Newton Polyhedron, Hilbert Polynomial, and Sums of Finite Sets

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§1. Statement of the Results

If A, B are subsets of a commutative semigroup G, then we can define the sum $A+B$ as the set of points $z=a+b$, where $a \in A$ and $b \in B$. Denote by $N*A$ the sum of N copies of A.

Theorem 1. *For arbitrary finite subsets A and B of G, the number of elements of the set $B+N*A$ with large enough natural N is a polynomial in N. The degree of this polynomial is less than the number of elements of A.*

Theorem 1 is a special case of Theorem 2 of § 2. The polynomials in Theorems 1 and 2 are Hilbert polynomials of certain graded modules over the ring of polynomials in several variables.

If a semigroup G is an abelian group without elements of finite order, then it is possible to calculate the degree and the leading coefficient of the polynomial in Theorem 1. Let us introduce some necessary notation. Denote by $G(A)$ the subgroup of the group G generated by the differences of the elements of a set A (the group $G(A)$ consists of the elements of the form $\sum n_i a_i$, where $a_i \in A$, $n_i \in \mathbb{Z}$, and $\sum n_i = 0$). Since G has no elements of finite order, $G(A)$ is isomorphic to the group $\mathbb{Z}^n$, where n is the rank of $G(A)$. The set $\overline{A} = A - a$, where a is an element of A, is contained in $G(A)$.

Definition. The *reduced Newton polyhedron* of a set $A \subset G$ is the convex hull, in the space $\mathbb{R}^n$ containing the lattice $\mathbb{Z}^n$, of the image of $\overline{A}$ under an isomorphism of the groups $G(A)$ and $\mathbb{Z}^n$.

The reduced Newton polyhedron is defined up to an affine transformation $x \mapsto z \pm Ux$, where $z \in \mathbb{Z}^n$ and U is a unimodular transformation. In particular, the volume of the reduced Newton polyhedron is well-defined. We denote by $i(A, B)$ the number of cosets of the group G modulo the subgroup $G(A)$ that contain points of the set B.

Theorem 4. *Let a semigroup G be an abelian group without elements of finite order. Then the ratio of the number of points in $B+N*A$ to N^n, where n is the rank of $G(A)$, tends, as $N \to \infty$, to the product of the volume of the reduced Newton polyhedron of A and the number $i(A, B)$.*

Theorem 4 is proved in § 3, which is devoted mainly to the group $G = \mathbb{Z}^n$. Let $A \subset \mathbb{Z}^n$ be a finite set such that the group $\mathbb{Z}^n(A)$ is equal to $\mathbb{Z}^n$, and let Δ be the convex hull of A in $\mathbb{R}^n$, $\mathbb{Z}^n \subset \mathbb{R}^n$. According to Theorem 3 of § 3, every integral point of the polyhedron $N \cdot \Delta$ lying far enough from its boundary belongs to the set $N*A$.

In § 4 we show how it is possible, using Hilbert polynomials and Theorem 3, to prove the Kushnirenko theorem (in this theorem, the number of solutions of a generic system of n polynomial equations in n unknowns is calculated for the case in which the polynomials have equal Newton's polyhedra). The results of the present paper arose from the attempts to find the simplest proof of this theorem.

In § 5 we calculate Grothendieck groups of the semigroup of compact subsets in $\mathbb{R}^n$ and of the semigroup of finite subsets in $\mathbb{Z}^n$.

The results of the paper were presented to the International Topological Conference in Baku, 1988, and since then have passed into the folklore, but they have never been published. Here we fill this gap.

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§2. Hilbert Polynomial and the Number of Points in the Sum of Finite Sets

Let $\overline{G}$ be a set, let $\overline{B}$ be a finite subset of it, and let $\overline{A} = (\overline{a}_{ij})$ be a set of m commuting maps of $\overline{G}$ into itself, $\overline{a}_i \circ \overline{a}_j = \overline{a}_j \circ \overline{a}_i$, $1 \le i, j \le m$. Denote by $\overline{B}(N)$ the set $\bigcup_{1\le i_j \le m} \overline{a}_{i_1} \circ \cdots \circ \overline{a}_{i_N}(\overline{B})$.

