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A RIEMANN-ROCH THEOREM FOR INTEGRALS AND SUMS OF QUASIPOLYNOMIALS OVER VIRTUAL POLYTOPES

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ABSTRACT. This paper is devoted to the proof of a theorem (which the authors call a Riemann-Roch theorem) connecting the integral and the lattice sum of a quasipolynomial over a convex chain belonging to some family. We show that there exists a linear differential operator (the Todd operator) transforming the integral to the sum. This gives a higher-dimensional generalization of the well-known Euler-Maclaurin formula.

INTRODUCTION

This paper is a direct continuation of [1], by the same authors. The theory developed in [1] concerning convex chains and finitely additive measures on them realizes one possible approach to studying them; in this paper we propose another, "transversal", approach, based on the systematic use of the idea of a conical representation of a convex chain, i.e., a representation of a chain as an integral linear combination of characteristic functions of cones. This theme was touched on once in [1]: when the question of the recovery of a convex chain from its support function was considered (§4, Proposition 2, where in essence the general lines of the technique on which the present paper is constructed were considered).

The object of our study is special measures of convex chains—integrals and lattice sums (i.e., sums over the points of a discrete lattice) of quasipolynomials. We give a rather detailed computation of these measures, which allows us to obtain the main result—a "Riemann-Roch theorem", connecting the lattice sum and the integral of the (same) quasipolynomial over a family of convex chains. The one-dimensional variant of this theorem is the old Euler-Maclaurin formula for a special class of functions (quasipolynomials), so our "Riemann-Roch theorem" can be interpreted as a higher-dimensional generalization of the Euler-Maclaurin formula (see [2], for example, for details).

We shall freely and without special references use the language of the previous paper [1], especially the concepts of a convex chain and its support function. Aside from this the present paper is almost independent of [1]—from the results proved in [1] we essentially need only the almost trivial Proposition 2 of §4, mentioned above.

The numbering scheme for assertions and definitions is the same as in [1].

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§1. Statement of the main theorem

In this section we introduce the concepts and constructions that will be needed for the statement of the Riemann-Roch theorem. In addition, we briefly explain the algebro-geometric origin of this theorem.

1. **Basic concepts and constructions.** Let (V, Λ) be an admissible pair [1], where V is an n-dimensional real linear space and Λ in this case is a discrete complete lattice. We fix an isomorphism $V \xrightarrow{\sim} \mathbb{R}^n$ relative to which Λ becomes a naturally embedded integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$; the points $x \in \Lambda$ will be referred to as *integral vectors*. Let $\Lambda^* = \{l \in V^* | l(\Lambda) \subset \mathbb{Z}\} \subset V^*$ be the dual lattice. Its elements will be called *integral covectors*.

Most of the concepts we shall use are generally accepted ones. However, to avoid possible confusion we give a list of them with their precise meaning.

(1) A *ray* is a closed affine half-line in V, the *vertex* of a ray is its origin, and a ray is a Λ-*ray* (or an *integer ray*) if it contains at least two (and hence infinitely many) points of Λ, one of which is its vertex.

(2) An affine subspace $W \subset \Lambda$ is a Λ-*space* (or a *lattice subspace*) if W is the affine hull of $W \cap \Lambda$.

(3) A *cone* $C \subset V$ is the convex hull of a finite set of rays $R_1, \dots, R_N$ with a common vertex; C is a Λ-*cone* (or *lattice cone*) if the rays $R_1, \dots, R_N$ can be chosen to be integer. We denote by $\langle C \rangle$ and $\mathrm{vs}(C)$ the affine hull and the vertex space, respectively, of the cone C. Obviously, if C is a Λ-cone, then $\langle C \rangle$ and $\mathrm{vs}(C)$ are Λ-spaces. Any face of a Λ-cone is obviously a Λ-cone.

(4) A cone C is said to be *developed* if $\dim \mathrm{vs}(C) \geq 1$. Otherwise the cone C is said to be *pointed*. In the latter case its vertex is the point $\mathrm{vs}(C)$ and edges (one-dimensional faces), where C is the convex hull of its edges.

(5) Let R be a ray with vertex x. The direction vector of the ray R is $y - x$, where $y \in R \backslash \{x\}$ is the closest integer point to x if R is a Λ-ray, and y is any point otherwise.

(6) A *simple* cone C is a pointed cone C for which the direction vectors of an edge are linearly independent. A *simple* Λ-*cone* (or a *simple lattice cone*) C is a simple cone C which is a Λ-cone such that the direction vectors of its edges form a basis of the lattice $\langle C \rangle \cap \Lambda$.

(7) The *dimension* of a cone is $\dim \langle C \rangle$, and the *interior* $\mathrm{Int}\, C$ of a cone is always understood as its interior in $\langle C \rangle$. We stress that a cone in our usage is always closed.

All of these concepts are of course applicable to any space and any discrete complete lattice (in fact to any admissible pair in the sense of [1]). Applying them to the pair (V^*, Λ^*), we give one more definition.

(8) A *decomposition* of the space V^* is a finite set of cones $\Sigma = \{C_i^* \subset V | i \in I\}$ with the following properties:

(a) $0 \in \mathrm{vs}(C_i^*)$, $i \in I$.

(b) $\mathrm{Int}\, C_a^* \cap \mathrm{Int}\, C_b^* = \varnothing$ for $a \neq b$, $a, b \in I$.

(c) $C_a^* \cap C_b^* = C_e^*$ for some $e \in I$.

(d) $C_a^* \subset C_b^*$ implies: C_a^* is a face of C_b^*.

(e) $V^* = \bigcup_{i \in I} \mathrm{Int}\, C_i^*$.

A decomposition is *simple* if all the cones C_i^* are simple, and *simple lattice* (or Λ-) if all the C_i^* are simple Λ-cones. The *edges* of a simple (lattice) decomposition are all the edges of the cones C_i^*, and the direction vectors of a decomposition are the direction vectors of the edges.

(9) A convex polytope $A \in \mathscr{P}(V)$ of full dimension (i.e., $\langle A \rangle = V$) is said to be simple, lattice (Λ-), or simple lattice if all the cones for its vertices are respectively

simple, lattice, or simple lattice. (We stress that in our terminology "simple lattice" is a stronger condition than "both simple and lattice"!) It is not hard to see that a simple lattice polytope A generates a simple lattice decomposition Σ_A of the space V^* (called the *dual* of A): let $\Gamma(A)$ be the set of faces of A, and let $C_{\Delta, A}$ for $\Delta \in \Gamma(A)$ be the cone for the face Δ (i.e., $\mathrm{vs}(C_{\Delta, A}) = \langle \Delta \rangle$ and for any $x \in \mathrm{Int}\,\Delta$ there exists a neighborhood $U \ni x$ such that $C_{\Delta, A} \cap U = A \cap U$); then $\Sigma_A = \{C^*_{\Delta, A} | \Delta \in \Gamma(A)\}$, where $C^*_{\Delta, A} \subset V^*$ is the cone dual to $C_{\Delta, A} \subset V$ translated so that the origin lies in its vertex space.

It is well known (in the theory of toric varieties this means that every complete toric variety extends to a nonsingular projective toric variety) that every lattice decomposition of the space V^* (i.e., decomposition with lattice cones) is refined to a simple lattice decomposition that is dual to a simple lattice polytope.

With each simple lattice decomposition Σ we associate the following four subsets of the group of convex chains $Z(V)$: $Z(V, \Sigma)$ are chains whose support functions are linear in Σ; $Z(\Lambda, \Sigma)$ are the integral chains: $Z(\Lambda, \Sigma) = Z(\Lambda) \cap Z(V, \Sigma)$; $\mathscr{P}^*(V, \Sigma) = \mathscr{P}^*(V) \cap Z(V, \Sigma)$ are virtual polytopes whose support functions are linear in Σ; and $\mathscr{P}^*(\Lambda, \Sigma) = \mathscr{P}^*(\Lambda) \cap Z(V, \Sigma)$ are the lattice virtual polytopes from $\mathscr{P}^*(V, \Sigma)$.

The groups of virtual polytopes $\mathscr{P}^*(V, \Sigma)$ and $\mathscr{P}^*(\Lambda, \Sigma)$ have a natural parametrization which is constructed canonically from a given decomposition Σ. Let $l_1, \dots, l_N \in \Lambda^*$ be the integral direction vectors of the edges of the decomposition Σ. Obviously the piecewise linear function $f\colon V^* \to \mathbb{R}$, linear in the decomposition Σ, is uniquely determined by its "coordinates" $z_i = f(l_i)$, $1 \le i \le N$. Moreover, from the condition that Σ is a simple lattice decomposition it follows easily that for any set $(z_1, \dots, z_N) \in \mathbb{R}^N$ there exists a piecewise linear function f, linear in Σ, such that $z_i = f(l_i)$, $1 \le i \le N$, and the virtual polytope corresponding to f and denoted by $\alpha(z_1, \dots, z_N)$ is lattice if and only if $z_i \in \mathbb{Z}$, $1 \le i \le N$. Thus, we have a diagram

$$\begin{array}{ccc} \mathbb{Z}^N & \hookrightarrow & \mathbb{R}^N \\ {\scriptstyle \alpha(\cdot)}\downarrow & & \downarrow{\scriptstyle \alpha(\cdot)} \\ Z(\Lambda, \Sigma) & \hookrightarrow & Z(V, \Sigma) \end{array}$$

The support function of the virtual polytope $\alpha(z_1, \dots, z_N)$ will be denoted by $f(z_1, \dots, z_N)$, and its value on a covector $\xi \in V^*$ by $f(z_1, \dots, z_N, \xi)$ or simply by $f(z, \xi)$.

2. Integrals and sums over convex chains.

Definition 1. (A) Let $h\colon V \to \mathbb{C}$ be a continuous function. Its *integral over a convex chain* $\alpha \in Z(V)$ is the number

$$I_h(\alpha) = \int_V \alpha(x) h(x)\, dx,$$

where the volume element dx is generated by the integer lattice $\Lambda \subset V$, and the integral always exists since the support of α is compact.

(B) Let $h\colon \Lambda \to \mathbb{C}$ be an arbitrary function. Its (*lattice*) *sum* over a chain $\alpha \in Z(V)$ is the number

$$S_h(\alpha) = \sum_{x \in \Lambda} \alpha(x) h(x),$$

which is defined since the support of α is compact, so that the sum has only a finite number of nonzero terms.

The main result of this paper is the determination of the connection between the mappings $I_h(\cdot)$ and $S_h(\cdot)$ (obviously these are measures on the group of convex chains) for a special class of functions h, restricted to suitable families of chains. More precisely, let $\beta \in Z(\Lambda, \Sigma)$ and $\alpha(z) \in \mathscr{P}^*(V, \Sigma)$, $(z) = (z_1, \dots, z_N)$; then for fixed h we have a pair of mappings

$$I_h(\beta * \alpha(\cdot)) \colon \mathbb{R}^N_z \to \mathbb{C}, \qquad S_h(\beta * \alpha(\cdot)) \colon \mathbb{Z}^N_z \to \mathbb{C}.$$

We wish to recover the second of these mappings from the first, where $h(x)$ is taken to be a polynomial on V. We recall that according to [1] (Corollaries 2.4 and 2.5) in this case both these mappings are polynomial in (z) of degree $\deg h + \dim V$. The problem is thus reduced to finding the coefficients of the polynomial $S_h(\beta * \alpha(z))$ from the coefficients of the polynomial $I_h(\beta * \alpha(z))$, or, more precisely, to constructing a linear operator mapping the first polynomial to the second. Such an operator (the Todd operator) exists and solves our problem. However, in order to prove this we need to consider a wider class of functions $h(x)$, the class of quasipolynomials, i.e., linear combinations of functions of the form $P(x) \exp \xi(x)$, where P is a polynomial and $\xi \colon V \to \mathbb{C}$ is a complex covector, $\xi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$. The action of the (complex) covector ξ on a vector $x \in V$ will be denoted $\xi(x)$ or $(\xi \cdot x)$.

