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FINITELY ADDITIVE MEASURES OF VIRTUAL POLYTOPES

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ABSTRACT. This paper is devoted to generalizing the classical theory of finitely additive measures of convex polytopes. The technique of integrating over the Euler characteristic is used to obtain simple proofs of much stronger facts than have been known until now. The concept of a support function is generalized, and a geometric construction of "subtraction of polyhedra in the sense of Minkowski" is given.

INTRODUCTION

Finitely additive measures of convex polytopes is an old and trusty topic. Related to this topic are such problems as the equidecomposability of polyhedra [1], and such results as the polynomiality of the number of lattice points in integer polyhedra relative to their Minkowski sum [2]–[4]. The interest in these classical questions, going back to elementary geometry, has recently increased in connection with the algebro-geometric theory of toric varieties, in which they arose in a natural way (in another language) [5]. The parallelism considered in [5] between objects of convex geometry and algebro-geometric objects has enabled new simple proofs to be given of some old theorems. For example, a polytope with integer vertices can be set into correspondence with an invertible sheaf on some smooth toric variety, and the number of lattice points of this polytope is equal to the Euler characteristic of the variety with respect to cohomology with coefficients in this given sheaf. Via the Riemann-Roch theorem we immediately get from this the classical theorem on the polynomiality of the number of lattice points [5].

The "algebro-geometric" approach to the theory of convex polytopes motivates the posing of new questions in the framework of the classical theory. Here is one of them. Under the above-mentioned correspondence of polytopes to invertible sheaves the Minkowski sum becomes the tensor product. But the invertible sheaves on an algebraic variety form a group (the Picard group), while convex polytopes form only a semigroup with a unique subtraction (if it can be realized). This generates a natural problem: extend the semigroup of convex polyhedra to a group.

This paper is devoted to constructing a theory in whose framework this and other problems have a simple and geometrically transparent solution. We generalize and strengthen the results of the theory of finitely additive measures of convex polytopes [3], and at the same time we shorten their proofs in an essential way. A very useful tool in this has turned out to be the technique of integration over the Euler characteristic. We note that, although all the work can be done by purely elementary methods, without references to this apparatus, the very idea of such a "nonclassical" integration to a large extent clarifies the heart of the matter.

The definitions, propositions, lemmas, theorems, and corollaries are numbered independently in each section, and a reference to "Definition (Proposition, etc.) a.b"

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means a reference to Definition (...) b of §a; and "§a.b" has the analogous meaning. We do not include the number of the section when referring to an assertion or subsection of the same section.

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§1. Finitely additive measures of convex polytopes

In this section we give the original definitions, recall some classical concepts and constructions and list a number of well-known theorems about finitely additive measures.

1. **Definition 1.** A pair (V, Λ), where V is a finite-dimensional real space and Λ is an additive subgroup of V, is said to be *admissible* if there exists an isomorphism $\varphi: V \to \mathbb{R}^n$ such that $\varphi(\Lambda) = F^n$, where either $F \subset \mathbb{R}$ is a subfield or $F = \mathbb{Z} \subset \mathbb{R}$.

If we define a morphism of admissible pairs $\Phi: (V, \Lambda) \to (W, \Sigma)$ as a linear mapping $\Phi \in \mathrm{Hom}_{\mathbb{R}}(V, W)$ such that $\Phi(\Lambda) \subset \Sigma$, then the set of admissible pairs becomes a category. The pair dual to (V, Λ) is a pair (V^*, Λ^*), where $\Lambda^* \subset V^*$ is the set of linear functionals that take values in F on Λ. This pair is obviously admissible.

We denote by $\mathscr{P}(\Lambda)$ the set of bounded closed convex polytopes with vertices in Λ. We call the elements of $\mathscr{P}(\Lambda)$ *Λ-polytopes.* If $\Lambda \cong \mathbb{Z}^n$, then we speak of *lattice polytopes.*

2. Recall that for $A, B \in \mathscr{P}(V)$ their Minkowski sum is the polytope $A \oplus B = \{x + y \mid x \in A, y \in B\}$.

Definition 2. Let (V, Λ) be an admissible pair. A *finitely additive measure* on $\mathscr{P}(\Lambda)$ is a mapping $\varphi: \mathscr{P}(\Lambda) \to M$, where M is an abelian group that satisfies the following additivity property: if $A_1, \dots, A_N \in \mathscr{P}(\Lambda)$ are such that $\bigcup_{i=1}^N A_i \in \mathscr{P}(\Lambda)$, and $A_{i_1} \cap \dots \cap A_{i_k} \in \mathscr{P}(\Lambda)$ for any $i_1 < \dots < i_k$, then the following "inclusion-exclusion" relation holds:

$$\varphi\left(\bigcup_{i=1}^N A_i\right) = \sum_i \varphi(A_i) - \sum_{i<j} \varphi(A_i \cap A_j) + \cdots .$$

A finitely additive measure on $\mathscr{P}(\Lambda)$ is said to be *Λ-invariant* if it does not vary under translations of the polytope by $\lambda \in \Lambda$.

In what follows we shall speak simply about measures, omitting the words "finitely additive".

Example 1. (i) $\Lambda = V$. The measure is the volume relative to some Euclidean structure (obviously this is invariant relative to translations).

(ii) $V = \mathbb{R}^n$ and $\Lambda = \mathbb{Z}^n$. The measure of a lattice polytope A is $\#\{A \cap \mathbb{Z}^n\}$, the number of lattice points. This is obviously $\mathbb{Z}^n$-invariant.

The classical results concerning Λ-invariant measures are summed up in the following two theorems of McMullen [3].

Theorem 1. *Let $\varphi: \mathscr{P}(\Lambda) \to M$ be a Λ-invariant measure, and let $A_1, \dots, A_N \in \mathscr{P}(\Lambda)$. Then the M-valued function*

$$\varphi(n_1 A_1 \oplus n_2 A_2 \oplus \cdots \oplus n_N A_N)$$

on $(\mathbb{Z}_+)^N$ *is a polynomial with respect to the* n_i *of total degree* $\leq \dim V$ *(see Definition* 2.1, *below).*

Theorem 2. *Let* φ *be as above,* $A \in \mathscr{P}(\Lambda)$, *and* $\Gamma(A)$ *the set of all faces of* A *(including* A *itself). Let* $f(t)$ *be a polynomial such that* $f(n) = \varphi(nA)$ *for* $n \geq 0$ *(which exists by Theorem* 1). *Then*

$$f(-1) = \sum_{\Delta \in \Gamma(A)} (-1)^{\dim \Delta} \varphi(\Delta).$$

If we apply Theorem 2 to the measure from Example 1(ii), then we obtain, as is not hard to check, the following assertion ("Ehrhardt's duality theorem" [3]): $f(-1) = (-1)^{\dim A} \times$(number of interior lattice points in A). Here and later on, when we talk about the interior of a polytope we mean its interior in its affine hull.

3. We recall three well-known constructions of convex geometry, which we shall need later on.

(1) The *support function* of a polytope $A \in \mathscr{P}(V)$ is a piecewise-linear function $\sigma_A \colon V^* \to \mathbb{R}$,

$$\sigma_A \colon V^* \ni l \mapsto \sup\{l(x) \mid x \in A\}.$$

It is well known that a convex polytope $A \in \mathscr{P}(V)$ generates a decomposition of the dual space V^* into cones, dual to the cones for the vertices and faces of A. On these cones σ_A is linear, and, moreover, the cones dual to the cones for the vertices of A are maximal among the cones with this property. We note that if $A \subset W$, where $W \subset V$ is an affine subspace, then σ_A is linear on $\operatorname{Ann} \widetilde{W} \subset V^*$, where $\widetilde{W} \subset V$ is the "carrier" linear subspace for W.

If $A \in \mathscr{P}(\Lambda)$, then the "linear pieces" of σ_A are represented by the vertices of the polytope A (i.e. by elements of Λ as functions on $V^* \supset \Lambda^*$), and the cones of the decomposition of V^* generated by A are spanned by the elements of Λ^*.

(2) The *trace of a polytope* $A \in \mathscr{P}(V)$ relative to a covector $0 \neq l \in V^*$ is the polytope

$$\operatorname{Tr}_l A = \{x \in A \mid l(x) = \sigma_A(l)\}.$$

We note that if $A \in \mathscr{P}(\Lambda)$, then we also have $\operatorname{Tr}_l A \in \mathscr{P}(\Lambda)$ for any l, and it is obvious that $\operatorname{Tr}_{l_1} A = \operatorname{Tr}_{l_2} A$ if $l_1 = \alpha l_2$, where $\alpha > 0$.

(3) *The operation* $*$. For a measure φ on $\mathscr{P}(\Lambda)$ we set

$$\varphi^*(A) = \sum_{\Delta \in \Gamma(A)} (-1)^{\dim \Delta} \varphi(\Delta).$$

It is classical and easily verified fact about the operator $*$ that φ^* is again a measure, and that $\varphi^{**} = \varphi$. Theorem 2, stated above, shows the importance of this operation.

§2. The algebra of convex chains

In this section we give a noninvariant definition of the algebra of convex chains and discuss its basic properties. Invariant constructions and proofs of the key facts are contained in §§3–5.