Theorem 2. *The number of elements of $\overline{B}(N)$ is a polynomial in N for sufficiently large natural N. The degree of this polynomial is less than m.*

Theorem 1 of § 1 is a special case of Theorem 2. To derive Theorem 1 from Theorem 2, it is sufficient to consider the set $\overline{G}$ of all the points of the semigroup G, the set $\overline{B}$ that is equal to B, and the set $\overline{A}$ of maps that are shifts $\overline{a}_i$ of G corresponding to $a_i \in A$ (i.e., $\overline{a}_i(g) = g + a_i$). In this case, the set $\overline{B}(N)$ coincides with $B + N * A$, and so Theorem 1 is reduced to Theorem 2.

Proof of Theorem 2. Let us consider a countable set of nonintersecting copies G_i, $i = 0, 1, \dots$, of $\overline{G}$ together with one-to-one correspondences $\pi_i : \overline{G} \to G_i$. Denote by X the union of G_i, $X = \bigcup G_i$, and by L the space of complex-valued linear functions on X, that are nonvanishing only on finite sets of points. The space L is generated by the functions δ_x that vanish at all points except $x \in X$ and are equal to 1 at x. It is possible to define a grading in L: a function belongs to the component of degree k if it does not vanish only on the copy G_k of the set $\overline{G}$. The space L can be transformed into a graded module over the ring $\mathbb{C}[x_1, \dots, x_m]$ of polynomials in m variables: by definition, the generator x_i transforms the functions δ_x, where $x = \pi_j(g)$, $g \in \overline{G}$, $j = 0, 1, \dots$, into the functions δ_y, where $y = \pi_{j+1}(\overline{a}_i(g))$. This action can be extended uniquely to the action of the ring $\mathbb{C}[x_1, \dots, x_m]$ on the space L. Let us consider the submodule $L_{\overline{B}}$ in L generated by the elements δ_x, where $x = \pi_0(\overline{b})$, $\overline{b} \in \overline{B}$. The component of the module $L_{\overline{B}}$ of degree N consists of all linear combinations of the functions δ_x corresponding to $x = \pi_N(g)$, $g \in \overline{B}(N)$. The dimension of this component is equal to the number of points in $\overline{B}(N)$. Therefore, Theorem 2 follows from Hilbert's theorem (see [1, Theorem 6.21]).

§3. Sums of Finite Subsets of the Integral Lattice

Let A be a finite subset of $\mathbb{Z}^n$, $\mathbb{Z}^n \subset \mathbb{R}^n$, such that the subgroup generated by the elements of A coincides with the group $\mathbb{Z}^n$.

Proposition 1. *There exists a constant C with the following property: for every linear combination $\sum \lambda_i a_i$ of vectors $a_i \in A$ with real coefficients λ_i such that $\sum \lambda_i a_i$ is an integral vector, there exists a linear combination $\sum n_i a_i$ of a_i with integer coefficients such that it is equal to $\sum \lambda_i a_i$ and $\sum |n_i - \lambda_i| < C$.*

Proof. For every x from the finite set X of integral vectors represented in the form $x = \sum \lambda_i a_i$ with $0 \le \lambda_i \le 1$, we fix a representation of the form $x = \sum n_i(x)\, a_i$, $n_i(x) \in \mathbb{Z}$. Such a representation exists because the elements $a_i \in A$ generate the group $\mathbb{Z}^n$. Now it is sufficient to take $C = m + q$, where m is the number of elements in A and $q = \max_{x\in X} \sum_{i=1}^m |n_i(x)|$. Indeed, for any integral vector $z \in \mathbb{Z}^n$ of the form $\sum \lambda_i a_i$, the vector $x = z - \sum [\lambda_i] a_i$ belongs to X. Hence, $x = \sum n_i(x)\, a_i$ and $z = \sum n_i a_i$, where $n_i = [\lambda_i] + n_i(x)$. Proposition 1 is proved.

Denote by Δ the convex hull of the set A in the space $\mathbb{R}^n$ containing the lattice $\mathbb{Z}^n$. If A contains the origin and N is a natural number, then the polyhedron $N \cdot \Delta$ can be defined as the set of linear combinations $\sum \lambda_i a_i$ of vectors $a_i \in A$ with $\lambda_i \ge 0$ and $\sum \lambda_i \le N$. Denote by $\Delta(N, C)$ the polyhedron consisting of linear combinations $\sum \lambda_i a_i$, where $\lambda_i \ge C$ and $\sum \lambda_i \le N - C$.

