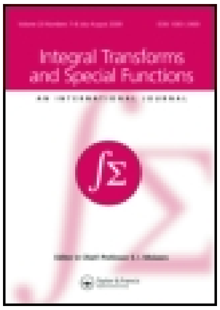

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INTEGRAL TRANSFORMS BASED ON EULER CHARACTERISTIC AND THEIR APPLICATIONS

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The basic notions and constructions of the theory of integration over the Euler characteristic are introduced and explained. A few examples from the theory of convex polytopes show usefulness of this non-classical integration. The analog of the Radon transform with respect to the Euler characteristic is constructed and illustrated by a topological example..

KEY WORDS: Euler characteristic, Convex polytope, Valuation, Minkowski addition, Radon transform

MSC (1991): 57H60, 52B45, 44A12

INTRODUCTION

The aim of the present paper is to explain the concept of integration over the Euler characteristic and to give a few examples of its use. The integral over the Euler characteristic can be discerned in many classic arguments (see example 2.1 below); the modern formalization of the theory is due to O. Ya. Viro [1]. Later the authors [2,3] found numerous applications of this technique to the theory of convex polytopes.

The concept of integration over the Euler characteristic is based upon the fact that the Euler characteristic has some properties of a measure. Let us consider two typical situations.

(1) Let X_1, X_2 and $X_1 \cap X_2$ be compact finite subcomplexes of a CW-complex X. Then $X_1 \cup X_2$ is also a compact subcomplex and there is the Mayer-Vietoris exact sequence $\cdots \to H^i(X_1 \cup X_2) \to H^i(X_1) \oplus H^i(X_2) \to H^i(X_1 \cap X_2) \to H^{i+1}(X_1 \cup X_2) \to \dots$ (for example, with integral coefficients). Thus for the Euler

characteristic

$$\chi(\cdot) = \sum_{i=0}^{\infty}(-1)^i rkH^i(\cdot)$$

we get the additivity $\chi(X_1 \cup X_2) + \chi(X_1 \cap X_2) = \chi(X_1) + \chi(X_2)$. Besides, for finite CW-complexes Y and Z, $\chi(Y \times Z) = \chi(Y)\chi(Z)$.

(2) For another example let X be a smooth connected manifold, X_1, X_2 and $X_1 \cap X_2$ its open submanifolds of finite type [4]. Then for cohomology with compact support there is the Mayer-Vietoris sequence $\cdots \to H_c^i(X_1 \cap X_2) \to H_c^i(X_1) \oplus H_c^i(X_2) \to H_c^i(X_1 \cup X_2) \to H_c^{i+1}(X_1 \cap X_2) \to \ldots$ whereof we get the additivity of the Euler characteristic again. The Euler characteristic with respect to cohomology with compact support also multiplies when the direct product is taken.

These examples show that the Euler characteristic looks very much like a measure, and the corresponding integration theory is going to be a full-fledged one in the sense of Fubini-type theorems – because of the multiplicativity of the Euler characteristic. But examples given above immediately point out the main obstacles to construction of the theory of integration over the Euler characteristic. There are two points of trouble: first, in both situations considered above the "measure" $\chi(\cdot)$ is defined for a certain system of subsets of the ambient space closed with respect to finite intersections and unions, whereas for an integration theory this system should be extended to an algebra of subsets. So the question is whether the topological Euler characteristic can be extended to a measure on this algebra. Then, in both situations there are certain conditions of finiteness which make sure that the Euler characteristic does exist, and this is a very strong restriction for an algebra of measurable sets. A situation when two algebras of χ-measurable sets are not both contained in any χ-measurable algebra, is typical (in particular there can be no universal slgebra of χ-measurable sets). At the same time, the χ-measures of a set belonging to both algebras coincide.

The first of the two difficulties can be easily overcome. But the second is a non-avoidable feature of the theory of integral over the Euler characteristic: every application of the theory should be preceded by pointing out a system of sets and verification of its "permissibility". In practice, though, this verification is usually trivial.

1. CONSTRUCTION OF INTEGRAL

Definition 1. *Let X be a topological space. A) If X is compact then a structure of a finite CW-complex*

$$X = \cup_{q\in\mathbb{Z}_+} \cup_{i\in I_q} e_i^q$$

is said to be a regular CW-structure if the following conditions are satisfied: (i) the characteristic mapping of any cell is a homeomorphism of the closed ball onto its closure, (ii) the boundary of any cell falls apart into a union of cells of smaller *dimension. The sets representable as a union of cells are said to be cellular. B) If X is arbitrary then the following data is said to be a regular CW-structure on X: (i) a dense injection $X \subset \widetilde{X}$, where $\widetilde{X}$ is compact, (ii) a regular CW-structure on $\widetilde{X}$ such that X is an open cellular sebset. The definition of cellular subsets of X is evident. C) We define a finitely-additive measure χ on the algebra of cellular subsets of a space X with a regular CW-structure setting for an open cell $e \subset X$ $\chi(e) = (-1)^{\dim e}$. This measure is said to be the Euler characteristic.*