3. **The Todd operator.** Let $z_1, \dots, z_N$ be independent variables, real or complex, and $(\partial/\partial z_i)$ the corresponding partial differential operators.

Definition 2. The *Todd mapping* is a function

$$\mathrm{Td}(z) \colon \mathbb{R}^N \to \mathbb{R} \quad \text{or} \quad \mathrm{Td}(z) \colon \mathbb{C}^N \dashrightarrow \mathbb{C},$$

real-analytic in the first case and meromorphic in the second, defined by the equality

$$\mathrm{Td}(z) = \prod_{i=1}^{N} \frac{z_i}{1 - \exp(-z_i)}.$$

Properties of the Todd mapping. (i) It is symmetric relative to permutations of the variables.

(ii) Let $(z'_1, \dots, z'_M) \subset (z_1, \dots, z_N)$, $M \le N$, be some subset of the variables. The mapping $\mathrm{Td}(z)$ on the plane of the variables (z') is $\mathrm{Td}(z')$. This follows from the fact that $\mathrm{Td}(0) = 1$.

(iii) The meromorphic function $\mathrm{Td}(z)$ has poles of multiplicity 1 along the hyperplanes $z_i = 2\pi\sqrt{-1}m$, $m \in \mathbb{Z}\setminus\{0\}$. In particular, the radius of convergence of the power series $\mathrm{Td}(z)$ at the origin is equal to 2π in each of the variables z_i, and the series itself has the form

$$\mathrm{Td}(z) = \prod_{i=1}^{N} \left(1 + \frac{1}{2} z_i + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z_i^{2k}\right),$$

where B_k is the kth Bernoulli number.

Removing the parentheses in the last formula, we write

$$\mathrm{Td}(z) = \sum_{I \in \mathbb{Z}^N_+} \tau_I z^I$$

in the usual multi-index notation: $I = (i_1, \dots, i_N)$, $|I| = \sum_{k=1}^N i_k$. Abusing the notation, we shall write this representation for different N: $\mathrm{Td}(z) = \sum_I \tau_I z^I$, without indicating the number N. This is legal by property (ii) and does not cause confusion.

In our understanding the Todd operator is the result of substituting the differentiation operator $\partial/\partial z_i$ for the variables z_i in this expansion. The functions to which it can be applied will be complex-valued analytic functions of the real variables z_i which are the restrictions of holomorphic functions on $D \subset \mathbb{C}^N$ to the real domain $D \cap \mathbb{R}^N$ (the coordinates are fixed and the embedding of $\mathbb{R}^N \subset \mathbb{C}^N$ is the natural one).

Definition 3. The function $f(z)$ is said to *admit the Todd operator* $\mathrm{Td}(\partial/\partial z)$ if the series

$$\sum_{I \in \mathbb{Z}_+^N} \tau_I \frac{\partial^{|I|}}{\partial z_I} f(z)$$

is absolutely convergent on the domain of definition of f, uniformly on compact subsets, to the function $h(z)$, and then $h(z) = \mathrm{Td}(\partial/\partial z) f(z)$.

We consider the action of the Todd operator on certain classes of functions.

(i) It is obvious that polynomials $P(z)$ admit the Todd operator, and its action does not require any comments. Unfortunately, we cannot restrict ourselves to the class of polynomials, for reasons that will become clear below.

(ii) Let p_i, $i = 1, \dots, N$, be complex coordinates. It is easy to see that for $|p_i| < 2\pi$ the exponential $\exp \sum_{i=1}^N p_i z_i$ admits the Todd operator and

$$\mathrm{Td}\left(\frac{\partial}{\partial z}\right) \exp \sum_{i=1}^N p_i z_i = \mathrm{Td}(p_1, \dots, p_N) \exp \sum_{i=1}^N p_i z_i .$$

Moreover, it is not hard to see that the series that realizes the action of the Todd operator converges uniformly on compact sets in $D_{2\pi} = \{(p) \mid |p_i| < 2\pi\}$.

(iii) We consider $\exp \sum_{i=1}^N p_i z_i$ as a function of (p, z). Then for any polynomial $P(z)$ we obviously have

$$P(z) \exp \sum_{i=1}^N p_i z_i = P\left(\frac{\partial}{\partial p_i}\right) \exp \sum_{i=1}^N p_i z_i .$$

All the concrete computations of the action of the Todd operator that we shall need below are contained in the following assertion.

Lemma 1. *The quasipolynomial* $P(z) \exp \sum_{i=1}^N p_i z_i$ *for* $|p_i| < 2\pi$ *admits the Todd operator, and the result of its action is given by*

$$\mathrm{Td}\left(\frac{\partial}{\partial z}\right) P(z) \exp \sum_{i=1}^N p_i z_i = P\left(\frac{\partial}{\partial q}\right) \left(\mathrm{Td}(q_1, \dots, q_N) \exp \sum_{i=1}^N q_i z_i \right)\Bigg|_{q_i = p_i} .$$

Proof. We apply the Todd operator, using the representation of the polynomial written above:

$$\begin{aligned} \mathrm{Td}\left(\frac{\partial}{\partial z}\right) P(z) \exp \sum_{i=1}^N p_i z_i &= \sum_I \tau_I \frac{\partial^{|I|}}{\partial z_I} P\left(\frac{\partial}{\partial p_i}\right) \exp \sum_{i=1}^N p_i z_i \\ &= \sum_I P\left(\frac{\partial}{\partial p_i}\right) \left(\tau_I p^I \exp \sum_{i=1}^N p_i z_i \right) . \end{aligned}$$

Since $\sum_I \tau_I p^I \exp \sum_{i=1}^N p_i z_i$ converges absolutely and uniformly on compact sets for p in $D_{2\pi}$, and its terms are holomorphic, this is also true for all its derivatives with respect to p, so that the required series also converges absolutely and uniformly on compact sets, the operator $P(\partial/\partial p_i)$ can be taken outside the summation sign, and we obtain the answer written in the lemma, a function that is holomorphic in $D_{2\pi}$

in the variable p and holomorphic everywhere with respect to z. This proves the lemma.

4. The main theorem and its origin. We shall now state the main result.

Riemann-Roch theorem for integrals and sums of quasipolynomials. *In the above notation, for a fixed simple lattice decomposition* Σ *there exists a neighborhood of the origin* $0 \in U \subset V^*$ *such that for any quasipolynomial*

$$f = \sum_{j \in J} P_j(x) \exp \xi_j(x),$$

where $\xi_j \in U$, $j \in J$, *the function*

$$I_f(\beta * \alpha(z_1, \dots, z_N)) \colon \mathbb{R}^N \to \mathbb{C}$$

admits the Todd operator and, for $(c_1, \dots, c_N) \in \mathbb{Z}^N$,

$$\operatorname{Td}\left(\frac{\partial}{\partial z}\right) I_f(\beta * \alpha(z_1, \dots, z_N))\Big|_{\substack{z_i = c_i, \\ 1 \le i \le N}} = S_f(\beta * \alpha(c_1, \dots, c_N)).$$

Roughly speaking, the Todd operator transforms the integration of a quasipolynomial (with sufficiently small exponents) over a convex chain into its lattice sum. For the proof of the theorem we need explicit formulas for $I_f(\beta * \alpha(z))$ and $S_f(\beta * \alpha(z))$, which will be obtained below. These formulas are also of independent interest. Such formulas have been obtained previously by other methods (see, for example, [3]). But the striking connection between integration and lattice summation has not been known until now.

We briefly explain the algebro-geometric origin of our theorem (and its name at the same time). It is well known (see [4] and [5]) that with each simple lattice decomposition Σ of the space V^* one can associate in a natural way a smooth toric variety X_Σ (the elements of the lattice Λ^* correspond to one-parameter subgroups of the torus $(\mathbb{C}\setminus\{0\})^n$, the elements of Λ to its characters). For a virtual polytope $\alpha \in \mathscr{P}^*(\Lambda, \Sigma)$ we denote the corresponding invertible sheaf by $\mathscr{F}(\alpha)$. We have (see [5])

$$\chi(X, \mathscr{F}(\alpha)) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, \mathscr{F}(\alpha)) = S_{\mathbb{1}}(\alpha),$$

where $\mathbb{1} \colon V \to \mathbb{R}$ is a constant function, i.e., the Euler characteristic of the sheaf is the "number of lattice points" of the corresponding virtual polytope. According to the usual algebro-geometric Riemann-Roch theorem [6], [7],

$$\chi(X, \mathscr{F}(\alpha)) = \deg(\operatorname{ch}(\mathscr{F}(\alpha)) \cdot \operatorname{td}(\mathscr{T}_X))_n,$$

where

$$\operatorname{ch}(\mathscr{F}(\alpha)) = \sum_{i=0}^{n} \frac{1}{i!} c_1^i(\mathscr{F}(\alpha))$$

is the exponential Chern character of the invertible sheaf $\mathscr{F}(\alpha)$, and $\operatorname{td}(\mathscr{T}_X)$ is the Todd class of the tangent sheaf of the variety X.

The method of computing the number of lattice points of a polyhedron using a Riemann-Roch theorem was proposed in [5]; in that paper this method was used to prove the polynomiality of the number of lattice points of the polyhedron nA relative to $n \in \mathbb{Z}_+$ and the corresponding special case of Ehrhardt's duality theorem (see Theorems 1.1 and 1.2 of [1]); the duality theorem followed from the algebro-geometric Serre duality theorem. Now, however, we can say considerably more. Let $v: \mathbb{R}_z^N \to \mathbb{R}$ be the volume function of the virtual polytope,

$$v(z_1, \ldots, z_N) = I_{\mathbb{1}}(\alpha(z_1, \ldots, z_N)) = \int_V \alpha(z, x)\, dx$$

(relative to the volume element defined by the lattice $\Lambda \subset V$). Obviously, $v(z)$ is a polynomial of degree $\dim V$. Let $J_v \subset \mathbb{Q}[x_1, \ldots, x_N]$ be the ideal consisting of the polynomials $p(x_1, \ldots, x_N)$ such that $p(\partial/\partial z_1, \ldots, \partial/\partial z_N)v(z_1, \ldots, z_N) = 0$. We have the following fact.

Theorem. *The Chow ring $A(X) \otimes \mathbb{Q} = \bigoplus_{i=0}^n A^i(X) \otimes \mathbb{Q}$ of algebraic cycles on the variety X modulo numerical equivalence, graded by codimension of cycles, is isomorphic as a graded algebra to the ring $\mathbb{Q}[x_1, \ldots, x_N]/J_v$.*

One can also show that the Chern class of the invertible sheaf $\mathscr{F}(\alpha)$, where $\alpha = \alpha(b_1, \ldots, b_N)$, is represented in $\mathbb{Q}[x_1, \ldots, x_N]/J_v$ by the polynomial $\sum_{i=1}^N b_i x_i$, and the corresponding exponential Chern character by a truncated series for the function $\exp(\sum_{i=1}^N b_i x_i)$, and finally, the Todd class of the tangent bundle $\mathscr{T}_X$ is represented by a truncated series of the function

$$\prod_{i=1}^N \frac{x_i}{1 - \exp(-x_i)}.$$

From this it is not hard to deduce that

$$S_{\mathbb{1}}(\alpha) = \chi(X, \mathscr{F}(\alpha)) = \prod_{i=1}^N \frac{\partial/\partial z_i}{1 - \exp(-\partial/\partial z_i)} v(z_1, \ldots, z_N)|_{z_i = b_i},$$

i.e., a special case (for $f = \mathbb{1}$) of our Riemann-Roch theorem (since $v(z) = I_{\mathbb{1}}(\alpha(z_1, \ldots, z_N))$). A detailed presentation of these arguments and facts will be published elsewhere.