Proposition 2. *Let the group $\mathbb{Z}^n(A)$ generated by the differences of the elements of A coincide with $\mathbb{Z}^n$, and let the origin belong to A. Then every integral point of the polyhedron $\Delta(N, C)$, where C is the constant occurring in Proposition 1, belongs to $N * A$.*

Proof. As the set A contains 0, the set $N * A$ consists of the points of the form $\sum n_i a_i$, where $a_i \in A$, $n_i \ge 0$, and $\sum n_i \le N$. According to Proposition 1, an arbitrary integral point of $\Delta(N, C)$ admits such a representation.

We fix an arbitrary Euclidean metric in $\mathbb{R}^n$. Let A be a finite subset of the group $\mathbb{Z}^n \subset \mathbb{R}^n$, and let Δ be its convex hull in $\mathbb{R}^n$.

Theorem 3. *Suppose that the group $\mathbb{Z}^n(A)$ coincides with $\mathbb{Z}^n$. Then there exists a constant ρ with the following property: for an arbitrary natural N, every integral point of the polyhedron $N\cdot\Delta$ whose distance from the boundary of the polyhedron is not less than ρ belongs to $N * A$.*

Proof. Without loss of generality, we can suppose that A contains the origin (otherwise we have to consider, instead of A, a shifted set $A - a$, where a is a vector belonging to A). Let us index the elements of A: $A = \{a_i\}$, $i = 1, \dots, m$. Consider the space $\mathbb{R}^m$, whose dimension is equal to the number of elements of A, and the map $\pi\colon \mathbb{R}^m \to \mathbb{R}^n$ that transforms the ith vector of the standard basis in $\mathbb{R}^m$ into the vector $a_i \in \mathbb{R}^n$. Denote by $K(N)$ and $K(N, C)$ the simplexes in $\mathbb{R}^m$ defined by the inequalities $\lambda_i \geq 0$, $\sum_{i=1}^m \lambda_i \leq N$ and $\lambda_i \geq C$, $\sum_{i=1}^m \lambda_i \leq N - C$, respectively. The polyhedra $N\cdot\Delta$ and $\Delta(N, C)$ are the images of the simplexes $K(N)$ and $K(N, C)$ under the map π. Each point of the simplex $K(N)$ whose distance from its boundary is not less than $C\sqrt{m}$ belongs to $K(N, C)$. Therefore, each point of the polyhedron $N\cdot\Delta$ whose distance from its boundary is not less than $\rho = C\sqrt{m}\,\|\pi\|$, where $\|\pi\|$ is the norm of the operator π, lies in $\Delta(N, C)$. Now, to complete the proof of Theorem 3, it remains to use Proposition 2.

Below we assume that the metric in $\mathbb{R}^n$ is normalized by the following condition: the volume of parallelepiped $\Pi(e_1, \dots, e_n)$ spanned by the basis $e_1, \dots, e_n$ of the lattice $\mathbb{Z}^n$ ($x \in \Pi(e_1, \dots, e_n) \iff x = \sum \lambda_i e_i$, $0 \leq \lambda_i \leq 1$) is equal to 1.

Corollary 1. *Suppose that the group $\mathbb{Z}^n(A)$ coincides with $\mathbb{Z}^n$. Then the ratio of the number of points lying in $N * A$ to the number $N^n V(\Delta)$, where $V(\Delta)$ is the volume of the convex hull Δ of A, tends to 1 as $N \to \infty$.*

Proof. The set $N * A$ is contained in the set of integral points of the polyhedron $N\cdot\Delta$. So we obtain an upper bound for the number of integral points lying in $N * A$. The lower bound is obtained from Theorem 3. Both bounds are of order $N^n V(\Delta)$ as $N \to \infty$.

Corollary 2. *Suppose that the group $\mathbb{Z}^n(A)$ has a finite index in the group $\mathbb{Z}^n$; denote it by $\operatorname{ind} A$. Then the ratio of the number of points lying in $N * A$ to N^n tends to $(\operatorname{ind} A)^{-1} V(\Delta)$ as $N \to \infty$.*

Proof. We may assume that the set A is contained in the group $\mathbb{Z}^n(A)$ (otherwise we consider a shifted set $A - a$, where a is an element of A). In this case we can use Corollary 1 with the lattice $\mathbb{Z}^n(A)$ instead of $\mathbb{Z}^n$. To do this, we have only to renormalize the metric: the ratio of the volumes of the parallelepipeds $\Pi(f_1, \dots, f_n)$ and $\Pi(e_1, \dots, e_n)$, where $f_1, \dots, f_n$ and $e_1, \dots, e_n$ are the bases of the lattices $\mathbb{Z}^n(A)$ and $\mathbb{Z}^n$, is equal to $\operatorname{ind} A$. Corollary 2 is proved.