Proposition 1. *Let $Z \subset X$ be a subset with compact closure $\overline{Z}$ and suppose that there exists at least one regular CW-structure on X such that Z is cellular. Then the Euler characteristic $\chi(Z)$ does not depend on the choice of a regular CW-structure.*

Proof. Suppose at first that Z is compact. If Z is cellular it is easy to see that $\chi(Z)$ in the sense of Definition 1 is the same as $\sum_{i=0}^{\infty}(-1)^i \dim H^i(Z, \mathbb{R})$ and thus does not depend on the choice of a regular CW-structure. If Z is not necessarily compact, let us construct the following series of cellular (with respect to a fixed regular structure) sets $Z^{(n)}$, $n \in \mathbb{Z}_+$: $Z^{(0)} = Z$, $Z^{(i+1)} = \overline{Z}^{(i)} \setminus Z^{(i)}$. Evidently, $\overline{Z}^{(i)}$ are compact and $Z^{(i)} = \varnothing$ for $i \gg 0$, because $Z^{(i+1)}$ consists of cells of dimension smaller than the maximum of dimensions of cells in $Z^{(i)}$. Thus $\chi(Z) = \sum_{i=0}^{\infty}(-1)^i \chi(\overline{Z}^{(i)})$, but for $\overline{Z}^{(i)}$ the invariance of χ has been already proved.

Definition 2. *Let X be a topological space with a regular CW-structure, A be an Abelian group. A function $f : X \to A$ is said to be cellular if $F^{-1}(a)$ is a cellular set for any $a \in A$. In particular, f is finite-valued. The integral of f over the Euler characteristic is set to be equal to*

$$\int_X f d\chi := \sum_{a \in A} \chi\left(f^{-1}(a)\right) a \in A$$

Definition 3. *A function $f : X \to A$ where X is a topological space, A is an Abelian group, is said to be permissible, if it is cellular with respect to some regular CW-structure on X.*

Corollary 1 (from Proposition 1). *For a permissible function $f : X \to A$ with a compact support its integral over the Euler characteristic $\int_X f d\chi$ does not depend on the choice of a regular CW-structure on X.*

Example 1. *Let $V \cong \mathbb{R}^n$ be a real vector space, $\mathcal{P}(V)$ be the set of (compact) convex polytopes in V. A function $\alpha : V \to \mathbb{Z}$ representable as $\alpha = \sum_{i \in I} n_i \mathbb{I}_{A_i}$, where $\#I < \infty$, $n_i \in \mathbb{Z}$, $A_i \in \mathcal{P}(V)$, $\mathbb{I}_A$ denotes the indicator of the set A, is said to be a (convex) chain. The additive group of chains is denoted by $\mathcal{Z}(V)$. Evidently chains are permissible and for α written out above*

$$\int_V \alpha d\chi = \sum_{i \in I} n_i.$$

In particular the latter integer does not depend on the representation of the chain. As in [2] we call it the degree and denote by $\deg \alpha$.

2. FUBINI THEOREM

In the notations of Sec. 1 let $f : X \to A$ be a permissible function, $\Phi : X \to Y$ be a continuous mapping of topological spaces. Suppose that for each fiber $X_y = \Phi^{-1}(y)$, $y \in Y$, a regular CW-structure is given, such that $f_y = f\big|_{X_y}$ is cellular. Then the "direct image" of f is defined as follows:

$$\Phi_* f : Y \to A, \quad \Phi_* f : y \to \int\limits_{X_y} f d\chi.$$

This operation makes sense if $\Phi_* f$ is permissible too.

Definition 1. *A) A continuous mapping $\Phi : X \to Y$ of spaces with regular CW-structures is said to be cellular if Φ maps each cell $e \subset X$ surjectively onto a cell $h \subset Y$. B) The direct product of spaces with regular CW-structure is defined by taking direct product of cells. C) Let $\Phi : X \to Y$ be a continuous mapping of spaces with regular CW-structures. The following data is said to be a fibration structure for Φ: for each cell $e \subset Y$ a space F_e with a regular CW-structure and a homeomorphism $\Phi_e : \Phi^{-1}(e) \to F_e \times e$ such that Φ_e and Φ_e^{-1} are cellular. D) A cellular function $f : X \to A$ is said to be compatible with the fibration structure for Φ, if for each cell $e \subset Y$ there is a cellular function $f_e : F_e \to A$ such that* $f\big|_{\Phi^{-1}(e)} = f_e \circ pr_1 \circ \Phi_e$.