5. **Plan of the proof of the Riemann-Roch theorem.** The idea of the proof of the Riemann-Roch theorem presented below (the remainder of the paper is devoted to it) is as follows. In the previous paper [1] we showed (Proposition 4.2) that a convex chain can be recovered from its support function as a linear combination of characteristic functions of cones, where as cones we can take the translations of the dual cones to the cones of the decomposition Σ. As $z_1, \ldots, z_N$ vary the chain $\beta * \alpha(z)$ itself varies in a complicated way, but the motion of each cone of it is simply its parallel transport, where the transport vector depends linearly on the coordinates (z). If the integral (sum) of a quasipolynomial over a concrete simple cone exists (thanks to the exponential), then for this separate cone and quasipolynomial the Riemann-Roch theorem (i.e., the connection between sum and integral) is stated and proved without difficulty (this is done in §2).

For this, in order to "glue" these facts into our Riemann-Roch theorem, we need to know how to decompose a convex chain into a linear combination of cones so that a quasipolynomial with given exponent $\xi \in V^*$ in the exponential can be integrated over each of them. This goal is achieved via the technique of conic representations of convex chains, considered in §3. The prototype of our technique is the well-known construction of Varchenko and Gel'fand [8].

Finally, combining the results of §§2 and 3 in §4, we shall prove that the integral (sum) of a quasipolynomial over suitable cones extends to a meromorphic-valued measure on the set of conical chains. This allows us to work as if we could integrate (sum) any quasipolynomial over any cone, without fear of divergences. Hence it is easy to compute rather explicit formulas for the integral and sum of a quasipolynomial over a convex chain and (independently of these formulas) to prove the Riemann-Roch theorem (§5).

From this superficial description it is already clear why we cannot restrict our attention to sums and integrals of polynomials over convex chains—for a cone they do not exist and an exponential is necessary to ensure convergence.

§2. Riemann-Roch theorem for a simple cone

1. **Universal exponential.** Let $V^*_{\mathbb{C}} = V^* \otimes \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ be the complexification of V^*, and $V^* = V^* \otimes 1 \hookrightarrow V^*_{\mathbb{C}}$ be the natural embedding of real vector spaces, so that over $\mathbb{R}$ we have $V^*_{\mathbb{C}} = V^* \otimes 1 \oplus V^* \otimes \sqrt{-1}$ and accordingly for $\xi \in V^*_{\mathbb{C}}$ we have the canonical decomposition $\xi = \mathrm{Re}\,\xi + \sqrt{-1}\,\mathrm{Im}\,\xi$. Every time when coordinate notation appears, it is understood that a coordinate system $x_1, \dots, x_n \in V^*$ is given on V, relative to which the discrete lattice $\Lambda \subset V$ is realized as the standard integer lattice (we shall call such coordinate systems "integral coordinate systems"), and that on V^* there is the dual coordinate system $\xi_1, \dots, \xi_n \in V$, $(\xi_i \cdot x_j) = \delta_{ij}$, where the ξ_i are extended naturally to $V^*_{\mathbb{C}}$ as complex coordinates,

$$\xi_i(\mathrm{Re}\,l + \sqrt{-1}\,\mathrm{Im}\,l) = \xi_i(\mathrm{Re}\,l) + \sqrt{-1}\,\xi_i(\mathrm{Im}\,l).$$

We call the function

$$\exp\colon V \times V^*_{\mathbb{C}} \to \mathbb{C}, \quad \exp\colon (x, \xi) \mapsto \exp \xi(x)$$

the *universal exponential*. Obviously, the universal exponential is analytic in (x, ξ) and, in particular, is holomorphic as a function of the complex variables ξ_i.

2. **Statement of the theorem.** We fix a pair (C, C^*) of dual simple lattice cones in V and V^* respectively, with vertices at the origin (it is easy to show that if one of the cones C or C^* is simple lattice, so is the other).

Definition 1. A cone $G \subset V$ is said to be *reduced relative to a covector* $\xi \in V^* \backslash \{0\}$ if it is pointed and the function $\xi|_G$ attains its maximum at exactly one point, the vertex of the cone.

For a given pointed cone G we denote by U_G the (obviously open) set of complex covectors ξ such that G is reduced relative to $\mathrm{Re}\,\xi$. We also denote by $G(x)$ the translation of the cone G by a vector $x \in V$, and we define the functions

$$i\colon V \times U_C \to \mathbb{C}, \qquad i\colon (x, \xi) \mapsto \int_{C(x)} \exp \xi(y)\, dy,$$
$$s\colon V \times U_C \to \mathbb{C}, \qquad s\colon (x, \xi) \mapsto \sum_{y \in \Lambda \cap C(x)} \exp \xi(y)$$

(obviously the integrals and series converge since the cone is reduced). Further, let $P\colon V \to \mathbb{C}$ be a polynomial function. We set

$$i(P, x, \xi) = \int_{C(x)} P(y) \exp \xi(y)\, dy, \qquad s(P, x, \xi) = \sum_{y \in \Lambda \cap C(x)} P(y) \exp \xi(y)$$

for $\xi \in U_C$ so that, in particular, $i(\mathbb{1}, x, \xi) = i(x, \xi)$ and $s(\mathbb{1}, x, \xi) = s(x, \xi)$. We note that the natural identification of $V^*_{\mathbb{C}}$ with the complexified tangent space $T_\xi V^* \otimes \mathbb{C}$ at any covector ξ allows us to interpret the polynomial P as a linear differential operator on $V_{\mathbb{C}}^*$ with constant coefficients. More precisely, let $(x_1, \ldots, x_n)$ and $(\xi_1, \ldots, \xi_n)$ be a pair of dual integral coordinates on V and $V^* \subset V_{\mathbb{C}}^*$; then the polynomial $P(x) = \sum_{|I| \leq K} a_I x^I$ corresponds to the linear differential operator

$$P\left(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n}\right) = \sum_{|I| \leq K} a_I \frac{\partial^{|I|}}{\partial \xi_I}$$

in the usual multi-index notation. We obviously have

$$P\left(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n}\right) \exp(\xi \cdot x) = P(x_1, \ldots, x_n) \exp(\xi \cdot x),$$

i.e., as in §1.3, we interpret the quasipolynomial on V as a result of applying a linear differential operator to the universal exponential.

Let $v_1, \ldots, v_n$ be the direction vectors of the edges of the cone C.

Proposition 1. (A) *For any $x \in V$ the function $i(x, \xi)\colon U_C \to \mathbb{C}$ extends to a meromorphic function on $V_{\mathbb{C}}^*$,*

$$\xi \mapsto \prod_{k=1}^{n} (-\xi(v_k))^{-1} \exp \xi(x),$$

with poles in the set of n hyperplanes $\xi(v_k) = 0$, $k = 1, \ldots, n$, and this function is also denoted $i(x, \xi)$. Outside of this set of hyperplanes for any polynomial $P\colon V \to \mathbb{C}$ the function $i(P, x, \xi)$ extends to a holomorphic (and meromorphic on $V_{\mathbb{C}}^*$) function $P(\partial/\partial\xi_1, \ldots, \partial/\partial\xi_n) i(x, \xi)$, which we also denote by $i(P, x, \xi)$.

(B) *For any $x \in \Lambda$ the function $s(x, \xi)\colon U_C \to \mathbb{C}$ extends to a meromorphic function on $V_{\mathbb{C}}^*$*

$$\xi \mapsto \prod_{k=1}^{n} (1 - \exp \xi(v_k))^{-1} \exp \xi(x),$$

with poles along the hyperplanes $\xi(v_k) = 2\pi\sqrt{-1}\, m$, $m \in \mathbb{Z}$, $1 \leq k \leq n$, which we denote by the same symbol $s(x, \xi)$. Outside of these hyperplanes for any polynomial $P\colon V \to \mathbb{C}$ the function $s(P, x, \xi)$ extends to a holomorphic (meromorphic on $V_{\mathbb{C}}^*$) function

$$P(\partial/\partial\xi_1, \ldots, \partial/\partial\xi_n) s(x, \xi),$$

which we also denote by $s(P, x, \xi)$.

Proof. We consider only part (A). Part (B) is completely analogous. The computations become completely trivial in a suitable coordinate system. We construct it.

With a pair (C, C^*) of dual simple lattice cones one can associate in a natural way a pair of dual integral coordinate systems (x_i), (ξ_i) on V and V^*, unique up to a permutation of the coordinate functions. Namely, it is uniquely determined by the conditions that the edges of C^* have as direction vectors the vectors $(\xi_j = \delta_{ij})$, $i = 1, \ldots, n$, i.e., the unit coordinate vectors (in other words, $\xi_i = -v_i \in V = V^{**}$), and the edges of C are the vectors $(x_j = -\delta_{ij})$, $i = 1, \ldots, n$ (in other words, the coordinate functions $x_i \in V^*$ are precisely the direction vectors of the cone C^*; there is a natural one-to-one correspondence between the edges of the cones C and C^*, as there is between the coordinates x_j and ξ_j). In the coordinates (x), (ξ), the lattices Λ and Λ^* are $\mathbb{Z}^n$, $C = \{(x_j) | x_j \leq 0\}$, and $C^* = \{(\xi_j) | \xi_j \geq 0\}$. In these coordinates we have for $\xi \in U_C$ $(\operatorname{Re}\xi \in \operatorname{Int} C^*)$

$$i(x, \xi) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \exp \sum_{i=1}^{n} \xi_i y_i \, dy_1 \cdots dy_n = \frac{1}{\xi_1 \cdots \xi_n} \exp \sum_{i=1}^{n} \xi_i x_i,$$

and the integral converges uniformly on compact sets in $U_C \subset V_{\mathbb{C}}^*$, whereas the integrand is holomorphic with respect to ξ. The remaining assertions are now obvious, which proves the proposition.

In passing we point out an important property of these meromorphic functions. We set $i_+(x, \xi) = i(x, \xi)$ and $i_+(P, x, \xi) = i(P, x, \xi)$; $i_-(x, \xi)$ and $i_-(P, x, \xi)$ are the integrals of the universal exponential and quasipolynomial over the cone C_-, translated by the vector x, spanned by the vectors $-v_1, v_2, \dots, v_n$, so that $C \cup C_-$ is the developed cone with vertex space the line $\langle v_1 \rangle$. We set C_0 equal to the simple lattice cone in $\langle v_2, \dots, v_n \rangle$ spanned by $v_2, \dots, v_n$, and $s_+(x, \xi) = s(x, \xi)$, $s_+(P, x, \xi) = s(P, x, \xi)$; $s_-(x, \xi)$ and $s_0(x, \xi)$ are meromorphic functions on $V_{\mathbb{C}}^*$, extending the functions

$$s_-(x, \xi) \colon \Lambda \times U_{C_-} \to \mathbb{C}, \qquad s_-(x, \xi) = \sum_{y \in C_-(x) \cap \Lambda} \exp \xi(y),$$

$$s_0(x, \xi) \colon \Lambda \times U_{C_0} \to \mathbb{C}, \qquad s_0(x, \xi) = \sum_{y \in C_0(x) \cap \Lambda} \exp \xi(y),$$

and analogously for the quasipolynomial we define $s_-(P, x, \xi)$ and $s_0(P, x, \xi)$.