Corollary 3. *Suppose that A and B are finite subsets of $\mathbb{Z}^n$ and the group $\mathbb{Z}^n(A)$ coincides with $\mathbb{Z}^n$. Then the ratio of the number of points lying in $N * A + B$ to $N^n V(\Delta)$, where $V(\Delta)$ is the volume of the convex hull of A, tends to 1 as $N \to \infty$.*

Proof. The set $N * A + B$ is contained in the set of integral points of the polyhedron $N\cdot\Delta + \Delta_B$, where Δ and Δ_B are the convex hulls of A and B. This gives an upper bound for the number of points of $N * A + B$. On the other hand, the number of points lying in $N * A + B$ is not less than the number of points lying in $N * A$. Both bounds are of order $N^n V(\Delta)$ as $N \to \infty$ (for the lower bound this follows from Corollary 2). Corollary 3 is proved.

Now we shall prove Theorem 4, which was stated in § 1.

First of all, note that the sets $b_1 + N * A$ and $b_2 + N * A$ are disjoint when the elements b_1 and b_2 belong to distinct cosets of the group G modulo the subgroup $G(A)$. Therefore, we can restrict ourselves to the case in which the whole set B is contained in one coset. Further, we can assume that B, as well as A, lies in $G(A)$. (Otherwise we can consider the sets $A - a$, $B = B - b$ instead of A and B, where a and b are elements of A and B, respectively.) But in this case Theorem 4 is reduced to Corollary 3; the group $G(A)$, according to the definition, has no elements of finite order and hence is isomorphic to $\mathbb{Z}^n$. Theorem 4 is proved.

At the end of this section we prove a property of semigroups in $\mathbb{Z}^n$, which follows from Proposition 1. Let A be a finite subset of $\mathbb{Z}^n$, and let $K(A)$ be the cone in $\mathbb{R}^n$ generated by this subset: $x \in K(A) \iff x = \sum \lambda_i a_i$, $\lambda_i \geq 0$, $a_i \in A$.

Proposition 3. *Suppose that the group generated by A coincides with $\mathbb{Z}^n$. Then every integral point of the shifted cone $K + x$, where $x = C \sum_{a_i \in A} a_i$ and C is the constant occurring in Proposition* 1, *belongs to the semigroup generated by A.*

Proof. If the vector $z - x$ lies in the cone K, then this vector can be represented in the form $z - x = \sum \lambda_i a_i$, $\lambda_i \geq 0$. Therefore, each integral vector z can be represented in the form $z = \sum(\lambda_i + C)a_i$, where $\lambda_i \geq 0$. According to Proposition 1, every integral vector z of this form can be represented as a linear combination of vectors a_i with natural coefficients.

Remark. The statement of Proposition 3 can be regarded as a multi-dimensional generalization of the following simple problem addressed to high school pupils: show that it is possible to pay without a change an arbitrary large natural sum of rubles (actually, an arbitrary sum greater than 7) using only 3-ruble and 5-ruble bank-notes. (Indeed, the group generated by the numbers 3 and 5 coincides with $\mathbb{Z}$, the cone generated by them coincides with the set of positive integers, and the shifted cone $K + C$ coincides with the set of integers not less than C.)

§4. The Number of Roots of a Generic System of Equations

First we remind of some known facts from algebraic geometry (see [1]). Then we derive the Kushnirenko theorem from these facts and Theorem 2.

4.1. Let X be an open subset in the Zariski topology of an irreducible n-dimensional complex algebraic manifold, and let L be a finite-dimensional linear space of regular functions on X. What could be said about the solutions of the system of n equations $f_1 = \cdots = f_n = 0$, where f_i are generic functions from L? This question was well investigated in algebraic geometry. Let us formulate a version of the answer. Denote by X_0 the subset of X where all the functions from L vanish.