1. The classical Theorems 1 and 2 of §1 can be generalized in different directions: a wider class of measures can be admitted and we can consider not only Minkowski summation of polytopes, but also "motion of the walls".

Definition 1. (A) A mapping $h \colon N \to L$ of abelian groups is called a *polynomial of degree* $\leq m$ if one of the following two conditions holds:

(i) $m = 0$ and h is a constant mapping, $h(N) = l \in L$;

(ii) $m \geq 1$ and for any $a \in N$ the mapping $h_a\colon N \to L$, $h_a\colon x \mapsto h(x+a) - h(x)$, is a polynomial of degree $\leq m-1$.

(B) A measure $\varphi\colon \mathscr{P}(\Lambda) \to M$ is said to be *polynomial of degree* $\leq m$ if for each $A \in \mathscr{P}(\Lambda)$ the function $\varphi(A+\lambda)\colon \Lambda \to M$ is a polynomial of degree $\leq m$.

Remarks. (i) Part (A) of Definition 1 can be reformulated as follows: for any $a_1, \dots, a_{m+1} \in N$

$$\sum_{\substack{0 \leq i \leq m+1 \\ 1 \leq k_1 < \cdots < k_i \leq m+1}} (-1)^i h(x + a_{k_1} + \cdots + a_{k_i}) \equiv 0.$$

(ii) If $h\colon N \to L$ is a polynomial of degree $\leq m$, then for any $a_1, \dots, a_k \in N$ there exist $b_1, \dots, b_r \in L$ and integer polynomials (i.e., polynomials that realize a mapping $\mathbb{Z}^k \to \mathbb{Z}$) $p_i(n_1, \dots, n_k)$, $1 \leq i \leq r$, of degree $\leq m$ such that

$$h\left(\sum_{j=1}^{k} n_j a_j\right) = \sum_{i=1}^{r} b_i p_i(n_1, \dots, n_k).$$

(iii) Polynomial measures of degree 0 are simply invariant measures.

Example 1. (i) $\Lambda = V$, and the measure φ is the integral of a polynomial $f\colon V \to \mathbb{R}$ over the polytope. Obviously φ is polynomial of degree $\leq \deg f$.

(ii) Λ is a discrete lattice, $f\colon V \to \mathbb{R}$ is a polynomial, and

$$\varphi(A) = \sum_{x \in A \cap \Lambda} f(x).$$

Again φ is polynomial of degree $\leq \deg f$ (as a measure on $\mathscr{P}(\Lambda)$).

Let $A \in \mathscr{P}(V)$, with $\langle A \rangle = W$, $\widetilde{W}$ the "carrier" linear subspace for W, and let $\{R_1, \dots, R_m\}$ be a set of rays in V^* that define cooriented linear hyperplanes of the edges of maximal dimension of the polytope A. In other words, let $0 \neq l_i \in R_i$; then

$$A = \{x \in W \mid l_i(x) \leq c_i\},$$

and none of these inequalities is redundant.

Definition 2. A polytope $A' \in \mathscr{P}(V)$ is obtained from A by a *motion of the walls* if its support function is linear on the cones of the decomposition of V^* generated by A (see §1.3).

Obviously, a polytope A' that is obtained from A by a motion of the walls is uniquely determined by the set of numbers $\xi_i = \sigma_{A'}(l_i)$ and the linear function $\sigma_{A'}|_{\operatorname{Ann}\widetilde{W}}$. If $A \in \mathscr{P}(\Lambda)$, then $\operatorname{Ann}\widetilde{W}$ is spanned by covectors from Λ^*; one can choose $l_i \in \Lambda^*$, and then $\xi_i \in F$ (recall that $F \subset \mathbb{R}$ is a subfield or $\mathbb{Z}$).

Now Theorem 1.1 can be generalized as follows.

Theorem 1. *For any polytope $A \in \mathscr{P}(\Lambda)$ and any measure $\varphi\colon \mathscr{P}(\Lambda) \to M$ which is polynomial of degree $\leq k$, there exists a polynomial h (in the above notations) of degree $\leq k + \dim V$,*

$$h\colon \bigoplus_{i=1}^{m} F_{\xi_i} \oplus \Lambda/(\Lambda \cap \widetilde{W}) \to M,$$

such that, for any $A' \in \mathscr{P}(\Lambda)$ obtained from A by a motion of the walls,

$$h((\xi_i = \sigma_{A'}(l_i),\ 1 \leq i \leq m),\ \sigma_{A'}|_{\operatorname{Ann}\widetilde{W}}) = \varphi(A').$$

It is not hard to see that Theorem 1.1 follows from Theorem 1, because the polytope $n_1A_1 \oplus \cdots \oplus n_kA_k$ for $n_i \geq 0$ is obtained from $A_1 \oplus \cdots \oplus A_k$ by a motion of the walls, and the parameters ξ_i and $\sigma_{\sum n_jA_j}|_{\operatorname{Ann}\widetilde{W}}$ depend linearly on n_j.

However we shall prove another, more general, fact, from which Theorem 1 follows automatically. Namely, we shall prove, in particular, that the measure φ can be extended in a natural way to objects of a wider class than convex polytopes, and will be polynomials relative to an additive structure on the set of these objects. Moreover, assertions of the type of Theorem 1.2 receive a transparent geometric interpretation; in particular, we attach a geometric meaning to formal sums $n_1A_1 \oplus \cdots \oplus n_kA_k$, where the n_i are arbitrary integers (including negative ones). We shall call such objects *virtual polytopes.* In the framework of our theory Theorems 1.1 and 1.2 are generalized by a single assertion about the polynomiality (of degree $\leq \dim V + m$) of a polynomial (of degree $\leq m$) measure relative to Minkowski summation of virtual polytopes.

2. **Definition of the algebra of convex chains.** Let (V, Λ) be an admissible pair.

Definition 4. A *convex* Λ*-chain* is a function $\alpha\colon V \to \mathbb{Z}$ of the form $\alpha = \sum_{i=1}^k n_i \mathbb{1}_{A_i}$, where $A_i \in \mathscr{P}(\Lambda)$, $\mathbb{1}_Y$ is the characteristic function of the set Y, and $n_i \in \mathbb{Z}$. We denote the additive group of convex Λ-chains by $Z(\Lambda)$; in particular, if $\Lambda = V$, then $Z(\Lambda) = Z(V)$, and in this case we speak simply of convex chains.

To keep the notation simple, we shall carry out the presentation for the case $\Lambda = V$. The naturality of the theory allows us to obtain both assertions and proofs in the general case from this by replacing V everywhere by (V, Λ) or Λ, by replacing the linear mapping $f\colon V \to W$ by a morphism of pairs $(V, \Lambda) \to (W, \Sigma)$, V^* by (V^*, Λ^*) or Λ^*, etc. These changes are obvious each time; sometimes we shall nevertheless make explanatory remarks. Finally, for brevity we shall omit the word "convex" and speak simply about chains.

Proposition-Definition 1. *For any chain* $\alpha = \sum n_i \mathbb{1}_{A_i}$, $A_i \in \mathscr{P}(V)$, *the number* $\sum n_i$ *depends only on* α *and is called the degree of the chain* α. *Notation*: $\deg\alpha$.

The proof is given in §3.

Obviously $\deg\colon Z(V) \to \mathbb{Z}$ is a homomorphism of additive groups.

Let $f\colon V \to W$ be a linear mapping of vector spaces.

Proposition-Definition 2. *For any chain* $\alpha = \sum n_i \mathbb{1}_{A_i} \in Z(V)$ *the chain* $\sum n_i \mathbb{1}_{f(A_i)} \in Z(W)$ *depends only on* α. *This homomorphism of additive groups is called the* ***direct image homomorphism*** *and is denoted by* f_*. *Moreover,* $\deg f_*\alpha = \deg\alpha$.

The proof is given in §4.

Proposition-Definition 3. *Minkowski summation of polytopes extends in a unique way to a bilinear operation on the group of convex chains*

$$*\colon Z(V) \times Z(V) \to Z(V),$$
$$\mathbb{1}_A * \mathbb{1}_B = \mathbb{1}_{A\oplus B} \quad \text{for } A, B \in \mathscr{P}(V),$$

which is called (***Minkowski***) ***multiplication of chains****. The group* $Z(V)$ *with this multiplication* (*which is commutative and associative*) *is called the* ***algebra of convex chains****. The direct image* f_* *and degree mapping* deg *are ring homomorphisms.*

The proof is given in §4.

Obviously $\mathbb{1}_{\{0\}}$ functions as an identity element of the algebra $Z(V)$. Minkowski multiplication also admits the following description: let $\mu\colon V \times V \to V$ be the summation mapping, $\alpha, \beta \in Z(V)$, and $\alpha \times \beta(x, y) = \alpha(x)\beta(y)$ the direct product of the chains α and β. Then $\alpha * \beta = \mu_*(\alpha \times \beta)$.