Proposition 2. *For any* $P \colon V \to \mathbb{C}$

$$i_+(P, x, \xi) + i_-(P, x, \xi) = 0, \qquad s_+(P, x, \xi) + s_-(P, x, \xi) = s_0(P, x, \xi).$$

Proof. The verification of these relations for $P \equiv 1$ is an elementary computation. An application of Proposition 1 then completes the proof.

Let (x_j) and (ξ_j) be the coordinate systems naturally associated with the cones C and C^*, introduced in the proof of the proposition. We define the Todd operator of the cone C as the operator $\mathrm{Td}_C = \mathrm{Td}(\partial/\partial x_j)$.

Theorem 1 (Riemann-Roch theorem for a simple cone). *For any polynomial* $P(x)$ *and any covector* $\xi = (\xi_1, \dots, \xi_n) \in (\mathbb{C}\backslash\{0\})^n$ *such that* $|\xi_i| < 2\pi$ *(i.e.,* $0 \neq |\xi(v_i)| < 2\pi$*) the function* $i(P, x_1, \dots, x_n, \xi)$ *admits the Todd operator* Td_C, *and for* $(x_1, \dots, x_n) \in \mathbb{Z}^n$ *we have*

$$\mathrm{Td}_C i(P, y_1, \dots, y_n, \xi)|_{y_i = x_i} = s(P, x_1, \dots, x_n, \xi),$$

where the series realizing the Todd operator converges with respect to ξ *uniformly on compact sets in the domain* $0 \neq |\xi_i| < 2\pi$, $i = 1, \dots, n$.

3. **Proof of the Riemann-Roch theorem.** Before we prove Theorem 1, which is not complicated, we give an argument (apparently analogous to the way in which Euler and Maclaurin were led to their formula) which, although not a proof, is nevertheless rather transparent and explanatory on the intuitional level for why this surprising transformation of an integral to a sum takes place. Let $f \colon V \to \mathbb{C}$ be a function, $F \colon V \to \mathbb{C}$ its "indefinite integral" over the cone $C(x)$ (the coordinates (x), (ξ) are as in the statement of the theorem): $F(x) = \int_{C(x)} f(y)\, dy$. Obviously,

$$\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} F(x) = f(x).$$

We apply the Todd operator Td_C to $F(x)$ in the following way:

$$\mathrm{Td}_C F(x) = \prod_{i=1}^{n} \left(1 - \exp\left(-\frac{\partial}{\partial x_i}\right)\right)^{-1} \left(\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}\right) F(x),$$

and the operator $(1-\exp(-\partial/\partial x_i))^{-1}$ is considered (formally!) as the sum of the infinite series $\sum_{k=0}^{\infty}\exp(-k\,\partial/\partial x_i)$. But the operator $\exp(a\,\partial/\partial z)$ can be interpreted (Taylor's formula) as the operator of translation by a, whence

$$\mathrm{Td}_C F(x) = \sum_{(k_1,\dots,k_n)\in\mathbb{Z}_+^n} f(x_1-k_1,\dots,x_n-k_n),$$

i.e., exactly what was needed!

We pass to the rigorous proof of the Riemann-Roch theorem. It is completely elementary and is obtained by a combination of the explicit formulas of Proposition 1 and the arguments of §1.3. For the universal exponential we have

$$i(x_1,\dots,x_n;\xi_1,\dots,\xi_n) = \frac{1}{\xi_1\cdots\xi_n}\exp\sum_{i=1}^{n} x_i\xi_i,$$

$$s(x_1,\dots,x_n;\xi_1,\dots,\xi_n) = \frac{1}{\prod_{i=1}^{n}(1-\exp(-\xi_i))}\exp\sum_{i=1}^{n} x_i\xi_i,$$

from which the theorem follows. The general case of arbitrary P follows from Lemma 1.1 and its proof. This completely proves the theorem.

§3. Conical representations of convex chains

1. **Definition of the group of conical chains.** Let (V,Λ) be an admissible pair [1] (i.e., there exists an isomorphism $V \xrightarrow{\sim} \mathbb{R}^n$, relative to which Λ is either $\mathbb{Z}^n$ or F^n, where $F\subset\mathbb{R}$ is a subfield). All the definitions of §1.1 are meaningful in this general case, so that in fact the general case will also be considered in this section. To the definitions and notation of §1.1 we add the following new concepts and symbols.

(1) The set of all Λ-cones will be denoted by $C(\Lambda)$. In particular, the set of all cones will be denoted by $C(V)$.

(2) The set of all Λ-cones $C\in C(\Lambda)$ such that $x\in \mathrm{vs}(C)$ will be denoted by $C(\Lambda, x)$.

(3) The set of all developed Λ-cones will be denoted by $\widetilde{C}(\Lambda)$.

(4) We set $\widetilde{C}(\Lambda,x) = C(\Lambda,x)\cap\widetilde{C}(\Lambda)$.

(5) A *conical chain* (resp. a *conical* Λ*-chain*) is a $\mathbb{Z}$-valued function $\alpha\colon V\to\mathbb{Z}$ with a representation of the form $\alpha=\sum_{i\in I} n_i\mathbb{1}_{C_i}$, where $\#I<\infty$, $C_i\in C(V)$ (resp. $C(\Lambda)$), $n_i\in\mathbb{Z}$, and $\mathbb{1}_\bullet$ is the characteristic function of the set $\bullet$. The additive group of conical chains (resp. of Λ-chains) is denoted by $ZC(V)$ (by $ZC(\Lambda)$).

(6) A conical chain $\alpha\in ZC(V)$ (resp. $ZC(\Lambda)$) is *developed* if it has the representation

$$\alpha = \sum_{i\in I} n_i\mathbb{1}_{C_i},$$

where the $C_i\in\widetilde{C}(V)$ (resp. $C(\Lambda)$) are developed cones. The subgroup of developed conical chains is denoted by $\widetilde{ZC}(V)$ (resp. $\widetilde{ZC}(\Lambda)$).

(7) A conical chain $\alpha\in\widetilde{ZC}(V)$ is *developed at a point* $x\in V$ if there exist a developed chain $\beta\in ZC(V)$ and a neighborhood $U\ni x$ such that $\alpha|_U=\beta|_U$.

(8) A point $x\in V$ is called a *vertex* of a chain $\alpha\in ZC(V)$ if α is not developed at x.

(9) For $x \in \Lambda$ we set

$$ZC(\Lambda, x) = \left\{ \alpha = \sum_{i \in I} n_i \mathbb{1}_{C_i} \,\middle|\, C_i \in C(\Lambda, x),\ i \in I \right\},$$

$$\widetilde{ZC}(\Lambda, x) = \{\alpha \in ZC(\Lambda, x) | \alpha \text{ is developed at } x\}.$$

(10) A nonzero covector $\xi \in V^*\backslash\{0\}$ is *in general position* with a cone $C \in C(V, x)\backslash\widetilde{C}(V, x)$ if the hyperplane $H_{\xi, x} = \{y \in V | \xi(y) = \xi(x)\}$ does not contain edges of the cone C. A covector ξ is in general position with a cone $C \in \widetilde{C}(V, x)$ if $H_{\xi, x} \not\supset \operatorname{vs}(C)$. A covector ξ is in general position with a chain $\alpha \in ZC(V, x)$ if there exists a representation $\alpha = \sum_{i \in I} n_i \mathbb{1}_{C_i}$ such that $C_i \in C(V, x)$ and ξ is in general position with C_i, $i \in I$. Finally, for an arbitrary conical chain $\alpha \in ZC(V)$ and a point $x \in V$ we set $\{\alpha\}_x \in ZC(V, x)$ to be a chain such that $\{\alpha\}_x|_U = \alpha|_U$ in some neighborhood $U \ni x$ (this condition determines $\{\alpha\}_x$ uniquely). A covector ξ is in general position with a conical chain $\alpha \in ZC(V)$ if ξ is in general position with $\{\alpha\}_x$ for each point $x \in V$.

Remark. For brevity we shall usually consider the case $\Lambda = V$. All the arguments and assertions automatically carry over to the general case with the obvious changes in the statements and notation.

Proposition 1. (A) *The set of vertices of a conical chain is finite.*

(B) *A chain that is developed at every point is developed.*

Proof. Part (A) is obvious. We prove (B). Let $\alpha = \sum_{i \in I} n_i \mathbb{1}_{C_i}$. Fix this representation. For $x \in V$ we set $I(x) = \{i \in I | \operatorname{vs}(C_i) = x\}$. If $I(x) = \varnothing$ for all $x \in V$, then all the cones C_i are developed and there is nothing to prove. Otherwise there exists a finite set of points $x_1, \dots, x_K \in V$ such that $I(x_k) \neq \varnothing$, $1 \le k \le K$. Obviously, $I(x_a) \cap I(x_b) = \varnothing$ for $a \neq b$. Since α is developed at each point, we have, for each k, $1 \le k \le K$,

$$\sum_{i \in I(x_k)} n_i \mathbb{1}_{C_i} = \sum_{j \in N_k} m_{kj} \mathbb{1}_{D_{kj}},$$

$m_{kj} \in \mathbb{Z}$, $D_{kj} \in \widetilde{C}(V, x)$. The chain on the right-hand side is developed. Setting $I' = I \backslash \bigcup_{k=1}^{K} I(x_k)$, we obtain

$$\alpha = \sum_{i \in I'} n_i \mathbb{1}_{C_i} + \sum_{k=1}^{K} \sum_{j \in N_k} m_{kj} \mathbb{1}_{D_{kj}},$$

i.e., α is a developed chain, as required.

2. Chains that are reduced relative to a covector.

Theorem 1. *Let $\alpha \in ZC(V, x)$ be a conical chain, $\xi \in V^*\backslash\{0\}$ a covector, and let ξ and α be in general position. Then there exists a unique chain $T(\alpha, \xi)$ with the following properties:*

(i) $T(\alpha, \xi) = \sum_{i \in I} n_i \mathbb{1}_{C_i}$, $C_i \in C(V, x)\backslash\widetilde{C}(V, x)$.

(ii) $\xi|_{C_i} \le \xi(x)$ *and* $H_{\xi, x} \cap C_i = \{x\}$ *for all* $i \in I$.

In particular, ξ is in general position with C_i, $i \in I$.

(iii) $\alpha - T(\alpha, \xi) \in \widetilde{ZC}(V, x)$.

(*Both the statement and the proof given below carry over without any changes to the case of Λ-chains.*)

Proof. Existence. It is obviously sufficient to establish the existence of $T(\alpha, \xi)$ for a chain of the form $\alpha = \mathbb{1}_C$, where $C \in C(V, x)\backslash\widetilde{C}(V, x)$ is a simple (pointed) cone and ξ is in general position with it. Let $R_1, \dots, R_K$ be the edges of the cone C. If $\xi|_{R_i} \le \xi(x)$ for all $k = 1, \dots, K$, then there is nothing to prove. Otherwise suppose, for example, that $\xi|_{R_1} \ge \xi(x)$. Let $\overline{R}_1$ be the ray with vertex x opposite to the ray R_1, and let C_1 be the convex hull of the rays $\overline{R}_1, R_2, \dots, R_K$, and C_1' the convex hull of $R_2, \dots, R_K$. Obviously $\mathbb{1}_{C_1} + \mathbb{1}_C - \mathbb{1}_{C_1'}$ is the characteristic function of the developed cone $C_1 \cup C$ whose vertex space is the straight line $R_1 \cup \overline{R}_1$. But the number of edges of the cones C_1 and C_1' for which $\xi \ge \xi(x)$ is less than this number for C. Acting in this way, in a finite $(\le \dim V)$ number of steps we obtain a chain of the required form.