Theorem 1 (Fubini theorem). *In the situation described in C) and D) of the last definition the function $\Phi_* f : Y \to A$, $\Phi_* f : y \to \int_{\Phi^{-1}(y)} f d\chi = \int_{F_e} f_e d\chi$, $y \in e \subset Y$, is cellular and*

$$\int\limits_Y \Phi_* f d\chi = \int\limits_X f d\chi.$$

In other words, integration of a function over the Euler characteristic can be represented as the composition of two operations: first, its integration over the fibers of the mapping, second, integration of the resulting function over the base.

The proof of the theorem is evident.

Clearly, the "direct image" operation is connected with a definite fibration structure in the existence aspect only, while its result does not depend on the fibration structure (Proposition 1).

Definition 2. *A permissible function is said to be compatible with the map $\Phi : X \to Y$ if there exists a fibration structure for Φ such that f is compatible with it.*

Example 1. *The Riemann-Hurwitz theorem.*

Let C, $\widetilde{C}$ be compact Riemann surfaces of genuses g and $\widetilde{g}$, respectively, $\pi : \widetilde{C} \to C$ be a holomorphic covering of the degree m, $B \subset \widetilde{C}$ be the ramification divisor (a point $b \in \widetilde{C}$ comes into B with multiplicity k if $k+1$ sheets of the covering meet in b). The classic Riemann-Hurwitz theorem asserts that

$$2\widetilde{g} - 2 = m(2g-2) + \deg B.$$

The standard topological proof [5] of the theorem can be interpreted as a computation of certain integral over the Euler characteristic [1]: let $f : \widetilde{C} \to \mathbb{Z}$ be equal to 1 identically then by the Fubini theorem

$$\chi(\widetilde{C}) = \int\limits_{\widetilde{C}} f d\chi = \int\limits_{C} \pi_* f d\chi.$$

But $\pi_* f(c) = \#\pi^{-1}(c)$ for $c \in C$, whereof we get the theorem.

Example 2. *Multiplication of chains [2].*

Minkowski addition of convex polytopes extends to a bilinear operation on the group of convex chains

$$* : \mathcal{Z}(V) \times \mathcal{Z}(V) \to \mathcal{Z}(V),$$

$\mathbb{I}_A * \mathbb{I}_B = \mathbb{I}_{A \oplus B}$ for A, $B \in \mathcal{P}(V)$, which can be represented as convolution with respect to the Euler characteristic:

$$\alpha * \beta(x) = \int\limits_{V} \alpha(z)\beta(x-z)d\chi(z)$$

(an easy check is left to the reader). The group $\mathcal{Z}(V)$ with the multiplication $*$ is said to be the algebra of (convex) chains. Denote by $\mu : V \times V \to V$ the addition mapping, and for α, $\beta \in \mathcal{Z}(V)$ set $\alpha \times \beta \in \mathcal{Z}(V \times V)$ to be the direct product of α, β: $\alpha \times \beta(x,y) = \alpha(x)\beta(y)$. Evidently, $\alpha * \beta = \mu_*(\alpha \times \beta)$. Let $f : V \to W$ be a linear map. Applying Theorem 1 to the diagram

$$\begin{array}{ccc} V \times V & \xrightarrow[(f,f)]{} & W \times W \\ {\scriptstyle \mu_V}\downarrow & & \downarrow{\scriptstyle \mu_W} \\ V & \xrightarrow[f]{} & W \end{array}$$

we get that the direct image mapping (i.e., fiberwise integration) $f_* : \mathcal{Z}(V) \to \mathcal{Z}(W)$ is a homomorphism of commutative rings.

Example 3. *Ideals in the algebra of chains [2].*

It is easy to see that $\deg : \mathcal{Z}(V) \to \mathbb{Z}$ is a ring homomorphism. Let $\mathcal{L} = \operatorname{Ker} \deg \subset \mathcal{Z}(V)$ be the ideal of chains of degree zero. Let $\mathcal{M} = \mathcal{L} \cap \mathbb{Z}[V]$ where the group algebra $\mathbb{Z}[V]$ is considered as a subalgebra (of zero-dimensional chains) of $\mathcal{Z}(V)$.

Theorem 2. *For $k \geq 1$*

$$\mathcal{L}^{\dim V+k} \subset \mathcal{M}^k \mathcal{Z}(V).$$

In the case $k = 1$ the theorem contains all the classic theory of valuations of convex polytopes ([2,6]).

The proof of the theorem see in [2].

Example 4. *"Minkowski subtraction" [2].*

For $A \in \mathcal{P}(V)$ denote by $\operatorname{Int} A$ its interior in its affine hull $< A >$.

Theorem 3. $(-1)^{\dim A}\mathbb{I}_{\operatorname{Int}(-A)} * \mathbb{I}_A = 1$, *where* $1 = \mathbb{I}_{\{0\}}$ *is the identity of the algebra* $\mathcal{Z}(V)$.