3. **Zero-dimensional and one-dimensional chains.** As usual, the support $\operatorname{Supp}\alpha$ of a chain α is the closure of the set $\{x \in V \mid \alpha(x) \neq 0\}$, and the dimension of a chain is the dimension of its support. Obviously, together with 0 the zero-dimensional chains form a subalgebra of $Z(V)$, which is isomorphic to the group algebra $\mathbb{Z}[V]$ (if $\Lambda \neq V$, then to $\mathbb{Z}[\Lambda]$). The elements of $\mathbb{Z}[V]$ are written as $\sum n_i[x_i]$, $x_i \in V$.

We consider chains on the line in more detail: suppose $\dim V = 1$. The chains on V have the form $\sum n_i \mathbb{1}_{A_i}$, where the A_i are closed intervals or points. We choose an orientation on V, i.e., we fix one of the two connected components of $V \backslash \{0\}$. Now for each interval we distinguish its right (left) end.

Proposition 4. *The association of its right (left) end to each interval,* $\mathbb{1}_{[a,b]} \mapsto [b]$, ($\mathbb{1}_{[a,b]} \mapsto [a]$) *extends uniquely to a homomorphism of algebras*

$$\sup(\inf)\colon Z(V) \to \mathbb{Z}[\mathbb{R}],$$

which commutes with deg. *There is a relation*

$$\inf = (-1)_* \circ \sup \circ (-1)_*.$$

Moreover, the following sequence is exact:

$$0 \to Z(V) \xrightarrow{(\sup, \inf)} \mathbb{Z}[\mathbb{R}] \oplus \mathbb{Z}[\mathbb{R}] \xrightarrow{(\deg, -\deg)} \mathbb{Z} \to 0.$$

The proof is obvious.

We note a simple method of recovering a chain α from its sup and its inf. Suppose $\sup\alpha = \sum n_i[x_i]$ and $\inf\alpha = \sum m_j[y_j]$. Now

$$\alpha = \sum n_i \mathbb{1}_{(-\infty, x_i]} + \sum m_j \mathbb{1}_{[y_j, \infty)} - \deg\alpha \mathbb{1}_{\mathbb{R}}.$$

4. **Support functions of convex chains.** The construction of a support function (§1.3) can be extended to convex chains by linearity.

Definition 4. A function $f\colon W \to \mathbb{Z}[\mathbb{R}]$, where W is a linear space, is a *piecewise-linear* positive support function if there exists a set of piecewise-linear positive support functions in the usual sense $f_i\colon W \to \mathbb{R}$ and numbers $n_i \in \mathbb{Z}$ such that $f(x) = \sum n_i[f_i(x)]$ for any $x \in W$.

Definition 5. A *support function* of a convex chain $\alpha \in Z(V)$ is a function

$$\Phi_\alpha\colon V^* \to \mathbb{Z}[\mathbb{R}], \qquad \Phi_\alpha\colon l \mapsto \sup \circ l_*\alpha \in \mathbb{Z}[\mathbb{R}],$$

where $l \in V^*$ is considered as a mapping of linear spaces $l\colon V \to \mathbb{R}$.

Explicitly, if $\alpha = \sum n_i \mathbb{1}_{A_i}$, then

$$\Phi_\alpha(l) = \sum n_i[\sigma_{A_i}(l)].$$

(If $\Lambda \neq V$, then in the definition of piecewise-linearity we must consider $\mathbb{Z}[F]$-valued functions on Λ^*.) Obviously Φ_α is piecewise-linear.

Proposition 5. *The association of a chain to its support function,* $Z(V) \ni \alpha \mapsto \Phi_\alpha$, *is an algebra isomorphism, commuting with* deg, *of* $Z(V)$ *and the algebra of all piecewise-linear functions from* V^* *into* $\mathbb{Z}[\mathbb{R}]$.

Proof. It is obvious that this association is a homomorphism and that it commutes with deg. The surjectivity follows from the well-known fact that a piecewise-linear function is the difference of two piecewise-linear convex functions, i.e., the support functions of convex polytopes, and from Proposition-Definition 6 and Theorem 2 below. The injectivity will be established in §4.

Corollary 1. *Let* $A_1, \dots, A_n$ *and* $B_1, \dots, B_m$ *be convex polytopes. Then* $\sum_{i=1}^n \mathbb{1}_{A_i} = \sum_{j=1}^m \mathbb{1}_{B_j}$ *if and only if, for each covector* $l \in V^*$*, the sets of numbers* $(\sigma_{A_1}(l), \dots, \sigma_{A_n}(l))$ *and* $(\sigma_{B_1}(l), \dots, \sigma_{B_m}(l))$ *differ only by a permutation.*

5. **Operations on convex chains.** The classical construction of §1.3 carry over to arbitrary chains. We consider three operations: "star", "trace", and "shadow".

The operation "star" in §1.3 was defined for finitely additive measures; however, it can be realized on the level of chains. As usual, for $A \in \mathscr{P}(V)$ we set $\Gamma(A) =$ {the set of all faces of A, including A}.

Proposition-Definition 6. *The association*

$$\mathbb{1}_A \mapsto (-1)^{\dim A}\mathbb{1}_{\operatorname{Int} A} = \sum_{\Delta \in \Gamma(A)} (-1)^{\dim \Delta}\mathbb{1}_\Delta$$

extends to an involutive automorphism of the additive group of convex chains, denoted $*: \alpha \mapsto *\alpha$.

The proof is given in §4, but we mention here that the equality

$$(-1)^{\dim A}\mathbb{1}_{\operatorname{Int} A} = \sum_{\Delta \in \Gamma(A)} (-1)^{\dim \Delta}\mathbb{1}_\Delta$$

can be verified directly.

Proposition-Definition 7. *The operation of taking the trace relative to a covector* $\xi \in V^* \backslash \{0\}$, $\operatorname{Tr}_\xi: A \mapsto \operatorname{Tr}_\xi A$, *extends by linearity to a ring homomorphism* $\operatorname{Tr}_\xi: Z(V) \to Z(V)$.

The proof is given in §4. Note that the relation

$$\operatorname{Tr}_\xi(A \oplus B) = \operatorname{Tr}_\xi A \oplus \operatorname{Tr}_\xi B$$

(multiplicative homomorphy) is obvious.

It is not hard to verify that the trace and star commute with deg: $\deg = \deg \circ \operatorname{Tr}_\xi$ and $\deg = \deg \circ *$.

Proposition-Definition 8. *The* ***shadow*** *of a polytope* $A \in \mathscr{P}(V)$ *in the direction of the vector* $v \neq 0$ *is the set*

$$T_v(A) = \{x \in A | x + tv \notin A \text{ for } t > 0\}.$$

Obviously $T_v(A) \in Z(V)$. *The operation of taking the shadow extends to a linear degree-preserving endomorphism of* $Z(V)$.

The proof is given in §4.

We note that all three of these operations—star, trace, and shadow—map the subgroup of Λ-chains into itself.

6. **The group of virtual polytopes.** The operation "star" has an interesting application, explaining Theorem 1.2 (the "duality theorem" for finitely additive measures). It turns out that it realizes "Minkowski inversion" of convex polytopes.

Theorem 2 (Minkowski inversion). *For* $A \in \mathscr{P}(V)$,

$$(-1)^{\dim A}\mathbb{1}_{\operatorname{Int}(-A)} * \mathbb{1}_A = \mathbb{1},$$

where $\mathbb{1} = \mathbb{1}_{\{0\}}$ *is the identity of the ring* $Z(V)$.

The proof will be given in §4.

Definition 7. An invertible element $\alpha \in Z(V)$ of degree 1 is called a *virtual polytope*. We denote the multiplicative group (relative to Minkowski multiplication) of virtual polytopes by $\mathscr{P}^*(V)$ (resp. $\mathscr{P}^*(\Lambda)$).

Corollary 2 (of Theorem 2). *$\mathscr{P}^*(V)$ is generated by the characteristic functions of polytopes $\mathbb{1}_A$, $A \in \mathscr{P}(V)$. In fact, the imbedding $\mathscr{P}(V) \subset \mathscr{P}^*(V)$ is a canonical extension of a commutative semigroup with identity and unique division to a group.*

Proof. Let $\alpha \in \mathscr{P}^*(V)$. We consider the corresponding support function Φ_α. Obviously, for any $l \in V^*$ we have $\Phi_\alpha(l) \in \mathbb{Z}[\mathbb{R}]^*$ and $\deg \Phi_\alpha(l) = 1$. But $\gamma \in \mathbb{Z}[\mathbb{R}]$ is invertible if and only if $\gamma = \pm[r]$ for $r \in \mathbb{R}$. Hence, $\Phi_\alpha = [f]$, where $f: V^* \to \mathbb{R}$ is a piecewise-linear function. As was noted above, f is the difference of support functions of polytopes, $f = \sigma_A - \sigma_B$, $A, B \in \mathscr{P}(V)$. But then we will obtain $\alpha = \mathbb{1}_A * (\mathbb{1}_B)^{-1}$, as required.