Uniqueness. This is a less trivial fact. We shall give a transparent proof. It obviously suffices to establish the following fact. Let $\alpha \in \widetilde{ZC}(V, x)$ be a developed chain in general position with the covector $\xi \in V^* \backslash \{0\}$. Let

$$V_{\xi, \pm} = \{y \in V \mid \pm\xi(y) > \xi(x)\}$$

be the half-spaces into which $H_{\xi, x}$ divides V. Then if $\alpha|_{V_{\xi,+}} \equiv 0$, then also $\alpha|_{V_{\xi,-}} \equiv 0$. We shall prove this assertion.

We write $\alpha = \sum_{i \in I} n_i \mathbb{1}_{C_i}$, where the $C_i \in \widetilde{C}(V, x)$ are developed cones, and the hyperplane $H_{\xi, x}$ does not contain their vertex spaces. Let $H_\pm = \{y \in V \mid \xi(y) = \xi(x) \pm 1\}$ be the hyperplanes parallel to $H_{\xi, x}$ "above" and "below" respectively.

Let W be a linear space of dimension $\dim H_{\xi, x}$, and $V \ni v \ne 0$ a vector such that $\xi(v) = 1$. We fix an isomorphism of affine spaces $\varphi\colon W \to H_{\xi, x}$, $\varphi(0) = x$, and we set $\varphi_\pm\colon W \to H_\pm$, $\varphi_\pm\colon w \mapsto \varphi(w) \pm v$. We consider conical chains on W:

$$\alpha_\pm = \varphi_\pm^*(\alpha|_{H_\pm}) \in ZC(W).$$

We know that $\alpha_+ \equiv 0$. We shall show that $\alpha_- \equiv 0$: this obviously completes the proof of the theorem.

On the cones $C \in C(W)$ we define an operation θ in the following way: $\theta(C)$ is the translation of the cone C by the vector $(-2v)$, $\theta(C) = -2v + C$, where $v \in \mathrm{vs}(C)$ is any vector from the vertex space of the cone C.

Lemma 1. *Let $n_i \in \mathbb{Z}$ and let $D_i \in C(W)$, $i \in I$, be such that $\sum_{i \in I} m_i \mathbb{1}_{D_i} \equiv 0$. Then $\sum_{i \in I} m_i \mathbb{1}_{\theta(D_i)} \equiv 0$.*

End of the proof of the theorem. Above we wrote $\alpha = \sum_{i \in I} n_i \mathbb{1}_{C_i}$, $C_i \in \widetilde{C}(V, x)$, $\mathrm{vs}(C_i) \not\subset H_{\xi, x}$. We set $D_i^\pm = \varphi_\pm^{-1}(C_i \cap H_\pm)$, so that the representation $\alpha_\pm = \sum_{i \in I} n_i \mathbb{1}_{D_i^\pm}$ holds. It is not hard to see that $D_i^- = \theta(D_i^+)$, so that we will obtain $\alpha_- \equiv 0$ in view of Lemma 1, as required.

Proof of Lemma 1. We prove it by decreasing induction on $M = \min\{\dim \mathrm{vs}(D_i) \mid i \in I\}$. If $M = \dim W$, then all the $D_i = W$, $\theta(D_i) = W$, and the lemma is obvious. Suppose the lemma has been proved for $M \ge m + 1$. We establish it for $M = m \ge 0$.

For an affine subspace $L \subset W$ of dimension m we set $I(L) = \{i \in I \mid L = \mathrm{vs}(D_i)\}$. Obviously, there exists a finite set $L_1, \dots, L_K$ of m-dimensional affine planes such that $I(L_k) \ne \varnothing$, $1 \le k \le K$. We see that $I(L_a) \cap I(L_b) = \varnothing$ for $a \ne b$. We set

$$I' = I \backslash \bigcup_{k=1}^{K} I(L_k).$$

Now I is partitioned into $K + 1$ disjoint subsets $I(L_1), \dots, I(L_K), I'$. We note that $\mathrm{vs}(\theta(D)) = -\mathrm{vs}(D)$ for any $D \in C(W)$. By hypothesis $\sum_{i \in I} m_i \mathbb{1}_{D_i} \equiv 0$. From this it follows that for any k, $1 \le k \le K$, there exists a representation of the form

$$\sum_{i\in I(L_k)} m_i \mathbb{1}_{D_i} = \sum_{j\in N_k} m_{kj} \mathbb{1}_{G_{kj}},$$

where $\dim \mathrm{vs}(G_{kj}) \ge m+1$ and $L_k \subset \mathrm{vs}(G_{kj})$. Thus

$$\sum_{i\in I'} m_i \mathbb{1}_{D_i} + \sum_{k=1}^{K} \sum_{j\in N_k} m_{kj} \mathbb{1}_{G_{kj}} \equiv 0.$$

But it is not hard to check that

$$\sum_{i\in I(L_k)} m_i \mathbb{1}_{\theta(D_i)} = \sum_{j\in N_k} m_{kj} \mathbb{1}_{\theta(G_{kj})},$$

since if $v_k \in L_k$ is an arbitrary vector, then for $i \in I(L_k)$ and $j \in N_k$ we have $\theta(D_i) = (-2v_k) + D_i$ and $\theta(G_{kj}) = (-2v_k) + G_{kj}$, i.e., all the cones occurring in the transformation of the chain "around" L_k, are shifted by the same vector. In view of this we have

$$\sum_{i\in I} m_i \mathbb{1}_{\theta(D_i)} = \sum_{i\in I'} m_i \mathbb{1}_{\theta(D_i)} + \sum_{k=1}^{K} \sum_{i\in I(L_k)} m_i \mathbb{1}_{\theta(D_i)} = \sum_{i\in I'} m_i \mathbb{1}_{\theta(D_i)} + \sum_{k=1}^{K} \sum_{j\in N_k} m_{kj} \mathbb{1}_{\theta(G_{kj})}.$$

But the last chain is equal to zero by the induction hypothesis. The lemma is proved.

Corollary 1. *If a nonzero conical chain $\alpha \in ZC(V)$ has compact support, i.e., $\alpha \in Z(V)$, then it has at least one vertex.*

The proof is not hard and is left to the reader.

Definition 2. (i) A chain $\alpha \in ZC(V, x)$ is *reduced relative to a covector* $\xi \in V^*\setminus\{0\}$ if ξ and α are in general position and $\alpha = T(\alpha, \xi)$.

(ii) Let $\alpha \in ZC(V)$ be a conical chain. A *reduced representation* of the chain α relative to a covector $\xi \in V^*\setminus\{0\}$ is a set of chains $\{\alpha_{x_i} \in ZC(V, x_i) | i \in I\}$ such that

(1) α_{x_i} is reduced relative to ξ for each $i \in I$, and

(2) $\alpha - \sum_{i\in I} \alpha_{x_i}$ is developed everywhere (and thus is developed).

Proposition 2. (A) *A reduced representation of the chain $\alpha \in ZC(V)$ relative to a generic covector $\xi \in V^*\setminus\{0\}$ exists and is unique.*

(B) *Suppose that the chain $\alpha \in ZC(V)$ has compact support ($\alpha \in Z(V)$) and $\{\alpha_{x_i} \in ZC(V, x_i) | i \in I\}$ is its reduced representation relative to the covector ξ. Then $\alpha = \sum_{i\in I} \alpha_{x_i}$.*

Proof. (A) is obvious in view of Theorem 1. We establish (B). It is not hard to check: for covectors $\eta \in V^*$ sufficiently close to ξ, $\{\alpha_{x_i} | i \in I\}$ will also be a reduced representation of α relative to η. We consider the chain $\beta = \alpha - \sum_{i\in I} \alpha_{x_i}$. By the above remark we may assume that this chain is in general position with the covector ξ. On the other hand, it is developed. Finally, since the support of α is compact and the chains α_{x_i} are reduced, we have $\beta|_{\{x\in V|\xi(x)\ge C\}} \equiv 0$ for $C \gg 0$. Assume that $\beta \equiv 0$. Let $\lambda = \sup\{\xi(z) | \beta(z) \ne 0\}$ and $x \in \mathrm{Supp}\,\beta \cap \{z \in V | \xi(z) = \lambda\}$. Then $\{\beta\}_x$ is a developed chain in general position with ξ and equal to zero over the hyperplane $\{\xi(z) = \lambda\}$. According to Theorem 1, $\{\beta\}_x \equiv 0$, i.e., $\beta \equiv 0$ in some neighborhood of x. This contradiction proves the proposition.

Definition 3. A chain $\alpha \in ZC(V)$ is said to be *reduced relative to a covector* $\xi \in V^*\setminus\{0\}$ if $\alpha = \sum_{i\in I} \alpha_{x_i}$, where $\{\alpha_{x_i} | i \in I\}$ is the reduced representation of the chain α relative to ξ. The chain α is said to be *pointed* if it is reduced relative to some covector.

3. Criterion for the compactness of a conical chain.

Definition 4. Suppose that the chain $\alpha \in ZC(V, x)$ is reduced relative to a covector $\xi \in V^*\backslash\{0\}$. A ray R with vertex x, looking strictly "down" from the hyperplane $H_{\xi, x}$, is called an *edge* of the chain α if for the hyperplane $H_- = \{z \mid \xi(z) = \xi(x) - 1\}$ the point $R \cap H_-$ is a vertex of the chain $\alpha|_{H_-} \in ZC(H_-)$.

Proposition 3 (Criterion for compact support). *Let $\alpha_i \in ZC(V, x_i)$, $i \in I$, be conical chains, reduced relative to the covector $\xi \in V^*\backslash\{0\}$. For an affine line $L \subset V$, set*

$$I(L) = \{i \in I \mid x_i \in L \text{ and } \alpha_i \text{ has an edge } R_{i,L} \subset L\}.$$

The chain $\alpha = \sum_{i \in I} \alpha_i$ has compact support (i.e., is a convex chain) if and only if, for all affine lines $L \subset V$ such that $I(L) \neq \varnothing$, the following condition holds: the chain $\sum_{i \in I(L)} \tau_{-x_i} \alpha_i \in ZC(V, 0)$ (where $\tau_h \alpha(x) = \alpha(x - h)$) has no edges lying on the line L shifted to 0.

The proof is not hard and is left to the reader.

4. To conclude this section we consider the question of the explicit recovery of a convex chain up to $\widetilde{ZC}(V)$ from its support function—in the form in which we need this to complete the proof of the Riemann-Roch theorem. Let $\Sigma = \{C_i^* | i \in I\}$ be a simple lattice decomposition of V^* (see §1.1 for the terminology and notation). We set $I = I^* \cup \widetilde{I}$, where $i \in I^*$ if $\dim\langle C_i^* \rangle = n$ and $i \in \widetilde{I}$ if $\dim\langle C_i^* \rangle < n$. Let $C_i \in C(\Lambda)$ be the cone dual to C_i^*, $i \in I$, $0 \in \mathrm{vs}(C_i)$. We note that the condition that C_i is pointed or developed is exactly equivalent to the condition $i \in I^*$ or $i \in \widetilde{I}$ respectively.

Let $\beta \in Z(V, \Sigma)$ be a convex chain; then for any $i \in I$ there exists a zero-dimensional chain $\beta_i \in \mathbb{Z}[V]$ such that, under the identification $V = V^{**}$, $\beta_i|_{C_i^*}$ realizes the support function of the chain β on C_i^* and $\deg \beta = \deg \beta_i$ (see [1], §4). If $i \in I^*$, then the chain β_i is uniquely determined, where $\beta_i \in \mathbb{Z}[\Lambda]$ for all $i \in I^*$ if and only if the chain β is integral, $\beta \in Z(\Lambda, \Sigma)$.