The proof is straighrforward and easy, and left to the reader.

3. RADON TRANSFORM

Let $X = \mathbb{R}P^n$, $X^* = \mathbb{R}P^{n*}$ be the dual projective space, so that points of X^* correspond to hyperplanes in X, and vice versa, $Z \subset X \times X^*$ be the graph of the incidence correspondence: $\{(x,h)|x \in h\}$. We say that a permissible function $f : X \to A$ permits the Radon transform if the function $\operatorname{res}_Z \circ pr_1^*(f) : Z \to A$ is compatible with the fibration $pr_2 : Z \to X^*$. If that is the case, the function $f^* : X^* \to A$,

$$f^* = (pr_2)_* \circ \operatorname{res}_Z \circ pr_1^*(f),$$

(so that $f^*(h) = \int_h f d\chi$ for a hyperplane $h \subset X$) is said to be the Radon transform of f (with respect to the integration over the Euler characteristic).

Theorem 1. *If f, f^* permit the Radon transform, then the following identity holds: $f^{**} + f = \int_X f d\chi = \int_{X^*} f^* d\chi$ for even $n = \dim X$ and $f^{**} = f$ for odd n.*

Proof. For $x \in X$ set

$$W_x = \{(y,h) \in X \times X^* | y \in h,\ x \in h\}.$$

Evidently,

$$f^{**}(x) = \int\limits_{\{h\in X^* | x\in h\}} f^*(h) d\chi(h) = \int\limits_{\{h\in X^* | x\in h\}} \int\limits_{\{y\in X | y\in h\}} f(y) d\chi(y) d\chi(h).$$

By the Fubini theorem, $f^{**}(x) = \int_{W_x} pr_1^*(f)d\chi$. On the other hand, the projection onto the first factor $pr_1 : W_x \to X$ is a fibration over $X \setminus \{x\}$ with the fiber $\mathbb{R}P^{n-2}$ and $pr_1^*(f)$ is evidently constant on the fibers of pr_1. Finally, $pr_1^{-1}(x) \cong \mathbb{R}P^{n-1}$. Applying Fubini theorem again, now to the map $pr_1 : W_x \to X$, we get

$$\begin{aligned} f^{**}(x) &= \int\limits_{W_x \setminus pr_1^{-1}(x)} pr_1^*(f)d\chi + \int\limits_{pr_1^{-1}(x)} pr_1^*(f)d\chi \\ &= \chi(\mathbb{R}P^{n-2}) \int\limits_{X\setminus\{x\}} f(y)d\chi(y) + \chi(\mathbb{R}P^{n-1})f(x) \\ &= \chi(\mathbb{R}P^{n-2}) \int\limits_X f d\chi + \left(\chi(\mathbb{R}P^{n-1}) - \chi(\mathbb{R}P^{n-2})\right) f(x). \end{aligned}$$

For even m $\chi(\mathbb{R}P^m) = 1$, for odd m - zero. Q.E.D.

Example 1. *Finite covers of* $\mathbb{R}P^2$.

Let S be a smooth connected compact (real) surface, $\pi : S \to \mathbb{R}P^2$ be a finite map unramified over $\mathbb{R}P^2 \setminus C$, where C is a smooth (possibly non-connected) curve, having the simple fold over C. In other words, for the branch curve $B \subset S$ $\pi : B \to C$ is an isomorphism and for any point $b \in B$ there are local parameters x, y in b and u, v in $p = \pi(b) \in C$ such that π can be written locally as $u = x$, $v = y^2$. For a point $p \in C$ which is not a point of inflexion, we define the index $i(p)$, setting it to be equal to $+1$, if (in the notations above) the tangent vector $\frac{\partial}{\partial v}$ and the curve C lie on the same side of the tangent line T_pC near p, and to (-1), in the other case.

Theorem 2. *Suppose that $x \in \mathbb{R}P^2$ does not lie on the union of tangent lines to C in all the points of inflexion. Then*

$$\chi(S) = \#\pi^{-1}(x) + \sum_{x \in T_pC} i(p).$$

Proof. Define a function $f : \mathbb{R}P^2 \to \mathbb{Z}$, setting $f(p) = \#\pi^{-1}(p)$. If the line L is not tangent to C, then $\pi^{-1}(L) \subset S$ is a smooth compact one-dimensional variety, i.e. a disjoint union of loops; consequently, $\chi(\pi^{-1}(L)) = 0$. Thus f^* vanishes outside the curve dual to C. If L is tangent to C in the points p_j, $j \in J$, which are not points of inflexion, then it checks easily that $\chi(\pi^{-1}(L)) = \sum_{j\in J} i(p_j)$. Now apply Theorem 1.

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