7. **Finitely additive measures of convex chains.**

Definition 8. A *finitely additive measure* on $Z(V)$ is an arbitrary homomorphism of additive groups $\varphi: Z(V) \to M$. (Analogously for $Z(\Lambda)$.) A measure is *invariant* (resp. *polynomial of degree* $\le m$) (relative to translations) if for each $\alpha \in Z(V)$ the function $\varphi(\tau_\lambda \alpha)$, $\lambda \in \Lambda$, does not depend (resp. depends polynomially of degree $\le m$) on $\lambda \in \Lambda$, where $\tau_h: Z(V) \to Z(V)$ is the operator of translation by a vector $h \in V$, $\tau_h \alpha(x) = \alpha(x - h)$.

In the last case we may assume, for example, that M is an F-module and $\varphi(\tau_\lambda \alpha): \Lambda \cong F^{\dim V} \to M_F$ is a polynomial in the usual sense.

Thus, we have two definitions of finitely additive measures, Definitions 1.2 and 2.7. It is understood that any measure on $Z(V)$ (or on $Z(\Lambda)$), bounded on $\mathscr{P}(V)$ (resp. on $\mathscr{P}(\Lambda)$), defines a measure in the sense of the first definition—the characteristic functions of the sets satisfy the inclusion-exclusion relation. Furthermore, a measure on $Z(\Lambda)$ is uniquely determined by restriction to $\mathscr{P}(\Lambda)$. Naturally the question arises: Does any measure extend from $\mathscr{P}(\Lambda)$ to $Z(\Lambda)$? The answer is affirmative, and Definitions 1.2 and 2.7 are thus essentially equivalent. For the case when $F \subset \mathbb{R}$ is a subfield this is not complicated to prove. For the case when $\Lambda \subset V$ is a discrete lattice, this is a nontrivial fact, which can be interpreted in the following way. Let $\mathbb{Z}\langle\mathscr{P}(\Lambda)\rangle$ be the free abelian group generated by the set of all polytopes with vertices in Λ. Then the kernel of the natural homomorphism $\pi: \mathbb{Z}\langle\mathscr{P}(\Lambda)\rangle \to Z(\Lambda)$, $\pi: \mathscr{P}(\Lambda) \ni A \mapsto \mathbb{1}_A$, is generated by the inclusion-exclusion relations (§1.2). The authors were able to prove this fact; the corresponding argument will be published elsewhere. It rests on the technique of conic representations of convex chains.

The algebra of convex chains was introduced by the authors as a "universal measure" $\varphi: \mathscr{P}(V) \to Z(V)$, $\varphi: A \mapsto \mathbb{1}_A$. What can be proposed as a "universal polynomial measure"? Let $J_k \subset Z(V)$ be the subgroup generated by chains of the form

$$(\tau_{h_1} - 1) \circ \cdots \circ (\tau_{h_k} - 1)(\alpha)$$

for all possible $(h_1, \ldots, h_k) \in V^{\times k}$ and $\alpha \in Z(V)$. We also define $\mathscr{L} \subset Z(V)$ to be the ideal of chains of degree 0, $\mathscr{L} = \operatorname{Ker} \deg$, and $\mathscr{M} \subset \mathbb{Z}[V]$ the ideal (in $\mathbb{Z}[V]$) of zero-dimensional chains of degree 0.

Proposition 9. *$J_k = \mathscr{M}^k Z(V)$. In particular, $\mathscr{L} \supset J_k$ is an ideal.*

Proof. $(\tau_h - 1)\alpha = ([h] - [0]) * \alpha$. ●

The measures that are polynomial of degree $\le k$ obviously vanish on J_{k+1}, and conversely. Hence the ring homomorphism $\pi_k: Z(V) \to Z(V)/J_{k+1}$ can be considered as a universal polynomial measure of degree $\le k$. We shall study it.

The central fact of our theory is

Theorem 3 (concerning ideals in the algebra of convex chains). *For* $k \geq 1$,

$$\mathscr{L}^{\dim V + k} \subset J_k.$$

(*Analogously for any admissible pair* (V, Λ).)

We devote §5 to the proof of this theorem.

Remark. $\bigcap_{k=1}^{\infty} J_k = \{0\}$, i.e. every nonzero chain $\alpha \notin J_k$ for sufficiently large k. This is almost obvious: for a chain $\alpha \neq 0$ it is sufficient to construct a polynomial measure φ such that $\varphi(\alpha) \neq 0$. Multiplication of α by an invertible element can be added to the fact that $\dim \operatorname{Supp} \alpha = \dim V$. Suppose the smooth function $\tilde{f}$ is chosen so that $\int_V \alpha(x) \tilde{f}(x)\, dx \neq 0$. Now if the polynomial f approximates $\tilde{f}$ sufficiently well in a bounded domain containing $\operatorname{Supp} \alpha$, then the polynomial measure $\varphi(\beta) = \int_V \beta(x) f(x)\, dx$ possesses the desired property.

Corollary 3 (of Theorem 3). *In the quotient ring* $Z(V)/J_k$ *the ideal of elements of degree zero is nilpotent of degree* $(\dim V + k)$.

Corollary 4. *Let* $\alpha_1, \dots, \alpha_r \in Z(V)/J_k$ *be classes of degree* 1, *and set* $\tilde{\alpha}_j = \alpha_j - 1$ *and* $\deg \tilde{\alpha}_j = 0$. *Then for arbitrary integers* $m_1, \dots, m_r \in \mathbb{Z}$ *the following equality holds in* $Z(V)/J_k$:

$$\alpha_1^{*m_1} * \cdots * \alpha_r^{*m_r} = \sum_{\substack{n_1, \dots, n_r \in \mathbb{Z}_+ \\ \sum_1^r n_j \leq \dim V + k - 1}} \prod_{j=1}^{r} \frac{m_j(m_j - 1) \cdots (m_j - n_j + 1)}{n_j!} \tilde{\alpha}_1^{*n_1} * \cdots * \tilde{\alpha}_r^{*n_r}.$$

Corollary 4 is obvious in view of the previous corollary.

When we consider that virtual polytopes have degree 1, we obtain

Corollary 5. *A finitely additive measure* $\varphi \colon Z(V) \to M$, *polynomial of degree* $\leq k$ *relative to translations, restricted to the group of virtual polytopes* $\varphi \colon \mathscr{P}^*(V) \to M$, *is a polynomial of degree* $\leq \dim V + k$.

The last assertion contains Theorem 1 concerning the polynomiality of the measure for a motion of the walls and, in particular, all of the classical theory of finitely additive measures.

All of the presentation carries over without changes to the case of arbitrary admissible pairs (V, Λ). As an application we obtain

Corollary 6. *Let* $\varphi \colon Z(V) \to M$ *be a finitely additive measure that is polynomial of degree* $\leq k$, *where* $\Lambda \subset V$ *is a full discrete lattice and* M *has the structure of a* $\mathbb{Q}$-*vector space. Then* φ *extends uniquely to a finitely additive measure* $\overline{\varphi} \colon Z(\overline{\Lambda}) \to M$, *polynomial of degree* $\leq k$, *where* $\overline{\Lambda} = \Lambda\mathbb{Q} \supset \Lambda$ *and* $\overline{\varphi}|_{Z(\Lambda)} = \varphi$.

Proof. We realize Λ as the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Let $A \in \mathscr{P}(\mathbb{R}^n)$ be a polytope with rational vertices (i.e., $A \in \mathscr{P}(\mathbb{Q}^n)$). Then $NA \in \mathscr{P}(\mathbb{Z}^n)$ for some $0 \neq N \in \mathbb{Z}_+$. But $\varphi(nNA) = p(n)$ for $n \in \mathbb{Z}_+$, where $p(t)$ is a polynomial. Set $\overline{\varphi}(A) = p(1/N)$. It is easy to see that $\overline{\varphi}$ is well defined and is a finitely additive measure on $Z(\mathbb{Q}^n)$, polynomial of degree $\leq k$, extending φ. This proves the corollary.

The following three sections are technical in nature and contain the proofs of the key assertions of the theory of convex chains that we have omitted so far.

§3. Integral with respect to the Euler characteristic

The idea of integrating with respect to the Euler characteristic has very old roots and is implicitly present in the proofs of many classical theorems (for example, see the proof of the Riemann-Hurwitz theorem in [6], Chapter II, §1). In a modern form this theory was developed by Viro [7]. Here we give the necessary definitions, facts, and constructions in a form that we need (simplified). This section does not contain new results and is included only for the convenience of the reader.

1. **Definition 1.** Let X be a topological space.

(A) If X is compact, then a *regular cellular structure* on X is the structure of a finite cell complex

$$X = \bigcup_{q\in\mathbb{Z}_+} \bigcup_{i\in I_q} e_i^q$$

with the following additional properties:

(i) The characteristic mapping of any cell is a homeomorphism of the closed ball onto its closure.

(ii) The boundary of any cell splits into a union of cells of smaller dimension.

Sets that can be represented as a union of cells are called *cellular.*

(B) If X is arbitrary, then a *regular cellular structure* on X is the following set of data:

(i) a dense imbedding $X \subset \widetilde{X}$, where $\widetilde{X}$ is compact, and

(ii) a regular cellular structure on $\widetilde{X}$, relative to which X is an open (dense) cellular subset.

Cellular subsets of X are defined in the obvious way.