We write the translation of the cone $C \in C(V)$ on the zero-dimensional chain $\gamma = \sum_{j \in J} m_j [v_j]$ (where $[v]$ for $v \in V$ denotes the generator of the group algebra corresponding to this vector) in terms of chains:

$$\sum_{j \in J} m_j \mathbb{1}_{C(v_j)} = \gamma * \mathbb{1}_C = \mathbb{1}_C * \gamma.$$

In [1] (Proposition 4.2) it was proved, in particular, that

$$\beta - \sum_{i \in I^*} \beta_i * \mathbb{1}_{C_i} \in \widetilde{ZC}(V)$$

is a developed conical chain. In particular, modulo $\widetilde{ZC}(V)$, $\beta * \alpha(z_1, \dots, z_N)$ is represented by the conical chain

$$\sum_{i \in I^*} (\beta_i * \alpha_i(z_1, \dots, z_N)) * \mathbb{1}_{C_i},$$

where the parentheses enclose zero-dimensional chains $\beta_i * \alpha_i(z_1, \dots, z_N) \in \mathbb{Z}[V]$. We explain what $\alpha_i(z_1, \dots, z_N)$ represents.

In the notation of §1.1, $l_1, \dots, l_N \in \Lambda^*$ are integral direction vectors of the edges of the decomposition Σ. Let $l_{i_1}, \dots, l_{i_n}$ be the direction vectors of the edges of the cone C_i^*, and $z_{i_1}, \dots, z_{i_n}$ the corresponding coordinates on the space of virtual polytope $\mathscr{P}^*(V, \Sigma)$. The covectors $l_{i_1}, \dots, l_{i_n}$ will be considered as an integral coordinate system on V. Now we have $\alpha_i(z_1, \dots, z_N) = [w_i(z_1, \dots, z_N)]$, where $w_i(z_1, \dots, z_N) \in V$ is a vector such that $l_{i_k}(w_i) = z_{i_k}$. We note that $w_i(z_1, \dots, z_N)$ really depends only on the variables $z_{i_1}, \dots, z_{i_n}$.

§4. Sums and integrals of quasipolynomials over conical chains

1. Let $C \in C(V)$ be a pointed cone with vertex $\mathrm{vs}(C) \in V$ and complex covector $\xi \in V_{\mathbb{C}}^*$ such that C is reduced relative to $\operatorname{Re}\xi$. Then the integral and lattice sum of the universal exponential over the cone C are defined: $\int_C \exp\xi(x)\,dx$ and $\sum_{x\in C\cap\Lambda}\exp\xi(x)$. We have obtained two functions of two arguments: the cone C and the covector $\xi \in V_{\mathbb{C}}^*$. In the second case (the sum) we shall only consider lattice cones $C \in C(\Lambda)$. The study of both functions is completely analogous, and the proofs are repeated word for word with the sole difference that for the sums all the work is carried out in the class of integral chains. We shall give the detailed presentation for integrals and only the statements for sums. The required changes in the arguments for sums reduce to obvious changes in concepts and notation ($ZC(V)$ by $ZC(\Lambda)$, the letter I for the integral by the letter S for sums, etc.). We shall denote by $\mathscr{M}(V_{\mathbb{C}}^*)$ and $\mathscr{O}(V_{\mathbb{C}}^*)$ the spaces of meromorphic and holomorphic functions, respectively, on $V_{\mathbb{C}}^* \cong \mathbb{C}^n$.

Proposition 1. (A) *The integral of the universal exponential over reduced lattice cones extends to a meromorphic-valued measure on the group of conical chains. More precisely, there exists a unique homomorphism of abelian groups* $I\colon ZC(V) \to \mathscr{M}(V_{\mathbb{C}}^*)$ (*the value of the function* $I(\alpha)$, $\alpha \in ZC(V)$, *on a complex covector* $\xi \in V_{\mathbb{C}}^*$ *will be denoted by* $I(\alpha, \xi)$ *to shorten the notation*) *such that*

(i) *if the pointed cone* C *is reduced relative to* $\operatorname{Re}\xi$, *then*

$$I(\mathbb{1}_C, \xi) = \int_C \exp\xi(x)\,dx.$$

The mapping I *possesses the following properties*:

(ii) $I(\tau_h\alpha, \xi) = \exp\xi(h) I(\alpha, \xi)$ *for* $h \in V$, *and*

(iii) I *is identically zero on developed chains.*

(B) *The lattice sum of the universal exponential over reduced lattice cones extends to a meromorphic-valued measure on the group of integral conical chains. More precisely, there exists a unique homomorphism of abelian groups* $S\colon ZC(\Lambda) \to \mathscr{M}(V_{\mathbb{C}}^*)$ (*the value of the function* $S(\alpha)$, $\alpha \in ZC(\Lambda)$, *on a covector* $\xi \in V_{\mathbb{C}}^*$ *will be denoted by* $S(\alpha, \xi)$) *such that*

(i) *if the pointed lattice cone* C *is reduced relative to* $\operatorname{Re}\xi$, *then*

$$S(\mathbb{1}_C, \xi) = \sum_{x\in C\cap\Lambda} \exp\xi(x).$$

The mapping S *possesses the following properties*:

(ii) $S(\tau_h\alpha, \xi) = \exp\xi(h) S(\alpha, \xi)$ *for* $h \in \Lambda$, *and*

(ii) S *is identically zero on developed integral chains.*

Proof. We shall prove part (A). The proof of (B) is exactly the same.

Let $\alpha \in ZC(V)$ be a conical chain, reduced relative to the covector $\xi_0 \in V^*\setminus\{0\}$. Then for $\xi \in V_{\mathbb{C}}^*$ such that $\operatorname{Re}\xi$ is close to ξ_0, the integral $\int_V \alpha(x)\exp\xi(x)\,dx$ converges absolutely and uniformly on compact subsets, defining a holomorphic function

$$V_{\mathbb{C}}^* \supset U_\alpha \ni \xi \mapsto \int_V \alpha(x)\exp\xi(x)\,dx,$$

where $U_\alpha \subset V_{\mathbb{C}}^*$ is the open subset consisting of the covectors ξ such that α is reduced relative to $\operatorname{Re}\xi$. It is clear that α has a representation of the form $\alpha = \sum_{i\in I} n_i \mathbb{1}_{C_i}$, where the C_i are simple cones, reduced relative to all the covectors $\operatorname{Re}\xi$, $\xi \in U_\alpha$. Hence,

$$\int_V \alpha(x)\exp\xi(x)\,dx = \sum_{i\in I} n_i \int_{C_i} \exp\xi(x)\,dx$$

for $\xi \in U_\alpha$. But on the right-hand side here, as shown in §2, we have a holomorphic function on U_α, extended to a meromorphic function on $V_{\mathbb{C}}^*$. Thus, we have proved that for any pointed chain the holomorphic function

$$U_\alpha \ni \xi \mapsto \int_V \alpha(x)\exp\xi(x)\,dx$$

extends to a meromorphic function on $V_{\mathbb{C}}^*$, which we denote by $I(\alpha)$. We shall show that the association (to pointed chains α) $\alpha \mapsto I(\alpha) \in \mathscr{M}(V_{\mathbb{C}}^*)$ extends in a unique way to a homomorphism of abelian groups $I\colon ZC(V) \to \mathscr{M}(V_{\mathbb{C}}^*)$ and that I possesses the property (iii). We note that because of the existence of the reduced representation, I is uniquely determined by properties (i) and (iii).

We note that this mapping on pointed chains $\alpha \mapsto I(\alpha)$ possesses a "local linearity" property: if α_i, $i \in \mathscr{I}$, is a finite set of chains such that all the α_i are simultaneously reduced relative to the covector ξ_0, then for any $n_i \in \mathbb{Z}$

$$I\left(\sum_{i\in\mathscr{I}} n_i\alpha_i\right) = \sum_{i\in\mathscr{I}} n_i I(\alpha_i).$$

The proposition will be proved if we establish that "local linearity" extends to "global linearity".

Lemma 1. *Let α_i, $i \in \mathscr{I}$, be a finite set of pointed chains such that the chain $\sum_{i\in\mathscr{I}} \alpha_i$ is developed. Then*

$$\sum_{i\in\mathscr{I}} I(\alpha_i) \equiv 0 \in \mathscr{M}(V_{\mathbb{C}}^*).$$

Lemma 2. *Let $C_i \in C(V, x)$, $i \in \mathscr{L}$, be pointed cones with vertex $x \in V$. If $\sum_{i\in\mathscr{L}} m_i\mathbb{1}_{C_i} \equiv 0$ for $m_i \in \mathbb{Z}$, then $\sum_{i\in\mathscr{L}} m_i I(\mathbb{1}_{C_i}) \equiv 0 \in \mathscr{M}(V_{\mathbb{C}}^*)$.*

Lemma 3. *Let $C \in \widetilde{C}(V, x)$ be a developed cone. Then there exist pointed cones $D_j \in C(V, x)$, $j \in \mathscr{N}$, such that $\sum_{j\in\mathscr{N}} r_j\mathbb{1}_{D_j} = \mathbb{1}_C$, $r_j \in \mathbb{Z}$, and $\sum_{j\in\mathscr{N}} r_j I(\mathbb{1}_{D_j}) \equiv 0$.*

Derivation of the proposition from Lemma 1. Let $\alpha \in ZC(V)$ be an arbitrary conical chain, and $\{\alpha_i \in ZC(V, x_i) \mid i \in I\}$ its reduced representation relative to some common covector. We set $I(\alpha) = \sum_{i\in\mathscr{I}} I(\alpha_i) \in \mathscr{M}(V_{\mathbb{C}}^*)$. Lemma 1 guarantees that this definition does not depend on the choice of the covector. Property (i) holds by construction, (iii) holds by Lemma 1, and (ii) is obvious. This proves the proposition.

Derivation of Lemma 1 *from Lemmas* 2 *and* 3. Obviously it suffices to prove Lemma 1 for the case when all the chains have a common vertex $x \in V$ and do not have other vertices. We shall assume this and understand below that the vertex spaces of all the cones occurring in the proof contain the point x.

Each chain α_i can be represented in the form of a linear combination of characteristic functions of pointed cones C_{ij}, $j \in \mathscr{W}_i$, $\alpha_i = \sum_{j\in\mathscr{W}_i} p_{ij}\mathbb{1}_{C_{ij}}$, such that $I(\alpha_i) = \sum_{j\in\mathscr{W}_i} p_{ij} I(\mathbb{1}_{C_{ij}})$. Furthermore, by hypothesis $\sum_{i\in J} \alpha_i = \sum_{k\in K} q_k\mathbb{1}_{D_k}$ with $D_k \in \widetilde{C}(V, x)$. We apply Lemma 3 to each of the developed cones D_k, $k \in K$: let

G_{ka}, $a \in A_k$, be pointed and such that

$$\sum_{a\in A_k} r_{ka}\mathbb{1}_{G_{ka}} = \mathbb{1}_{D_k} \quad \text{and} \quad \sum_{a\in A_k} r_{ka} I(\mathbb{1}_{G_{ka}}) \equiv 0.$$

Now it suffices to verify that

$$\sum_{i\in J}\sum_{j\in\mathscr{W}_i} p_{ij} I(\mathbb{1}_{C_{ij}}) - \sum_{k\in K}\sum_{a\in A_k} (q_k r_{ka}) I(\mathbb{1}_{G_{ka}}) \equiv 0.$$

Since

$$\sum_{i\in J}\sum_{j\in\mathscr{W}_i} p_{ij}\mathbb{1}_{C_{ij}} - \sum_{k\in K}\sum_{a\in A_k} (q_k r_{ka})\mathbb{1}_{G_{ka}} \equiv 0$$

by construction, the required equality is a direct consequence of Lemma 2.