Denote by π_L the map of the set $X \setminus X_0$ to the projective space $\mathbb{C}P^{|L|-1}$ defined by the relation $\pi_L(x) = e_1(x) : e_2(x) : \cdots : e_m(x)$, where $e_1, \ldots, e_m$ form the basis of L (the map π_L is defined up to a projective transformation). If the image of $\pi_L(X \setminus X_0)$ is of dimension less than n, then a generic system of equations $f_1 = \cdots = f_n = 0$ on $X \setminus X_0$ is inconsistent. Suppose that the image $\pi_L(X \setminus X_0)$ is of dimension n. In this case the roots of an almost arbitrary system $f_1 = \cdots = f_n = 0$ lie only at smooth points of the set $X \setminus X_0$ and are nondegenerate. Almost all such systems have the same number of roots. This number is equal to the product of the degree of the map $\pi_L \colon X \setminus X_0 \to \pi_L(X \setminus X_0)$ and the degree of the closure of the set $\pi_L(X \setminus X_0)$ in the projective space. The degree of the map $\pi_L \colon X \setminus X_0 \to \pi_L(X \setminus X_0)$ is equal to the number of preimages $\pi_L^{-1}(y)$ for an almost arbitrary point $y \in \pi_L(X \setminus X_0)$. The degree of the closure of the set $\pi_L(X \setminus X_0)$ in the projective space can be calculated using Hilbert polynomials. Let us consider the linear space L^N consisting of linear combinations of functions $f = f_1 \cdots f_N$, where $f_i \in L$. According to Hilbert's theorem, the dimension of the space L^N for a sufficiently large natural number N is a polynomial in N. The degree d of this polynomial coincides with the dimension of the closure of the set $\pi_L(X \setminus X_0)$, and the product of $d!$ and the coefficient of N^d in this polynomial coincides with the degree of the closure of the set $\pi_L(X \setminus X_0)$ in the projective space.

4.2. Now we turn to the Kushnirenko theorem. Let us introduce necessary definitions and notation. To every character $\chi = x_1^{m_1} \cdots x_n^{m_n}$ of the torus $(\mathbb{C} - 0)^n$ there corresponds the point $m = m_1, \ldots, m_n$ of the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. A finite linear combination of characters $P = \sum \lambda_i \chi_i$ is called a *Laurent polynomial* P on the torus $(\mathbb{C} - 0)^n$. The support $\operatorname{supp}(P)$ of a Laurent polynomial P is, by definition, the finite subset of the lattice $\mathbb{Z}^n$ corresponding to the characters χ_i occurring in the polynomial P with nonzero coefficients λ_i.

Let us fix a finite subset A of $\mathbb{Z}^n$. How many roots in $(\mathbb{C} - 0)^n$ has a generic system of equations $P_1 = \cdots = P_n = 0$, where P_i are Laurent polynomials with support A? The answer to this question is given by the following theorem.

Theorem (Kushnirenko [2]). *The number of roots in* $(\mathbb{C}-0)^n$ *of a generic system of equations* $P_1 = \cdots = P_n = 0$ *with* $\operatorname{supp}(P_j) = A$ *is* $n!$ *times the volume of the convex hull of the set* A.

Proof. To prove this theorem, we use the facts of algebraic geometry formulated in 4.1 for the case $X = (\mathbb{C}-0)^n$ and the space L consisting of linear combinations of the characters corresponding to the points of A. The dimension of the image of the space $(\mathbb{C}-0)^n$ under the map π_L to the projective space equals the rank of the group $\mathbb{Z}^n(A)$ generated by the differences of the elements of A. The rank of the group $\mathbb{Z}^n(A)$ is less than n if and only if the volume of the convex hull of A is zero. It is obvious that in this case the corresponding generic system of equations is inconsistent. If the rank of the group $\mathbb{Z}^n(A)$ is equal to n, then the degree of the map of the torus $(\mathbb{C}-0)^n$ onto its image coincides with the index $\operatorname{ind} A$ of the subgroup $\mathbb{Z}^n(A)$ in $\mathbb{Z}^n$ (every point of the image has exactly $\operatorname{ind} A$ pre-images in $(\mathbb{C}-0)^n$). The space L^N consists of linear combinations of the characters corresponding to the points in $N * A$. The dimension of L^N is equal to the number of points of $N * A$. According to Corollary 2 of Theorem 3 (see § 3), the leading coefficient of the Hilbert polynomial is equal to the volume $V(\Delta)$ of the convex hull Δ of A divided by $\operatorname{ind} A$. Therefore, the number of roots of a generic system of equations $P_1 = \cdots = P_n = 0$ in $(\mathbb{C}-0)^n$ is equal to $n! \operatorname{ind} A \cdot (V(\Delta)/\operatorname{ind} A) = n!\, V(\Delta)$.

§5. Grothendieck Groups of the Semigroups of Finite Subsets of $\mathbb{Z}^n$ and Compact Subsets of $\mathbb{R}^n$

Let A be a compact subset of a finite-dimensional linear space L and Δ its convex hull of dimension k.