(C) On the algebra of cellular subsets of a space X with a regular cellular structure we define a finitely additive measure χ by setting $\chi(e) = (-1)^{\dim e}$ for an open cell $e \subset X$. This measure is called the *Euler characteristic.*

This definition of the Euler characteristic depends on an arbitrariness in the choice of a regular cellular structure. However, this arbitrariness is formal.

Proposition 1. *Let $Z \subset X$ be a subset whose closure $\overline{Z}$ is compact, and on X let there exist at least one regular cellular structure relative to which Z is cellular. Then the Euler characteristic $\chi(Z)$ is defined and does not depend on the choice of this structure.*

Sketch of the proof. First suppose that Z is compact. If Z is cellular, then it is not hard to see that $\chi(Z)$ in the sense of Definition 1 is $\sum_{i=0}^{\infty}(-1)^i \dim H^i(Z, \mathbb{R})$ and therefore does not depend on the choice of a regular cellular structure. If Z is not necessarily compact, then we construct a series of cellular sets (relative to some fixed regular structure) $Z^{(n)}$, $n \in \mathbb{Z}_+$: $Z^{(0)} = Z$, $Z^{(i+1)} = \overline{Z^{(i)}}\backslash Z^{(i)}$. The $\overline{Z^{(i)}}$ are obviously compact, and $Z^{(i)} = \varnothing$ for $i \gg 0$ (because $Z^{(i+1)}$ consists of cells of smaller dimension than the maximum of the dimensions of the cells that make up $Z^{(i)}$). From this we will find that $\chi(Z) = \sum_{i=0}^{\infty}(-1)^i \chi(\overline{Z}^{(i)})$, and the invariance of χ has already been proved for $\overline{Z}^{(i)}$.

Definition 2. Let X be a topological space with a regular cellular structure, A an abelian group. A function $f\colon X \to A$ is said to be *cellular* if $f^{-1}(a)$ is a cellular set for any $a \in A$ (in particular, f is finite-valued), and its *integral with respect to the Euler characteristic* is

$$\int_X f\,d\chi := \sum_{a\in A} \chi(f^{-1}(a)), \qquad a \in A.$$

Definition 3. A function $f\colon X \to A$, where X is a topological space and A is an abelian group, is said to be *admissible* if it is cellular relative to some regular cellular structure on X.

Corollary 1 (of Proposition 1). *If $f: X \to A$ is an admissible function with compact support, then its integral with respect to the Euler characteristic $\int_X f\,d\chi$ does not depend on the choice of a regular cellular structure on X.*

2. Let $f: X \to A$ be an admissible function, $\Phi: X \to Y$ a continuous mapping of topological spaces. If a regular cellular structure is given on each fiber $X_y = \Phi^{-1}(y)$, $y \in Y$, relative to which $f_y = f|_{X_y}$ is cellular, then the "direct image" of f is defined: $\Phi_* f: Y \to A$, $\Phi_* f(y) = \int_{X_y} f\,d\chi$. This operation will be well defined if $\Phi_* f$, the result of "integration over the fibers", is again admissible. We give sufficient conditions for this.

Definition 4. (A) A continuous mapping $\Phi: X \to Y$ of spaces with a regular cellular structure is said to be *cellular* if Φ is an epimorphic map of each cell $e \subset X$ onto some cell $h \subset Y$.

(B) The *Cartesian product* of spaces with a regular cellular structure is defined by the multiplication of cells.

(C) Let $\Phi: X \to Y$ be a continuous mapping of spaces with a regular cellular structure. The following set of data is called a *bundle structure* for Φ: for each cell $e \subset Y$ a space F_e with a regular cellular structure and a homeomorphism $\Phi_e: \Phi^{-1}(e) \to F_e \times e$ such that Φ_e and Φ_e^{-1} are cellular.

(D) A cellular function $f: X \to A$ is compatible with a bundle structure for Φ if, for each cell $e \subset Y$, there exists a cellular function $f_e: F_e \to A$ such that $f|_{\Phi^{-1}(e)} = f_e \circ \mathrm{pr}_1 \circ \Phi_e$.

Proposition 2 ("Fubini's Theorem"). *In the situation described in parts* (C) *and* (D) *of the preceding definition, the function $\Phi_* f: Y \to A$,*

$$\Phi_* f(y) = \int_{\Phi^{-1}(y)} f\,d\chi = \int_{F_e} f_e\,d\chi, \qquad y \in e \subset Y,$$

is cellular and $\int_Y \Phi_* f\,d\chi = \int_X f\,d\chi$.

In other words, a function can be integrated first over the fibers of the mapping, and then the result of this operation, a function on the base, integrated over the base.

The proof of Proposition 2 is obvious.

We see that the "direct image" operation is connected with the concrete bundle structure only by its realizability, but the result of this operation does not depend on this structure (Proposition 1).

Definition 5. We say that an admissible function $f: X \to A$ is *compatible* with a bundle $\Phi: X \to Y$ if there exists a bundle structure for Φ with which f is compatible.

3. **Radon transform for an integral over the Euler characteristic.** Let $X = \mathbb{R}P^n$, and let $X^* = \mathbb{R}P^{n*}$ be the dual projective space (i.e., the points of X^* are hyperplanes in X); let $Z \subset X \times X^*$ be the graph of the incidence relation $\{(x, h) | \, x \in h\}$. We say that an admissible function $f: X \to A$ *admits a Radon transform* if the function $\mathrm{res}_Z \circ \mathrm{pr}_1^*(f): Z \to A$ is compatible with the bundle $\mathrm{pr}_2: Z \to X^*$. If this is so, its *Radon transform* is a function $f^*: X^* \to A$,

$$f^* = (\mathrm{pr}_2)_* \circ \mathrm{res}_Z \circ \mathrm{pr}_1^*(f),$$

i.e., $f^*(h) = \int_h f\,d\chi$ for a hyperplane $h \subset X$.

Theorem 1. *If f admits a Radon transform and f^* does too, then*

$$f^{**} + f\int_X f\,d\chi = \int_{X^*} f^*\,d\chi \quad \textit{for even } n = \dim X,$$
$$f^{**} = f \quad \textit{for odd } n = \dim X.$$

In particular, a function can be recovered from its Radon transform.

Sketch of the proof. For $x \in X$ we set $W_x = \{(y, h) \in X \times X^* \mid y \in h \ni x\}$. Obviously,

$$f^{**}(x) = \int_{\{h\in X^* | x\in h\}} f^*(h)\,d\chi(h)$$
$$= \int_{\{h\in X^* | x\in h\}} \int_{\{y\in X | y\in h\}} f(y)\,d\chi(y)\,d\chi(h).$$

By Fubini's theorem $f^{**}(x) = \int_{W_x} \mathrm{pr}_1^*(f)\,d\chi$. On the other hand, the projection onto the first factor $\mathrm{pr}_1\colon W_x \to X$ fibers $W_x \setminus \mathrm{pr}_1^{-1}(x)$ over $X \setminus \{x\}$ with fiber $\mathbb{RP}^{n-2}$, and obviously $\mathrm{pr}_1^*(f)$ is constant on the fibers of pr_1. Finally, $\mathrm{pr}_1^{-1}(x) \cong \mathbb{RP}^{n-1}$. Applying Fubini's theorem again (now for the mapping $\mathrm{pr}_1\colon W_x \to X$), we obtain

$$f^{**}(x) = \int_{W_x \setminus \mathrm{pr}_1^{-1}(x)} \mathrm{pr}_1^* f\,d\chi + \int_{\mathrm{pr}_1^{-1}(x)} \mathrm{pr}_1^* f\,d\chi$$
$$= \chi(\mathbb{RP}^{n-2}) \int_{X\setminus\{x\}} f(y)\,d\chi(y) + \chi(\mathbb{RP}^{n-1}) f(x)$$
$$= \chi(\mathbb{RP}^{n-2}) \int_X f\,d\chi + (\chi(\mathbb{R}P^{n-1}) - \chi(\mathbb{RP}^{n-1})) f(x).$$

For even m we have $\chi(\mathbb{RP}^m) = 1$, and for odd m we have $\chi(\mathbb{RP}^m) = 0$. This proves the theorem.

4. The above "integration" is very specific. It is not hard to see that all of the complicated sets of data defining cellularity or admissibility, etc., are necessary only in order to justify the possibility of integration, and the result of integrating is completely independent of these data. The very use of the term "integral" is related to the fact that this operation is well in agreement with the usual idea of integration; it can be used in correspondence with the usual intuition. For example, it is not hard to see that the integral over the Euler characteristic is a linear operation:

$$\int_X (\alpha f_1 + \beta f_2)\,d\chi = \alpha \int_X f_1\,d\chi + \beta \int_X f_2\,d\chi,$$

but one must remember that this formula makes sense only if there exists a regular cellular structure on X such that f_1 and f_2 (and thus $(\alpha f_1 + \beta f_2)$) are admissible. These circumstances force us to make a choice: either to restrict the (inevitably rather narrow) class of sets and functions under consideration, so that any finite set of functions would automatically be admissible, or to stipulate in each theorem the conditions under which its statement and proof make sense. We took the first path, considering only functions having the form $f = \sum n_i \mathbb{1}_{A_i}$, where $A_i \in \mathscr{P}(V)$. We see that for these functions the justification of the possibility of integrating with respect to the Euler characteristic (including a suitable compactification of V) is trivial, and

$$\int_V f\,d\chi = \sum n_i = \deg f$$

in the sense of Proposition-Definition 2.1, which is therefore proved.