Proof of Lemma 2. It is not hard to construct a set of pointed cones $F_j \in C(V, x)$, $j \in \mathscr{J}$, such that

(i) $\operatorname{Int} F_a \cap \operatorname{Int} F_b = \varnothing$ if $a \neq b$, and

(ii) for each $i \in \mathscr{L}$ there is a distinguished subset $\mathscr{J}(i) \subset \mathscr{J}$ and there exists a representation $\mathbb{1}_{C_i} = \sum_{j\in\mathscr{J}(i)} m_{ij}\mathbb{1}_{F_j}$ where $F_j \subset C_i$ for $j \in \mathscr{J}(i)$.

By the "local linearity" we now have $I(\mathbb{1}_{C_i}) = \sum_{j\in\mathscr{J}(i)} m_{ij} I(\mathbb{1}_{F_j})$, so that

$$\sum_{i\in\mathscr{L}} m_i I(\mathbb{1}_{C_i}) \equiv \sum_{i\in\mathscr{L}}\sum_{j\in\mathscr{J}(i)} (m_i m_{ij}) I(\mathbb{1}_{F_j}).$$

On the other hand, by the hypothesis of the lemma

$$\sum_{i\in\mathscr{L}}\sum_{j\in\mathscr{J}(i)} (m_i m_{ij})\mathbb{1}_{F_j} \equiv 0.$$

It is not hard to verify that thanks to property (i) of our set of cones for any fixed $j \in \mathscr{J}(i)$ we have $\sum_{i\in\mathscr{L}, j\in\mathscr{J}(i)} m_i m_{ij} = 0$ and thus Lemma 2 has been proved.

Lemma 3 is derived without difficulty from Proposition 2.2. We leave the detailed arguments to the reader. This completes the proof of Proposition 1.

Now let $\alpha \in Z(V)$ be a conical chain with compact support. Then the meromorphic function $I(\alpha, \xi)$ is holomorphic everywhere on $V^*_{\mathbb{C}}$ and

$$I(\alpha, \xi) = \int_V \alpha(x)\exp\xi(x)\,dx.$$

Moreover, $I(\alpha) \in \mathscr{O}(V^*_{\mathbb{C}})$ if and only if $\alpha \in Z(V)$. In particular, $I(\alpha, 0)$ is the "volume" of the chain α.

2. **Explicit formulas for the integrals and sums of exponentials.** The technique developed above allows us to write down rather explicit formulas for the integral and sum of the exponential over a convex chain by means of its support function. Let $\Sigma = \{C_i^* | i \in I\}$ be a simple lattice decomposition. In the notation of §§1.1 and 3.4, for a chain $\beta \in Z(V, \Sigma)$ we have $\beta - \sum_{i\in I^*} \beta_i * \mathbb{1}_{C_i} \in \widetilde{ZC}(V)$, so that

$$I(\beta, \xi) = \sum_{i\in I^*} I(\beta_i * \mathbb{1}_{C_i}, \xi).$$

We define the value of the exponential on a zero-dimensional chain $\gamma = \sum m_j[v_j]$ by linearity: $\exp\xi(\gamma) = \sum m_j \exp\xi(v_j)$. Now $I(\beta, \xi) = \sum_{i\in I^*} \exp\xi(\beta_i) I(\mathbb{1}_{C_i}, \xi)$, where the "coefficients" $I(\mathbb{1}_{C_i}, \xi)$ depend only on the decomposition Σ, but not on the chain $\beta \in Z(V, \Sigma)$. Analogously, for an integral chain $\beta \in Z(\Lambda, \Sigma)$ we have $S(\beta, \xi) = \sum_{i\in I^*} \exp\xi(\beta_i) S(\mathbb{1}_{C_i}, \xi)$. But if $v_{i1}, \dots, v_{in} \in \Lambda$ are the direction vectors of the edges of the cone C_i, $i \in I^*$, then, according to the results of §2,

$$I(\mathbb{1}_{C_i}, \xi) = \frac{(-1)^n}{\xi(v_{i1})\cdots\xi(v_{in})} \quad \text{and} \quad S(\mathbb{1}_{C_i}, \xi) = \frac{1}{\prod_{k=1}^{n}(1-\exp\xi(v_{ik}))},$$

so that we eventually have

$$I(\beta, \xi) = \sum_{i\in I^*} \frac{\exp\xi(\beta_i)(-1)^n}{\xi(v_{i1})\cdots\xi(v_{in})} \quad \text{and} \quad S(\beta, \xi) = \sum_{i\in I^*} \frac{\exp\xi(\beta_i)}{\prod_{k=1}^{n}(1-\exp\xi(v_{ik}))}$$

(in the second case $\beta \in Z(\Lambda, \Sigma)$). These formulas are true for a common covector $\xi \in V^*_{\mathbb{C}}$. We know, however, that for a convex chain β the functions $I(\beta, \xi)$ and $S(\beta, \xi)$ are globally holomorphic on $V^*_{\mathbb{C}}$. Therefore, in order to obtain the value of $I(\beta, \xi_0)$ and $S(\beta, \xi_0)$ for an arbitrary covector ξ_0, we can proceed as follows. Let $t \in \mathbb{C}$ be a complex parameter, and $\eta \in V^*_{\mathbb{C}}$ a common covector. We carry out the construction for the integral; for the sum they are entirely analogous. We consider

$$I(\beta, \xi_0 + t\eta) = \sum_{i\in I^*} \frac{(-1)^n \exp((\xi_0 + t\eta)\cdot\beta_i)}{\prod_{k=1}^{n}(\xi_0(v_{ik}) + t\eta(v_{ik}))}$$

as a function of the parameter t. As $t \to 0$ the function remains bounded (a removable singularity). Therefore in each term of the above sum we must separate the term of degree zero in t and sum all such terms over $i \in I^*$. We write the resulting expressions in a more expanded form. Suppose ξ_0 vanishes on the vectors v_{ik}, $k \in K_i$, and suppose $\xi_0(v_{ik}) \neq 0$ if $k \in K_i^*$. We set $\nu_i = \#K_i$. We write an explicit representation for the zero-dimensional chain $\beta_i = \sum_{a\in A_i} m_{ia}[b_{ia}]$, $b_{ia} \in V$. Now, acting according to the scheme described above, we find that $I(\beta, \xi_0)$ is a sum $\sum_{i\in I^*}\sum_{a\in A_i}$ whose (i, a)th term appears as

$$(-1)^n \frac{m_{ia}\exp\xi_0(b_{ia})}{\prod_{k\in K_i}\eta(v_{ik})\prod_{k\in K_i^*}\xi_0(v_{ik})} \times [\cdots],$$

where the square brackets contain the coefficient of t^{ν_i} in the series

$$\left(\sum_{l=0}^{\infty}\frac{t^l}{l!}\eta^l(b_{ia})\right)\prod_{k\in K_i^*}\left(\sum_{l=0}^{\infty}(-1)^l t^l \frac{\eta^l(v_{ik})}{\xi_0^l(v_{ik})}\right).$$

This representation depends on the choice of the common covector η. An analogous representation can be obtained for $S(\beta, \xi_0)$. The above formulas are the source of many more special formulas and some assertions. We note some of them.

Corollary 1. *If $\beta = \beta(y_1, \dots, y_M)$ depends linearly on arbitrary coordinates (y), i.e., all the b_{ia}, $i \in I^*$, $a \in A_i$, depend linearly on (y), then, for fixed ξ, $I(\beta(y), \xi)$ is a quasipolynomial of the form*

$$\sum_{a\in A}\exp[\gamma_a(y)]Q_a(y_1, \dots, y_M),$$

where the $\gamma_a(y)$ *are linear functions and the* $Q_a(y)$ *are polynomials of degree at most* $(n - \min\{\dim C_i^* \mid i \in I\,,\ \xi \in C_i^*\})$.

If $\beta \in \mathscr{P}^*(V, \Sigma)$ is a virtual polytope, then $\beta_i = [b_i]$, $b_i \in V$, $i \in I^*$, so that the formulas are simplified: the sum over the $a \in A_i$ vanishes. If $\xi_0 = 0$, then the formula can be written explicitly.

Corollary 2. *For a common covector* $\eta \in V_{\mathbb{C}}^*$ *and* $\beta \in \mathscr{P}^*(V, \Sigma)$

$$I(\beta, 0) = \sum_{i \in I^*} \frac{(-1)^n \eta(b_i)^n}{n!\eta(v_{i1}) \cdots \eta(v_{in})}$$

is the "volume" of the virtual polytope β.

The last formula has a particularly transparent form for a simple complex polytope (in the usual sense), since in this case the b_i, $i \in I^*$, are precisely its vertices, and the v_{ik} are the direction vectors of edges of cones for the vertices.

3.. **Integrals and sums of quasipolynomials.** Let $(x_1, \dots, x_n)$ and $(\xi_1, \dots, \xi_n)$ be a pair of dual integral coordinates on V and V^*, and consider the (ξ) as complex coordinates on $V_{\mathbb{C}}^*$. As in §2, we interpret the quasipolynomial

$$P(x) \exp \xi(x), \qquad P(x) = \sum_{|I| \le K} a_I x^I, \qquad a_I \in \mathbb{C},$$

as the result of applying the linear differential operator $P(\partial/\partial\xi)$ to the universal exponential.

Let $\alpha \in ZC(V, x)$ be a reduced conical chain relative to the covector $\xi_0 \in V^* \backslash \{0\}$. As above we set $U_\alpha = \{\xi \in V_{\mathbb{C}}^* \mid \alpha \text{ is reduced relative to } \operatorname{Re}\xi\}$. For $\xi \in U_\alpha$ we have $I(\alpha, \xi) = \int_V \alpha(x) \exp \xi(x)\, dx$, where the integral converges uniformly with respect to ξ on each compact subset in U_α. Therefore, on U_α we have

$$P\left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n}\right) I(\alpha, \xi) = \int_V \alpha(x) P(x_1, \dots, x_n) \exp \xi(x)\, dx.$$

What we have said carries over in an obvious way to lattice sums. Now from Proposition 1, using the uniqueness theorem for analytic functions, we obtain

Proposition 2. (A) *Let* $P(x)$ *be a polynomial on* V *with complex coefficients; the integral of the quasipolynomial* $P(x)\exp\xi(x)$ *with respect to conical chains reduced relative to* $\operatorname{Re}\xi$ *extends to a meromorphic-valued measure on the group of conical chains. More precisely, there exists a unique homomorphism of abelian groups (linear with respect to* P*)* $I(P)\colon ZC(V) \to \mathscr{M}(V_{\mathbb{C}}^*)$ *(the image of the chain* α *and its value on a covector* ξ *are denoted by* $I(P, \alpha)$ *and* $I(P, \alpha, \xi)$ *respectively) such that if the pointed cone* C *is reduced relative to* $\operatorname{Re}\xi$, *then*

$$I(P, \mathbb{1}_C, \xi) = \int_C P(x) \exp \xi(x)\, dx.$$

The mapping $I(P)$ *is identically zero on developed chains. Moreover,* $I(P, \alpha, \xi) = P(\partial/\partial\xi) I(\alpha, \xi)$, *and if* $\alpha \in Z(V)$ *is a chain with compact support, then* $I(P, \alpha) \in \mathscr{O}(V_{\mathbb{C}}^*)$ *is a global holomorphic function and*

$$I(P, \alpha, \xi) = \int_V \alpha(x) P(x) \exp \xi(x)\, dx.$$

In particular, $I(P, \alpha, 0)$ *is the "integral of the polynomial* P *over the chain* α".