Lemma. *For each point* x *of the polyhedron* $\lambda\cdot\Delta$, *where* λ *is an arbitrary number non less than* $k+1$, *there exists a point* $a \in A$ *such that the vector* $x - a$ *lies in the polyhedron* $(\lambda-1)\cdot\Delta$.

Proof. The point $y = \lambda^{-1}x$ lies in the polyhedron Δ. Since the dimension of Δ is equal to k, the point y lies in one of the k-dimensional simplexes whose vertices $a_0, \ldots, a_k$ lie in the set A, $y = \sum_{0\le i\le k} \lambda_i a_i$, $\lambda_i \ge 0$, $\sum_{0\le i\le k} \lambda_i = 1$. One of the λ_j's is not less than $(k+1)^{-1}$. The point $x - a_j$ belongs to $(k-1)\cdot\Delta$.

Corollary 1. *For an arbitrary homomorphism of the semigroup (with respect to addition) of compact subsets of the linear space in an abelian group, the images of a finite-dimensional subsets having common convex hull coincide.*

Proof. Let Δ be the convex hull of compact subsets A and B. Then the sets $A+k\cdot\Delta$ and $B+k\cdot\Delta$, where k is equal to the dimension of the linear space, coincide. Indeed, according to the Lemma, both sets coincide with $(k+1)\cdot\Delta$. Since $A + k\cdot\Delta = B + k\cdot\Delta$, the images of elements of A and B under a homomorphism into an abelian group coincide.

The formal differences $\Delta_1 - \Delta_2$ of convex sets Δ_1 and Δ_2 with the natural equivalence relation $\Delta_1 - \Delta_2 \sim \Delta_3 - \Delta_4 \iff \Delta_1 + \Delta_4 = \Delta_2 + \Delta_3$ form an abelian group. Corollary 1 shows that this group is the Grothendieck group of the semigroup (with respect to addition) of compact subsets of L.

Let A and B be finite subsets of $\mathbb{Z}^n$, let Δ be the convex hull of A, and let B be the set of integral points of the polyhedron $K\cdot\Delta$, where $K \in \mathbb{Z}$, $K \ge n$.

The Lemma immediately implies the following corollary.

Corollary 2. *The set* $A+B$ *consists of all integral points of the polyhedron* $(K+1)\cdot\Delta$.

Corollary 3. *For every integral polyhedron* Δ *the number of integral points of the polyhedron* $N\cdot\Delta$ *is a polynomial on the set of sufficiently large natural number* N.

Corollary 3 follows from Theorem 1, applied to subsets A and B of the lattice $\mathbb{Z}^n$, where A is the set of integral points of the polyhedron Δ and B is the set of integral points of the polyhedron $n\cdot\Delta$ (see Corollary 2).

Remark. A stronger theorem due to McDonald [3] is valid: the number of integral points of the polyhedron $N\cdot\Delta$ is a polynomial on the set of all natural numbers N. The algebraic proof [4] of the McDonald theorem uses not only the Hilbert polynomial, but also the fact that a torus manifold has zero cohomologies with coefficients in a specially constructed sheaf depending on the polyhedron Δ.

Corollary 4 can be proved in the same manner as Corollary 1. The formal differences of integral polyhedra with the natural equivalence relation form an abelian group. Corollary 4 shows that this group is the Grothendieck group of the semigroup of finite subsets of the lattice.

Remark. Corollary 4 clarifies why discrete characteristics of a hypersurface $P = 0$ in the torus $(\mathbb{C} - 0)^n$ depend not on the support $\mathrm{supp}(P)$ of the Laurent polynomial P, but on the convex hull $\Delta(P)$ of this support. The proof of this fact is based on different ideas, which can be found in [4].

References

1. D. Mumford, Algebraic Geometry, I, Complex Projective Varieties, Springer-Verlag, Berlin–Heidelberg–New York (1976).
2. A. G. Kouchnirenko, *Polyedres de Newton et nombres de Milnor*, Inv. Math., **32**, 1–31.
3. P. McMullen, *Valuations and Euler-type relations on certain classes of convex polytopes*, Proc. London Math. Soc. Ser. 3, **35**, No. 1, 113–135 (1977).
4. A. G. Khovanskii, *Newton polyhedron and torus manifolds*, Funkts. Anal. Prilozhen., **11**, No. 4, 56–67 (1977).

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