We note that it is not hard to give another, direct proof of Proposition-Definition 2.1: to each set of hyperplanes $H_1, \dots, H_n \subset V$ one associates the natural decomposition of V into open cells of different dimension, and, for a function that is constant on each cell, we can define an integral, setting the measure of each cell Δ equal to $(-1)^{\dim \Delta}$. Now induction on the number of hyperplanes forming the decomposition gives the desired assertion. "Fubini's theorem" for linear mappings of linear spaces becomes completely trivial when we take convex chains as functions. Thus, we could in general not use the integral with respect to the Euler characteristic, but carry out all the proofs on an elementary level. However, we have used the technique we did for two reasons: first, the proofs (see the following section) become particularly transparent if we use the intuition of "integration", and second, the constructions and assertions of this paper are valid in a much more general situation than the one considered, namely, whenever the technique of integration over the Euler characteristic is applicable. For example, the "support function", obtained by the recipe of Definition 2.5, can be associated to a geometric figure belonging to a very wide class, and the original figure can be recovered from its support function. The proofs that we give automatically carry over to the general case, but with numerous stipulations needed every time the integral with respect to the Euler characteristic is used.

Thus, the authors' desire to simplifying the presentation has led to the fact that we speak practically only about convex chains, implying generalizations, however: to the case of Λ-chains and to the case of functions of a much more general form.

§4. Invariant constructions

This section is of a technical nature. Using the apparatus of integration over the Euler characteristic described in §3, we shall prove the basic facts of the theory of complex chains (except for Theorem 2.3).

1. As we noted in the preceding section, to prove Proposition-Definition 2.1 it suffices to note that the degree of a chain is its integral over the Euler characteristic. Similarly, in order to prove Proposition-Definition 2.2 it suffices to consider that (in the notation of §2.2)

$$f_*\alpha(y) = \int_{f^{-1}(y)} \alpha \, d\chi$$

for any $y \in W$, so that the equality $\deg f_*\alpha = \deg \alpha$ is a corollary of Fubini's theorem.

Proof of Proposition-Definition 2.3. On $Z(V)$ we define a convolution operation relative to integration over the Euler characteristic:

$$\alpha * \beta(x) = \int_V \alpha(z)\beta(x-z)\, d\chi(z),$$

and we show that this invariant definition coincides with the noninvariant one from §2.2, and verify that the properties of this multiplication hold. Let $\mu: V \times V \to V$ be the summation mapping, as in §2; for $\alpha, \beta \in Z(V)$ we set $\alpha \times \beta(x, y) = \alpha(x)\beta(y)$ for $\alpha \times \beta \in Z(V \times V)$. Obviously, $\alpha * \beta = \mu_*(\alpha \times \beta)$. If $A, B \in \mathscr{P}(V)$, then $\mathbb{1}_A \times \mathbb{1}_B = \mathbb{1}_{A \times B}$. We have

$$\mathbb{1}_A * \mathbb{1}_B = \begin{cases} 1 & \text{if } A \times B \cap \mu^{-1}(x) \neq \varnothing, \text{ i.e. } x \in A \oplus B, \\ 0 & \text{otherwise, i.e. } x \notin A \oplus B. \end{cases}$$

Fubini's theorem implies that f_* and deg are homomorphisms. We verify this for the direct image. Let $f\colon V \to W$ be linear; then there is a commutative diagram

$$\begin{array}{ccc} V \times V & \xrightarrow{(f,f)} & W \times W \\ {\scriptstyle \mu_V}\downarrow & & \downarrow{\scriptstyle \mu_W} \\ V & \xrightarrow{f} & W \end{array}$$

Let $\alpha, \beta \in Z(V)$; then $\alpha * \beta$ is obtained by integrating $\alpha \times \beta$ on the fibers of μ_V, and $f_*(\alpha * \beta)$ by integration of $\alpha * \beta$ on the fibers of f, so that as a result

$$f_*(\alpha * \beta)(x) = \int_{(f \circ \mu_V)^{-1}(x)} \alpha \times \beta \, d\chi, \qquad x \in W$$

(Fubini's theorem). Analogously

$$\begin{aligned} (f_*\alpha) * (f_*\beta)(x) &= \int_{\mu_W^{-1}(x)} f_*\alpha \times f_*\beta \, d\chi \\ &= \int_{\mu_W^{-1}(x)} (f, f)_*(\alpha \times \beta) d\chi = \int_{(\mu_W \circ (f,f))^{-1}(x)} \alpha \times \beta \, d\chi, \end{aligned}$$

as required. Proposition-Definition 2.3 is proved.

2. In §2.4 the support function of a convex chain was defined. The injectivity of the homomorphism

$$Z(V) \ni \alpha \mapsto \Phi_\alpha \colon V^* \to \mathbb{Z}[\mathbb{R}],$$

associating a support function to a chain, remains unproved. We give some arguments that explain the connection between these two objects.

The Radon transform gives the first method of proof. Each chain $\alpha \in Z(V)$ can be considered as a function $\alpha\colon \mathbb{RP}^n \to \mathbb{Z}$, $V \subset \mathbb{RP}^n$, $n = \dim V$, defining it as zero at the hyperplane at infinity of $\mathbb{RP}^n \backslash V$. The Radon transform for α is always admissible and its result is a function $\alpha^*\colon \mathbb{RP}^n \to \mathbb{Z}$, also admitting a Radon transform, and α can be recovered from α^*. We explain the connection between α^* and Φ_α. We consider a line $\langle l \rangle \subset V^*$, $l \in V^* \backslash \{0\}$. The mapping $l\colon V \to \mathbb{R}$ fibers V into the hyperplanes $\{l = \text{const}\}$. It is clear that $l_*\alpha(t) = \alpha^*(l^{-1}(t))$ for $t \in \mathbb{R}$. Thus, α^* defines a support function. On the other hand, according to Proposition 2.4, applied to $\Phi_\alpha(l)$ and $\Phi_\alpha(-l)$, the converse is also true: the Radon transform can be recovered from the support function, and thus the original chain can also be recovered.

The second method of proving the injectivity of the association $\alpha \mapsto \Phi_\alpha$ is to construct the inverse map explicitly. Let

$$I_{\geq}\colon \mathbb{Z}[\mathbb{R}] \times \mathbb{Z}[\mathbb{R}] \to \mathbb{Z}$$

be a bilinear map such that

$$I_{\geq}([a], [b]) = \begin{cases} 1 & \text{if } a \geq b, \\ 0 & \text{if } a < b. \end{cases}$$

Then

$$\alpha(x) = (-1)^{\dim V} \int_{V^*} I_{\geq}(\Phi_\alpha(l), [x(l)]) \, d\chi$$

for $x \in V$, $x(l) = l(x)$. The simple verification is left to the reader.

As a third method of proving the one-to-one correspondence between chains and their support functions we give an explicit geometric construction for recovering a chain from a support function. First we consider an example.

Let $A \in \mathscr{P}(V)$ be a convex polytope, $\langle A \rangle = V$, and $F \subset A$ an arbitrary face. With this face one associates a closed convex cone $C(F)$, containing the affine space $\langle F \rangle$, defined by the condition that in a small neighborhood of any point $x \in \operatorname{Int} F$, say $U \ni x$, we have $C(F) \cap U = A \cap U$. From "Euler's identity" it is easy to deduce that, for $x \in A$,

$$\sum_{F \in \Gamma(A)} (-1)^{\dim F} \mathbb{1}_{C(F)}(x) = 1.$$

In fact, the assertion

$$\sum_{F \in \Gamma(A)} (-1)^{\dim F} \mathbb{1}_{C(F)} = \mathbb{1}_A$$

is true. We shall prove a substantially more general assertion. For this we consider as admissible functions the characteristic functions (unbounded) of closed convex cones in V and linear combinations of them. Let $C \subset V$ be a cone. We use the following notational and conceptual conventions: $\operatorname{vs}(C) \subset C$ is the vertex subspace of the cone, and $\widehat{\operatorname{vs}}(C)$ is the translation of $\operatorname{vs}(C)$ by any vector belonging to $(-\operatorname{vs}(C))$, so that C is invariant relative to translations by vectors from $\widehat{\operatorname{vs}}(C)$. Also $L^{\perp} \subset V^*$ is the annihilator of the linear subspace $L \subset V$, $\widehat{C} \subset V$ is the translation of the cone C by any vector belonging to $-\operatorname{vs}(C)$, and $\widehat{C}^* = \{l \in V^* \mid (l \cdot v) \leq 0$ for all $v \in \widehat{C}\}$ is the dual cone to $\widehat{C}$; obviously $\widehat{C}^* \subset \widehat{\operatorname{vs}}(C)^{\perp}$. Finally, $\operatorname{Int} C$ is the relative interior of C (in $\langle C \rangle$).

