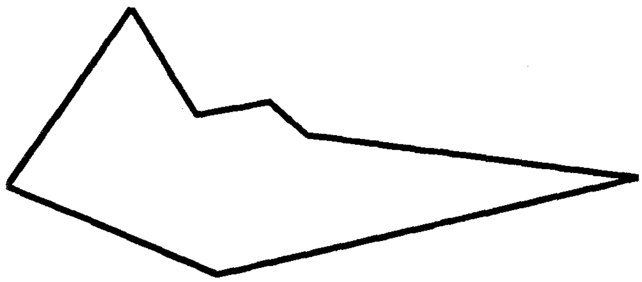

FIGURE 1. First integrable case

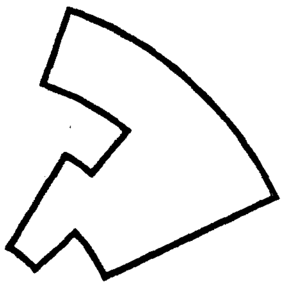

FIGURE 2. Second integrable case

All germs of $\bar{f} = f_{\overline{G}}$ at a nonsingular point c can be obtained by applying $L(\overline{G})$ to a fixed germ $\bar{f}_c$ at c:

$$\bar{f}_c \to a\bar{f}_c, \text{ or } \bar{f}_c \to b/\bar{f}_c.$$

The germ

$$R_c = (\bar{f}'_c/\bar{f}_c)^2$$

is invariant under the action of $L(\overline{G})$ and hence is the germ of a single-valued function R. The singularities of R can be only poles (cf. the proposition in Sec. 6.1), hence R is a rational function.

The equation

$$R = (\bar{f}'/\bar{f})^2$$

is solvable by quadrature.

Third integrable case. The group $H(G)$ is finite. This means that G can be mapped by a fractional linear transformation onto a polygon $\overline{G}$ whose sides lie on a finite network of great circles. The group $L(G)$ is finite, so f_G is finite-valued. Since all the singularities of f_G are of a power kind (cf. the proposition in Sec. 6.1), f_G is an algebraic function.

We now consider the case of a finite solvable $H(G)$ in more detail. This is the case when G can be mapped by a fractional linear transformation onto a polygon $\overline{G}$ whose sides lie on a finite network of great circles that is not dodecahedral-icosahedral. In this case $L(G)$ is solvable. By applying Galois theory, it is easy to show that in this case f_G is representable via rational functions by arithmetic operations and radicals.

We have thus shown the following.

Theorem on polygons bounded by arcs of circles. *If G is a polygon not covered by the three cases above, then not only is f_G not representable by generalized quadratures, but f_G is not representable via single-valued S-functions by generalized quadratures, compositions, and meromorphic operations.*

7. Classes of singular sets

In this paper we considered S-functions, i.e., multi-valued analytic functions of one complex variable with an at most countable number of singular points. Let S be the class of all at most countable subsets of the Riemann sphere S^2. We now enumerate those properties of S that we used:

(1) if $A \in S$, then $S^2 \backslash A$ is everywhere dense and locally path-connected;
(2) there is a nonempty set A such that $A \in S$;
(3) if $A \in S$ and $B \subseteq A$, then $B \in S$;
(4) if $A_i \in S$, $i = 1, 2, \ldots$, then

$$\bigcup_1^\infty A_i \in S,$$

(5) let U_1 and U_2 be open sets in S^2 and

$$f\, U_1 \to U_2$$

an invertible analytic map; if $A \subseteq U_1$, and $A \in S$, then $f(A) \in S$

We shall call a class of subsets of S^2 satisfying (1)–(5) above a *complete class of sets*. A multi-valued analytic function f will be called a *Q-function* if the set of singular points of f belongs to some complete class Q of sets. All definitions and results on S-functions extend to Q-functions. For example, we have the following variant of the main theorem:

Finite networks of circles

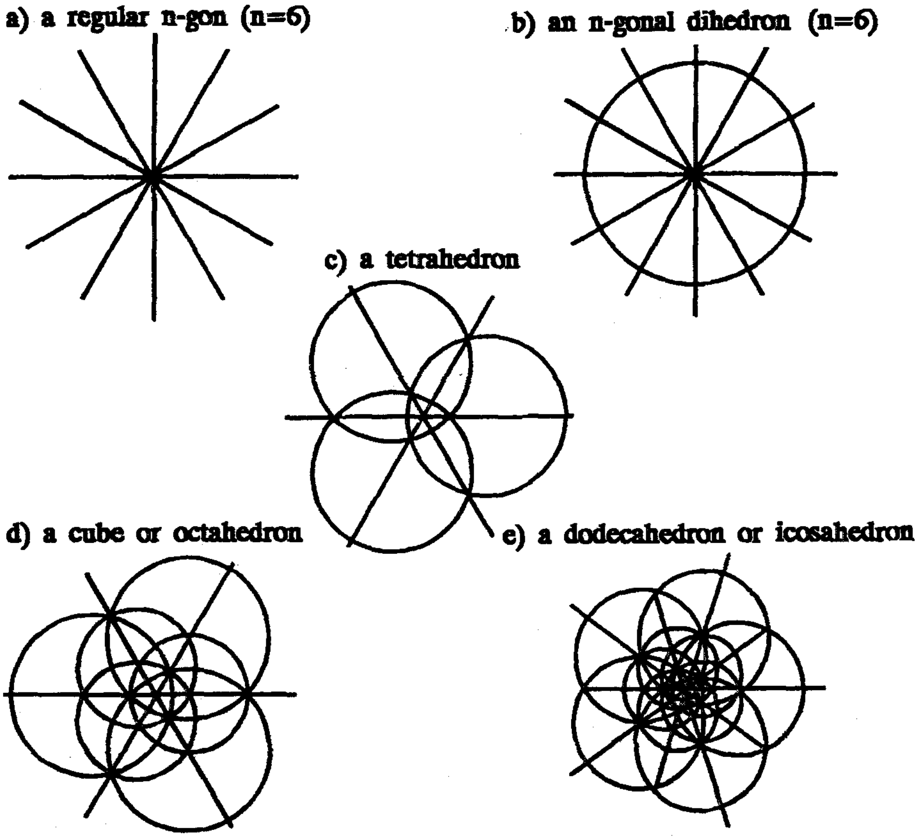

FIGURE 3. Third integrable case

Variant of the main theorem. *For each complete class Q of sets and complete class $\mathcal{M}$ of pairs, the class $\widehat{\mathcal{M}}$ consisting of all Q-functions f for which $[f] \in \mathcal{M}$ is closed with respect to differentiation, compositions, and meromorphic operations. If in addition,*

(1) $\mathbb{C} \in \mathcal{M}$, *then $\widehat{\mathcal{M}}$ is closed under integration;*
(2) $S(n) \in \mathcal{M}$, *then $\widehat{\mathcal{M}}$ is closed under solutions of algebraic equations of degree $\leq n$.*

Here is an example of a complete class of sets. Let X_α be the set of all subsets of S^2 whose Hausdorff measure of weight α is equal to 0. It is not difficult to show that for $\alpha \leq 1$, X_α is a complete class of subsets of S^2.

Note that the new formulation of the main theorem allows us to strengthen all negative results. So, for example, we have the following:

Corollary. *If a polygon G is not covered by one of the three integrable cases, f_G is not representable via single-valued X_1-functions by generalized quadratures, compositions, and meromorphic operations.*

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(Received 17.10.1994)

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