(B) *The lattice sum of the quasipolynomial* $P(x)\exp\xi(x)$ *with respect to integral conical chains reduced relative to* $\operatorname{Re}\xi$ *extends to a meromorphic-valued measure on the group of integral conical chains: there exists a unique homomorphism* $S(P)\colon ZC(\Lambda) \to \mathscr{M}(V_{\mathbb{C}}^*)$ *of abelian groups, linear with respect to* P *(the image of the chain* α *and its value on a covector* ξ *are denoted by* $S(P, \alpha)$ *and* $S(P, \alpha, \xi)$ *respectively), such that if the pointed cone* $C \in C(\Lambda)$ *is reduced relative to* $\operatorname{Re}\xi$, *then*

$$S(P, \mathbb{1}_C, \xi) = \sum_{x \in C \cap \Lambda} P(x)\exp\xi(x).$$

The mapping $S(P)$ *is identically zero on developed chains. Moreover,* $S(P, \alpha, \xi) = P(\partial/\partial\xi)S(\alpha, \xi)$, *and if* $\alpha \in Z(\Lambda)$ *is a chain with compact support, then* $S(P, \alpha) \in \mathscr{O}(V_{\mathbb{C}}^*)$ *is a holomorphic function and*

$$S(P, \alpha, \xi) = \sum_{x \in \Lambda} \alpha(x)P(x)\exp\xi(x).$$

In particular, $S(P, \alpha, 0)$ *is the "lattice sum of the polynomial* P *over the chain* α".

Explicit formulas for the integrals and sums of quasipolynomials over convex chains, analogous to the formulas of §4.2, are obtained from the latter by applying the differential operator $P(\partial/\partial\xi)$ and, as a consequence, lend themselves badly to development. We note only some features of the computations. For a chain $\beta \in Z(V, \Sigma)$ in the notation of §4.2 we have

$$(-1)^n I(P, \beta, \xi) = \sum_{i \in I^*} P\left(\frac{\partial}{\partial\xi}[\exp\xi(\beta_i)\xi^{-1}(v_{i1})\cdots\xi^{-1}(v_{in})]\right)$$

(and correspondingly for sums).

Corollary 3. *If* $\beta = \beta(y_1, \dots, y_M)$ *depends linearly on the coordinates* (y), *then, for fixed* ξ, $I(P, \beta(y), \xi)$ *is a quasipolynomial of the form*

$$\sum_{a \in A} \exp[\gamma_a(y)]Q_a(y_1, \dots, y_M),$$

where the $\gamma_a(y)$ *are linear functions and the* $Q_a(y)$ *are polynomials of degree not exceeding*

$$\deg P + (n - \min\{\dim C_i^* \mid i \in I, \ \xi \in C_i^*\}).$$

The analogous result is also true for lattice sums.

Corollary 4. *Let* $P(x)$ *be a homogeneous polynomial of degree* p, *and* $\beta \in \mathscr{P}^*(V, \Sigma)$ *a virtual polytope. In the notation of* §4.2,

$$I(P, \beta, 0) = \sum_{i \in I^*} \frac{(-1)^n}{(n+p)!} P\left(\frac{\partial}{\partial\eta}\right) \frac{\eta(b_i)^{n+p}}{\eta(v_{i1})\cdots\eta(v_{in})}\bigg|_{\eta=\xi},$$

where $\xi \in V_{\mathbb{C}}^*$ *is a common covector. In particular, if* β *(i.e., all the* b_i, $i \in I^*$) *depends linearly on the coordinates* (y), *then the integral of the polynomial* P *over the virtual polytope* β *depends polynomially on* (y) *of degree* $\le n + p$.

We note that we have re-proved a special case of Proposition 2.5 of [1] for the finitely additive measure which is the integral of a polynomial over a virtual polytope.

§5. Proof of the Riemann-Roch theorem

1. We shall use the notation of the preceding sections without special reference. As was shown in §3.4, modulo $\widetilde{ZC}(V)$ the chain $\beta * \alpha(z_1, \dots, z_N)$ has the representation

$$\sum_{i\in I^*}(\beta_i * [w_i(z_1, \dots, z_N)]) * \mathbb{1}_{C_i},$$

where the vector $w_i(z_1, \dots, z_N) \in V$ is defined by the relations $l_{i_k}(w_i) = z_{i_k}$, $k = 1, \dots, n$; the l_{i_k} are the direction vectors of the edges of the cone C_i^*. As was shown in §4, for covectors $\xi \in V_{\mathbb{C}}^*$ lying outside the hyperplanes $(\xi \cdot v_{ik}) = 2\pi\sqrt{-1}m$, $m \in \mathbb{Z}$, we have the representations

$$I(P, \beta * \alpha(z), \xi) = (-1)^n P\left(\frac{\partial}{\partial \xi}\right) \sum_{i\in I^*} \frac{\exp\xi(\beta_i * [w_i])}{\xi(v_{i1})\cdots\xi(v_{in})}$$

and

$$S(P, \beta * \alpha(z), \xi) = \sum_{i\in I^*} P\left(\frac{\partial}{\partial \xi}\right) \frac{\exp\xi(\beta_i * [w_i])}{\prod_{k=1}^n(1 - \exp\xi(v_{ik}))},$$

the last of these for $\beta, \alpha \in Z(\Lambda, \Sigma)$. We know that the ith term in each of these sums depends only on $z_{i_1}, \dots, z_{i_n}$. On the other hand, developing $\beta_i^v \sum_{a\in A_i} m_{ia}[b_{ia}]$ and applying the results of §2, we obtain that for small ξ for each $i \in I^*$ the ith term of the above representation for $I(P, \beta * \alpha(z), \xi)$ admits the Todd operator $\mathrm{Td}(\partial/\partial z_{i_1}, \dots, \partial/\partial z_{i_n})$, and the result of this procedure for $(z_{i_1}, \dots, z_{i_n}) \in \mathbb{Z}^n$ is likewise the ith term of the representation for $S(P, \beta * \alpha(z), \xi)$. Using the properties of the Todd mapping (§1.3), we will obtain the Riemann-Roch theorem for complex covectors ξ outside some set of hyperplanes, which are sufficiently small in modulus. We formulate what we have proved as follows: there exists a small neighborhood of zero $U \subset V_{\mathbb{C}}^*$, not intersecting the affine hyperplanes $(\xi \cdot v_{ik}) = 2\pi\sqrt{-1}m$, $m \in \mathbb{Z}\backslash\{0\}$, such that if

$$X = U \cap \bigcup_{i\in I^*,\, 1\le k\le n} \{(\xi \cdot v_{ik}) = 0\},$$

then for $\xi \in U\backslash X$ and an arbitrary polynomial $P(x)$ for $f = P\exp\xi(x)$ the function $I_f(\beta * \alpha(z_1, \dots, z_N))\colon \mathbb{R}^N \to \mathbb{C}$ admits the Todd operator $\mathrm{Td}(\partial/\partial z)$, and the result for integer sets (z) is the function $S_f(\beta * \alpha(z_1, \dots, z_N))$. According to Lemma 1.1, the series realizing the action of the Todd operator converges uniformly on compact sets in $U\backslash X$. Furthermore, *its terms*

$$\tau_I \frac{\partial^{|I|}}{\partial z_I} I_f(\beta * \alpha(z))$$

are holomorphic functions of ξ *on* U *(and even on all of* $V_{\mathbb{C}}^*$*).*

Indeed, reducing the sum of fractions

$$\sum_{i\in I^*} \frac{\exp\xi(\beta_i * [w_i(z_1, \dots, z_N)])}{\xi(v_{i1})\cdots\xi(v_{in})}$$

to a common denominator, we obtain a function of the form $Q(z, \xi)/L(\xi)$, where $L(\xi)$ is a product of linear forms in ξ with constant coefficients, and the numerator has the series expansion

$$Q(z, \xi) = \sum_{p=0}^{\infty}\sum_{|I|=p} q_I(z_1, \dots, z_N)\xi^I = \sum_{p=0}^{\infty} Q_p(z, \xi),$$

where $\deg q_I \le |I| + M$, M is some constant, and the coefficients of the series for

$z^J\xi^I$ have order $(c/(|I|+|J|))^{|I|+|J|}$ as $|I|+|J| \to \infty$. Since $I(\beta*\alpha(z), \xi)$ is globally holomorphic with respect to ξ for any (z), we obviously have $L(\xi)|Q_p(z, \xi)$ for any $p \in \mathbb{Z}_+$. Hence $I(\beta * \alpha(z), \xi)$ expands in a power series in (z, ξ) with infinite radius of convergence, and hence the same is true for $I(P, \beta * \alpha(z), \xi)$, and we obtain our claim.

The sum $S_f(\beta * \alpha(z))$ of the Todd series is also holomorphic as a function of ξ for fixed $(z_1, \dots, z_N)$. From this, by elementary methods of complex analysis it is easy to deduce that $I_f(\beta * \alpha(z))$ admits $\mathrm{Td}(\partial/\partial z)$ everywhere on $U \ni \xi$, and the corresponding series converges uniformly on compact subsets of U (with respect to ξ), and its sum for $(z_1, \dots, z_N) \in \mathbb{Z}^N$ is $S_f(\beta * \alpha(z))$.

This completes the proof of the Riemann-Roch theorem.

2. **Concluding remarks.** Some themes that could naturally have been included remain beyond the scope of this paper. These include the algebro-geometric topic, only mentioned in §1, of describing relations in the group of lattice polytopes (see §2 of [1]), and the more detailed study of the integrals and lattice sums of polynomials over virtual polytopes. Moreover, similar to the way a holomorphic function can be expanded in a power series around any point at which it is defined, the series $\mathrm{Td}(\partial/\partial z)$ can be written at any point of holomorphy of the Todd mapping. This allows us to "analytically continue" the Todd operator so that the Riemann-Roch theorem will be true not for small $\xi \in V_{\mathbb{C}}^*$ but for $\xi \in V_{\mathbb{C}}^*\backslash X$, where X is the union of a countable set of complex hyperplanes, $0 \notin X$. These questions will be considered elsewhere. Here we touch upon another theme, linking this paper with its predecessor [1].

The basic motif of our arguments in this paper is the "decomposition" of concrete measures (integrals and lattice sums of quasipolynomials) over cones for the vertices of chains. The following question arises: can one do this for an arbitrary finitely additive measure φ that is polynomial relative to a translation? The answer is "yes" in some sense. We outline the main ideas. Let $\Sigma = \{C_i^* | i \in I\}$ be a partition of the space V^*. Then there exists a covector $\xi \in V^*\backslash\{0\}$ relative to which all the chains $\alpha \in Z(V, \Sigma)$ are reduced. Therefore for a measure of a chain α to be expanded in the sum of the measures of the chains $\{\alpha\}_x$ over all the vertices x of the chain α, it is sufficient to define a suitable measure on cones that are reduced relative to ξ. Let $H_{\xi,0}$ be a hyperplane defined by an equation $\xi(y) = 0$. Then for a cone C that is reduced relative to ξ, whose vertex lies above $H_{\xi,0}$, we set $\varphi(C) = \varphi(C \cap \{x|\xi(x) \geq 0\}$. It is not hard to verify that φ will be polynomial relative to translations of C by a vector h such that $\xi(h) + \xi(\mathrm{vs}(C)) \geq 0$. Hence, φ can be extended in a unique way, with polynomiality being preserved, to all cones of the form $h + C$, $h \in V$. This measure on reduced cones relative to ξ can be uniquely extended to the set of conical chains in general position with ξ, by requiring it to vanish on developed chains. The resulting measure is polynomial relative to translations, and solves the problem posed above for chains in general position with the covector ξ. This series of problems will be considered in more detail elsewhere.

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