We remark that all of the technique of integration over the Euler characteristic is applicable without changes to the given wider class of functions, including the characteristic functions of cones. We first compute the Radon transform of the function $\mathbb{1}_C$.

Proposition 1. *Let $H \subset V$ be an affine hyperplane. Then*

$$\int_H \mathbb{1}_C \, d\chi = \chi(C \cap H) = (-1)^{\dim \operatorname{vs}(C)},$$

if H can be defined by the equation $h = \text{const}$, $h \in \operatorname{Int} \widetilde{C}^*$, and $H \cap C \neq \varnothing$. Otherwise $\chi(C \cap H) = 0$.

The proof is obvious.

For $l \in \widetilde{C}^*$ the number $\max_{v \in C} l(v)$ is defined, and it obviously equals the value of l on any vector belonging to the vertex space $\operatorname{vs}(C)$. To the cone $C \subset V$ we associate the pair $(\widetilde{C}^*, f_C)$, where $f_C \colon \widetilde{C}^* \ni l \mapsto l(\operatorname{vs}(C))$. This pair plays the role of the support function of the cone C. By the above, the Radon transform of the characteristic function of the original cone is $\mathbb{1}_C^* = (-1)^{\dim \operatorname{vs}(C)}$ if the hyperplane is defined by the equation $h = \gamma$, $h \in \operatorname{Int} \widetilde{C}^*$, $\gamma \leq f_C(h)$, and is equal to 0 otherwise.

Thus, we have established a one-to-one correspondence between cones in V and linear functions on cones in V^*. The procedure for recovering C from the pair $(\widetilde{C}^*, f_C)$ goes as follows: we take the cone $(\widetilde{C}^*)^* \subset V$ and translate it by a vector $x \in V$ such that $x|_{\widetilde{C}^*} = f_C$ under the identification of V with V^{**}. In other words, the vertex space $\operatorname{vs}(C)$ is a solution of the system of equations $\{l(x) = f_C(l) | l \in \widetilde{C}^*\}$.

Now let $f \colon V^* \to \mathbb{Z}[\mathbb{R}]$ be a continuous piecewise-linear (Definition 2.5) function. There exists a set of closed convex cones F_i, $i = 1, \ldots, N$, that satisfy the following conditions:

(i) The interiors $\operatorname{Int} F_i$ of the different cones do not intersect.

(ii) $V^* = \bigcup_{i=1}^{N} \operatorname{Int} F_i$ and $0 \in \operatorname{vs}(F_i)$, $1 \leq i \leq N$ (such sets of cones are called *decompositions* of the dual space V^*).

(iii) $f|_{F_i}$ is linear for each i; that is, there exists a zero-dimensional chain $x_i \in \mathbb{Z}[V]$ such that $f|_{F_i} = x_i|_{F_i}$, i.e., if $x_i = \sum t_{ik}[x_{ik}]$, $x_{ik} \in V = V^{**}$, then for a covector $l \in F_i$ we have $f(l) = \sum t_{ik}[l(x_{ik})] \in \mathbb{Z}[\mathbb{R}]$.

If conditions (i)–(iii) hold, then we say that the function f is *linear* in the decomposition $\{F_i | i = 1, \dots, N\}$ of V^*.

Let $C_i \subset V$ be the closed cone dual to F_i.

According to Proposition 2.5, which has already been proved, f is the support function of some convex chain $\alpha \in Z(V)$.

Proposition 2.

$$(-1)^{\dim V}\alpha = \sum_{i=1}^{N} x_i * \mathbb{1}_{C_i} \cdot (-1)^{\dim F_i}.$$

The proof is almost trivial: from the support function f we recover the Radon transform of the chain α and compute the Radon transform of the chain on the right-hand side, according to Proposition 1. Our assertion will be valid if they are equal. But in turn it suffices to prove that they are equal on each line of the form $\{l = t \mid t \in \mathbb{R}\}$, $l \in V^* \backslash \{0\}$. We remark that each such line intersects exactly three of the open cones $\operatorname{Int} F_i$: $\{0\}$ and two of positive dimension (the openness is taken in the affine hull). Furthermore, it suffices to prove the proposition for a single-valued piecewise-linear function $f \colon V^* \to \mathbb{R}$. We leave the details to the reader: we have already essentially reduced it to the one-dimensional case, and there it is obvious (see §2.3).

To conclude our discussion of support functions we make a remark, a technical adjunction to the preceding construction. The characteristic functions of cones are suitable in order to describe the (Minkowski) multiplicative structure of the algebra of convex chains. They give a kind of "partition of unity". Let $C_1, C_2 \subset V$ be closed convex cones. In the notation adopted above we have

Proposition 3. (i) $\mathbb{1}_{C_1} * \mathbb{1}_{C_2} = 0$ *if and only if* $\widetilde{C}_1 \cap (-\widetilde{C}_2) \neq \widehat{\mathrm{vs}}(C_1) \cap \widehat{\mathrm{vs}}(C_2)$ (*i.e., the left-hand side strictly contains the right-hand side*). *The latter is equivalent to the fact that* $\operatorname{Int} \widetilde{C}_1^* \cap \operatorname{Int} \widetilde{C}_2^* = \varnothing$.

(ii) *If* $\widetilde{C}_1 \cap (-\widetilde{C}_2) = \widehat{\mathrm{vs}}(C_1) \cap \widehat{\mathrm{vs}}(C_2)$, *then*

$$\mathbb{1}_{C_1} * \mathbb{1}_{C_2} = (-1)^{\dim \widehat{\mathrm{vs}}(C_1) \cap \widehat{\mathrm{vs}}(C_2)} \mathbb{1}_{C_1 \oplus C_2},$$

where $\oplus$ *as usual denotes Minkowski sum.*

Proof. These are simple, geometrically almost obvious facts. The cone C_i is the direct product of $\widehat{\mathrm{vs}}(C_i)$ by a cone whose vertex is a point; this reduces everything to cones of the latter type. If $\widetilde{C}_1 \cap (-\widetilde{C}_2) \neq \{0\}$ (we assume that the vertices of C_i are points), then $C_1 \cap (x - C_2)$ for any $x \in V$ is either $\varnothing$ or a closed convex unbounded set; its Euler characteristic is equal to zero. But if $\widetilde{C}_1 \cap (-\widetilde{C}_2) = \{0\}$, then $C_1 \cap (x - C_2)$ for any $x \in V$ is either empty ($x \notin C_1 \oplus C_2$) or a closed convex bounded polytope. The rest is obvious.

Corollary 1. *Let* $\alpha = \sum_{i=1}^{N} (-1)^{\dim L_i} n_i \mathbb{1}_{C_i}$ *and* $\beta = \sum_{i=1}^{N} (-1)^{\dim L_i} m_i \mathbb{1}_{D_i}$ *be functions, where* $L_i = \widehat{\mathrm{vs}}(C_i) = \widehat{\mathrm{vs}}(D_i)$, $\widetilde{C}_i = \widetilde{D}_i$ *for* $i = 1, \dots, N$, *and the cones* $\widetilde{C}_i^*$, $1 \leq i \leq N$, *give a decomposition of the space* V^*. *Then*

$$\alpha * \beta = \sum_{i=1}^{N} (-1)^{\dim L_i} n_i m_i \mathbb{1}_{C_i \oplus D_i}.$$

3. We give an invariant definition of the three operations on convex chains: "star", "trace", and "shadow".

"Star". We let $B_\varepsilon(x)$ be an open ball of radius ε with center at a point x relative to some Euclidean metric on V. Then

$$*\alpha(x) = \lim_{\varepsilon\to 0}\int_{B_\varepsilon(x)} \alpha\, d\chi.$$

The verification of the identity $*\mathbb{1}_A = (-1)^{\dim A}\mathbb{1}_{\operatorname{Int} A}$ for $A \in \mathscr{P}(V)$ is trivial. This proves Proposition-Definition 2.6.

"Trace". For $\xi \in V^*\backslash\{0\}$ we set

$$U_{\xi,\varepsilon} = B_\varepsilon(x) \cap \{y \mid \xi(y) \geq \xi(x)\},$$

a half-ball. Then

$$*\operatorname{Tr}_\xi \alpha(x) = \lim_{\varepsilon\to 0}\int_{U_{\xi,\varepsilon}(x)} \alpha\, d\chi.$$

This proves Proposition-Definition 2.7.

"Shadow". For a vector $v \in V\backslash\{0\}$ we set

$$L_{v,\varepsilon}(x) = \{tv + x \mid 0 \leq t < \varepsilon\},$$

a half-open interval. Then

$$T_v\alpha(x) = \lim_{\varepsilon\to 0}\int_{L_{v,\varepsilon}(x)} \alpha\, d\chi.$$

It is not hard to verify that the shadow of a polytope is homeomorphic to a closed ball (the true shadow if the "illumination" is parallel rays), and therefore its Euler characteristic is equal to 1. This completely proves Proposition-Definition 2.8.

We consider some applications of these constructions. We shall prove Theorem 2.2 on Minkowski inversion: for $A \in \mathscr{P}(V)$

$$*\mathbb{1}_{(-A)} * \mathbb{1}_A = 1.$$

In fact,

$$\begin{aligned} 1_A * 1_{\operatorname{Int}(-A)}(x) &= \int_{v+w=x} \mathbb{1}_A(v)\mathbb{1}_{\operatorname{Int}(-A)}(w)\, d\chi \\ &= \int_V 1_{(-A+x)} 1_{\operatorname{Int}(-A)}\, d\chi = \chi((-A+x)\cap \operatorname{Int}(-A)). \end{aligned}$$

If $x = 0$, we obtain $(-1)^{\dim A}$. If $x \neq 0$, we obtain 0: if $(-A+x)\cap\operatorname{Int}(-A) \neq \varnothing$, then

$$(-A+x)\cap\operatorname{Int}(-A) = [\operatorname{Int}(-A+x)\cap\operatorname{Int}(-A)] \cup [\partial(-A+x)\cap\operatorname{Int}(-A)].$$

It is easy to see that the Euler characteristic of the set in the first set of the square brackets is $(-1)^{\dim A}$, and that of the second is $(-1)^{\dim A+1}$, as required.

We return to the operation of taking the trace. We remark that the trace is connected with the support function: for a chain $\alpha \in Z(V)$ and a general covector $\xi \in V^*\backslash\{0\}$ we have $\Phi_\alpha(\xi) = \xi_*\operatorname{Tr}_\xi\alpha$. An interesting application of the operation Tr arises in the following problem.

Definition 1. We say that two polytopes $A, B \in \mathscr{P}(V)$ are *equivalent* if $\mathbb{1}_A - \mathbb{1}_B \in J_1$ (see §2.7).

Informally speaking, this means that A and B can be decomposed into pieces so that the pieces of one are translations of the pieces of the other. Pieces of arbitrary dimension are permitted here.

It turns out that this concept of equivalence has a trivial meaning.

Proposition 4. *The condition* $\mathbb{1}_A - \mathbb{1}_B \in J_1$ *for* $A, B \in \mathscr{P}(V)$ *holds if and only if* A *and* B *differ by a translation*: $A = v + B$, $v \in V$.

Proof. For any covector $l \in V^*\backslash\{0\}$, we obviously have $\mathrm{Tr}_l(J_1) \subset J_1$, which allows us to apply induction on the dimension, taking account of the fact that the trace of a polytope is a polytope of smaller dimension. We leave the details to the reader.

The concept of the shadow also has important applications: using it we prove the main theorem of this paper (Theorem 2.3 on ideals in the algebra of convex chains): see the following section. Other than that theorem, we have now proved all the assertions stated in §2.

We stress that all the constructions and arguments of this section carry over without any changes to the case of Λ-chains for an arbitrary admissible pair (V, Λ).

§5. Proof of the theorem on ideals

1. We use all the notations of §§2.7 and 4.

Definition 1. The vectors $v_i \in V_i$, $i \in I$, are said to be *positively dependent* if there exist $\lambda_i \geq 0$, $i \in I$, not all equal to zero, such that $\sum_{i\in I} \lambda_i v_i = 0$.

The theorem on ideals in the algebra of convex chains is deduced from the following important assertion.

Theorem 1 (theorem on shadows). *Let* $v_i \neq 0$, $i \in I$, *be positively dependent. Then*

$$\mathop{\ast}_{i\in I}(T_{v_i} - \mathrm{id})\alpha_i = 0$$

for any set of chains $\alpha_i \in Z(V)$, $i \in I$.

Proof of the theorem on shadows. As usual, let V^I denote the linear space of V-valued functions on I. We write

$$\mathop{\ast}_{i\in I}(T_{v_i}\alpha_i - \alpha_i)(x) = \int_{V_x^I} \prod_{i\in I}(T_{v_i}\alpha_i - \alpha_i)\, d\chi,$$

where $V_x^I \subset V^I$ is $\{(w_i | i \in I) | \sum_{i\in I} w_i = x\}$. Let $L_i = \langle v_i \rangle \subset V$, let $H_i \subset V$ be an arbitrarily chosen complementary subspace, and $V = L_i \oplus H_i$. Let $\pi_i \colon V \to H_i$ be the projection along L_i, and $\nu_i \colon V \to L_i$ the projection along H_i, so that $\pi_i + \nu_i = \mathrm{id}_V$. Also let

$$\gamma = \prod_{i\in I}(T_{v_i}\alpha_i - \alpha_i) \in Z(V^I)$$

be the chain that must be integrated over V_x^I.

Consider the mapping

$$\pi = \prod_{i\in I} \pi_i \colon V_x^I \to \prod_{i\in I} H_i.$$

By Fubini's theorem we can first integrate γ over the fibers of π, and then integrate the resulting function over $\prod_{i\in I} H_i$. Hence, it suffices to prove that

$$\int_{\pi^{-1}(h)} \gamma\, d\chi = 0, \quad \text{where } h = (h_i | i \in I) \in \prod_{i\in I} H_i.$$

Now the mapping

$$\nu = \prod_{i\in I} \nu_i \colon V^I \to \prod_{i\in I} L_i$$

embeds $\pi^{-1}(h)$ as the subspace defined by $\sum_{i\in I} u_i = x - \sum_{i\in I} h_i$. Here ν is in fact translation by h, so that

$$\int_{\pi^{-1}(h)} \gamma\, d\chi = \int_{\prod_{i\in I} L_i} \tau_h \gamma\, d\chi,$$

and since translation obviously commutes with the shadow operation, we find that it suffices to prove that

$$\int_{\prod_{i\in I} L_i} \prod_{i\in I} (T_{v_i} - \mathrm{id})\alpha_i\, d\chi = 0,$$

where $\alpha_i \in Z(L_i)$ are chains on the lines L_i. By multilinearity with respect to the α_i it suffices to assume that the α_i are intervals; then $(T_{v_i} - \mathrm{id})\alpha_i$ is simply an interval without the end where the vector v_i is directed.

Lemma 1 (on positive dependence). *Let $w_i \in V$, $i \in I$, be nonzero positively dependent vectors. Then*

$$\mathop{\ast}_{i\in I} \mathbb{1}_{[0, w_i)} = 0.$$

The lemma obviously completes the proof of the theorem on shadows.

Proof of the lemma. We have

$$\mathop{\ast}_{i\in I} \mathbb{1}_{[0, w_i)}(x) = \chi\left(\left\{ (t_i | i\in I) \middle| 0 \le t_i < 1,\ \sum_{i\in I} t_i w_i = x \right\}\right).$$

We replace the set $(w_i | i \in I)$ by a subset of it, still all positively dependent, and such that any (proper) subset of it is linearly independent. Then the set

$$T_x = \left\{ (t_i | i\in I) \middle| 0 \le t_i < 1,\ \sum_{i\in I} t_i w_i = x \right\}$$

is at most one-dimensional. We shall show that if it is not empty, then it is a half-open interval. Let $\sum_{i\in I} \lambda_i w_i = 0$, $\lambda_i \ge 0$, $(\lambda) \ne 0$; then

$$T_x = \{ (\bar{t}_i + \varepsilon\lambda_i | i \in I) | 0 \le \bar{t}_i + \varepsilon\lambda_i < 1 \},$$

where $(\bar{t}_i) \in T_x$ is an arbitrary point. Hence T_x is the intersection of a finite number of half-open intervals, directed to the same side, and thus T_x is itself a half-open interval. This proves the lemma.

Finally, we come to establishing the main theorem.

Lemma 2. *Let $\widetilde{A}_i \in \mathscr{P}(V)$, $i \in I$, where the linear "carrier" subspaces $\langle \widetilde{A}_i \rangle$ of their affine hulls $\langle A_i \rangle$ do not form a direct sum. Then the chain $\ast_{i\in I}(\mathbb{1}_{A_i} - \mathbb{1})$ can be decomposed into an integer linear combination of chains of the form $\ast_{i\in I}(\mathbb{1}_{B_i} - \mathbb{1})$, where $B_i \in \mathscr{P}(V)$, and*

$$\sum_{i\in I} \dim\langle \widetilde{B}_i \rangle < \sum_{i\in I} \dim\langle \widetilde{A}_i \rangle.$$

Theorem 2.3 is derived from the lemma in the obvious way. We prove the lemma. For this we note that if the $\langle \widetilde{A}_i \rangle$ do not form a direct sum, then we can always find a set $v_i \in \langle \widetilde{A}_i \rangle \setminus \{0\}$ of positively dependent vectors. Now, by the theorem on shadows,

$$\mathop{\ast}_{i\in I} (\mathbb{1}_{A_i} - T_{v_i}(\mathbb{1}_{A_i})) = 0$$

or

$$\mathop{\ast}_{i\in I} [(\mathbb{1}_{A_i} - \mathbb{1}) - (T_{v_i}(\mathbb{1}_{A_i}) - \mathbb{1})] = 0.$$

Removing the square brackets and considering that the shadow of a polytope relative to a vector of its linear "carrier" subspace is a linear combination of polytopes of smaller dimension, we obtain the lemma.

Thus the main Theorem 2.3 is completely proved. The arguments carry over without changes to the case of Λ-chains.

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